## PHD

## Critical Exponents in Sandpiles

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# Critical Exponents in Sandpiles 

submitted by

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September 2021

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#### Abstract

This thesis has both numerical and theoretical aspects in studying Abelian sandpiles. We start with a numerical approach, using exact sampling. Our methods use a combination of Wilson's algorithm to generate uniformly distributed spanning trees, and Majumdar and Dhar's bijection. We study the probability of topplings of individual vertices in avalanches starting at the centre of large cubic lattices in 2,3 and 5 dimensions. Based on these, we estimate the values of the toppling probability exponent in the infinite volume limit in $d=2,3$, and find good agreement with theoretical results on the mean-field value of the exponent in $d \geq 5$. We also study the distribution of the number of waves in 2 dimension. Our simulation method, combined with a variance reduction concept, is well suited for analyzing various problems, even in very high dimensions. We demonstrate this by estimating the single-site height probability distribution in 32 dimension, and compare it to the asymptotic behaviour as $d \rightarrow \infty$. Then we prove an asymptotic formula for the single-site height distribution with error estimates in terms of Poisson(1) probabilities.

We continue with studying the following problem arising in the simulation context. We consider a simple random walk on $\mathbb{Z}^{d}$ started at the origin and stopped on its first exit time from $(-L, L)^{d} \cap \mathbb{Z}^{d}$. Write $L$ in the form $L=m N$ with $m=m(N)$ and $N \rightarrow \infty$ so that $L^{2} \sim A N^{d}$ for some real positive constant $A$. Our main result is that for $d \geq 3$, the projection of the stopped trajectory to the $N$-torus locally converges, away from the origin, to an interlacement process at level $A d \sigma_{1}$, where $\sigma_{1}$ is the exit time of a Brownian motion from the unit cube $(-1,1)^{d}$ that is independent of the interlacement process. The above problem is a variation on results of Windisch (2008) and Sznitman (2009).


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To Mum and Dad
for always loving and supporting me.
They told me: "Be healthy and happy first, then make every day count!"

## Publications

The work done in this thesis is based on two peer-reviewed publications and one manuscript submitted for publication as follows:

- Toppling and height probabilities in sandpiles

Antal A. Járai, Minwei Sun
Journal of Statistical Mechanics: Theory and Experiment, Vol.2019, 113204.

- Asymptotic height distribution in high-dimensional sandpiles

Antal A. Járai, Minwei Sun
Journal of Theoretical Probability, Vol.34(1), p.349-362.

- Interlacement limit of a stopped random walk trace on a torus

Antal A. Járai, Minwei Sun, submitted.
Some adaptations have been made to them so that the thesis reads better when viewed as a whole.

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## Notation

We use $c, c^{\prime}$ and $C$ to denote positive constants, depending only on dimension $d$. It may change from line to line, but we keep the same notation for simplicity. If a constant is to depend on some other parameters, this will be made explicit.

If $f(x), g(x)$ are functions, we write $f \sim g$ if they are asymptotic, i.e.

$$
\lim _{x \rightarrow \infty} \frac{f(x)}{g(x)}=1
$$

We write $f \approx g$ if they are logarithmically equivalent, i.e.

$$
\lim _{x \rightarrow \infty} \frac{\log f(x)}{\log g(x)}=1
$$

We write $f(x)=O(g(x))$ if $f(x) \leq c g(x)$, as $x \rightarrow \infty$, for some constant $c$, where the constant $c$ depends only on dimension $d$ as well. We write $f(x)=o(g(x))$ if $f(x) / g(x) \rightarrow 0$, as $x \rightarrow \infty$. Similarly, the rate of convergence depends on no other quantities, except dimension $d$.

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## Chapter 1

## Introduction

### 1.1 Motivation

In the theoretical physics literature, the sandpile model appears in connection with the self-organized criticality. Before explaining what the self-organized criticality is, we first explain the meaning of criticality in the example of bond percolation on the d-dimensional integer lattice $\mathbb{Z}^{d}$. While percolation will not concern us later in the thesis, the two exponents we discuss have analogues for the sandpile model.

Let $0<p<1$ and set each edge of $\mathbb{Z}^{d}$ occupied with probability $p$ and not occupied with probability $1-p$, independently. An edge of $\mathbb{Z}^{d}$ is also called a bond of $\mathbb{Z}^{d}$. The clusters are defined as the connected components of the random subgraph of $\mathbb{Z}^{d}$ induced by the occupied bonds. Let $\mathcal{C}_{o}$ denote the cluster containing the origin (denoted by o) and we write $\left|\mathcal{C}_{o}\right|$ for the size of the cluster $\mathcal{C}_{o}$ (the number of vertices). Let $P_{p}$ denote the probability measure with the parameter $p$.

Broadbent and Hammersley [8, 21] studied the critical probability $p_{c}$ in percolation theory and proved the following fundamental result. In $d \geq 2$, there exists a critical probability $0<p_{c}=p_{c}(d)<1$ such that

$$
\begin{aligned}
& P_{p}\left[\left|\mathcal{C}_{o}\right|<\infty\right]=1, \text { if } p<p_{c} \\
& P_{p}\left[\left|\mathcal{C}_{o}\right|=\infty\right]>0, \text { if } p>p_{c} .
\end{aligned}
$$

This implies that in the case $p<p_{c}$ there is no infinite cluster anywhere in the lattice a.s. by translation invariance. In the case $p>p_{c}$ there exists a unique infinite cluster in the lattice a.s.[20]. We can say that a phase transition occurs at the critical value $p_{c}$,
which separates the subcritical phase $p<p_{c}$ where all cluster in the lattice are finite a.s. and the supercritical phase $p>p_{c}$ where there exists an infinite cluster a.s.

Percolation at the critical probability $p_{c}$ has many properties which are different from the case $p \neq p_{c}$ and are typically more challenging to establish. Indeed, analysing the critical behaviour is considered to be one of the most challenging problems to prove in probability. For example, it is widely believed that there is no infinite component almost surely in critical percolation on d-dimensional lattices for $d \geq 2$. In the planar case $d=2, P_{p_{c}}\left[\left|\mathcal{C}_{o}\right|<\infty\right]=1$ was proved by Harris [28] and Kesten [44]. This result was also proved in high dimensions ( $d \geq 19$ for the nearest-neighbour model on $\mathbb{Z}^{d}$ or $d>6$ for a class of "spread-out" models of independent bond percolation on $\mathbb{Z}^{d}$ ) by Hara and Slade [23]. Later, Fitzner and van der Hofstad [16] extended this result to $d \geq 11$ for the nearest-neighbour model on $\mathbb{Z}^{d}$. In the case $p \neq p_{c}$, it is known that the probabilities $P_{p}\left[\left|\mathcal{C}_{o}\right|=n\right]$ decay faster than a power of $n[20]$, which are already not trivial to establish.

It is conjectured that there exist critical exponents $\eta=\eta(d)>0$ and $\delta=\delta(d)>0$ such that, for all $d \geq 2$,

$$
\begin{aligned}
& P_{p_{c}}\left[x \in \mathcal{C}_{o}\right]=\frac{1}{|x|^{d-2+\eta+o(1)}}, \text { as }|x| \rightarrow \infty ; \\
& P_{p_{c}}\left[\left|\mathcal{C}_{o}\right| \geq n\right]=n^{-\frac{1}{\delta}+o(1)}, \text { as } n \rightarrow \infty .
\end{aligned}
$$

Most progress on this conjecture has been made in the case $d=2$ and in high dimensions. We have $\eta=\frac{5}{24}$ and $\delta=\frac{91}{5}$ for the critical site percolation on the triangular lattice [64, 45]. $\eta=0$ is derived in high dimensions by Hara [22] (for the nearestneighbour model on $\mathbb{Z}^{d}$ with $d \geq 19$ ) and by Hara, van der Hofstad and Slade [26] (for the spread-out model on $\mathbb{Z}^{d}$ with $d>6$ ). Later, Fitzner and van der Hofstad [16] reduced the dimensions from $d=19$ to $d \geq 11$ for the nearest-neighbour model on $\mathbb{Z}^{d}$. We also know that $\delta=2$ in sufficiently high dimensions [24, 25].

It seems that the interesting properties of percolation are very sensitive to the fact that $p=p_{c}$, that is, we are at the critical value of the parameter $p$.

Bak, Tang and Wiesenfeld [3] showed that certain extended dissipative dynamical systems naturally evolve into a critical state, with no intrinsic time or length scales. This phenomenon is called the self-organized criticality (SOC), since there is no parameter turned to a critical value. In the same paper, Bak, Tang and Wiesenfeld used the sandpile dynamics to further explain their idea. Dhar [10] established the com-
mutativity of the sandpile model, which has many nice consequences and makes the model more amenable to study. While not intended to be a realistic model of sand, the Abelian sandpile model has important qualitative features of avalanche-like phenomena and has very non-trivial behaviour. Hence, the Abelian sandpile model became the primary theoretical example of SOC [11].

The Abelian sandpile model $[11,33]$ is a system of particles subjected to local distribution rules called topplings, that give rise to non-local, scale-invariant dynamical events called avalanches. These will be defined in Chapter 2. Establishing the power law exponents rigorously in $d=2,3$ is challenging, study of which has largely been limited to simulations. Analogues of the percolation cluster will be the set of vertices toppled in the avalanche.

### 1.2 Thesis outline

We start by discussing the motivation of the thesis. In this thesis, we work on critical exponents in sandpiles both analytically and numerically. My research involved: numerical simulation of sandpile critical exponents, as well as two theoretical questions inspired by these simulations. The first of these is about the asymptotic height distribution in high-dimensional sandpiles. The second gives a heuristic understanding of the hashing algorithm used for the high-dimensional simulations and it is related to random interlacements. The rest of this chapter presents an outline of the content of the thesis in different chapters.

Chapter 2 introduces the key definitions and background of the Abelian sandpile model. Some of the simulation methods used in Chapter 3 are also present here. Other definitions and properties used in this thesis will be stated in the corresponding chapters. Very briefly, the sandpile model in a finite set, a subset of $\mathbb{Z}^{d}$, describes a dynamics of particles (grains of sand, etc) where the basic rule is that when a vertex has at least $2 d$ particles, it topples by sending one particle to each neighbour. Particles are added to the system one by one, and at each step the system is stablized until no topplings are possible. Hence, when the dynamics reaches stationarity, there will be some fixed probability that a particle added at the origin causes a toppling at a vertex $x$. We call this the toppling probability of $x$.

In Chapter 3, we explain the simulation method based on Wilson's algorithm and the burning bijection between recurrent sandpiles and spanning trees. We also used the hashing function when computing the avalanche in high dimensions $(d \geq 5)$ to reduce memory we use. We discuss the simulation results both in low dimensions $(d=2,3)$
and high dimensions ( $d \geq 5$ ).
The main results of Chapter 3 are:

- In $d=2$, the simulation suggests that the toppling probability at a vertex $x$ satisfies a power law of the form:

$$
\mathbf{P}[x \text { topples }]=|x|^{-\eta+o(1)},
$$

where $\eta=\eta(2)=0.4$, to 1 d.p.

- In $d=3$, the simulation suggests that the behaviour of the toppling probability at $x$ is $|x|^{-1-\eta(3)+o(1)}$ with $\eta(3) \approx 0.1$.
- The simulation in very high dimensions $(d=32)$ gives a close agreement with a asymptotic formula for the height probability at the origin $o$ (the probability at stationarity to see a specific number of particles between 0 and $2 d-1$ at the origin).

We state the asymptotic formula (see Theorem 3.5.1 below) in Chapter 3, and prove this formula in Chapter 4. The asymptotic formula is for the single site height distribution of Abelian sandpiles on $\mathbb{Z}^{d}$ as $d \rightarrow \infty$, in terms of Poisson(1) probabilities with error estimates. This chapter involves detailed random walk estimates. We introduce the statement of Theorem 3.5.1 below, and we will explain more details in Chapter 4.

Let $p_{d}(i)=\mathbf{P}[\eta(o)=i], i=0, \ldots, 2 d-1$, denote the height probabilities at the origin in $d$ dimensions.

Theorem (Theorem 3.5.1). (i) For $0 \leq i \leq d^{1 / 2}$, we have

$$
p_{d}(i)=\sum_{j=0}^{i} \frac{e^{-1} \frac{1}{j!}}{2 d-j}+O\left(\frac{i}{d^{2}}\right)=\frac{1}{2 d} \sum_{j=0}^{i} e^{-1} \frac{1}{j!}+O\left(\frac{i}{d^{2}}\right) .
$$

(ii) If $d^{1 / 2}<i \leq 2 d-1$, we have

$$
p_{d}(i)=p_{d}\left(d^{1 / 2}\right)+O\left(d^{-3 / 2}\right) .
$$

(iii) As a consequence of (i) and (ii), we have $p_{d}(i) \sim(2 d)^{-1}$, if $i, d \rightarrow \infty$.

The main result of Chapter 5 (see Theorem 5.1.1 below) is motivated by the hashing algorithm used in Chapter 3 to study high-dimensional sandpiles. While it looks difficult to bound the running time of the algorithm directly, we can obtain a heuristic bound
by making some simplifications. This leads us to prove that the trace of the random walk in an $N$-torus converges (away from the origin) to an interlacement process at a random level expressed in terms of the exit time of Brownian motion from the cube $(-1,1)^{d}$. We introduce the statement of Theorem 5.1.1 below, and we will explain more details in Chapter 5.

Let us first define some notations used in the theorem. $L=m N$, where $L^{2} \sim A N^{d}$ as $N \rightarrow \infty$ for some constant $A \in(0, \infty)$. Let $\mathbb{T}_{N}=[-N / 2, N / 2)^{d} \cap \mathbb{Z}^{d}, d \geq 3$ and $\varphi: \mathbb{Z}^{d} \rightarrow \mathbb{T}_{N}$. We write vertices and subsets of the torus in bold, i.e. $\mathrm{x} \in \mathbb{T}_{N}$ and $\mathbf{K} \subset \mathbb{T}_{N}$. Let $T=\inf \left\{t \geq 0: Y_{t} \notin(-L, L)^{d}\right\}$, the first exit time from $(-L, L)^{d}$. Let $\sigma_{1}$ denote the exit time from $(-1,1)^{d}$ of a standard Brownian motion started at $o$. For $K^{0} \subset \mathbb{Z}^{d}$, let $\operatorname{Cap}\left(K^{0}\right)$ denote the capacity of $K^{0}[48]$. For any $0<R<\infty$ and $x \in \mathbb{Z}^{d}$, let $B_{R}(x)=\left\{y \in \mathbb{Z}^{d}:|y-x|<R\right\}$, where $|\cdot|$ is the Euclidean norm. Let $\mathcal{K}_{R}$ denote the collection of all subsets of $B_{R}(o)$. Given $\mathbf{x} \in \mathbb{T}_{N}$, let $\tau_{\mathbf{x}}: \mathbb{T}_{N} \rightarrow \mathbb{T}_{N}$ denote the translation of the torus by $\mathbf{x}$. Let $g: \mathbb{N} \rightarrow \mathbb{R}$ be any function satisfying $g(N) \rightarrow \infty$.

Theorem (Theorem 5.1.1). Let $d \geq 3$. For any $0<R<\infty$, any $K^{0} \in \mathcal{K}_{R}$, and any $\mathbf{x}$ satisfying $\tau_{\mathbf{x}} \varphi\left(B_{R}(o)\right) \cap \varphi\left(B_{g(N)}(o)\right)=\emptyset$ we have

$$
\mathbf{P}_{o}\left[\varphi\left(Y_{t}\right) \notin \tau_{\mathbf{x}} \varphi\left(K^{0}\right), 0 \leq t<T\right]=\mathbf{E}\left[e^{-d A \sigma_{1} \operatorname{Cap}\left(K^{0}\right)}\right]+o(1) \quad \text { as } N \rightarrow \infty .
$$

The error term depends on $R$ and $g$, but is uniform in $K^{0}$ and $\mathbf{x}$.
Chapter 6 concludes the thesis by briefly summarising the research findings in the preceding chapters and stating the possible future open questions arising from the simulation results.

A note on the references. Chapters 3, 4, and 5 that are based on published or submitted papers have their own references sections at the end of those chapters. References of the remaining part of the thesis are at the end of the thesis.

## Chapter 2

## Definitions and background

In this chapter, we introduce the definitions and basic properties used throughout this thesis. Other definitions and properties, which are only used in the individual chapters, will be introduced in the corresponding chapters.

### 2.1 Abelian sandpile model

We start with the definition of and some fundamental facts about the Abelian sandpile model on a finite graph $G$. Sandpiles are a lattice model of self-organized criticality, introduced by Bak, Tang and Wiesenfeld [3], and have been studied in both physics and mathematics. See the surveys [33], [51], [63], [30], [11].

After discovering the Abelian group structure of addition operators in this model, Dhar [10] generalized it to arbitrary finite graphs and called it the Abelian sandpile model. He studied the self-organized critical nature of the stationary measure and gave an algorithmic characterization of recurrent configurations, the so-called "burning algorithm". This algorithm gives a one-to-one correspondence between the recurrent configurations of the Abelian sandpile models and rooted spanning trees [55]. This bijection is essential for our numerical simulations.

### 2.1.1 Definitions

Let $G=(V \cup\{\rho\}, E)$ be a finite, connected graph, where we allow multiple edges between vertices. $V$ is a finite set of vertices and the distinguished vertex $\rho$ is called the sink. $E$ is the set of edges. Loop-edges are excluded for simplicity (their presence would involve only trivial modifications). Let $\operatorname{deg}_{G}(x)$ be the degree of the vertex $x$ in
the graph $G$ and let $x \sim y$ denote that vertices $x$ and $y$ are connected by at least one edge.

Two examples we will be concerned with are as follows. Let $V \subset \mathbb{Z}^{d}$ be a finite $d$ dimensional box: $V=V_{L}=[-L, L]^{d} \cap \mathbb{Z}^{d}$. All vertices in $V^{c}=\mathbb{Z}^{d} \backslash V$ are identified to the sink, $\rho$. All loop-edges created at $\rho$ are removed. This is called the wired graph induced by $V$. A second example is obtained, if we take $V=V_{L} \backslash\{\rho\}$, where $\rho=(L, \ldots, L)$, with periodic boundary conditions. This is called the torus graph. Instead of $\mathbb{Z}^{d}$, we can start from any locally finite, infinite, connected graph.

A sandpile is a collection of indistinguishable grains on the vertices in $V$. A sandpile is specified by a map $\eta: V \rightarrow\{0,1,2, \ldots\}$. We say that $\eta$ is stable at $x \in V$, if $\eta(x)<\operatorname{deg}_{G}(x)$ (the latter being $=2 d$ when $V \subset \mathbb{Z}^{d}$ ). We say that $\eta$ is stable, if $\eta(x)<\operatorname{deg}_{G}(x)$, for all $x \in V$. Sometimes, especially in physics, a sandpile is specified by a map $\eta^{*}: V \rightarrow\{1,2, \ldots\}$. A stable sandpile is then defined as having one of the values $1,2, \ldots, \operatorname{deg}_{G}(x)$ at all $x$. This defines the same model after a trivial shift of coordinates.

If $\eta$ is unstable (i.e. $\eta(x) \geq \operatorname{deg}_{G}(x)$ for some $x \in V$ ), $x$ is allowed to topple which means that $x$ passes one grain along each edge to its neighbours. When the vertex $x$ topples, the grains are re-distributed as follows:

$$
\begin{align*}
& \eta(x) \rightarrow \eta(x)-\operatorname{deg}_{G}(x) \\
& \eta(y) \rightarrow \eta(y)+n_{x y}, \quad y \in V, y \neq x . \tag{2.1}
\end{align*}
$$

where $n_{x y}$ is the number of edges between $x$ and $y$. In the examples we are concerned with, we have $n_{x y}=1$ for all $x, y \in V$. Grains arriving at $\rho$ are lost, so we do not keep track of them. Toppling a vertex may generate further unstable vertices.

