Appendix. Omitted proofs

Proof of Lemma 1. The attack takes place during the time interval $J = [M, M + \alpha]$. Since S satisfies the unit speed condition (1), we have that $\lambda(S(J)) \leq |J| = \alpha$, where $|J|$ is the length of J. By the definition of the uniform attack strategy, the probability that the attack takes place in $S(J)$, and is thus intercepted, does not exceed $\lambda(S(J))/\mu$, giving the claimed bound. □

Proof of Lemma 2. First observe that the new metric d' will still have speed one. If S is a patrol on Q, then it satisfies (1) so

$$d'(S(t), S(t')) \leq d(S(t), S(t')) \leq |t - t'|,$$

which means that S is still a patrol on Q', d'. On the other hand, attacks on Q', d' are the same as the attacks on Q, d. So the Patroller might have additional strategies whereas the Attacker has no new strategies. Thus the new game can only be the same or better for the Patroller, giving the main inequality. If the length of an arc is decreased then the new metric satisfies the assumption $0 \leq d'(x, y) \leq d(x, y)$. Finally suppose x and y are identified, so that Q', d' has the quotient topology. For any points z and w in $Q - \{x, y\}$ we have

$$d'(z, w) = \min\{d(z, w), d(z, x) + d(x, w), d(z, y) + d(y, w)\} \leq d(z, w),$$

so the result follows from the first part of the proof. □

Proof of Lemma 3 This proof mimics the usual proof of Euler's Theorem. We first construct a circuit C satisfying condition (2), which we call a $*$-circuit, using the following rules:

1. Start at any node x and leave by any passage P (we let P' be the paired passage of P).
2. Always leave a node by an untraversed passage not paired with the arriving passage.
3. If, after arriving at a node, there are three untraversed passages with exactly two of them paired, leave by one of this pair.
4. If, after arriving at node x, there are two untraversed passages, leave by passage P', if it is untraversed.
5. If there are no remaining untraversed passages after arriving at a node, stop.

To simply obtain a circuit (not necessarily satisfying (2)) starting and ending at x, we would follow the usual method of simply leaving a node by any *untraversed passage*, a simpler form of Rule 2. The existence of an untraversed passage (at any node other than the starting node x) follows from the fact the after arriving at a node an odd number of passages will have been traversed, so an odd number (hence not 0) are untraversed. We show that the full form of Rule 2 along with the other rules ensure that we can always leave a node in a way that satisfies (2) whether the node is the initial node x or another node y.

We first check that after arriving at a node y other than the starting node x, there cannot be only one remaining untraversed passage which is paired with the arriving passage. Since every node has even degree and there are no degree two nodes, the node y must have been previously arrived at. After this previous arrival at y, there must have been three untraversed passages with exactly two of them paired. But Rule 3 ensures the circuit left by one of those two passages, so after arriving by the other one on the final visit, the last untraversed passage must have a different label.

To check that the final arriving passage at the initial node x is not P', note that if P' had not been traversed before the penultimate visit to x, Rule 4 ensures that it will be traversed on that visit, and it will not be the final arriving passage.

If C is a tour (contains all the arcs), we are done. Otherwise, since Q is connected, there is a node z with some passages in C and some not in C (see Figure 1). Suppose that C leaves z beginning via passage a and ends at z via passage b. We create a new $*$-circuit starting at z, called C', using the same rules and using only passages not in C. Suppose C' begins with a passage called d (which we can choose) and ends with a passage called e (which we cannot control). The combined circuit CC' which starts at z and traverses C and then C' will satisfy (2) except possibly for the transitions b, d and e, a between the two circuits, so we need $d \neq b'$ and $e \neq a'$ (this means d is not paired with b and e is not paired with a). The arc d is chosen as follows.

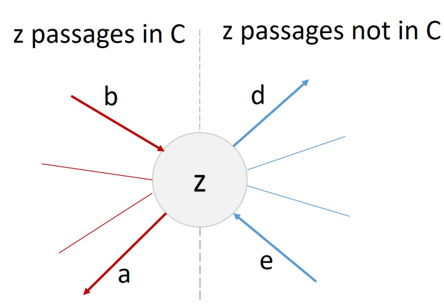

Figure 1 How to join two $*$-circuits at node z.

1. If a' is not in C, take $d = a'$. This ensures that $d = a' \neq b'$ since $a \neq b$. Also $e \neq d = a'$.
2. If a' is in C, take $d \neq b'$. We know that also $e \neq a'$ because a' is in C.

If the circuit CC' is not a tour, we iteratively continue to add new circuits until we end up with a tour, noting that the process is guaranteed to end since every new circuit contains at least one new arc and there are a finite number of arcs. $\square$

Proof of Lemma 8. We fix a best response S to the E-attack strategy, and show that the probability of interception is no more than v^*. To do this, we will define a new network Q' of total length $\mu + \lambda(E)$ and a patrol S' of Q', and show that the probability S intercepts the E-attack strategy on Q is equal to the probability that S' intercepts the uniform attack strategy (starting at time $t = M$) on Q'. The latter probability is at most v^*, by Lemma 1, so this will complete the proof.