Regarding $\eta$ as a row vector, the re-distribution rule (2.1) can be written as

$$
\eta \rightarrow \eta-\Delta_{x, \cdot}^{\prime}
$$

where

$$
\begin{aligned}
& \Delta_{x y}^{\prime}=\operatorname{deg}_{G}(x), \quad \text { if } x=y \\
& \Delta_{x y}^{\prime}=-n_{x y}, \quad \text { if } x \neq y .
\end{aligned}
$$

$\Delta_{x,}^{\prime}$, is the row $x$ of $\Delta^{\prime}$, that is, if $\Delta$ is the graph Laplacian of $G$ then $\Delta^{\prime}$ is the restriction of the graph Laplacian $\Delta$ to $V \times V$.

Since $\eta(x) \geq \operatorname{deg}_{G}(x)$ is required before toppling $x$, the number of grain at $x$ is nonnegative after toppling. Hence, we still have a sandpile configuration after toppling. In this case, we a say that toppling $x$ is legal in $\eta$. Given a sandpile $\xi$ on V , we define its stabilization

$$
\xi^{\circ} \in \Omega_{G}:=\{\text { all stable sandpiles on } V\}=\prod_{x \in V}\left\{0,1, \ldots, \operatorname{deg}_{G}(x)-1\right\}
$$

by carrying out all possible legal topplings, in any order, until a stable sandpile is reached. It was shown by Dhar [10] that the map $\xi \mapsto \xi^{\circ}$ is well-defined, in the sense that no matter what the order of topplings carried out, the same stable configuration will be reached.

Theorem 2.1.1. [10, 33] The map $\xi \mapsto \xi^{\circ}$ is well-defined.

We include the proof here that follows in [33, Theorem 2.3].

Proof. We need to show:
(a) Only finitely many topplings can occur, regardless of how we choose to topple vertices.
(b) The final stable configuration is independent of the sequence of topplings chosen.

In order to see (a), observe that if $x \sim r h o$ then $x$ can topple only finitely many times (the system loses particles to the sink $\rho$ on each toppling of $x$ ). It follows by induction that for all $k \geq 1$,if $x \sim x_{k-1} \sim \cdots \sim x_{1} \sim \rho$, then $x$ can topple only finitely many times. Since $G$ is connected, we are done.

We now prove (b) in two steps:
(i) Topplings commute: If $x, y \in V, x \neq y$ and $\eta$ is unstable at both $x$ and $y$, then writing $T_{x}$ to denote the effect of toppling $x$ we claim that

$$
T_{y} T_{x} \eta=T_{x} T_{y} \eta .
$$

Observe that in either order, both topplings are legal. Then the claim is immediate from observing that both sides equal $\eta-\Delta_{x, \cdot}^{\prime}-\Delta_{y, \cdot}^{\prime}$.
(ii) Suppose now that and

$$
\begin{equation*}
x_{1}, x_{2}, \ldots, x_{k} \tag{2.2}
\end{equation*}
$$

and

$$
y_{1}, y_{2}, \ldots, y_{\ell}
$$

are two sequences of vertices that are both possible stabilizing sequences of $\eta$. That is, when carried out in order from left to right, in both sequences each toppling is legal, and the final results are stable configurations.

If $\eta$ is already stable, then $k=\ell=0$ and there is nothing to prove.
Otherwise, we have $k, \ell \geq 1$ and $\eta\left(x_{1}\right) \geq \operatorname{deg}_{G}\left(x_{1}\right)$. Therefore, $x_{1}$ must occur somewhere in the second sequence, otherwise the second sequence would never reduce the number of particles at $x_{1}$. Let $x_{1}=y_{i}, 1 \leq i \leq \ell$, and suppose that $i$ is the smallest such index. By part (i), the toppling of $y_{i}=x_{1}$ can be moved to the front of the second stabilizing sequence. Precisely, we have

$$
\begin{aligned}
T_{x_{1}} T_{y_{i-1}} \ldots T_{y_{1}} \eta & =T_{y_{i-1}} T_{x_{1}} T_{y_{i-2}} \ldots T_{y_{1}} \eta \\
& =T_{y_{i-1}} T_{y_{i-2}} T_{x_{1}} \ldots T_{y_{1}} \eta \\
& \vdots \\
& =T_{y_{i-1}} T_{y_{i-2}} \ldots T_{y_{1}} T_{x_{1} \eta} \eta .
\end{aligned}
$$

It follows that the sequence

$$
\begin{equation*}
x_{1}, y_{1}, \ldots, y_{i-1}, y_{i+1}, \ldots, y_{\ell} \tag{2.3}
\end{equation*}
$$

also stabilizes $\eta$. We now remove $x_{1}$ from the beginning of the sequences (2.2) and (2.3) and repeat the argument for $T_{x_{2}} \eta$. Iterating gives that $k=\ell$ and the multisets $\left[x_{1}, \ldots, x_{k}\right]$ and $\left[y_{1}, \ldots, y_{\ell}\right]$ are permutations of each other. That is, each vertex topples the same number of times in the two stabilizing sequences, and hence they reach the same final configuration. This completes the proof that the stabilization $\xi \mapsto \xi^{\circ}$ is well-defined.

Adding a single particle to the sandpile and stabilizing can induce a highly complex sequence of topplings, see for example the early simulations in [3, Figure 1]. The sequence of topplings carried out is called an avalanche.

Definition 2.1.1. Given a configuration $\eta$, the avalanche started at $x$ is the multi-set of topplings resulting from adding a particle at $x$ in $\eta$ and stabilizing. Note the multi-set is well-defined by the proof of Theorem 2.1.1. The avalanche cluster started at $x$ is the set of sites that topples at least once: $\operatorname{Av}_{x}=\{y \in V: y$ topples after adding a particle at $x\}$.

We denote $\mathrm{Av}_{o}$ by Av .
We can use a decomposition of the avalanche into waves, introduced by Ivashkevich, Ktitarev and Priezzhev [32]. Waves are defined as follows. After we added a particle at $x$, topple $x$, and all other vertices that can be toppled, but do not allow $x$ to topple a second time. It is not difficult to see that each vertex topples at most once under this restriction. The set of vertices that toppled is called the first wave. After the first wave, if $x$ is still unstable (this will be the case if and only if all of its neighbours were in the first wave), topple $x$ a second time and topple all other vertices that can be toppled, not allowing $x$ to topple a third time. This is called the second wave, etc.

### 2.1.2 Basic properties

Definition 2.1.2. The addition operators $A_{x}$ are maps from the set of all sandpiles to the set of all stable sandpiles, $\Omega_{G}$, defined by adding one particle at $x$ and stabilizing. $A_{x} \eta=\left(\eta+\mathbf{1}_{\mathbf{x}}\right)^{\circ}$, where $\mathbf{1}_{\mathbf{x}}$ is the row vector with 1 in $x$ and 0 elsewhere.

Lemma 2.1.1. [10, 33] The addition operators commute. More formally, $A_{x} A_{y}=$ $A_{y} A_{x}$ for all $x, y \in V$.

We include the proof here that follows in [33, Lemma 2.8].

Proof. We have

$$
\begin{equation*}
A_{x} A_{y} \eta=\left(\left(\eta+\mathbf{1}_{\mathbf{y}}\right)^{\circ}+\mathbf{1}_{\mathbf{x}}\right)^{\circ} \tag{2.4}
\end{equation*}
$$

and

$$
\begin{equation*}
A_{y} A_{x} \eta=\left(\left(\eta+\mathbf{1}_{\mathbf{x}}\right)^{\circ}+\mathbf{1}_{\mathbf{y}}\right)^{\circ} . \tag{2.5}
\end{equation*}
$$

We show that both expressions equal

$$
\begin{equation*}
\left(\eta+\mathbf{1}_{\mathbf{x}}+\mathbf{1}_{\mathbf{y}}\right)^{\circ} . \tag{2.6}
\end{equation*}
$$

To see this, start with the configuration $\eta+\mathbf{1}_{\mathbf{x}}+\mathbf{1}_{\mathbf{y}}$, and carry out topplings as in the stabilization of $\eta+\mathbf{1}_{\mathbf{y}}$. The extra particle present at $x$ does not affect the legality of any of the topplings in the stabilization of $\eta+\mathbf{1}_{\mathbf{y}}$. Hence with the extra particle at $x$ present, we arrive at the configuration $\left(\eta+\mathbf{1}_{\mathbf{y}}\right)^{\circ}+\mathbf{1}_{\mathbf{x}}$. Now carry out any further topplings that are possible, arriving at the right hand side of (2.4). Due to Theorem 2.1.1, the final configuration also equals (2.6). Equality of (2.6) and (2.5) is seen similarly.

The reason that the addition operators commute in this model is that the toppling
conditions depend on local heights, and not on discrete gradients of these heights. Since the addition operators commute in this model, we call it the Abelian sandpile model. In other possible models of self-organized criticality, the toppling conditions may depend on gradients of heights, in this case the addition operators may not commute [10]. The dynamics of the sandpile can be considered as a toy model of avalanche-like phenomena.

The sandpile model is not meant to be a realistic representation of sand. A more suitable condition for toppling in order to model sand grains moving down a slope could be that the discrete gradient exceeds some fixed critical value. However, topplings do not commute in such models, as can be seen. Later, we will see how commutativity in the Abelian sandpile model has a lot of nice consequences that make it easier to study. The point is that the Abelian sandpile model already has important qualitative characteristics of avalanche-type phenomena and has very nontrivial behaviour.

We will now define the sandpile Markov chain with inital state $\eta_{0}$. The state space is the set of stable sandpiles, $\Omega_{G}$. Fix a positive probability distribution $p$ on $V$, i.e. $\sum_{x \in V} p(x)=1$ and $p(x)>0$ for all $x \in V$. Starting at $\eta_{0} \in \Omega_{G}$, choose a random vertex $X \in V$ according to $p$, add one particle at $X$ and stabilize. The one step transition of the Markov chain moves $\eta$ to $A_{X} \eta=\left(\eta+\mathbf{1}_{\mathbf{X}}\right)^{\circ}$. By Theorem 2.1.1, we can write the time evolution of the sandpile Markov chain as $\eta_{n}=\left(\eta_{0}+\sum_{i=1}^{n} \mathbf{1}_{\mathbf{X}_{\mathbf{i}}}\right)^{\circ}$, where $n \geq 1$ and $X_{1}, X_{2}, \ldots$ are i.i.d. random variables with distribution $p$. The stationary distribution of the Markov chain does not depend on $p$, and equals to the uniform distribution on the set of recurrent states [58, Corollary 3.2] and [30].

Considering the sandpile Markov chain on a finite connected graph $G$, there is only one recurrent class [30]. We denote the set of recurrent sandpiles by $\mathcal{R}_{G}$.

Although the Abelian sandpile model can be defined on an arbitrary finite connected graph, in this thesis we mainly focus on $V \subset \mathbb{Z}^{d}$. Let us now introduce the sandpile model on subsets of $\mathbb{Z}^{d}$.

Given a finite set $V \subset \mathbb{Z}^{d}$ (the wired graph induced by $V$ ), a sandpile on $V$ is given by $\eta: V \rightarrow\{0,1,2, \ldots\}$. Recall that we identify all vertices in $V^{c}=\mathbb{Z}^{d} \backslash V$ into a single vertex that becomes the sink $\rho$. Then remove all loop-edges at $\rho$. This is called the wired graph induced by $V$. We say that $\eta$ is stable at $x \in V$, if $\eta(x)<2 d . \eta$ is unstable at $x \in V$, if $\eta(x) \geq 2 d$, then $x$ is allowed to topple. When topping $x, x$ sends one particle along each edge adjacent to $x$ and the particles at $x$ and its neighbours are
re-distributed as follows:

$$
\begin{aligned}
& \eta(x) \rightarrow \eta(x)-2 d ; \\
& \eta(y) \rightarrow \eta(y)+1, y \in V, y \sim x .
\end{aligned}
$$

Consider the stationary sandpile in the box $V_{L}=[-L, L]^{d} \cap \mathbb{Z}^{d}$ with Dirichlet boundary conditions. We write $\mathbf{P}_{L}$ for the stationary distribution of sandpiles in $V_{L}$. It was shown in [2] that $\mathbf{P}_{L}$ has a weak limit $\mathbf{P}$ in $d \geq 2$. That means that for any event $E$ that only depends on finitely many sandpile heights, we have $\mathbf{P}_{L}[E]$ converges to $\mathbf{P}[E]$, as $L \rightarrow \infty$. Let us add a grain at the origin $o$ and carry out the resulting avalanche. We are going to abbreviate the event 'when a grain is added at $o$, vertex $x$ topples in the resulting avalanche' to simply ' $x$ topples'. Thus let $\mathbf{P}_{L}[x$ topples $]$ denote the probability that $x$ topples in the avalanche initiated at $o$ in volume $V_{L}$. Note that the event ' $x$ topples' is not local, however it still holds that $\mathbf{P}[x$ topples $]=\lim _{L \rightarrow \infty} \mathbf{P}_{L}[x$ topples $]$. This result was shown in the case for $d \geq 3$ in Proposition 3.11 and $d=2$ in Lemma 5.10 in [6].

It is still an open problem to prove that in $d=2$

$$
\mathbf{P}[\text { finitely many vertices topple in an avalanche }]=1,
$$

and the corresponding problem in $d \geq 3$ was shown by Járai and Redig [34, Theorem 3.11]. In Section 3.3.2, our wave simulation data in $d=2$ suggests that the number of waves decays as a power law, and this lends heuristic support to the above open problem. Extending the Markov chains dynamics to $\mathbb{Z}^{d}$ has only been done in $d \geq 3$ [34]. In $d=2$, it hinges on the above open problem.

### 2.1.3 Height probability in 2D

Dhar and Majumdar [12] studied the Abelian sandpile model on the Bethe lattice and the exact expressions for various distribution functions including the height distribution at a vertex were obtained using combinatorial methods. However, on d-dimensional cubic lattices of $d \geq 2$, exact results for the height probability are only known for $d=2$. Under the stationary distribution in a box $V_{L}=[-L, L]^{d}$, the height of a sandpile at any vertex takes values $0,1, \ldots, 2 d-1$. The height probabilities of the d-dimensional Abelian sandpile model are the limits of the probabilities of the origin having heights $0,1, \ldots, 2 d-1$ as $L \rightarrow \infty$, denoted by $p(0), p(1), \ldots, p(2 d-1)$. Existence of these limits follow from [2].

In dimension 2, the height of a sandpile at any vertex of a square lattice takes values in $0,1,2$ and 3 . The first numerical estimation of the height probabilities was made by Zhang [70]. Zhang introduced a continuous-energy model, which has continuous heights, and obtained $p(0)=0.10, p(1)=0.16, p(2)=0.32$ and $p(3)=0.42$.

Erzan and Sinha [15] obtained $p(0)=0.07 \pm 4 \%, p(1)=0.17 \pm 7 \%, p(2)=0.31 \pm 9 \%$ and $p(3)=0.45 \pm 3 \%$ for the discrete sandpile model on the lattice of linear sizes 30 and 40. Manna [56] obtained $p(0)=0.073, p(1)=0.174, p(2)=0.307$ and $p(3)=0.446$ on the square lattice of size 672 . Typical errors in these estimates are of the order of 0.003 . Grassberger and Manna [19] performed simulations on some even larger lattices (up to size $1400 \times 1400$ ), but this did not allow to draw conclusions with the same accuracy as Manna's results [56]. Their results for the lattice of size 672 give $p(0)=0.0736$, $p(1)=0.1740, p(2)=0.3062$ and $p(3)=0.4462$. Both of the last two values of $p(0)$ are very consistent with the exact value $p(0)=2 / \pi^{2}-4 / \pi^{3}=0.07363 \ldots$.. [54].

Exact results for $p(1), p(2)$, and $p(3)$ were derived in the infinite volume limit by Priezzhev [61]. The formulas for $p(1), p(2)$ and $p(3)$ involved rational polynomials in terms of $1 / \pi$ and two multiple integrals. A high-precision numerical evaluation of these two different integrals seemed to give the average height $\sum_{i=0}^{3} i p(i)$ exactly equal to the simple fraction $17 / 8$ [11]. Jeng, Piroux and Ruelle [41] extended Priezzhev's work. By assuming the average height was $17 / 8$ exactly, they obtained the following formulas:

$$
\begin{align*}
& p(0)=\frac{2}{\pi^{2}}-\frac{4}{\pi^{3}} ; \\
& p(1)=\frac{1}{4}-\frac{1}{2 \pi}-\frac{3}{\pi^{2}}+\frac{12}{\pi^{3}} ;  \tag{2.7}\\
& p(2)=\frac{3}{8}+\frac{1}{\pi}-\frac{12}{\pi^{3}} ; \\
& p(3)=1-p(0)-p(1)-p(2)=\frac{3}{8}-\frac{1}{2 \pi}+\frac{1}{\pi^{2}}+\frac{4}{\pi^{3}} .
\end{align*}
$$

The average height, which is indeed $17 / 8$, was proved independently by Kenyon and Wilson [42], and Poghosyan, Priezzhev and Ruelle [60]. This implies (2.7), which we will use in Chapter 3 to check our code.

### 2.2 The burning bijection and Wilson's algorithm

The Abelian sandpile is closely related with other probability models on graphs such as spanning trees. The connection between the spanning trees and the abelian sandpile
model was discovered by Majumdar and Dhar [55]. In this section, we present two connections to spanning trees, which feature in many parts of my thesis.

In 1990, Dhar [10] introduced an efficient way called the burning algorithm to check whether a particular stable sandpile $\eta$ is in $\mathcal{R}_{G}$. Application of this algorithm leads to the burning bijection established by Majumdar and Dhar [55] between $\mathcal{R}_{G}$ and the set of all spanning trees of $G$, denoted by $\mathcal{T}_{G}$.

The burning algorithm is introduced as follows. Given a stable sandpile $\eta \in \Omega_{G}$, at time $t=0$, the sink, $\rho$, burns. We set $B_{0}=\{\rho\}$, the set of vertices burnt at time 0 , and set $U_{0}=V$, the set of vertices unburnt at time 0 .

At time $t=1$, all $x \in U_{0}$ such that $\eta(x) \geq \operatorname{deg}_{U_{0}}(x)$ burns. Then we set $B_{1}=\{x \in$ $\left.U_{0}: \eta(x) \geq \operatorname{deg}_{U_{0}}(x)\right\}$, the set of vertices burnt at time 1 , and set $U_{1}=U_{0} \backslash B_{1}$, the set of vertices unburnt at time 1 .

Generally, for $t \geq 1$, all $x \in U_{t-1}$ such that $\eta(x) \geq \operatorname{deg}_{U_{t-1}}(x)$ burns. Then we set $B_{t}=\left\{x \in U_{t-1}: \eta(x) \geq \operatorname{deg}_{U_{t-1}}(x)\right\}$ and $U_{t}=U_{t-1} \backslash B_{t}$.

We have $U_{T}=U_{T+1}=\ldots$ for some $1 \leq T<\infty$ and we know $\eta$ is recurrent if and only if $U_{T}=\emptyset[10,30,33]$.

The burning bijection is described as follows. We use the burning algorithm to define a map $\psi: \mathcal{R}_{G} \rightarrow \mathcal{T}_{G}$. For every $x \in V$, fix an arbitrary ordering $\prec_{x}$ of the edges incident to $x$. Let $\eta \in \mathcal{R}_{G}$. The spanning tree $\psi(\eta)$ is defined by assigning to $x \in V$ an edge adjacent to $x$, and oriented outwards from $x$. The construction guarantees that the edges form a spanning tree directed towards $\rho$.

Given any vertex $x$, there is a unique $t=t(x) \geq 1$ such that $x \in B_{t}$. Let

$$
\begin{aligned}
F_{x} & =\left\{f: \operatorname{tail}(f)=x, \operatorname{head}(f) \in B_{t-1}\right\}, \\
m_{x} & =\left|\left\{f: \operatorname{tail}(f)=x, \operatorname{head}(f) \in \bigcup_{i \leq t-1} B_{i}\right\}\right| .
\end{aligned}
$$

$F_{x}$ is the set of edges leading to the neighbours of $x$ burnt one step before $x$, and $m_{x}$ is the number of edges connecting $x$ to its neighbours burnt before $x . F_{x}$ is not empty. Since we start with a stable sandpile $\eta$ in the algorithm, a number of neighbours of $x$ need to burn to satisfy $\eta(x) \geq \operatorname{deg}_{U_{t-1}}(x)$. Hence there is at least one neighbour of $x$ burnt at time $t-1$.

Since $x$ burns at time $t$ and we declare $x$ burnt when $\eta \geq \operatorname{deg}_{U_{t-1}}=\operatorname{deg}_{G}(x)-m_{x}$,
and $\eta(x)<\operatorname{deg}_{G}(x)-m_{x}+\left|F_{x}\right|$, otherwise $x$ should have burnt before time $t$, we have $\operatorname{deg}_{G}(x)-m_{x} \leq \eta(x)<\operatorname{deg}_{G}(x)-m_{x}+\left|F_{x}\right|$.

Assuming $\eta(x)=\operatorname{deg}_{G}(x)-m_{x}+i$, for some $0 \leq i<\left|F_{x}\right|$, we add the i-th edge in $F_{x}$ in the ordering to $\psi(\eta):=\tau$. We call this edge $e_{x}$. Since the burning algorithm on a recurrent sandpile burns all vertices, $\tau$ spans. For any edge $e_{x} \in \tau, \operatorname{head}\left(e_{x}\right) \in B_{t(x)-1}$, where $t(x)$ denotes the time when $x$ is burnt. All paths orient towards $\rho$, so $\tau$ does not contain loops. Hence $\tau$ is a spanning tree of $G$.

The orientation of the edges is obtained uniquely by following paths towards $\rho$, so we can remove the orientation to obtain an unoriented spanning tree without loss of information.

Lemma 2.2.1. [55, 33] The map $\psi$ is injective.

Proof. Let $\eta_{1}, \eta_{2} \in \mathcal{R}_{G}$ and assume $\eta_{1} \neq \eta_{2}$. Then there is a first time $t$ such that either $B_{t}^{1} \neq B_{t}^{2}$, or $B_{t}^{1}=B_{t}^{2}$ but $\left.\eta_{1}\right|_{B_{t}^{1}} \neq\left.\eta_{2}\right|_{B_{t}^{2}}$, where $B_{t}^{1}$ and $B_{t}^{2}$ are sets of vertices burnt at time $t$ for $\eta_{1}$ and $\eta_{2}$ respectively.

In the case that $B_{t}^{1} \neq B_{t}^{2}$, there exists some vertices $x_{1}, x_{2}, \ldots, x_{i}$, belonging to $B_{t}^{1}$, but not in $B_{t}^{2}$ or vice versa, so the edges $e_{x_{1}}, e_{x_{2}}, \ldots, e_{x_{i}}$ are assigned differently for $\eta_{1}$ and $\eta_{2}$, where recall $e_{x}$ from the definition of the map $\psi$ above.

In the case that $\left.\eta_{1}\right|_{B_{t}^{1}} \neq\left.\eta_{2}\right|_{B_{t}^{2}}$, there exists some vertices $x_{1}, x_{2}, \ldots, x_{j}$ such that $\eta_{1}\left(x_{1}\right) \neq \eta_{2}\left(x_{1}\right), \ldots, \eta_{1}\left(x_{j}\right) \neq \eta_{2}\left(x_{j}\right)$. This implies that the edges $e_{x_{1}}, e_{x_{2}}, \ldots, e_{x_{j}}$ are different in the corresponding spanning trees.

Therefore, in both cases, the spanning trees $\psi\left(\eta_{1}\right)$ and $\psi\left(\eta_{2}\right)$ are constructed differently.

We have that $\left|\mathcal{R}_{G}\right|=\operatorname{det}\left(\Delta_{G}^{\prime}\right)[33]$ and the Matrix-Tree Theorem [7] in combinatorics states that $\left|\mathcal{T}_{G}\right|=\operatorname{det}\left(\Delta_{G}^{\prime}\right)$. Therefore Lemma 2.2 .1 implies the $\operatorname{map} \psi: \mathcal{R}_{G} \rightarrow \mathcal{T}_{G}$ is bijective.

The map $\psi: \mathcal{R}_{G} \rightarrow \mathcal{T}_{G}$ is called the burning bijection and toppling procedure used to construct $\psi$ is usually referred to as the burning procedure.

Recall the unique stationary measure $\nu_{G}$ is the uniform distribution on $\mathcal{R}_{G}$ [10]. The burning bijection $\psi$ maps $\nu_{G}$ to the uniform distribution on $\mathcal{T}_{G}$. This uniform distribution is called the uniform spanning tree measure, denoted $\mu_{G}$.

Wilson [68] introduced a simple and efficient algorithm to generate a uniformly random element in $\mathcal{T}_{G}$ (a sample from $\mu_{G}$ ). We need to state the procedure of loop-erasure before introducing the Wilson's algorithm. Given a path $\pi=\left[w_{0}, w_{1}, \ldots, w_{k}\right]$ in $G$, we erase loops from $\pi$ chronologically, as they are created. We trace $\pi$ until the first time $t$, if any, when $w_{t} \in\left\{w_{0}, w_{1}, \ldots, w_{t-1}\right\}$, i.e. there is a loop. We suppose $w_{t}=w_{i}$, for some $i \in\{0,1, \ldots, t-1\}$ and remove the loop $\left[w_{i}, w_{i+1}, \ldots, w_{t}=w_{i}\right]$. We repeat this process as long as there are loops. This gives the loop-erasure $L E(\pi)$ of $\pi$, which is a self-avoiding path [53]. If $\pi$ is generated from a random walk process on $G$, the loop-erasure of $\pi$ is called the loop-erased random walk (LERW).

In general, fix a vertex $r$ of $G$ and let $v_{1}, v_{2}, \ldots, v_{n}$ be an arbitrary enumeration of the remaining vertices in $G$. In the sandpile context, we normally choose $r=\rho$. Let $\tau_{0}=\{r\}$. We start a simple random walk $\pi_{1}$ at $v_{1}$ on $G$ and $\pi_{1}$ stops when $r$ is first hit. $L E\left(\pi_{1}\right)$ is attached to $\tau_{0}$ and the resulting path is denoted by $\tau_{1}$. Then we start a second simple random walk $\pi_{2}$ at $v_{2}$ and stop $\pi_{2}$ when it hits $\tau_{1}$. We attach $L E\left(\pi_{2}\right)$ to $\tau_{1}$ and denote the resulting tree by $\tau_{2}$. We continue the same procedure until all the vertices are visited. We construct a random sequence of trees $\tau_{1} \subset \tau_{2} \subset \ldots \subset \tau_{n}$, where $\tau_{n}$ is a spanning tree of $G$. Wilson's theorem [68] implies that $\tau_{n}$ is uniformly distributed over all spanning trees of $G$.

## Chapter 3

## Simulation results

## Chapter Overview

This chapter is based on a joint work by Járai and me [11]. The presentation here is slightly different and some additional commentary is included. Due to this there will be some repetition of material from the previous chapters.

This chapter is mainly motivated by the critical exponent of toppling probability in sandpiles. We will be interested in the following question: if a particle is added at the origin, what is the probability that a distant vertex topples? The set of vertices that topple can be viewed as an analogue of the critical percolation cluster; see Chapter 1. The exponent describing the toppling probability is the analogue of the exponent $\eta$ of the percolation, which was introduced in Chapter 1. The toppling probability is well-understood in dimensions $d \geq 5$, but only limited information was known in lower dimensions [3]. We also examine a number of related numerical questions.

The simulation data is freely available from the University of Bath Research Data Archive and the simulation code is available upon request [12].

## Statement of Authorship

| Toppling and height probabilities in sandpiles |  |  |  |  |  |
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#### Abstract

We study Abelian sandpiles numerically, using exact sampling. Our method uses a combination of Wilson's algorithm to generate uniformly distributed spanning trees, and Majumdar and Dhar's bijection with sandpiles. We study the probability of topplings of individual vertices in avalanches initiated at the centre of large cubic lattices in dimensions $d=2,3$ and 5 . Based on these, we estimate the values of the toppling probability exponent in the infinite volume limit in dimensions $d=2,3$, and find good agreement with theoretical results on the mean-field value of the exponent in $d \geq 5$. We also study the distribution of the number of waves in 2 -dimensional avalanches.

Our simulation method, combined with a variance reduction idea, lends itself well to studying some problems even in very high dimensions. We illustrate this with an estimation of the single site height probability distribution in $d=32$, and compare this to the asymptotic behaviour as $d \rightarrow \infty$. We give an asymptotic formula for the single site height distribution of Abelian sandpiles on $\mathbb{Z}^{d}$ as $d \rightarrow \infty$, in terms of Poisson(1) probabilities with error estimates. The proof of this formula is given in Chapter 4.


### 3.1 Introduction

The definitions of and some fundamental facts about the Abelian sandpile model has given in Chapter 2. In this chapter, we briefly recalled them and then begin by introducing toppling probability exponents and discussing previous related work.

### 3.1.1 Abelian sandpile model

We start with the definition of and some fundamental facts about the Abelian sandpile model on a finite graph $G$. Sandpiles are a lattice model of self-organized criticality, introduced by Bak, Tang and Wiesenfeld [2], and have been studied in both physics and mathematics. We refer to [5] for an overview. After discovering the Abelian group structure of addition operators in this model, Dhar [4] generalized it to arbitrary finite graphs and called it the Abelian sandpile model. He studied the self-organized critical nature of the stationary measure and gave an algorithmic characterization of recurrent
configurations, the so-called "burning algorithm". This algorithm gives a one-to-one correspondence between the recurrent configurations of the Abelian sandpile models and rooted spanning trees [19]. This bijection is essential for our numerical simulations.

### 3.1.1.1 Basic properties

Let $G=(V \cup\{\rho\}, E)$ be a finite, connected graph, where we allow multiple edges between vertices. $V$ is a finite set of vertices and the distinguished vertex $\rho$ is called the sink. $E$ is the set of edges and loop-edges are excluded for simplicity. Let $\operatorname{deg}_{G}(x)$ be the degree of the vertex $x$ in the graph $G$ and let $x \sim y$ denote that vertices $x$ and $y$ are connected by at least one edge.

Two examples we will be concerned with are as follows. Let $V \subset \mathbb{Z}^{d}$ be a finite $d$ dimensional box: $V=V_{L}=[-L, L]^{d} \cap \mathbb{Z}^{d}$. All vertices in $V^{c}=\mathbb{Z}^{d} \backslash V$ are identified to the sink, $\rho$. All loop-edges created at $\rho$ are removed. This is called the wired graph induced by $V$. A second example is obtained, if we take $V=V_{L} \backslash\{\rho\}$, where $\rho=(L, \ldots, L)$, with periodic boundary conditions. This is called the torus graph.

A sandpile is a collection of indistinguishable grains on the vertices in $V$. A sandpile is specified by a map $\eta: V \rightarrow\{0,1,2, \ldots\}$. We say that $\eta$ is stable at $x \in V$, if $\eta(x)<\operatorname{deg}_{G}(x)$ (the latter being $=2 d$ when $V \subset \mathbb{Z}^{d}$ ). We say that $\eta$ is stable, if $\eta(x)<\operatorname{deg}_{G}(x)$, for all $x \in V$. Sometimes, especially in physics, a sandpile is specified by a map $\eta^{*}: V \rightarrow\{1,2, \ldots\}$. A stable sandpile is then defined as having one of the values $1,2, \ldots, \operatorname{deg}_{G}(x)$ at all $x$. This defines the same model after a trivial shift of coordinates.

If $\eta$ is unstable (i.e. $\eta(x) \geq \operatorname{deg}_{G}(x)$ for some $x \in V$ ), $x$ is allowed to topple which means that $x$ passes one grain along each edge to its neighbours. When the vertex $x$ topples, the grains are re-distributed as follows:

$$
\begin{aligned}
& \eta(x) \rightarrow \eta(x)-\operatorname{deg}_{G}(x) ; \\
& \eta(y) \rightarrow \eta(y)+n_{x y}, \quad y \in V, y \neq x .
\end{aligned}
$$

where $n_{x y}$ is the number of edges between $x$ and $y$. In the examples we are concerned with, we have $n_{x y}=1$ for all $x, y \in V$. Grains arriving at $\rho$ are lost, so we do not keep track of them.

Toppling a vertex may generate further unstable vertices. Given a sandpile $\xi$ on V , we
define its stabilization

$$
\xi^{\circ} \in \Omega_{G}:=\{\text { all stable sandpiles on } V\}=\prod_{x \in V}\left\{0,1, \ldots, \operatorname{deg}_{G}(x)-1\right\}
$$

by carrying out all possible topplings, in any order, until a stable sandpile is reached. It was shown by Dhar [4] that the map $\xi \mapsto \xi^{\circ}$ is well-defined, that is, the order of topplings does not matter.

We now define the sandpile Markov chain with inital state $\eta_{0}$. The state space is the set of stable sandpiles, $\Omega_{G}$. Fix a positive probability distribution $p$ on $V$, i.e. $\sum_{x \in V} p(x)=1$ and $p(x)>0$ for all $x \in V$. Starting at $\eta \in \Omega_{G}$, choose a random vertex $X \in V$ according to $p$, add one grain at $X$ and stabilize. The one step transition of the Markov chain moves from $\eta$ to $\left(\eta+\mathbf{1}_{\mathbf{X}}\right)^{\circ}$. Considering the sandpile Markov chain on a finite connect graph $G$, there is only one recurrent class [4]. We denote the set of recurrent sandpiles by $\mathcal{R}_{G}$.

### 3.1.2 Toppling probability exponent

Consider the stationary sandpile in the box $V_{L}=[-L, L]^{d} \cap \mathbb{Z}^{d}$ with Dirichlet boundary conditions. We write $\mathbf{P}_{L}$ for the stationary distribution of sandpiles in $V_{L}$. It was shown in [1] that, in $d \geq 2, \mathbf{P}_{L}$ has a weak limit $\mathbf{P}$. That means that for any event $E$ that only depends on finitely many sandpile heights we have $\mathbf{P}_{L}[E]$ converges to $\mathbf{P}[E]$, as $L \rightarrow \infty$. Let us add a grain at the origin $o$, and carry out the resulting avalanche. We are going to abbreviate the event 'when a grain is added at $o$, vertex $x$ topples in the resulting avalanche' to simply ' $x$ topples'. Thus let $\mathbf{P}_{L}[x$ topples $]$ denote the probability that $x$ topples in the avalanche initiated at $o$ in volume $V_{L}$. Note that the event $x$ topples is not local, however it still holds that $\mathbf{P}[x$ topples $]=\lim _{L \rightarrow \infty} \mathbf{P}_{L}[x$ topples $]$. This result was shown the case when $d \geq 3$ in Proposition 3.11 and the case when $d=2$ in Lemma 5.10 in [3].

It was shown by Dhar [4], [5] that in the stationary sandpile in the box $V_{L}=[-L, L]^{d} \cap$ $\mathbb{Z}^{d}$, the expected number of topplings at $x$, when a grain is added at $o$ is given by the Green function $G_{L}(o, x)$ (the inverse of the graph Laplacian). In dimensions $d \geq 3$, this has infinite volume limit

$$
\lim _{L \rightarrow \infty} G_{L}(o, x)=G(o, x) \sim c_{d}|x|^{2-d}, \quad \text { as }|x| \rightarrow \infty,
$$

where we write $|x|$ for the Euclidean distance of $x$ from $o$. Due to Markov's inequality, we have $\mathbf{P}[x$ topples] $\leq G(o, x)$. It was shown in [10] that in $d \geq 5$ it also holds
that $\mathbf{P}[x$ topples $] \geq c G(o, x)$ with some constant $c=c(d)>0$, and hence in these dimensions

$$
\begin{equation*}
\mathbf{P}[x \text { topples }] \approx|x|^{2-d} . \tag{3.1}
\end{equation*}
$$

In analogy with other statistical physics models at criticality (such as percolation at the critical threshold), the authors of [10] conjecture that in all dimensions $d \geq 2$ one has the behaviour

$$
\begin{equation*}
\mathbf{P}[x \text { topples }] \approx|x|^{2-d-\eta} \tag{3.2}
\end{equation*}
$$

with a critical exponent $\eta=\eta(d) \geq 0$. Then (3.1) shows that the mean-field value of $\eta$ equals $0(d>4)$, and one expects that $\eta(d)>0$ in dimensions $d=2,3$, and that $\eta(4)=0$ with a logarithmic correction.

In this chapter we carry out a numerical study of the conjecture (3.2) in dimensions $d=2,3$, and also study the behaviour of $\mathbf{P}_{L}[x$ topples $]|x|^{d-2}$ in $d=5$. We also consider the scaling limit of the toppling probability $\mathbf{P}_{L}[x$ topples $]$ in finite volumes when $|x| / L$ is bounded away from 0 .

### 3.1.3 Related work

To the best of our knowledge, individual toppling probabilities were not studied numerically previously in the literature. Manna [20] and Grassberger and Manna [7] studied average 'cluster sizes' related to our findings. In order to explain what these are, let

$$
t_{L}(x ; z)=\mathbf{P}_{L}[x \text { topples if a grain is added at } z],
$$

where $\mathbf{P}_{L}$ refers to probabilities in the stationary state in volume $[-L, L]^{d}$, with Dirichlet boundary conditions. Let us write $n(z, x)$ for the random variable that is the number of topplings at $x$, given a grain is added at $z$. The above papers considered numerical estimates of the expected number of topplings in an avalanche under stationarity initiated at a randomly chosen site. Denoting this expectation by $\langle s\rangle$, we have

$$
\begin{equation*}
\langle s\rangle=\frac{1}{\left|V_{L}\right|} \sum_{z \in V_{L}} \mathbf{E}_{L}\left[\sum_{x \in V_{L}} n(z, x)\right]=\frac{1}{\left|V_{L}\right|} \sum_{z \in V_{L}} \sum_{x \in V_{L}} G_{L}(z, x) \sim c(d) L^{2}, \tag{3.3}
\end{equation*}
$$

as $L \rightarrow \infty$.

They also considered numerical estimates of the expected number of distinct sites toppled in such avalanches. This is given by an average over the vertex $z \in V_{L}$ where the avalanche is initiated of the expected number of $x \in V_{L}$ that topple at least once,
that is, where $n(z, x) \geq 1$. Thus

$$
\begin{equation*}
\left\langle s_{\text {distinct }}\right\rangle=\frac{1}{\left|V_{L}\right|} \sum_{z \in V_{L}} \mathbf{E}_{L}\left[\sum_{x \in V_{L}} \mathbf{1}_{n(z, x) \geq 1}\right]=\frac{1}{\left|V_{L}\right|} \sum_{z \in V_{L}} \sum_{x \in V_{L}} t_{L}(x ; z) \tag{3.4}
\end{equation*}
$$

In [7] it was found that this scales as $\approx L^{1.64}$ in $d=2$. In order to confirm that our methods give results consistent with earlier work, we checked both of the exponents (3.3) and (3.4) with our simulation methods for lattice sizes comparable to those in [20] and $[7](L=64,128,256,512)$, and found very close agreement with the above exponents. Another test we performed was to check that our methods yield the exactly known height probabilities in 2D [23].