The network Q' is derived from Q by replacing each component E_i of E with a loop L_i of length $2e_i$, where $e_i = \lambda(E_i)$. This is possible by the Leaf Condition, and clearly $\lambda(Q') = \mu + \lambda(E)$. Note that the probability the attack takes place on E_i under the E-attack strategy is equal to the probability the attack takes place on L_i under the uniform attack strategy. Since S is a best response, we can assume that whenever it enters some component E_i, it proceeds directly to the leaf node of E_i, arriving at some time t_1, then leaves at a later time t_2, and returns directly to E^c. We will show later that we can assume $t_1 = t_2$, so that S performs tours of the components of E. We define the Patroller strategy S' on Q' by setting it equal to S when S is in E^c, and replacing any tour that S performs of a component E_i of E in Q with a tour of the loop L_i in Q'.

Let p_0 be the probability the attack on Q is intercepted by S, conditional on it taking place on E^c and let q_0 be the corresponding conditional probability for S' and Q'. Clearly, $p_0 = q_0$. For every component E_i of E, we also define p_i to be the probability that the attack on Q is intercepted by S, conditional on it taking place at the leaf node x_i in the closure of E_i. Similarly, we define q_i to be the probability that the attack on Q' is intercepted by S' conditional on it taking place in L_i. It is sufficient to show that $p_i = q_i$ for each i.

The timing of the attacks on Q is shown in Figure 2. The first attack at x_i finishes at time $M - e_i + \alpha$ and the last attack starts at time $M + e_i$. By the Leaf Condition, $M + e_i \leq M - e_i + \alpha$, so $p_i = 1$ if and only if the patrol visits x_i in the time interval $[M + e_i, M - e_i + \alpha]$. Recall that t_1 and t_2 are the respective times that S arrives at and leaves node x_i.

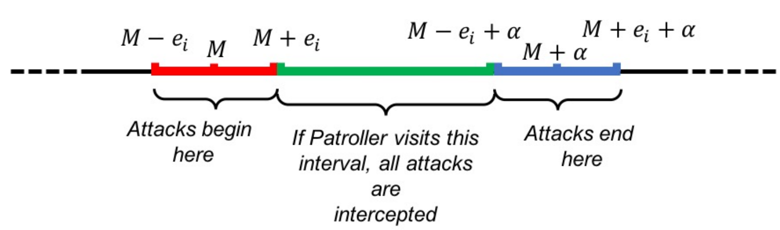

Figure 2 Timing of the attacks on Q.

First suppose $p_i = 1$. In this case, there is some time $t_0 \in [M + e_i, M - e_i + \alpha]$ when the Patroller is at x_1, so we may as well assume that $t_1 = t_2 = t_0$ (otherwise we can replace S with a patrol that dominates it). So that S performs a tour of E_i during the time interval $[t_0 - e_i, t_0 + e_i] \subset [M, M + \alpha]$. This means that S' also performs a tour of L_i during this time interval, and therefore $q_i = 1$.

Now suppose that $p_i < 1$. In this case, we must have either $t_1 > M - e_i + \alpha$ or $t_2 < M + e_i$. In the former case, the probability of an attack starting at x_i after time t_1 is zero, so we can assume that $t_2 = t_1$. In other words, S performs a tour of E_i in the time interval $[t_1 - e_i, t_1 + e_i]$, and S' performs a tour of L_i in the same time interval. If $t_1 \geq M + e_i + \alpha$ then $p_i = q_i = 0$. If $t_1 \leq M + e_i + \alpha$, then S intercepts the attack if it starts at x_i in the time interval $[t_1 - \alpha, M + e_i]$, so $p_i = (M + e_i + \alpha - t_1)/(2e_i)$. Furthermore, S' intercepts the attack if it takes place in $S([t_1 - e_i, M + \alpha]) \subset L_i$, so $q_i = p_i$.

A similar argument holds for the case of $t_2 < M + e_i$ and this completes the proof. □

Proof of Theorem 8. It is enough to show the statement is true for the case that $x > n$ and $2n \leq \alpha \leq 2x$. Assume the Attacker uses the symmetric-skewed attack strategy. Let p_i be the probability that the attack is intercepted, conditional on it taking place at node a_i for $i = 1, ..., n$. Let q_j be the probability that the attack is intercepted, conditional on it taking place at E^c at time $\alpha/2 + 2j$, for $j = 0, ..., n-1$. Let p_b be the probability that the attack is intercepted, conditional on it taking place at node b.

It is easy to see that if $p_b = 1$, then the patrol must either stay at node b until time $\alpha + 2(n-1)$ or arrive at node b at time α or earlier (then stay there). In both cases, we have $q_j = 0$ for all j since the patrol cannot be in $\lambda(E^c)$ during the time interval $[\alpha/2, 3\alpha/2 + 2(n-1)]$. Similarly, we have $p_i = 0$ for all i. Thus, the probability of interception is $\alpha/(\mu + \lambda(E))$.