In the present chapter we restrict to avalanches started at the origin $o$, which yield the somewhat modified average cluster sizes:

$$
\begin{equation*}
\langle s\rangle_{o}:=\mathbf{E}_{L}\left[\sum_{x \in V_{L}} n(o, x)\right]=\sum_{x \in V_{L}} G_{L}(o, x) \sim c^{\prime}(d) L^{2}, \quad \text { as } L \rightarrow \infty \tag{3.5}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\langle s_{\text {distinct }}\right\rangle_{o}:=\mathbf{E}_{L}\left[\sum_{x \in V_{L}} \mathbf{1}_{n(o, x) \geq 1}\right]=\sum_{x \in V_{L}} t_{L}(x ; o) . \tag{3.6}
\end{equation*}
$$

For the latter, we find an exponent somewhat different from that of (3.4), namely $\approx L^{1.58}$ in $d=2$, when considering lattice sizes $L=2^{n}, 6 \leq n \leq 13$; see Figure 3-1. The difference could be due to large avalanches started closer to the boundary having significantly smaller size than those started at the center of the box.

In 3D, Grassberger and Manna [7] found the behaviour of (3.4) to be very close to $L^{2}$, and suggest that the difference from (3.3) could be only a logarithmic correction. When we restricted to avalanches started at $o$, we found that (3.6) also differs very little from (3.5); see Figure 3-2. However, our analysis of individual toppling probabilities in 3D suggest that $\eta(3)>0(\approx 0.09)$. The reason why this positive exponent does not affect the average number of topplings could be that the averages are dominated by very large avalanches.

We also collected data on the number of waves in 2D avalanches initiated at the origin. Let $w_{L}(n)$ be the probability of observing $n$ waves in a box of radius $L$. It has been pointed out in [16] that although the expected scaling (in the limit $L \rightarrow \infty$ ) is $w(n) \sim$ $n^{-2}$, the data is better fit with an exponent larger than 2 (about 2.1 in [16]). In Section 3.3.2, we present data restricting to avalanches initiated at the middle of the box, and see that there is better agreement with the theoretically predicted exponent


Figure 3-1: Log-log plot of average cluster size $\langle s\rangle_{o}$ (black) and average number of distinct sites toppled $\left\langle s_{\text {distinct }}\right\rangle_{o}$ (red) versus lattice size $L$ with Dirichlet boundary conditions in $d=2$, when considering $L=2^{n}$ with $6 \leq n \leq 13$ with sample sizes $10^{7}$, $10^{7}, 10^{7}, 10^{7}, 10^{7}, 2.5 \times 10^{6}$ and $6 \times 10^{5}, 6 \times 10^{5}$ respectively. The dots show the data points and the straight lines are the least squares fits to these data points. The slope of the black line is 2 , and the slope of the red line is 1.58 .


Figure 3-2: Average cluster size $\langle s\rangle_{o}$ and average number of distinct sites toppled $\left\langle s_{\text {distinct }}\right\rangle_{o}$ rescaled by $L^{2}$ versus $\log L$ with Dirichlet boundary conditions in $d=3$. We considered the values $L=32,64,128,256$ (with sample sizes $10^{7}, 10^{7}, 4 \times 10^{6}$ and $4 \times 10^{5}$ ).

2 [19, Eqn. (5.11)].
Grassberger and Manna [7] observed that convergence to stationarity is faster, especially in high dimensions, starting from a uniformly random sandpile compared to an empty sandpile. As a partial explanation, in the present chapter we state an asymptotic formula for the single site marginals in stationarity (and in the infinite volume limit) that approaches a uniform distribution as $d \rightarrow \infty$. We give the exact asymptotic formula for the probabilities of small heights given in terms of the Poisson distribution. Our asymptotic formula, whose proof will be given in the next chapter, coincides with the asymptotics of the exact results of Dhar and Majumdar [6] on regular trees of high degree (see formula (8.2) in [6]).

### 3.2 Simulation methods

### 3.2.1 Overview

We use an exact sampling method. By this we mean that we use an algorithm that, given perfectly random numbers as input, will output a recurrent sandpile configuration
that is exactly uniformly distributed (which is the steady-state of the model). First, we generate a uniformly distributed spanning tree (or part thereof) on the underlying graph using Wilson's algorithm [25]. This is an efficient algorithm, and good estimates on its running time are available on cubic lattices. Note that since the number of spanning trees grows exponentially in the number of vertices, it is not a straightforward task to sample one uniformly at random. However, the algorithm in [25], described below, achieves this in polynomial time. Second, we convert the spanning tree into a sandpile configuration (or part thereof) based on a version of Majumdar and Dhar's burning bijection [19]. The sandpile configuration thus obtained is an unbiased sample from the stationary distribution of the model. This allows us to avoid any issues arising from having to estimate mixing properties of the underlying Markovian dynamics. A grain is added to the sampled configuration, and the resulting avalanche computed. The above is repeated a large number of times to obtain independent samples of avalanches in the steady-state. Independence allows us to estimate sampling errors accurately, and avoid issues arising from unknown effects due to correlated samples.

As random number generator, we used the 32 bit version of the Mersenne Twister [21], that is known to have a very large period $\left(>2^{19,000}\right)$. An additional advantage of this generator is that it allows one to 'jump ahead' by a given number of steps in the pseudorandom sequence [8], which allowed us to run computations in parallel with sequences that were guaranteed to be disjoint (we used jump-ahead with different multiples of $2^{100}$ steps on each node).

In 2D only about $44 \%$ of configurations yield an avalanche, and in high dimensions only about fraction $1 / 2 d$. Therefore, in dimensions $d=5$ and higher we used an importance sampling technique that allows us to sample only those configurations that yield an avalanche, and thereby increase the accuracy of our estimates compared to simple sampling. This is described in Section 3.2.3.

### 3.2.2 Wilson's algorithm

Let $G=(V \cup\{\rho\}, E)$ be a finite connected graph, where $\rho$ plays the role of the sink for the sandpile. A loop-erased random walk (LERW) from vertex $x \in V$ with target set $F \ni \rho$ is defined as follows. Consider a simple random walk on $G$ started from $x$ and stopped at the first time it hits $F$. Then erase the loops in the path chronologically, as they are created, yielding a simple path between $x$ and $F$. (When $x \in F$, we define this as the trivial path of zero steps.)

Wilson's algorithm generates a random spanning tree of $G$ as follows. Enumerate the
vertices in $V$ as $V=\left\{x_{1}, \ldots, x_{n}\right\}$, and set $F_{0}=\{\rho\}$. Run a LERW from $x_{1}$ with target set $F_{0}$, and let $\gamma_{1}$ be the path of the LERW. Set $F_{1}=F_{0} \cup \gamma_{1}$. Next, run a LERW from $x_{2}$ with target set $F_{1}$, and let $\gamma_{2}$ be the path of the LERW. Set $F_{2}=F_{1} \cup \gamma_{2}$, etc. The union of the loop-erased walks $\gamma_{1}, \ldots, \gamma_{n}$ form a random spanning tree of $G$. Wilson proved that the tree is uniformly distributed over all spanning trees of $G$ [25], regardless of the chosen enumeration of $V$. Wilson also showed that the running time of the above algorithm is the mean commute time between $\rho$ and a randomly chosen vertex that is distributed according to the stationary distribution of the simple random walk on $G$. Here the commute time between vertices $x$ and $y$ of $G$ is defined to be $\mathbf{E}_{x} T_{y}+\mathbf{E}_{y} T_{x}$, where $T_{x}$ is the first hitting time of $x$, and $\mathbf{E}_{x}$ is expectation over random walk started at $x$.

In our 2D simulations we used two different boundary conditions: (i) $G$ is the $2 L \times 2 L$ torus with $\rho$ equal one of the vertices (periodic boundary conditions); (ii) $G$ is given by the box $V_{L}=[-L+1, L-1]^{2}$, with $\rho$ equal to the entire boundary of this box (Dirichlet boundary conditions). In case (i), the mean commute time between $\rho$ and a random vertex of the torus is of order $L^{2} \log L$; see [18, Proposition 10.21], and it is of the same order with the boundary condition (ii); see [18, Proposition 10.7]. Hence the entire spanning tree can be generated in time $O\left(L^{2} \log L\right)$.

In 3D, we only used the Dirichlet boundary condition, and in this case the entire tree can be generated in time $O\left(L^{3}\right)$; see [18, Proposition 10.21].

In 5D and higher, avalanches typically take place on a small subset of the box. (The upper critical dimension for the model is $d_{c}=4 ;[24]$.) Hence on high-dimensional lattices we only generated those LERWs that were necessary to compute the avalanche; see Sections 3.2.3 and 3.2.4 below. The time required to compute a single LERW from the bulk of the lattice to the boundary is $O\left(L^{2}\right)$, as this is the number of random walk steps required. We used hashing [15, Sections $6.5,6.6]$ to store the generated random walk steps, and the resulting LERW, so the memory requirement for a single LERW is also $O\left(L^{2}\right)$. This method of simulation allowed us to investigate the height distribution at the origin in very high dimensions ( $d=32$ ), on lattices of radius up to $L=128$, as this only requires running LERWs from the origin and its neighbours.

### 3.2.3 Bijection and importance sampling

We first recall Majumdar and Dhar's burning bijection [19]. Given a sandpile configuration $\eta$ in volume $V$, first burn the sink vertex $\rho$. Then at each step $t \geq 1$, burn all
vertices $x$ such that

$$
\eta(x) \geq \#\{\text { unburnt neighbours after step } t-1\} .
$$

Let $B_{t}=\{x \in V: x$ burnt at step $t\}$. Connect a vertex in $B_{t}$ to a neighbour in $B_{t-1}$ by an edge. If there are more than one such neighbours, the choice can be made depending on the value of $\eta(x)$, in a bijective fashion. This maps the sandpile $\eta$ to a spanning tree. In order to invert the map at a vertex $x$, it is sufficient to know the length of the paths in the spanning tree from $x$ and its neighbours to $\rho$. For this purpose, when we generate our LERWs, we also record their lengths. Then the tree-distance dist $(x, \rho)$ from any vertex $x$ to $\rho$ is given by the sum of the length of the LERW $\gamma_{x}$ from $x$ to its endpoint $y$ in its target set and the tree-distance $\operatorname{dist}(y, \rho)$ from $y$ to $\rho$. (This is already available when $\gamma_{x}$ is generated, if we record the tree-distance along each LERW after they were generated.) We checked the one-site marginals obtained with the above method in 2D against the exactly known values [23], [13], [22], [14] and there was very close agreement.

We will need the following modification of the above burning rule [23]. Let us burn vertices as above, with the exception that the origin $o$ is not allowed to burn. This way there will be a set $W \subset V$, such that $o \in W$, and $W$ did not burn yet. Once only $W$ is left unburnt, we burn $o$ and complete the process by burning $W$. The following fact will be important. Let

$$
q_{d}(i)=\mathbf{P}\left[\operatorname{deg}_{W}(o)=i\right], \quad i=0, \ldots, 2 d-1,
$$

where $\operatorname{deg}_{W}(o)$ denotes the degree of vertex $o$ in the subgraph of $V$ induced by $W$, in other words, $\operatorname{deg}_{W}(o)=\#\{y \in W: y \sim o\}$. Then conditioned on the random variable $\operatorname{deg}_{W}(o)$, we have that the random variable $\eta(o)$ is uniformly distributed over the set $\left\{\operatorname{deg}_{W}(o), \ldots, 2 d-1\right\}[23]$. Then we have

$$
p_{d}(i)=\mathbf{P}[\eta(o)=i]=\sum_{j=0}^{i} \frac{q_{d}(j)}{2 d-j} .
$$

Let us modify the bijection based on the above burning process, as follows: from vertices in $V \backslash W$, we choose an outgoing edge of the spanning tree as in Majumdar and Dhar's original bijection. We choose an outgoing edge from $o$ to the set $V \backslash W$ according to the value of $\eta(o) \in\left\{\operatorname{deg}_{W}(o), \ldots, 2 d-1\right\}$ in a bijective fashion. Finally, we choose outgoing edges from vertices in $W$, again as in the standard burning bijection. In the resulting spanning tree we have that $W$ equals the set of descendants of $o$.

For the purposes of simulating $\operatorname{deg}_{W}(o)$, we would like to be able to distinguish during the simulation which vertices are descendants of $o$. We first note that we are not able to use the graph distance in the tree to do this. For example, we have

$$
\{y \sim o: y \in W\} \subset\{y \sim o: \operatorname{dist}(y, \rho)>\operatorname{dist}(o, \rho)\},
$$

but the containment may be strict: there can exist $z \sim o, z \notin W$ such that $\operatorname{dist}(z, \rho)>$ $\operatorname{dist}(o, \rho)$.

Let us define a new distance function dist' in the tree under which the edge pointing from $o$ to its neighbour outside $W$ has a new length SHIFT, but all other edges of the tree still have length 1 , where SHIFT will be a sufficient large integer. We choose SHIFT large enough, so that

$$
\{y \in W: y \sim o\}=\left\{y \sim o: \operatorname{dist}^{\prime}(y, \rho)>\operatorname{dist}^{\prime}(o, \rho)\right\} .
$$

Therefore, $\operatorname{deg}_{W}(o)$ will be readily available if the function $d^{\prime}(x)=\operatorname{dist}^{\prime}(x, \rho)$ can be easily simulated.

To find $d^{\prime}$, we first generated the LERW $\gamma_{o}$ from $o$ to $\rho$. We added the large constant SHIFT to its length, and set $d^{\prime}(o)=\operatorname{dist}(o, \rho)+\operatorname{SHIFT}$, and $d^{\prime}(y)=\operatorname{dist}(y, \rho)$ for all other vertices on $\gamma_{o}$. Then we generated the remaining LERWs. For a LERW started at $x$ that hits its target set at $y$, we computed $d^{\prime}(x)=\left|\gamma_{x}\right|+d^{\prime}(y)$. Note here $d^{\prime}(y)$ is already available, and $\left|\gamma_{x}\right|$ is available from the newly simulated LERW. Similarly, for other vertices $z$ on $\gamma_{x}$, the distance $d^{\prime}(z)$ can be computed. The added shift at $o$ ensures that if $x \notin W$, then $d^{\prime}(x)=\operatorname{dist}(x, \rho)$, while for $x \in W$, we have $d^{\prime}(x)=\operatorname{dist}(x, \rho)+$ SHIFT. Choosing SHIFT sufficiently large ensures that

$$
\{y \in W: y \sim o\}=\left\{y \sim o: d^{\prime}(y)>d^{\prime}(o)\right\}
$$

and hence $\operatorname{deg}_{W}(o)$ is readily available from the simulated spanning tree. In $d=2,3$ we chose SHIFT to be the volume of the box $(2 L)^{d}$, and in $d \geq 5$ we chose it to be the size of the hashtable. Note that $d^{\prime}$ can be used directly to find the sandpile heights under the bijection, regardless whether $x$ is in $W$ or not.

### 3.2.3.1 5D variance estimate

In dimension $d=5$, in order to only sample configurations where avalanches occur, we disregard the value of $\eta$ at $o$, and set it equal to $2 d-1$. This biases the toppling probabilities in a computable way.

Let $Q=\operatorname{deg}_{W}(o)$ and $P=\eta(o)$ be random variables, then, according to the bijection, the follow three objects are conditionally independent given $W$ : $P$, the restriction of $\eta$ to $W \backslash\{o\}$, the restriction of $\eta$ to $V \backslash W$. In particular, the conditional probability of $P$ given $W$

$$
\begin{equation*}
\mathbf{P}_{L}[P=j \mid W]=\frac{1}{2 d-Q}, \quad 0 \leq Q \leq j \leq 2 d-1 \tag{3.7}
\end{equation*}
$$

Let the avalanche cluster be $\mathrm{Av}:=\{x \in V: x$ topples at least once after adding at $o\}$. Let $\mathrm{Av}^{*}:=\{x \in V: x$ topples at least once after setting $\eta(o)=2 d-1\}$, the resulting avalanche of the original configuration $\eta$ after setting $\eta(o)=2 d-1$. Let $p_{L}(i, x):=$ $\mathbf{P}_{L}\left[x \in \mathrm{Av}^{*} \mid Q=i\right]$.

Since in order for vertex $x$ to be toppled in Av, the origin $o$ has to be toppled, and hence when $x \in$ Av we have $P=2 d-1$. On the event $P=2 d-1$, the events $x \in \operatorname{Av}$ and $x \in \mathrm{Av}^{*}$ are the same. Let $\mathcal{W}_{i}$ be the collection of possible values of $W$ when $Q=i$. We have the toppling probability by the law of total probability

$$
\begin{aligned}
\mathbf{P}_{L}[x \in \mathrm{Av}] & =\mathbf{P}_{L}[x \in \mathrm{Av}, P=2 d-1]=\mathbf{P}_{L}\left[x \in \mathrm{Av}^{*}, P=2 d-1\right] \\
& =\sum_{i=0}^{2 d-1} \sum_{w \in \mathcal{W}_{i}} \mathbf{P}_{L}[W=w] \mathbf{P}_{L}[P=2 d-1 \mid W=w] \mathbf{P}_{L}\left[x \in \mathrm{Av}^{\star} \mid W=w\right] .
\end{aligned}
$$

The last term of the right-hand side holds since conditional on $\{W=V\},\{P=2 d-1\}$ and $\left\{x \in \mathrm{Av}^{*}\right\}$ are independent, as $x \in \mathrm{Av}^{*}$ only depends on the restriction of $\eta$ to $V \backslash\{o\}$. We have $\mathbf{P}_{L}[P=2 d-1 \mid W=w]=1 /(2 d-i)$, for $w \in \mathcal{W}_{i}$.

Then the toppling probability

$$
\begin{aligned}
\mathbf{P}_{L}[x \in \mathrm{Av}] & =\sum_{i=0}^{2 d-1} \frac{1}{2 d-i} \sum_{w \in \mathcal{W}_{i}} \mathbf{P}_{L}[W=w] \mathbf{P}_{L}\left[x \in \mathrm{Av}^{\star} \mid W=w\right] \\
& =\sum_{i=0}^{2 d-1} \frac{1}{2 d-i} \mathbf{P}_{L}\left[x \in \mathrm{Av}^{*}, Q=i\right] .
\end{aligned}
$$

In order to estimate this, we check whether $x$ toppled or not in $\mathrm{Av}^{*}$. Grouping the result according to the values of $Q$, this gives us an estimate of $p_{L}(i, x)$. Summing over all possible values of $Q$, we estimate the toppling probability.

In order to estimate the standard error of the toppling probability estimate, let $Z_{x}=$ $I[x \in \mathrm{Av}]$ be a random variable, $\mathbf{P}_{L}\left[Z_{x}=1 \mid P=2 d-1, Q=i\right]=p_{L}(i, x)$. Conditioned on $\{Q=i\} \cap\{P=2 d-1\}$, we define the random variable $Y_{x}=\frac{1}{2 d-i} Z_{x}$. Then the
adjusted toppling probability

$$
\mathbf{P}_{L}\left[\left.Y_{x}=\frac{1}{2 d-i} \right\rvert\, Q=i\right]=p_{L}(i, x)
$$

with the expectation and variance conditioned on $Q$

$$
\mathbf{E}_{L}\left[Y_{x} \mid Q\right]=\frac{p_{L}(Q, x)}{2 d-Q} \quad \text { and } \quad \operatorname{Var}_{L}\left(Y_{x} \mid Q\right)=\frac{p_{L}(Q, x)\left(1-p_{L}(Q, x)\right)}{(2 d-Q)^{2}} .
$$

The toppling probability $\mathbf{P}_{L}[x \in \mathrm{Av}] \simeq \frac{1}{n} \sum_{k=1}^{n} Y_{x}^{(k)}$, where each $Y_{x}^{(k)}$ takes values 0 or $1 /(2 d-Q)$. Comparing with $Z_{x}$, we expect that the variance of $Y_{x}$ is smaller than that of $Z_{x}$.

Denote $q_{L}(i)=\mathbf{P}_{L}[Q=i]$. Since $\mathbf{E}_{L}\left[Y_{x}\right]=\mathbf{E}_{L}\left[\mathbf{E}_{L}\left[Y_{x} \mid Q\right]\right]=\mathbf{P}_{L}[x \in \mathrm{Av}]=t_{L}(x ; o)=:$ $t_{L}(x)$ and $t_{L}(x)=\sum_{i=0}^{2 d-1} q_{L}(i) \frac{p_{L}(i, x)}{2 d-i}$, we have

$$
\begin{aligned}
& \operatorname{Var}_{L}\left(\mathbf{E}_{L}\left[Y_{x} \mid Q\right]\right)=\mathbf{E}_{L}\left[\frac{p_{L}(Q, x)^{2}}{(2 d-Q)^{2}}\right]-t_{L}(x)^{2}, \\
& \mathbf{E}_{L}\left[\operatorname{Var}_{L}\left(Y_{x} \mid Q\right)\right]=\mathbf{E}_{L}\left[\frac{p_{L}(Q, x)\left(1-p_{L}(Q, x)\right)}{(2 d-Q)^{2}}\right], \\
& \operatorname{Var}_{L}\left(Y_{x}\right)=\mathbf{E}_{L}\left(\operatorname{Var}_{L}\left(Y_{x} \mid Q\right)\right)+\operatorname{Var}_{L}\left(\mathbf{E}_{L}\left[Y_{x} \mid Q\right]\right)=\mathbf{E}_{L}\left[\frac{p_{L}(Q, x)}{(2 d-Q)^{2}}\right]-t_{L}(x)^{2} \\
& \quad=\sum_{i=0}^{2 d-1} q_{L}(i) \frac{p_{L}(i, x)}{(2 d-i)^{2}}-t_{L}(x)^{2} \leq t_{L}(x)-t_{L}(x)^{2}=t_{L}(x)\left(1-t_{L}(x)\right) .
\end{aligned}
$$

Based on the above, we recorded $\sum_{i=0}^{2 d-1} \hat{q}_{L}(i) \frac{\hat{p}_{L}(i, x)}{(2 d-i)^{2}}-\hat{t}_{L}(x)^{2}$, where $\hat{p}, \hat{q}$ and $\hat{t}$ denote simulation estimates. This gives an approximation to the variance of the toppling probability estimate at $x$. We obtained $\hat{q}_{L}(i)$ as the number of sample $Q=i$ dividing by the total number of samples. $\hat{p}_{L}(i, x)$ is the number of samples in which $x$ topples and $Q=i$ divided by the number of samples in which $Q=i$. Finally, we obtained

$$
\hat{t}_{L}(x)=\sum_{i=0}^{2 d-1} \hat{p}_{L}(i, x) \hat{q}_{L}(i) .
$$

As a comparison with simple sampling we note the following. Let $d=5, L=32$, and let $x$ be a neighbour of the origin. Then simple sampling with $1.5 \times 10^{6}$ avalanches (on 64 nodes) took 16,901 seconds of CPU time per node, resulting in the estimate
$\hat{t}_{L}(x)=0.0157 \pm 0.0001$. With variance reduction, the same precision was obtained with $1.5 \times 10^{5}$ avalanches (on 64 nodes) and took 3455 seconds of CPU time per node, with a time save of a factor 4.89 .

### 3.2.3.2 Variance of height probability estimates in $d=32$

We recorded the estimated probabilities $\hat{q}_{d}(i)$ for $i=0, \ldots, 2 d-1$ while simulating the height probabilities $\hat{p}_{d}(i)$ in $d=32$. Then we can compute the variance of the height probability estimates as follows. We have

$$
\hat{p}_{d}(i)=\sum_{j=0}^{i} \frac{\hat{q}_{d}(j)}{2 d-j}, \quad i=0, \ldots, 2 d-1 .
$$

Since the estimates $\hat{q}_{d}(j)$ are almost independent, except for the constraint $\sum_{j=0}^{2 d-1} \hat{q}_{d}(j)=$ 1 , the variance of the height probability estimates is

$$
\operatorname{Var}\left(\hat{p}_{d}(i)\right) \simeq \sum_{j=0}^{i} \frac{1}{(2 d-j)^{2}} \operatorname{Var}\left(\hat{q}_{d}(j)\right)=\sum_{j=0}^{i} \frac{1}{(2 d-j)^{2}} \frac{\hat{q}_{d}(j)\left(1-\hat{q}_{d}(j)\right)}{n},
$$

where $i=0,1, \ldots, 2 d-1$ and $n$ is the number of samples generated.

### 3.2.4 Avalanche simulation

Using the Abelian property of the model, Ivashkevich, Ktitarev and Priezzhev [9] introduced a special order of topplings of non-stable vertices during an avalanche. We use this to generate the avalanche in $d=2,3$ and 5 as follows. First, adding a grain to $o$, if $o$ is unstable, we topple it once and then topple all possible vertices without toppling $o$ a second time. The toppled vertices form the first wave of topplings in the avalanche. Then, we allow the vertex $o$ to topple a second time creating a second wave and so on. The process terminates when $o$ becomes stable. Hence, we obtain the representation of the avalanche as a sequence of waves. In each wave, no vertex can topple more than once. We made use of this property as follows: whenever a vertex reached the height $2 d$, we pushed it onto a stack containing vertices to be toppled, and popped from this stack until it became empty.

The avalanches are expected to behave differently in dimensions $d \geq 5$, compared to $d=2,3$. Long loops are unlikely, and the loop erased random walk behaves similarly to the random walk, in particular it scales diffusively [17, Section 7.7]. Also, independent random walks starting from two neighbouring vertices are likely to either meet after a
few steps or not to meet at all, i.e. they are likely to connect to the sink with disjoint paths. Considering the Dirichlet boundary conditions, the number of vertices in $d$ dimensions is $O\left(L^{d}\right)$ and the number of steps that the random walk takes to exit a box is $O\left(L^{2}\right)$. In high dimensions, the order of the number of vertices grows much faster than that of the number of steps of the random walk. Therefore, it is very inefficient to store the entire box, since we are likely to only use a small part of the box. The idea is to only generate loop erased random walks when they are needed. The way to do this is the following.

First, the loop erased random walks starting from $o$ and its neighbours are generated. This allows the computation of the random variable $Q$. We then set $\eta(o)=2 d-1$. We then compute the sand heights of the neighbours by running loop erased random walks starting from their neighbours. This allows the toppling of $o$ to be carried out. We repeat the above steps as long as there are topplings.

It would be time-consuming to search every step of previous loop-erased walks to check whether they were hit or not. Instead, we do the following to see when a random walk hits previous paths. We used the technique of hashing [15, Sections 6.5, 6.6] to store the steps of the walk in such a way that it is easy to locate intersections. For a hash function $f: V_{L} \rightarrow\{0,1, \ldots$, HASHSIZE -1$\}$, if we have a vertex $x \in V_{L}$, we apply a function $f(x)$ that assigns a memory location. The first walk will occupy some part of this memory space, and when we run a second walk and so on, we have to compute $f$ of the current location of the walk and check if that memory location is already used or not. If yes, the current walk hits a previous walk. If not, we will continue the walk. One of the difficulties is that it can happen that $f(x)=f(y)$ for distinct $x, y$, that is, the random walk passing through another space point has the same location assigned to it in the memory. We use linking and a separate table to keep track of the used memory spaces. This will be discussed in more detail after explaining the hash functions. See [15, Sections 6.5,6.6] for a detailed description of the idea of hashing.

We first used a hashing function $g$ based on projecting the box $[-L, L]^{d}$ onto a torus with size length $N$, with a linear ordering of the vertices of the torus.

$$
g(x)=\sum_{i=1}^{d}\left(x_{i} \bmod N\right) \times N^{i-1},
$$

where the size of the hashtable was HASHSIZE $=N^{d}$.
Experimentally, the following hash function $f$ tends to be more efficient than the simple projection method. Hence, our simulation used a hash function (called function hash)
of the form

$$
f(x)=\sum_{i=1}^{d} x_{i} * m^{i-1}(\bmod \text { HASHSIZE })
$$

for a point $x$ in the box $\{1, \ldots, 2 L-1\}^{d}$, where $L, m$ and HASHSIZE are powers of 2. Setting $L, m$ and HASHSIZE as powers of 2 and writing them in binary numbers allows one to compute $f$ efficiently.

We used two arrays to manage the hashing algorithm. One is called hashtab with size HASHSIZE to store the hash values already used by the hash function $f$. The other one is the vertexdata with size HASHSIZE to store the positions, sandpile and random walk data of the corresponding vertices. vertexdata is a list of all vertices visited by random walks so far.

For instance, see Figure 3-3. For vertex 1 (the origin o), we first use the hash function $f$ to compute a hash value, and store the hash value in the array hashtab. Then we install vertex 1 in the first node of the array vertexdata. For vertex 2, we repeat similar steps by first using the hash function $f$ to compute a hash value, and storing the hash value in the array hashtab. Then we install vertex 2 in the second node of the array vertexdata. When a collision happens, i.e. $f(x)=f(y)$ for some vertices $x \neq y$, we need to distinguish $x$ and $y$ in the table vertexdata. This is shown in the third picture of Figure 3-3. Say that vertex $k$ has the same hash value as vertex 2 . Then vertex $k$ will map to the same value as vertex 2 in hashtab. Checking whether the actual coordinates are the same or not, we find that $y$ is different from $x$. We place a link from vertex 2 in vertexdata to the next available free node. When a further vertex, say vertex $l$, is mapping to the same place as vertices $2, k$ in hashtab, we place an additional link from vertex $k$ to the next available free node in vertexdata. We repeat the above steps until all the random walk steps needed for the avalanche have been computed, unless the vertexdata is full, in this case, the sample will be discarded.

In high dimension, avalanches are 4 dimensional and an avalanche that reaches all the way to the boundary has about $O\left(L^{4}\right)$ vertices. Each of these $L^{4}$ vertices will have its own random walk. This means we need at most $O\left(L^{6}\right)$ random walk steps to be stored, independently of the dimension. We will discuss this further in Section 6.2.


Figure 3-3: This figure explains how the hashing algorithm is implemented.
$\log \mathrm{P}[\mathrm{x}$ topples]


Figure 3-4: A heat-plot of the logarithm of the toppling probability with Dirichlet boundary conditions in $d=2$ for a system with $L=4096$. The values are shown for vertices in the box $[-256,256]^{2}$.

### 3.3 2D results

### 3.3.1 Toppling probabilities in the bulk

We simulated the toppling probability both with Dirichlet and periodic boundary conditions. We found similar behaviour in different radial directions; see Figure 3-4 and $3-5$. In this thesis, we included an extra rainbow plot that demonstrates the phenomenon more clearly at various radial distances from the origin $o$. It appears that asymptotically, the toppling probability only depends on the Euclidean distance from the origin (in the infinite volume limit). In the case of periodic boundary conditions, with the largest system size considered $L=4096$, we occasionally encountered some extremely long avalanches.

Assuming the behaviour $t(x):=\mathbf{P}[x$ topples $] \sim c|x|^{-\eta}$ for $|x| \gg 1$ (that is, in the infinite volume limit $L \rightarrow \infty$ ), we want to estimate $\eta=\eta(2)$. Let $\hat{t}_{L}(x)$ be the number of samples in which $x$ topples divided by the total number of samples. A log-log plot of the numerical estimates $\hat{t}_{L}(x)$ are shown in Figure 3-6.

We have attempted to fit a finite-size scaling form $t_{L}(x) \approx c|x|^{-a} f_{2}\left(|x| / L^{1 / \nu}\right)$, with


Figure 3-5: A rainbow-plot of the logarithm of the toppling probability with Dirichlet boundary conditions in $d=2$ for a system with $L=4096$. The values are shown for vertices in the box $[-256,256]^{2}$.

Toppling probabilities with Dirichlet b.c.


Figure 3-6: The logarithm of the toppling probabilities against the logarithm of $|x|$ 's with Dirichlet boundary conditions in $d=2$ for systems with $L=512$ (blue), 1024 (yellow), 2048 (green), 4096 (red), and 8192 (black) with sample sizes $6 \times 10^{7}, 3 \times 10^{7}$, $7.5 \times 10^{6}, 4 \times 10^{6}$ and $10^{6}$ respectively. The probabilities are shown for vertices in the positive $x$-axis up to $L-1$.


Figure 3-7: The estimates $\hat{t}_{L}(x)$ rescaled by the $|x|^{\hat{a}}$, where $\hat{a}=0.43$ is obtained from finite-size scaling for $50 / 512 \leq|x / L| \leq 1$, when considering Dirichlet boundary conditions in $d=2$ for systems with $L=512$ (blue), 1024 (yellow), 2048 (green), 4096 (red), and 8192 (black). Sample sizes are as in Figure 3-6.
a scaling function $f_{2}$, to the data. For this, we minimized the sum of squares of the pairwise differences between $t_{L}(x)|x|^{a}$ and $t_{L^{\prime}}\left(x^{\prime}\right)\left|x^{\prime}\right|^{a}$, with $|x| / L^{1 / \nu}=\left|x^{\prime}\right| /\left(L^{\prime}\right)^{1 / \nu}$, normalized by the standard error of the difference. First, this clearly showed that we must have $\nu=1$. Second, we obtained a reasonable collapse of the data for $\hat{a}=0.43$, when small $|x|$ values $(|x / L|<50 / 512)$ were excluded from the least squares sum; see Figure 3-7.

For all pairs of $L$ and $L^{\prime}$, where $L \neq L^{\prime}$ and $L, L^{\prime} \in\left\{2^{9}, \ldots, 2^{13}\right\}$, we sum over $x$ on the positive $x$-axis where $|x / L| \geq 50 / 512$ (excluded small $|x|$ values)

$$
\sum_{L \neq L^{\prime}} \sum_{x} \frac{\left(t_{L}(x)|x|^{a}-t_{L^{\prime}}\left(x^{\prime}\right)\left|x^{\prime}\right|^{a}\right)^{2}}{\sqrt{\sigma_{x, L}^{2}+\sigma_{x^{\prime}, L^{\prime}}^{2}}}
$$

where $\sigma_{x, L}$ is the standard error for $t_{L}(x)|x|^{a}$ and $\sigma_{x^{\prime}, L^{\prime}}$ is the standard error for $t_{L^{\prime}}\left(x^{\prime}\right)\left|x^{\prime}\right|^{a} .\left|x^{\prime}\right|$ is the nearest integer to $|x|\left(L^{\prime}\right)^{1 / \nu} / L^{1 / \nu}$.

We considered the above sum is dependant on $\nu$ and $a$. First we found when $\nu$ is different from $1(\nu<1$ or $\nu>1)$ the sum is greater than that when $\nu=1$. Setting $\nu=1$, we found the minimum as a function of $a$, for $\hat{a}=0.43$.

Least squares fit


Figure 3-8: The logarithm of the toppling probability against the logarithm of $|x|$ with Dirichlet boundary conditions in $d=2$ for a systems with $L=8192$ (black dots). The probabilities are shown for vertices in the positive $x$-axis up to $L-1$. The line of best fit with slope 0.42 is from the least squares method (red line).

Alternatively, a least squares fit of $\log \hat{t}_{L}(x)$ against $\log |x|$, for vertices $x$ along the positive $x$-axis with $20 \leq|x| \leq 150$ gives the estimate $\eta(2) \approx \hat{\eta}(2)=0.42$; see Figure $3-8$. This minimizes the sum

$$
\sum_{20 \leq|x| \leq 150}\left(\log \hat{t}_{L}(x)-\eta(2) \log |x|-c\right)^{2},
$$

where we do not care about the value of $c$. We choose the range of $|x|$ by visualising the graph plot. We used the function lsfit in R to obtain the least squares estimate.

Multiplying $\hat{t}_{L}(x)$ by $|x|^{\hat{\eta}}$ we found little deviation from a constant; see Figure 3-9.
We also simulated the toppling probability with periodic boundary conditions, and this gave similar results. The agreement with the power law appears to extends to a longer interval, and the estimated exponent, based on a least squares fit to the log-log data over $20 \leq|x| \leq 500$, gave $\hat{\eta}(2) \approx 0.41$. Figure $3-10$ shows the toppling probabilities rescaled by $|x|^{0.41}$ in systems of size $L=512,1024,2048,4096$. Figure $3-11$ compares the $L=4096$ data rescaled with varying $\eta$.

The least squares fit of $\log \hat{t}_{L}(x)$ against $\log |x|$, for all vertices $x$ with Euclidean distance


Figure 3-9: The toppling probability with Dirichlet boundary conditions in a system with $L=8192$, rescaled by $|x|^{\hat{\eta}}(\hat{\eta}=0.42)$ against the logarithm of $|x|$, where $x$ is taken along the positive $x$-axis between 1 and $L-1$. The black curve connects all integer points with $x$ between 0 and $L-1$. The error bars show $\pm 2$ standard deviations (for every power of 2 only, for readability). The number of samples taken was $10^{6}$.

## Rescaled toppling probability



Figure 3-10: Rescaled toppling probabilities against the logarithm of $|x|$ with periodic boundary conditions in $d=2$ for systems with $L=512$ (yellow), 1024 (green), 2048 (red), and 4096 (black) (with sample sizes $2 \times 10^{7}, 1.5 \times 10^{7}, 3 \times 10^{6}$ and $7.5 \times 10^{5}$ ). The probabilities are shown for vertices in the positive $x$-axis up to $L$.
$20 \leq|x| \leq 100$ gives an estimate $\hat{\eta}^{\prime}(2) \approx \hat{\eta}(2)$, that is, $\hat{\eta}^{\prime}(2) \approx 0.42$. Multiplying $\hat{t}_{L}(x)$ by $|x|^{\hat{\eta}^{\prime}}$ the graph settles to be horizontal for moderate values of $|x|$; see Figure 3-12.

In summary, we have estimates of toppling probabilities along the positive $x$-axis, both for Dirichlet and periodic boundary conditions. The least squares fit for the moderate values of $x$ with Dirichlet boundary condition gives the exponent 0.42 ; see Figure 3-8. In addition, the finite-size scaling with Dirichlet boundary condition gives the exponent 0.43 ; see Figure 3-7. For periodic boundary conditions, rescaling the toppling probabilities with $\eta=0.41$ and $L=512,1024,2048,4096$ is shown in Figure $3-10$. As there is little dependence on $L$ for moderate values of $x$, this suggests that this plot shows the results close to the infinite volume limit. Rescaling the toppling probabilities with different values of $\eta=0.38,0.405,0.425$ and $L=4096$ is shown in Figure 3-11. Both the curves for $\eta=0.405,0.425$ are reasonably flat for moderate values of $x$.

Based on the above, we make the following conjecture for the toppling probability in the infinite volume limit $L \rightarrow \infty$ :

$$
\mathbf{P}[x \text { topples }]=|x|^{-\eta+o(1)},
$$



Figure 3-11: Toppling probabilities against the logarithm of $|x|$ with periodic boundary conditions in $d=2$ for a system with $L=4096$ rescaled by $|x|^{\eta}$ with $\eta=\hat{\eta} \approx 0.425$ (black), $\eta=0.405$ (blue), $\eta=0.38$ (red). The probabilities are shown for vertices in the positive $x$-axis up to $L$. Error bars (green) shown for $|x|$ a power of 2 .


Figure 3-12: The toppling probability with Dirichlet boundary conditions in $d=2$ for a system with $L=4096$, rescaled by $|x|^{\hat{\eta}}(\hat{\eta}=0.42)$. The probabilities are shown for all vertices in a disk of radius 100 .


Figure 3-13: The difference of logarithms for $L=512$ (green), 1024 (red), and 2048 (black) in $d=2$. A horizontal line with $y=\hat{a} \approx 0.44$ (black line).
where $\eta=\eta(2)=0.4$, to 1 d.p. However, we cannot make a conclusion for the second decimal place just based on numerical results.