Next, suppose $p_b < 1$ and the patrol stays at node b until some time $t < \alpha + 2(n-1)$. We will split this case into two subcases: $2n \leq \alpha \leq \mu$ and $\mu \leq \alpha \leq 2x$.

Considering the first subcase, $2n \leq \alpha \leq \mu$, we assume the patrol arrives node v (the right side of E^c) at time $r = t + \alpha/2$, and q_0 will be bounded by the function $\gamma(r)$ as below.

If $\lambda(E^c) \geq \alpha$,

$$\gamma(r) = \begin{cases} \frac{r + \lambda(E^c) - \alpha/2}{\lambda(E^c)} & \text{if } 0 \leq r \leq 3\alpha/2 - \lambda(E^c), \\ \frac{\alpha}{\lambda(E^c)} & \text{if } 3\alpha/2 - \lambda(E^c) \leq r \leq \alpha/2, \\ \frac{3\alpha/2 - r}{\lambda(E^c)} & \text{if } \alpha/2 \leq r \leq 3\alpha/2, \\ 0 & \text{if } 3\alpha/2 \leq r. \end{cases}$$

If $\lambda(E^c) \leq \alpha$,

$$\gamma(r) = \begin{cases} \frac{r + \lambda(E^c) - \alpha/2}{\lambda(E^c)} & \text{if } \alpha/2 - \lambda(E^c) \leq r \leq \alpha/2, \\ 1 & \text{if } \alpha/2 \leq r \leq 3\alpha/2 - \lambda(E^c), \\ \frac{3\alpha/2 - r}{\lambda(E^c)} & \text{if } 3\alpha/2 - \lambda(E^c) \leq r \leq 3\alpha/2, \\ 0 & \text{if } 3\alpha/2 \leq r. \end{cases}$$

Thus, for $j = 1, ..., n-1$, the probability q_j is bounded by $\gamma(r - 2j)$.

Without loss of generality, we assume the patrol arrives at node a_1 at time $s = t + x + 1$ then moves within all a_i $(i = 1, ..., n)$ thereafter. If the patrol visits every other node a_i $(i \neq 1)$ before returns to a_1, it takes time $2n \leq \alpha$. So, all attacks at node a_1 happening from time $s - \alpha$ will be intercepted and p_1 is bounded above by

$$\delta(s) = \begin{cases} 1 & \text{if } 0 \leq s \leq \alpha + n - 1, \\ \frac{(2\alpha+n-1)-s}{\alpha} & \text{if } \alpha + n - 1 \leq s \leq 2\alpha + n - 1. \end{cases}$$

If the patrol moves directly from a_1 to a_2, then p_2 is bounded above by $\delta(s+2)$. Upper bounds for the other p_i can be calculated in the same way.

A patrol that leaves node b at time t, arrives at E^c at time $t + \alpha/2$, crosses E^c to reach node a_1 at time $t + x + 1$, then moves within a_i thereafter has interception probability $p(t)$ given by

$$\begin{aligned} p(t) =& \frac{\alpha}{\mu + \lambda(E)} f(t) + \frac{\lambda(E^c)}{n(\mu + \lambda(E))} \sum_{j=0}^{n-1} \gamma(t + \frac{\alpha}{2} - 2j) \\ &+ \frac{\alpha}{n(\mu + \lambda(E))} \sum_{i=1}^{n} \delta(t + x + 1 + 2(i-1)) \\ &= \frac{\alpha}{\mu + \lambda(E)}. \end{aligned}$$

We now consider the second subcase, $\mu \leq \alpha \leq 2x$. In this case, $\lambda(E) = \mu$ so that E^c is empty and there are no middle attacks. We assume the patrol leaves node b at time t, visits all nodes a_i at time $t + x + 1 + 2(i-1)$ then returns to x at time $t + 2\mu$.

Let s be the time the patrol arrives at node a_i. Then p_i is bounded by the function $g(s)$ given below.

$$g(s) = \begin{cases} \frac{s-(\alpha-x-1)}{2(x+n)-\alpha} & \text{if } \alpha - x - 1 \leq s \leq x + 2(n-1) + 1, \\ 1 & \text{if } x + 2(n-1) + 1 \leq s \leq 2\alpha - x - 1, \\ \frac{\alpha+x+2(n-1)+1-s}{2(x+n)-\alpha} & \text{if } 2\alpha - x - 1 \leq s \leq \alpha + x + 2(n-1) + 1. \end{cases}$$

Since the patrol stays leaves node b at time t and returns at $t + 2\mu$, the upper bound $h(t)$ of p_b is

$$h(t) = f(t) + 1 - f(t + 2\mu - \alpha).$$

So, the interception probability $p(t)$ of the patrol is

$$p(t) = \frac{\alpha}{2\mu} h(t) + \frac{2\mu - \alpha}{n\mu} \sum_{i=1}^{n} g(t + x + 1 + 2(i-1)) = \frac{\alpha}{2\mu} = \frac{\alpha}{\mu + \lambda(E)}.$$

□