Toppling probabilities in the scaling limit. Finally, we comment on the toppling probability of those vertices $x$ whose distance from the origin is of the same order as the system size $L$. Assume that the toppling probability for $x$ within a system of size $L$ scales as $t_{L}(x)=|x|^{-a} f_{2}(x / L)$ with some exponent $a$ and a scaling function $f_{2}$. Then we can estimate $\hat{a}$ as follows. The toppling probability for $2 x$ in a system of size $2 L$ is $\hat{t}_{2 L}(2 x)=|2 x|^{-\hat{a}} f_{2}(2 x / 2 L)=2^{-\hat{a}}|x|^{-\hat{a}} f_{2}(x / L)=2^{-\hat{a}} \hat{t}_{L}(x)$. A plot of the difference of logarithms

$$
\frac{1}{\log 2}\left(\log t_{L}(x)-\log t_{2 L}(2 x)\right) \approx \hat{a}
$$

is shown in Figure $3-13$ for $L=2048,1024,512$. The data is affected by imprecise estimates of avalanches near the boundary. Hence, in estimating $\hat{a}$, we considered the mean difference over $1 / 2 \leq|x| / L \leq 3 / 4$. Using the data for $L=2048$ this gives $\hat{a}=\hat{a}(2)=0.44 \pm 0.05$, which is not far from the exponent obtained in the bulk.

Rescaling by $|x|^{\hat{a}}$, where $\hat{a}=0.43$ was obtained from the finite-size scaling analysis, and plotting against $|x| / L$, yields the graph in Figure 3-14. The graph suggests that as long as $|x| / L$ is bounded away from 0 , the rescaled quantity $t_{L}(x)|x|^{a}$ has a scaling


Figure 3-14: The rescaled toppling probability with Dirichlet boundary conditions in $d=2$ for systems with $L=512$ (yellow), 1024 (green), 2048 (red), and 4096 (black). The probabilities are shown for points on the positive $x$-axis with $50 / 512 \leq|x / L| \leq 1$.
limit $f_{2}(y)$, as $x / L \rightarrow y \in[-1,1]^{2}$.

### 3.3.2 The number of waves in 2D

In this section we present results on the number of waves in avalanches initiated at the origin. The distribution of the number of waves is the most interesting in 2D, since the average number of waves diverges logarithmically as $L \rightarrow \infty$ :

$$
\begin{equation*}
\mathbf{E}_{L}[\text { number of waves }]=G_{L}(o, o) \sim \frac{1}{2 \pi} \log L, \quad \text { as } L \rightarrow \infty . \tag{3.8}
\end{equation*}
$$

Recalling that $n(o, o)$ denotes the number of topplings at the origin $o$ caused by addition of a grain at the origin $o$, let us put

$$
w_{L}(n)=\mathbf{P}_{L}[n(o, o)=n], \quad w(n)=\lim _{L \rightarrow \infty} w_{L}(n) .
$$



Figure 3-15: Log-log plot of the probability of observing $n$ waves in a box of radius $L$, $w_{L}(n)$, versus $1 \leq n \leq 50$ with Dirichlet boundary conditions in $d=2$, when considering $L=8192$.
(The existence of the limit defining $w(n)$ is known rigorously; see [3].) Let us define the complementary distribution functions

$$
\begin{aligned}
W_{L}(n) & =\mathbf{P}_{L}[n(o, o) \geq n]=\sum_{m \geq n} w_{L}(m) \\
W(n) & =\mathbf{P}[n(o, o) \geq n]=\sum_{m \geq n} w(m)
\end{aligned}
$$

Assuming that $w(n)$ decays as a power law: $w(n) \approx n^{-\delta}$, and from the divergence of the mean as in (3.8), it has been predicted that $W(n) \approx n^{-1}$ (and hence $\delta=2$ ); see [19, Eqn. (5.11)]. Our simulation results found close agreement with this exponent. For $L=8192$ and $5 \leq n \leq 50$, a least square fit to $\log \hat{w}(n) / \log n$ gives a slope approximately -2.04 ; see Figure 3-15.

In Figure 3-16 we show for $L=8192$ the rescaled quantity $w_{L}(n) n^{2}$. Error bars are shown up to $n=200$. Beyond this bound, avalanches with particular values of $n$ are too infrequent to estimate from our data. There is no clear indication of the rescaled values settling down to a constant for moderate $n$.

The data is a lot smoother for $W_{L}(n)$, and the errors for the rescaled quantity $W_{L}(n) n$


Figure 3-16: $w_{L}(n) n^{2}$ with Dirichlet boundary conditions in $d=2$, when considering $L=8192$, against $\log n$ for $1 \leq n \leq 2000$. Error bars show $\pm 2$ standard deviations. The sample size is $10^{6}$.
are also smaller. In Figure $3-17$ we show for $L=8192$ the rescaled quantity $W_{L}(n) n$ against $\log n$ for $1 \leq n \leq 2000$. It is apparent that the graph does not settle to a constant value for moderate values of $n$. Therefore, if $W(n)$ indeed satisfies an asymptotic of the form $W(n) \sim c n^{-1}$, convergence to this asymptotic is reached only for very large values of $L$ and $n$. It has been pointed out in [7] that the simulation data in that paper better fits with $\delta \approx 2.1$. However, any exponent $>2$ can be ruled out, as

$$
\begin{equation*}
\sum_{n \geq 1} n w(n)=\lim _{L \rightarrow \infty} G_{L}(o, o)=\infty \tag{3.9}
\end{equation*}
$$

An alternative possibility is that the scaling behaviour of $W_{L}(n)$ depends in a more complicated way on $L$ and $n$. For example, it is consistent with (3.9) to have $W(n) \sim$ $c n^{-1}(\log n)^{-\beta}$ with some $0<\beta<1$. Note that we cannot have $W_{L}(n)$ satisfy this behaviour with a cut-off at some $L^{\sigma}$, since

$$
\sum_{n=2}^{L^{\sigma}} \frac{1}{n \log ^{\beta} n} \sim c(\sigma)(\log L)^{1-\beta}
$$

Hence a logarithmic correction of the form above would require that for finite $L$ there is sufficient weight on very large avalanches (whose size diverges with $L$ ) to yield


Figure 3-17: $\quad W_{L}(n) n$ with Dirichlet boundary conditions in $d=2$, when considering $L=8192$, against $\log n$ for $1 \leq n \leq 2000$. Error bars show $\pm 2$ standard deviations. The sample size is $10^{6}$.
$G_{L}(o, o) \sim(2 \pi)^{-1} \log L$. In Figure 3-18, we show the effect of scaling the data with different powers $(\log n)^{\beta}$. Scaling with $\beta=0.4$ describes the data fairly well for moderate values of $n$. However, our conclusion from the above is that understanding the scaling of $w(n)$ or $W(n)$ requires further work.

### 3.4 3D results

In 3D we found that with periodic boundary conditions there were some extremely long avalanches. In this chapter we only include our data with Dirichlet boundary conditions. Assuming the behaviour $t(x) \sim c|x|^{-1-\eta}$ in the infinite volume limit $L \rightarrow$ $\infty$, we want to estimate $\eta=\eta(3)$. A log-log plot of the numerical estimates $\hat{t}_{L}(x)$ are shown in Figure 3-19.

Fitting a finite size-scaling form $t_{L}(x)=|x|^{-1-a} f_{3}(|x| / L)$ yielded the estimated exponent $\hat{a} \approx 0.0$, when small values of $x$ (those with $|x| / L<5 / 32$ ) were excluded; see Figure 3-20.

On the other hand, for the largest system size $(L=256)$, the least squares fit of $\log \hat{t}_{L}(x)$ against $\log |x|$, for vertices $x$ along the positive $x$-axis with $7 \leq|x| \leq 55$ gives


Figure 3-18: $\quad W_{L}(n) n(\log n)^{\beta}$, with Dirichlet boundary conditions in $d=2$, when considering $L=8192$, for $\beta=0.3$ (red), $\beta=0.4$ (black), $\beta=0.5$ (blue). The horizontal axis shows $\log n$ for $1 \leq n \leq 2000$.

Toppling probabilities in 3D


Figure 3-19: The logarithm of the toppling probabilities against the logarithm of $|x|$ 's with Dirichlet boundary conditions in $d=3$ for systems with $L=32$ (yellow), 64 (green), 128 (red), and 256 (black) (with sample sizes $8 \times 10^{7}, 2 \times 10^{7}, 4.5 \times 10^{6}$, $4 \times 10^{6}$ ).


Figure 3-20: The estimates $\hat{t}_{L}(x)$ rescaled by the $|x|^{1+\hat{a}}$, where $\hat{a}=0.0$ is obtained from finite scaling for $5 / 32 \leq|x / L| \leq 1$, when considering Dirichlet boundary conditions in $d=3$ for systems with $L=32$ (yellow), 64 (green), 128 (red), and 256 (black).
the somewhat different estimate $\eta(3) \approx \hat{\eta}(3)=0.09$; see Figure 3-21.
The estimate $\hat{t}_{L}(x)$ rescaled by $|x|^{1+\hat{\eta}}$ appear to approach a constant with little deviation; see Figure 3-22.

Based on the above, we believe that the exponent describing the toppling probability in the infinite volume limit differs from the one describing it in the scaling limit $x / L \rightarrow y$. We make the following conjecture:

$$
\mathbf{P}[x \text { topples }]=|x|^{-1-\eta+o(1)}
$$

with $\eta=\eta(3) \approx 0.1$.
Toppling probabilities in the scaling limit. Next we consider the toppling probability in the scaling limit (at $x$ whose distance from the origin is of the same order as $L$ ). Rescaling by $|x|^{1+\hat{a}}$, where $\hat{a}=0.0$ was obtained from the finite-size scaling analysis, and plotting against $|x| / L$, yields the graph in Figure 3-23. The graph suggests that as long as $|x| / L$ is bounded away from 0 , the rescaled quantity $t_{L}(x)|x|^{1+a}$ has a scaling limit $f_{3}(y)$, as $x / L \rightarrow y \in[-1,1]^{3}$.

From the assumed scaling form $t_{L}(x)=|x|^{-1-a} f_{3}(x / L)$, with some exponent $a$ and

## Least squares fit in 3D



Figure 3-21: The logarithm of the toppling probability against the logarithm of $|x|$ with Dirichlet boundary conditions in a systems with $L=256$ (black dots). The line of best fit with slope $1+\hat{\eta}$, where $\hat{\eta}=0.09$, from the least squares method (red line).
a scaling function $f_{3}$. We can estimate $\hat{a}$ similarly as in $d=2$. The difference of logarithms

$$
\frac{1}{\log 2}\left(\log \hat{t}_{L}(x)-\log \hat{t}_{2 L}(2 x)\right) \approx \hat{a}
$$

We show these differences in Figure 3-24 for $L=128,64,32$, together with the horizontal line corresponding to $\hat{a}=0.0$.

The above raises the question: if you rescale with $|x|$, does the limit exist? In other words: is there a function $f_{3}:[-1,1]^{3} \rightarrow \mathbb{R}$ such that

$$
|x| \mathbf{P}_{L}[x \text { topples }] \sim f_{3}(x / L), \quad \text { as } L \rightarrow \infty ?
$$

### 3.5 High-dimensional results

### 3.5.1 Toppling probability simulations in 5D

In dimensions $d \geq 5$, it has been proved by Járai, Redig and Saada [10] that $\eta=\eta(d)=$ 0 , in the sense that

$$
c|x|^{2-d} \leq \mathbf{P}[x \text { topples }] \leq C|x|^{2-d}
$$



Figure 3-22: The toppling probability with Dirichlet boundary conditions in systems with $L=32$ (yellow), 64 (green), 128 (red), and 256 (black), rescaled by $|x|^{1+\hat{\eta}}$, where $x$ is taken along the positive $x$-axis between 1 and 55 , and $\hat{\eta} \approx 0.1$. The error bars show $\pm 2$ standard deviations of the toppling probability with $L=256$ (for every 3-rd point only, for readability). Sample sizes are same as in Figure 3-19.

Toppling probabilities in the scaling limit in 3D


Figure 3-23: The rescaled toppling probability $t_{L}(x)|x|$ with Dirichlet boundary conditions in $d=3$ for systems with $L=32$ (yellow), 64 (green), 128 (red), and 256 (black). The probabilities are shown for points on the positive $x$-axis with $5 / 32 \leq|x / L| \leq 1$.


Figure 3-24: The difference of logarithms in $d=3$ for $L=32$ (green), 64 (red), and 128 (black). A horizontal line with $y=1$ (black line), corresponding to $\hat{a}=0.0$.

Based on this we can expect that the toppling probability (in the infinite volume limit $L \rightarrow \infty$ ) rescaled by $|x|^{d-2}$ is asymptotic to a constant as $|x| \rightarrow \infty$.

Figure 3-25 shows our simulation results which appear consistent with this conjecture. There are approximately 400 samples discarded out of $4 \times 10^{7}$ due to a full hashtable. The discarded data affects the toppling probability estimate by at most $10^{-5}$ as an additive error. This estimated error $10^{-5}$ is an upper bound only, and we expect the actual error to be smaller since the discarded avalanche has to hit the vertices on the positive $x$-axis. The effects of discarded samples on small values of $|x|$ is small, but the effects will increase as $|x|$ increases. The discarded data may affect the probability more when $|x|$ is near the boundary.

In the following, we argue the effect heuristically due to discarded data is small, and $10^{-5}$ may be taken as a relative error, at least for a moderate value of $|x|$. We denote the number of random walk steps used to generate the whole avalanche by $|R W|$, just in this section. In the simulation, we used the size of the hashtable as $L^{5}$, which is $1 / 32$ times the size of the box $[-L, L]^{5}$. Then we have

$$
\mathbf{P}_{L}[x \in \mathrm{Av}]=\mathbf{P}_{L}\left[x \in \mathrm{Av},|\mathrm{RW}| \leq L^{5}\right]+\mathbf{P}_{L}\left[x \in \mathrm{Av},|\mathrm{RW}|>L^{5}\right]
$$

When $x$ is not too close to the boundary, let us consider the correlation between the events that the vertex $x$ is in the avalanche and that the number of random walk steps $|R W|$ is greater than $L^{5}$. As a simplification, we replace the event that $x$ topples in the whole avalanche with the event $x$ topples in a typical wave. Then using Wilson's algorithm to generate a wave, we first have a loop-erased random path from $x$ to the origin $o$. The typical number of vertices connected to this path is $O\left(|x|^{4}\right)$; this case typically involves $O(|x|)^{6}$ random walk steps in total. Hence, for the event that $|\mathrm{RW}|>L^{5}$ to occur we need $L^{5}=|x|^{6}$. Assuming this is not the case for $|x|$, we have the events $\mathbf{P}_{L}[x \in \mathrm{Av}]$ and $\mathbf{P}_{L}\left[|\mathrm{RW}|>L^{5}\right]$ are roughly independent and the event $\mathbf{P}_{L}\left[|\mathrm{RW}|>L^{5}\right]$ has probability $10^{-5}$ from our simulation results.

Since $G_{L}(x) \asymp L^{2-d} k / L$ where $x=L-k$ and $k=o(L)$, we expect the toppling probability rescaled by $|x|^{d-2}$ tends to 0 as $|x| / L$ tends to 1 . Proposition 1.5.9 in [17] states the results for a ball with radius $L$. We expect similar results in a box $[-L, L]^{d}$ for the part near a flat boundary. The probability would be even smaller when near a corner of a box. Hence we expect a drop in the rescaled toppling probabilities when $|x|$ is near the boundary in Figure 3-25, and the discarded samples will accentuate this.


Figure 3-25: The toppling probability in the box with radius $L=32$ in 5 D , rescaled by $|x|^{d-2}$, where $x$ is taken along the first coordinate axis. The error bars show $\pm 2$ standard deviations. The number of samples taken was $4 \times 10^{7}$, with approximately 400 samples discarded due to a full hashtable.

### 3.5.2 Asymptotic height probabilities

Recall that $p_{d}(i)=\mathbf{P}[\eta(o)=i]$ denotes the height probability in $d$ dimensions (in the infinite volume limit $L \rightarrow \infty$ ). The following theorem states the asymptotic form of the height probabilities as $d \rightarrow \infty$. The proof of this theorem, that relies on analyzing Wilson's algorithm on the infinite graph $\mathbb{Z}^{d}$, will be stated in Chapter 4.

Theorem 3.5.1. (i) For $0 \leq i \leq d^{1 / 2}$, we have

$$
p_{d}(i)=\sum_{j=0}^{i} \frac{e^{-1} \frac{1}{j!}}{2 d-j}+O\left(\frac{i}{d^{2}}\right)=\frac{1}{2 d} \sum_{j=0}^{i} e^{-1} \frac{1}{j!}+O\left(\frac{i}{d^{2}}\right) .
$$

(ii) If $d^{1 / 2}<i \leq 2 d-1$, we have

$$
p_{d}(i)=p_{d}\left(d^{1 / 2}\right)+O\left(d^{-3 / 2}\right)
$$

(iii) As a consequence of (i) and (ii), we have $p_{d}(i) \sim(2 d)^{-1}$, if $i, d \rightarrow \infty$.

The asymptotic formula appearing in part (i) of the theorem is the same as obtained by Dhar and Majumdar [6] on the Bethe lattice with large coordination number. Figure

Height probabilties in $\mathrm{d}=32$


Figure 3-26: Simulated height probabilities (black dots) in $d=32$ for a system with $L=128$ (with sample size $4 \times 10^{6}$ ), and the asymptotic formula (red pluses).

3-26 compares the formula to simulations in $d=32$ in the finite volume $L=128$.

### 3.6 Chapter outlook

While in $d=32$ we only looked at the height probability, one could consider the entire avalanche using the same methods with a smaller box size $L$. As we discuss in Chapter 6 , the memory requirement is of the order $L^{6}$ in high dimensions, where the constant multiple depends on $d$ as well. The exit of the random walk is of the order $d L^{2}$.

Some theoretical questions are inspired by these simulations discussed in Chapter 4 and 5 .

Our simulation method was based on the Wilson's algorithm and the burning bijection between recurrent sandpiles and spanning trees. Mersenne Twister [21] was used to generate a 32 bits long random number was involved in generating the loop erased random walks. The High Performance Computing Cluster (HPC) was used to compute models and simulations which desktop computers or laptops do not have the capacity or capability to compute.

The method could extend to 4D with slight modification. Further possible future work will be discussed in Chapter 6.

Chapter 4 will prove for the asymptotic height formula in Theorem 3.5.1. Chapter 5 will consider the aspect of the hashing. In high dimensions, the loop-erased random walk visits on $O\left(L^{2}\right)$ vertices, the same as the simple random walk generating it. Consider a simplified hashing function that projects this walk to the $N$-torus where $N^{d}$ is constant times $L^{2}$. We prove for the projected random walk that it converges to random interlacement.

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## Chapter 4

## Asymptotic height distribution

## Chapter overview

This chapter is based on [11]. According to Aldous' results [1], in the uniform spanning tree on certain $d$-regular graphs, the degree for a vertex can be approximated as $1+$ Poisson(1) for $d$ large. This suggests that the sandpile height on $\mathbb{Z}^{d}$ can also be approximated using Poisson(1). We explore this in this chapter. We saw in Chapter 3 that the asymptotic formula is very close to the numerical results already in a finite box in $d=32$.

## Statement of Authorship



## Abstract

We prove the asymptotic formula for the single site height distribution of Abelian sandpiles on $\mathbb{Z}^{d}$ as $d \rightarrow \infty$ in terms of Poisson(1) probabilities. We provide error estimates.

### 4.1 Introduction

We consider the Abelian sandpile model on the nearest neighbour lattice $\mathbb{Z}^{d}$; see Section 4.1.1 for definitions and background. Let $\mathbf{P}$ denote the weak limit of the stationary distributions $\mathbf{P}_{L}$ in finite boxes $[-L, L]^{d} \cap \mathbb{Z}^{d}$. Let $\eta$ denote a sample configuration from the measure $\mathbf{P}$. Let $p_{d}(i)=\mathbf{P}[\eta(o)=i], i=0, \ldots, 2 d-1$, denote the height probabilities at the origin in $d$ dimensions.

We now recall Theorem 3.5.1, which is our main result that states the asymptotic form of these probabilities as $d \rightarrow \infty$.

Theorem 4.1.1. (i) For $0 \leq i \leq d^{1 / 2}$, we have

$$
\begin{equation*}
p_{d}(i)=\sum_{j=0}^{i} \frac{e^{-1} \frac{1}{j!}}{2 d-j}+O\left(\frac{i}{d^{2}}\right)=\frac{1}{2 d} \sum_{j=0}^{i} e^{-1} \frac{1}{j!}+O\left(\frac{i}{d^{2}}\right) . \tag{4.1}
\end{equation*}
$$

(ii) If $d^{1 / 2}<i \leq 2 d-1$, we have

$$
p_{d}(i)=p_{d}\left(d^{1 / 2}\right)+O\left(d^{-3 / 2}\right) .
$$

(iii) As a consequence of (i) and (ii), we have $p_{d}(i) \sim(2 d)^{-1}$, if $i, d \rightarrow \infty$.

The appearance of the Poisson(1) distribution in the above formula is closely related to the result of Aldous [1] that the degree distribution of the origin in the uniform spanning forest in $\mathbb{Z}^{d}$ tends to 1 plus a Poisson(1) random variable as $d \rightarrow \infty$. Indeed our proof of (4.1) is achieved by showing that in the uniform spanning forest of $\mathbb{Z}^{d}$, the number of neighbours $w$ of the origin $o$, such that the unique path from $w$ to infinity passes through $o$ is asymptotically the same as the degree of $o$ minus 1 , that is, Poisson(1).

In Chapter 3 we compared the formula (4.1) to numerical simulations in $d=32$ on a finite box with $L=128$, and there is excellent agreement with the asymptotics already for these values.

Other graphs where information on the height distribution is available are as follows.

Dhar and Majumdar [7] studied the Abelian sandpile model on the Bethe lattice and the exact expressions for various distribution functions including the height distribution at a vertex were obtained using combinatorial methods. For the single site height distribution they obtained (see [7, Eqn. (8.2)])

$$
p_{\text {Bethe }, d}(i)=\frac{1}{\left(d^{2}-1\right) d^{d}} \sum_{j=0}^{i}\binom{d+1}{j}(d-1)^{d-j+1} .
$$

If one lets the degree $d \rightarrow \infty$ in this formula, one obtains the form in the right hand side of (4.1) for any fixed $i$ (with $2 d$ replaced by $d$ ).

Exact expressions for the distribution of height probabilities were derived by Papoyan and Shcherbakov [20] on the Husimi lattice of triangles with an arbitrary coordination number $q$. However, on $d$-dimensional cubic lattices of $d \geq 2$, exact results for the height probability are only known for $d=2$; see section 2.1.3 [18], [22], [13], [14], [21].

### 4.1.1 Definitions and background

Sandpiles are a lattice model of self-organized criticality, introduced by Bak, Tang and Wiesenfeld [3], and have been studied in both physics and mathematics. See the surveys [10], [15], [23], [9], [6]. Although the model can easily be defined on an arbitrary finite connected graph, in this chapter we will restrict to subsets of $\mathbb{Z}^{d}$.

Recall that we denote $V_{L}=[-L, L]^{d} \cap \mathbb{Z}^{d}$ as a box of radius $L$, where $L \geq 1$. For simplicity, we suppress the $d$-dependence in our notation. We let $G_{L}=\left(V_{L} \cup\{\rho\}, E_{L}\right)$ denote the graph obtained from $\mathbb{Z}^{d}$ by identifying all vertices in $\mathbb{Z}^{d} \backslash V_{L}$ that becomes $\rho$, and removing loop-edges at $\rho$. We call $\rho$ the sink. A sandpile $\eta$ is a collection of indistinguishable particles on $V_{L}$, specified by a map $\eta: V_{L} \rightarrow\{0,1,2, \ldots\}$.

We say that $\eta$ is stable at $x \in V_{L}$, if $\eta(x)<2 d$. We say that $\eta$ is stable, if $\eta(x)<2 d$, for all $x \in V_{L}$. If $\eta$ is unstable (i.e. $\eta(x) \geq 2 d$ for some $x \in V_{L}$ ), $x$ is allowed to topple which means that $x$ passes one particle along each edge to its neighbours. When the vertex $x$ topples, the particles are re-distributed as follows:

$$
\begin{aligned}
& \eta(x) \rightarrow \eta(x)-2 d ; \\
& \eta(y) \rightarrow \eta(y)+1, \quad y \in V_{L}, y \sim x .
\end{aligned}
$$

Particles arriving at $\rho$ are lost, so we do not keep track of them. Toppling a vertex may generate further unstable vertices.

Given a sandpile $\xi$ on $V_{L}$, we define its stabilization

$$
\xi^{\circ} \in \Omega_{L}:=\left\{\text { all stable sandpiles on } V_{L}\right\}=\{0,1, \ldots, 2 d-1\}^{V_{L}}
$$

by carrying out all possible topplings, in any order, until a stable sandpile is reached. It was shown by Dhar [5] that the map $\xi \mapsto \xi^{\circ}$ is well-defined, that is, the order of topplings does not matter.

We now recall the sandpile Markov chain. The state space is the set of stable sandpiles $\Omega_{L}$. Fix a positive probability distribution $p$ on $V_{L}$, i.e. $\sum_{x \in V_{L}} p(x)=1$ and $p(x)>0$ for all $x \in V_{L}$. Given the current state $\eta \in \Omega_{L}$, choose a random vertex $X \in V$ according to $p$, add one particle at $X$ and stabilize. The one step transition of the Markov chain moves from $\eta$ to $(\eta+\mathbf{1} \mathbf{X})^{\circ}$. Considering the sandpile Markov chain on $G_{L}$, there is only one recurrent class [5]. The set of recurrent sandpiles is denoted by $\mathcal{R}_{L}$ and the invariant distribution $\mathbf{P}_{L}$ of the Markov chain is uniformly distributed on $\mathcal{R}_{L}$.

Majumdar and Dhar [19] gave a bijection between $\mathcal{R}_{L}$ and spanning trees of $G_{L}$. This maps the uniform measure $\mathbf{P}_{L}$ on $\mathcal{R}_{L}$ to the uniform spanning tree measure $\mathrm{UST}_{L}$. A variant of this bijection was introduced by Priezzhev [22], and is described in more generality in [12], [8]. The latter bijection enjoys the following property, that we will exploit in this chapter. Orient the spanning tree towards $\rho$, and let $\pi_{L}(x)$ denote the oriented path from a vertex $x$ to $\rho$. Let

$$
W_{L}=\left\{x \in V_{L}: o \in \pi_{L}(x)\right\} .
$$

Then we have that
conditional on $\operatorname{deg}_{W_{L}}(o)=i$, the height $\eta(o)$ is uniformly
distributed over the values $i, i+1, \ldots, 2 d-1$.
This has the following consequence for the height probabilities. Let $q^{L}(i)=\mathrm{UST}_{L}\left[\operatorname{deg}_{W_{L}}(o)=\right.$ $i], i=0, \ldots, 2 d-1$. Then

$$
p^{L}(i):=\mathbf{P}_{L}[\eta(o)=i]=\sum_{j=0}^{i} \frac{q^{L}(j)}{2 d-j} .
$$

The measures $\mathbf{P}_{L}$ have a weak limit $\mathbf{P}=\lim _{L \rightarrow \infty} \mathbf{P}_{L}[2]$, and hence $p(i)=\lim _{L \rightarrow \infty} p^{L}(i)$ exist, $i=0, \ldots, 2 d-1$. Although the $q^{L}(i)$ depend on the non-local variable $W_{L}$, one also has that $q(i)=\lim _{L \rightarrow \infty} q^{L}(i)$ exist, $i=0, \ldots, 2 d-1$; see [12]. In fact, $q(i)$ is
given by the following natural analogue of its finite volume definition. Consider the uniform spanning forest measure USF on $\mathbb{Z}^{d}$; defined as the weak limit of $\mathrm{UST}_{L}$; see [16, Chapter 10]. Let $\pi(x)$ denote the unique infinite self-avoiding path in the spanning forest starting at $x$, and let

$$
W=\left\{x \in \mathbb{Z}^{d}: o \in \pi(x)\right\} .
$$

Then $q(i)=\operatorname{USF}\left[\operatorname{deg}_{W}(o)=i\right], i=0, \ldots, 2 d-1$.

Therefore, we have

$$
\begin{equation*}
p(i):=\mathbf{P}[\eta(o)=i]=\sum_{j=0}^{i} \frac{q(j)}{2 d-j} . \tag{4.3}
\end{equation*}
$$

### 4.1.2 Wilson's method

We recall Wilson's method here with slightly different focus from Chapter 3. Given a finite path $\gamma=\left[s_{0}, s_{1}, \ldots, s_{k}\right]$ in $\mathbb{Z}^{d}$, we erase loops from $\gamma$ chronologically, as they are created. We trace $\gamma$ until the first time $t$, if any, when $s_{t} \in\left\{s_{0}, s_{1}, \ldots, s_{t-1}\right\}$, i.e. there is a loop. We suppose $s_{t}=s_{i}$, for some $i \in\{0,1, \ldots, t-1\}$ and remove the loop $\left[s_{i}, s_{i+1}, \ldots, s_{t}=s_{i}\right]$. Then we continue tracing $\gamma$ and follow the same procedure to remove loops until there are no more loops to remove. This gives the loop-erasure $\pi=L E(\gamma)$ of $\gamma$, which is a self-avoiding path [17]. If $\gamma$ is generated from a random walk process, the loop-erasure of $\gamma$ is call the loop-erased random walk (LERW).

When $d \geq 3$, the USF on $\mathbb{Z}^{d}$ can be sampled via Wilson's method rooted at infinity [4], [16, Section 10], that is described as follows. Let $s_{1}, s_{2}, \ldots$ be an arbitrary enumeration of the vertices and let $\mathcal{T}_{0}$ be the empty forest with no vertices. We start a simple random walk $\gamma_{n}$ at $s_{n}$ and $\gamma_{n}$ stops when $\mathcal{T}_{n-1}$ is hit, otherwise we let it run indefinitely. $\operatorname{LE}\left(\gamma_{n}\right)$ is attached to $\mathcal{T}_{n-1}$ and the resulting forest is denoted by $\mathcal{T}_{n}$. We continue the same procedure until all the vertices are visited. The above gives a random sequence of forests $\mathcal{T}_{1} \subset \mathcal{T}_{2} \subset \ldots$, where $\mathcal{T}=\cup_{n} \mathcal{T}_{n}$ is a spanning forest of $\mathbb{Z}^{d}$. The extension of Wilson's theorem [24] to transient infinite graphs proved in [4] implies that $\mathcal{T}$ is distributed as the USF.

### 4.2 Proof of the main theorem

Let $\left(S_{n}^{x}\right)_{n \geq 0}$ be a simple random walk started at $x$ (independent between $x$ 's on $\mathbb{Z}^{d}$ ) and let $\pi(x)$ be the path in the USF from $x$ to infinity. We introduce the events:

$$
\begin{aligned}
E_{i} & =\{\mid\{w \sim o: \pi(w) \text { passes through } o\} \mid=i\}, \quad i=0, \ldots, 2 d-1 \\
E_{i}\left(x_{1}, x_{2}, \ldots, x_{i}\right) & =\left\{\{w \sim o: \pi(w) \text { passes through } o\}=\left\{x_{1}, x_{2}, \ldots, x_{i}\right\}\right\}
\end{aligned}
$$

Then recall that

$$
\begin{equation*}
q(i)=\mathbf{P}\left[\operatorname{deg}_{W}(o)=i\right]=\mathbf{P}\left[E_{i}\right]=\sum_{\substack{x_{1}, \ldots, x_{i} \sim o \\ \text { distinct }}} \mathbf{P}\left[E_{i}\left(x_{1}, \ldots, x_{i}\right)\right] \tag{4.4}
\end{equation*}
$$

We denote $q(i)$ by $q_{d}(i)$ from now on to emphasize the dependence on $d$.

### 4.2.1 Preliminary

Lemma 4.2.1. We have

$$
\begin{aligned}
& \mathbf{P}\left[S_{n}^{o}=o \text { for some } n \geq 2\right]=O(1 / d) \\
& \mathbf{P}\left[S_{n}^{o}=o \text { for some } n \geq 4\right]=O\left(1 / d^{2}\right)
\end{aligned}
$$

as $d \rightarrow \infty$.

Proof. Let $\hat{D}(k)=\frac{1}{d} \sum_{j=1}^{d} \cos \left(k_{j}\right), k_{j} \in[-\pi, \pi]^{d}$, be the Fourier transform in $d$ dimensions of the one-step distribution of RW. Lemma A. 3 in [17] states that for all non-negative integers $n$ and all $d \geq 1$ we have

$$
\left\|\hat{D}^{n}\right\|_{1}=(2 \pi)^{-d} \int_{[-\pi, \pi]^{d}}\left|\hat{D}(k)^{n}\right| d^{d} k \leq\left(\frac{\pi d}{4 n}\right)^{d / 2}
$$

Based on above, we have

$$
\begin{align*}
\mathbf{P}\left[S_{n}^{o}=o \text { for some } n \geq 4\right] & \leq \frac{1}{(2 \pi)^{d}} \sum_{n=4}^{\infty} \int \hat{D}^{n}(k) d k \\
& \leq \frac{1}{(2 \pi)^{d}} \sum_{n=4}^{d-1} \int \hat{D}^{n}(k) d k+\sum_{n=d}^{\infty}\left(\frac{\pi d}{4 n}\right)^{d / 2} \tag{4.5}
\end{align*}
$$

Since $\int \hat{D}^{4}(k) d k$ and $\int \hat{D}^{6}(k) d k$ state the probability that $S^{o}$ returns to $o$ in 4 and 6 steps each, by counting the number of ways to return, they are bounded by dimension-
independent multiples of $1 / d^{2}$ and $1 / d^{3}$ respectively. We have $\int \hat{D}^{n}(k) d k=0$ with odd $n$, and for $6<n \leq d-1$ and $n$ even, we have $\int \hat{D}^{n}(k) d k \leq \int \hat{D}^{6}(k) d k$. Hence,

$$
\frac{1}{(2 \pi)^{d}} \int \hat{D}^{n}(k) d k=O\left(\frac{1}{d^{3}}\right), \quad 6 \leq n \leq d-1 .
$$

The last sum in (4.5) can be bounded as:

$$
\begin{aligned}
\left(\frac{\pi d}{4}\right)^{d / 2} \sum_{n=d}^{\infty} n^{-d / 2} & \leq\left(\frac{\pi d}{4}\right)^{d / 2} \int_{d-1}^{\infty} x^{-d / 2} d x=\left(\frac{\pi d}{4}\right)^{d / 2} \frac{(d-1)^{1-\frac{d}{2}}}{d / 2-1} \\
& =\left(\frac{d-1}{d / 2-1}\right)\left(\frac{d}{d-1}\right)^{\frac{d}{2}}\left(\frac{\pi}{4}\right)^{\frac{d}{2}} \leq C e^{-c d},
\end{aligned}
$$

since we can take $d>4$ and $\frac{\pi}{4}<1$.
Hence, we have the required results

$$
\begin{aligned}
\mathbf{P}\left[S_{n}^{o}=o \text { for some } n \geq 4\right] & \leq \int \hat{D}^{4}(k) d k+d \int \hat{D}^{6}(k) d k+C e^{-c d} \\
& =O\left(\frac{1}{d^{2}}\right)+d \times O\left(\frac{1}{d^{3}}\right)=O\left(\frac{1}{d^{2}}\right), \\
\mathbf{P}\left[S_{n}^{o}=o \text { for some } n \geq 2\right] & \leq\left(\frac{1}{2 d}\right)+\mathbf{P}\left[S_{n}^{o}=o \text { for some } n \geq 4\right]=O\left(\frac{1}{d}\right) .
\end{aligned}
$$

### 4.2.2 Lower bounds

Let us fix the vertices $x_{1}, \ldots, x_{i} \sim o$. Let

$$
A_{0}=\left\{S_{1}^{o} \notin\left\{x_{1}, \ldots, x_{i}\right\}, S_{n}^{o} \notin \mathcal{N} \text { for } n \geq 2\right\}
$$

where $\mathcal{N}=\left\{y \in \mathbb{Z}^{d}:|y| \leq 1\right\}$.
Lemma 4.2.2. We have

$$
\left(1-\mathbf{P}\left[A_{0}\right]\right)=O(i / d) .
$$

## Proof.

$$
\mathbf{P}\left[A_{0}\right]=\mathbf{P}\left[S_{1}^{o} \neq x_{1}, \ldots, x_{i}\right] \mathbf{P}\left[S_{n}^{o} \notin \mathcal{N} \text { for } n \geq 2 \mid S_{1}^{o} \neq x_{1}, \ldots, x_{i}\right] .
$$

We have $\mathbf{P}\left[S_{1}^{o} \neq x_{1}, \ldots x_{i}\right]=1-O(i / d)$ and the probability for the remaining steps is at least $1-O(1 / d)$, shown as follows. The probabilities $\mathbf{P}\left[S_{2}^{o} \neq o \mid S_{1}^{o} \neq x_{1}, \ldots, x_{i}\right]$ and $\mathbf{P}\left[S_{3}^{o} \notin \mathcal{N} \mid S_{2}^{o} \neq o, S_{1}^{o} \neq x_{1}, \ldots, x_{i}\right]$ are both equal to $1-O(1 / d)$. Considering the s.r.w starting at the position $S_{3}^{o}$, it hits at most three neighbours of $o$ in two further steps, the remaining neighbours will need at least 4 steps to hit, so, by Lemma 4.2.1, we have

$$
\begin{aligned}
\sum_{\text {at most } 3 \text { neighbours }} \sum_{x_{j}} P_{k \geq 1}\left(S_{3}^{o}, x_{j}\right) & \leq O\left(\frac{1}{d}\right), \\
\sum_{\text {the remaining neighbours }} \sum_{x_{j^{\prime}}} P_{2 k}\left(S_{3}^{o}, x_{j^{\prime}}\right) & \leq O(d) O\left(\frac{1}{d^{2}}\right)=O\left(\frac{1}{d}\right),
\end{aligned}
$$

since $P_{2 k}(x, y) \leq P_{2 k}(o, o)$ for all $x, y$. Therefore, combining above results together, we get $\mathbf{P}\left[S_{n}^{o} \notin \mathcal{N}\right.$ for $\left.n \geq 2 \mid S_{1}^{o} \neq x_{1}, \ldots, x_{i}\right] \geq 1-O(1 / d)$ as required.

Let us label the neighbours of $o$ different from $x_{1}, \ldots, x_{i}$ as $x_{i+1}, \ldots, x_{2 d}$, in any order. On the event $A_{0}$, the first step of $\pi(o)$ is to a neighbour of $o$ in $\left\{x_{i+1}, \ldots, x_{2 d}\right\}$ and we could assume $x_{2 d}$ to be the first step of $\pi(o)$. Then $\pi(o)$ does not visit other vertices in $\mathcal{N} \backslash\{o\}$. Define $A_{j}=\left\{S_{1}^{x_{j}}=o\right\}$ for $j=1,2, \ldots, i$ and then $\mathbf{P}\left[A_{j}\right]=1 / 2 d$.

Using Wilson's algorithm, consider random walks first started at $o, x_{1}, . ., x_{i}$ and then started at $x_{i+1}, \ldots, x_{2 d-1}$. We obtain the following:

$$
\begin{align*}
\mathbf{P}\left[E_{i}\left(x_{1}, \ldots, x_{i}\right)\right] & \geq \mathbf{P}\left[A_{0}\right] \times \prod_{j=1}^{i} \mathbf{P}\left[A_{j}\right] \times \mathbf{P}\left[E_{i}\left(x_{1}, . ., x_{i}\right) \mid A_{0} \cap A_{1} \cap \cdots \cap A_{i}\right]  \tag{4.6}\\
& \geq\left(1-O\left(\frac{i}{d}\right)\right)\left(\frac{1}{2 d}\right)^{i} \mathbf{P}\left[E_{i}\left(x_{1}, . ., x_{i}\right) \mid A_{0} \cap A_{1} \cap \cdots \cap A_{i}\right] .
\end{align*}
$$

Define $B_{k}=\left\{S_{1}^{x_{k}} \neq o, S_{n}^{x_{k}} \notin\left\{x_{1}, \ldots, x_{i}\right\}\right.$ for $\left.n \geq 2\right\}$ for $k=i+1, \ldots, 2 d-1$.

## Lemma 4.2.3.

$$
\mathbf{P}\left[B_{k}\right] \geq 1-1 / 2 d-O\left(i / d^{2}\right), \quad \text { where } i+1 \leq k \leq 2 d-1 .
$$

Proof. We have $\mathbf{P}\left[S_{1}^{x_{k}} \neq o\right]=1-1 / 2 d$. If the first step is not to $o$, the first step could be in one of the $e_{1}, \ldots, e_{i}$ directions, say $e_{j}$, with probability $i / 2 d$. Then the probability to hit $x_{j}$ is $1 / 2 d+O\left(1 / d^{2}\right)$. Hence, the probability that $S^{x_{k}}$ hits $\left\{x_{1}, \ldots, x_{i}\right\}$ is $O\left(i / d^{2}\right)$.

## Lemma 4.2.4.

$$
q_{d}(i) \geq e^{-1} \frac{1}{i!}\left(1+O\left(\frac{i^{2}}{d}\right)\right) .
$$

Proof. By (4.6) and Lemma 4.2.3, we have

$$
\mathbf{P}\left[E_{i}\left(x_{1}, \ldots, x_{i}\right)\right] \geq\left(1-O\left(\frac{i}{d}\right)\right)\left(\frac{1}{2 d}\right)^{i}\left(1-\frac{1}{2 d}+O\left(\frac{i}{d^{2}}\right)\right)^{2 d-1-i}
$$

Then by (4.4), we have

$$
\begin{aligned}
q_{d}(i) & \geq\binom{ 2 d}{i}\left(1-O\left(\frac{i}{d}\right)\right)\left(\frac{1}{2 d}\right)^{i}\left(1-\frac{1}{2 d}+O\left(\frac{i}{d^{2}}\right)\right)^{2 d-1-i} \\
= & \frac{2 d(2 d-1) \ldots(2 d-i+1)}{i!(2 d)^{i}}\left(1-O\left(\frac{i}{d}\right)\right)\left(1-\frac{1}{2 d}+O\left(\frac{i}{d^{2}}\right)\right)^{2 d}\left(1+O\left(\frac{i}{d}\right)\right) .
\end{aligned}
$$

For the first term of the right-hand side, we have

$$
\begin{aligned}
\frac{2 d(2 d-1) \ldots(2 d-i+1)}{i!(2 d)^{i}} & =\frac{1}{i!} 1\left(1-\frac{1}{2 d}\right)\left(1-\frac{2}{2 d}\right) \ldots\left(1-\frac{i}{2 d}+\frac{1}{2 d}\right) \\
& =\left(1+O\left(\frac{i^{2}}{d}\right)\right)
\end{aligned}
$$

For the third term of the right-hand side, using the Taylor series expansion for exponential and logarithm, and since $i \leq d$, we have

$$
\begin{aligned}
\left(1-\frac{1}{2 d}+O\left(\frac{i}{d^{2}}\right)\right)^{2 d} & =\exp \left(2 d \times \log \left(1-\frac{1}{2 d}+O\left(\frac{i}{d^{2}}\right)\right)\right) \\
& =\exp \left(2 d\left(-\frac{1}{2 d}+O\left(\frac{i}{d^{2}}\right)\right)\right)=\exp \left(-1+O\left(\frac{i}{d}\right)\right) \\
& =e^{-1}\left(1+O\left(\frac{i}{d}\right)\right)
\end{aligned}
$$

Then the result follows

$$
\begin{aligned}
q_{d}(i) & \geq e^{-1} \frac{1}{i!}\left(1+O\left(\frac{i}{d}\right)\right)\left(1+O\left(\frac{i^{2}}{d}\right)\right)\left(1+O\left(\frac{i}{d}\right)\right)\left(1-O\left(\frac{i}{d}\right)\right) \\
& =e^{-1} \frac{1}{i!}\left(1+O\left(\frac{i^{2}}{d}\right)\right)
\end{aligned}
$$

The above lemma gives a lower bound for $q_{d}$ and we now prove an upper bound.

### 4.2.3 Upper bounds

Recall that $\pi(o)$ denotes the unique infinite self-avoiding path in the spanning forest starting at $o$ and let $\bar{A}_{o}=\{\pi(o)$ visits only one neighbour of $o\}$.

## Lemma 4.2.5.

$$
\mathbf{P}[\pi(o) \text { visits more than one neighbour of o }]=P\left[\bar{A}_{o}^{c}\right]=O(1 / d) .
$$

Proof. The first step of $\pi(o)$ must visit a neighbour of $o$, denoted by $w$, then $P\left[\bar{A}_{o}^{c}\right]$
$=\mathbf{P}\left[\right.$ The 2nd step of $\pi(o)$ visits $x \neq 2 w$, the 3rd step visits $\left.w^{\prime} \sim o, w^{\prime} \neq w\right]+O\left(\frac{1}{d^{2}}\right)$
$=\left(\frac{1}{2 d}\right)\left(\frac{2 d-1}{2 d}\right)+O\left(\frac{1}{d^{2}}\right)=O\left(\frac{1}{d}\right)$.

Let $\bar{A}_{\text {all }}=\{\forall w \sim o$ : either $\pi(w)$ does not visit $o$ or $\pi(w)$ visits $o$ at the first step $\}$.

## Lemma 4.2.6.

$$
\mathbf{P}[\exists w \sim o: \pi(w) \text { visits o but not at the first step }]=\mathbf{P}\left[\bar{A}_{\text {all }}^{c}\right]=O(1 / d) .
$$

Proof. For a given $w, w \sim o$, use Wilson's algorithm with a walk started at $w$. Consider that if $S_{1}^{w} \neq o$, or $S_{1}^{w}=o$ but $S^{w}$ returns to $w$ subsequently and then this loop starting from $w$ in $S^{w}$ is erased, $\pi(w)$ does not visit $o$ at the first step. Hence, we have the inequality:

$$
\begin{align*}
& \mathbf{P}[\pi(w) \text { visits } o \text { but not at the first step }] \\
& \leq \mathbf{P}\left[S^{w} \text { visits } o \text { but not at the first step }\right]+\mathbf{P}\left[S_{1}^{w}=o, S_{n}^{w}=w \text { for some } n \geq 2\right] . \tag{4.7}
\end{align*}
$$

We bound the two terms as follows. For the first term, let us append a step from $o$ to $w$ at the beginning of the walk, and analyze it as if the walk started at $o$. Since $S_{1}^{o} \in \mathcal{N} \backslash\{o\}$, by symmetry, we may assume $S_{1}^{o}=w$. Then if $S_{2}^{o} \neq o, S^{o}$ will need at least 2 more steps to return to $o$.

For the second term in the right hand side of (4.7), we first note that we have $\mathbf{P}\left[S_{1}^{w}=\right.$ $\left.o, S_{2}^{w}=w\right]=1 /(2 d)^{2}$. If $S^{w}$ does not return to $w$ in the first two steps, $S^{w}$ will need
at least 4 steps to return to $w$. Then, we have that the right hand side of (4.7) is

$$
\begin{aligned}
& \leq \mathbf{P} {\left[S^{o} \text { returns to } o \text { in at least } 4 \text { steps }\right]+\frac{1}{(2 d)^{2}} } \\
&+\mathbf{P}\left[S^{w} \text { returns to } w \text { in at least } 4 \text { steps }\right] \\
&=2 \times \mathbf{P}\left[S^{o} \text { returns to } o \text { in at least } 4 \text { steps }\right]+O\left(\frac{1}{d^{2}}\right) .
\end{aligned}
$$

Therefore, by Lemma 4.2.1, we have the required result
$\mathbf{P}[\exists w \sim o: \pi(w)$ visits $o$ but not at the first step $]$
$=2 d \times \mathbf{P}[\pi(w)$ visits $o$ but not at the first step for a fixed $w \sim o]=O\left(\frac{1}{d}\right)$.

Due to Lemmas 4.2.5 and 4.2.6, we have

$$
q_{d}(i) \leq O\left(\frac{1}{d}\right)+\mathbf{P}\left[\bar{A}_{o} \cap \bar{A}_{\text {all }} \cap E_{i}\right]=O\left(\frac{1}{d}\right)+\sum_{\substack{x_{1}, \ldots, x_{i} \sim o \\ \text { distinct }}} \mathbf{P}\left[\bar{A}_{o} \cap \bar{A}_{\text {all }} \cap E_{i}\left(x_{1}, \ldots, x_{i}\right)\right] .
$$

Here,

$$
\begin{align*}
& \bar{A}_{o} \cap \bar{A}_{\text {all }} \cap E_{i}\left(x_{1}, \ldots, x_{i}\right)  \tag{4.8}\\
& \subset \bar{A}_{o} \cap \bar{A}_{\text {all }} \cap\left\{\text { the first step of } \pi\left(x_{j}\right) \text { is to } o, j=1, \ldots, i\right\} \cap F_{i}\left(x_{1}, \ldots, x_{i}\right),
\end{align*}
$$

where

$$
F_{i}\left(x_{1}, \ldots, x_{i}\right)=\left\{\pi\left(x_{j}\right) \text { does not go through } o, j=i+1, \ldots, 2 d\right\} .
$$

The right hand side of (4.8) is contained in the event

$$
\bar{A}_{o} \cap\left\{\pi(o) \text { does not visit } x_{1}, \ldots, x_{i}\right\} \cap \bar{A}_{\text {rest }} \cap \bigcap_{1 \leq j \leq i} H_{j} \cap F_{i}\left(x_{1}, \ldots, x_{i}\right),
$$

where
$\bar{A}_{\text {rest }}=\left\{\pi\left(x_{j}\right)\right.$ goes through at most one $\left.x_{j^{\prime}}, j=i+1, \ldots, 2 d, i+1 \leq j^{\prime} \leq 2 d, j^{\prime} \neq j\right\}$ and $H_{j}=\left\{\right.$ the first step of $\pi\left(x_{j}\right)$ is to $\left.o\right\}$ for $j=1, \ldots, i$.

We denote $\bar{A}_{o} \cap\left\{\pi(o)\right.$ does not visit $\left.x_{1}, \ldots, x_{i}\right\}$ by $\bar{A}_{o, x_{1}, \ldots, x_{i}}$. Then

$$
\begin{aligned}
& \mathbf{P}\left[\bar{A}_{o, x_{1}, \ldots, x_{i}} \cap \bar{A}_{\text {rest }} \cap \bigcap_{1 \leq j \leq i} H_{j} \cap F_{i}\left(x_{1}, \ldots, x_{i}\right)\right] \\
& =\mathbf{P}\left[\bar{A}_{\left.o, x_{1}, \ldots, x_{i}\right]}\right] \prod_{j=1}^{i} \mathbf{P}\left[H_{j} \mid \bigcap_{1 \leq j^{\prime}<j} H_{j^{\prime}} \cap \bar{A}_{o, x_{1}, \ldots, x_{i}}\right] \\
& \quad \times \mathbf{P}\left[F_{i}\left(x_{1}, \ldots, x_{i}\right) \cap \bar{A}_{r e s t} \mid \bar{A}_{o, x_{1}, \ldots, x_{i}} \cap \bigcap_{1 \leq j \leq i} H_{j}\right] .
\end{aligned}
$$

Therefore, we have

$$
\begin{align*}
q_{d}(i) \leq O & \left(\frac{1}{d}\right)+\sum_{\substack{x_{1}, \ldots, x_{i} \sim o \\
\text { distinct }}}\left(\prod_{j=1}^{i} \mathbf{P}\left[H_{j} \mid \bigcap_{1 \leq j^{\prime}<j} H_{j^{\prime}} \cap \bar{A}_{o, x_{1}, \ldots, x_{i}}\right]\right)  \tag{4.9}\\
& \times \mathbf{P}\left[F_{i}\left(x_{1}, \ldots, x_{i}\right) \cap \bar{A}_{\text {rest }} \mid \bar{A}_{o, x_{1}, \ldots, x_{i}} \cap \bigcap_{1 \leq j \leq i} H_{j}\right] .
\end{align*}
$$

## Lemma 4.2.7.

$$
\mathbf{P}\left[H_{j} \mid \bar{A}_{o, x_{1}, \ldots, x_{i}} \cap \bigcap_{1 \leq j^{\prime}<j} H_{j^{\prime}}\right]=1 / 2 d+O\left(1 / d^{2}\right) \text {, where } j=1, \ldots, i .
$$

Proof. Given that $\pi(o)$ visits only one neighbour of $o$ which is not in $\left\{x_{1}, \ldots, x_{i}\right\}$ and the first steps of $\pi\left(x_{1}\right), \ldots, \pi\left(x_{j-1}\right)$ are all to $o$, the probability that $H_{j}$ happens is $\mathbf{P}\left[S_{1}^{x_{j}}=o\right]=1 / 2 d$ with the error term of $O\left(1 / d^{2}\right)$ due to the loop-erasure.

## Lemma 4.2.8.

$$
\begin{equation*}
\mathbf{P}\left[F_{i}\left(x_{1}, \ldots, x_{i}\right) \cap \bar{A}_{r e s t} \mid \bar{A}_{o, x_{1}, \ldots, x_{i}} \cap \bigcap_{1 \leq j \leq i} H_{j}\right] \leq \mathbf{E}\left[\left(1-\frac{1}{2 d}+O\left(\frac{1}{d^{2}}\right)\right)^{2 d-i-1-N} \mathbf{1}_{\bar{A}_{\text {rest }}}\right], \tag{4.10}
\end{equation*}
$$

where $N=\mid\left\{i+1 \leq j \leq 2 d-1: \exists i+1 \leq j^{\prime}<j\right.$ s.t. $\pi\left(x_{j^{\prime}}\right)$ goes through $\left.x_{j}\right\} \mid$.

Proof. Consider Wilson's algorithm with random walks started at the remaining neighbours $x_{i+1}, \ldots, x_{2 d}$. Assume $x_{2 d}$ to be the neighbour of $o$ that $\pi(o)$ goes through. The probability that $\pi\left(x_{k}\right)$ does not go through $o$ is $1-1 / 2 d+O\left(1 / d^{2}\right)$ for $k \in$ $\{i+1, \ldots, 2 d-1\}$.

If $\pi\left(x_{k}\right)$ visits $x_{k^{\prime}}$, where $k<k^{\prime} \leq 2 d-1$, the probability that $\pi\left(x_{k^{\prime}}\right)$ does not go through
$o$ is 1 instead of $1-1 / 2 d+O\left(1 / d^{2}\right)$, since the LERW from $x_{k^{\prime}}$ stops immediately and $\pi\left(x_{k^{\prime}}\right) \subset \pi\left(x_{k}\right)$, which does not go through $o$.

Lemma 4.2.9. On the event $\bar{A}_{\text {rest }}, N \leq B$, where $B \sim \operatorname{Binom}(2 d-i-1, p), p=$ $1 / 2 d+O\left(1 / d^{2}\right)$.

Proof. Since we have $(2 d-i-1)$ trials with probability at most $1 / 2 d+O\left(1 / d^{2}\right)$.
Due to Lemma 4.2.9, we have that the right hand side of (4.10) is

$$
\begin{equation*}
\leq\left(1-\frac{1}{2 d}+O\left(\frac{1}{d^{2}}\right)\right)^{2 d}\left(1+O\left(\frac{i}{d}\right)\right) \mathbf{E}\left[\frac{1}{\left(1-\frac{1}{2 d}+O\left(\frac{1}{d^{2}}\right)\right)^{B}}\right] \tag{4.11}
\end{equation*}
$$

where $\mathbf{E}\left[z^{B}\right]=\sum_{j=0}^{2 d-i-1} z^{j}\binom{2 d-i-1}{j} p^{j}(1-p)^{2 d-i-1-j}=(1-p-z p)^{2 d-i-1}$.
Hence (4.11) is

$$
\begin{align*}
& \leq e^{-1}\left(1+O\left(\frac{1}{d}\right)\right)\left(1+O\left(\frac{i}{d}\right)\right)\left(1-\frac{1}{2 d}+O\left(\frac{1}{d^{2}}\right)+\frac{\frac{1}{2 d}+O\left(\frac{1}{d^{2}}\right)}{1-\frac{1}{2 d}+O\left(\frac{1}{d^{2}}\right)}\right)^{2 d-i-1} \\
& =e^{-1}\left(1+O\left(\frac{1}{d}\right)\right)\left(1+O\left(\frac{i}{d}\right)\right)\left(1+O\left(\frac{1}{d^{2}}\right)\right)^{2 d-i-1}=e^{-1}\left(1+O\left(\frac{i}{d}\right)\right) \tag{4.12}
\end{align*}
$$

## Lemma 4.2.10.

$$
q_{d}(i) \leq O\left(\frac{1}{d}\right)+e^{-1} \frac{1}{i!}\left(1+O\left(\frac{i}{d}\right)\right) .
$$

Proof. Due to Lemma 4.2.7, (4.9) and (4.12), we have

$$
\begin{aligned}
q_{d}(i) & \leq O\left(\frac{1}{d}\right)+\binom{2 d}{i}\left(\frac{1}{2 d}+O\left(\frac{1}{d^{2}}\right)\right)^{i} e^{-1}\left(1+O\left(\frac{i}{d}\right)\right) \\
& =O\left(\frac{1}{d}\right)+e^{-1} \frac{2 d(2 d-1) \ldots(2 d-i+1)}{i!}\left(\frac{1}{2 d}\right)^{i}\left(1+O\left(\frac{1}{d}\right)\right)^{i}\left(1+O\left(\frac{i}{d}\right)\right) \\
& \leq O\left(\frac{1}{d}\right)+e^{-1} \frac{1}{i!}\left(1+O\left(\frac{i}{d}\right)\right) .
\end{aligned}
$$

Lemma 4.2.11. For $k=1, \ldots, 3$ and distinct $w_{1}, \ldots, w_{k} \sim o$, we have

$$
\mathbf{P}\left[\pi\left(w_{i}\right) \text { passes through o for } i=1, \ldots, k\right]=\left(\frac{1}{2 d}\right)^{k}+O\left(d^{-k-1}\right)
$$

This lemma can be proved using ideas used to prove Lemma 4.2.7.

### 4.2.4 Proof of the asymptotic formula

Proof of Theorem 3.5.1. We first prove part (i). By Wilson's algorithm,

$$
p_{d}(i)=\sum_{j=0}^{i} \frac{q_{d}(j)}{2 d-j} .
$$

Due to Lemmas 4.2.4 and 4.2.10, we have

$$
\begin{equation*}
p_{d}(i) \geq \sum_{j=0}^{i} \frac{e^{-1} \frac{1}{j!}\left(1+O\left(\frac{j^{2}}{d}\right)\right)}{2 d-j}=\sum_{j=0}^{i} \frac{e^{-1} \frac{1}{j!}}{2 d-j}+\sum_{j=0}^{i} \frac{\frac{1}{j!} O\left(\frac{j^{2}}{d}\right)}{2 d-j}, \tag{4.13}
\end{equation*}
$$

and

$$
\begin{equation*}
p_{d}(i) \leq \sum_{j=0}^{i} \frac{O\left(\frac{1}{d}\right)+e^{-1} \frac{1}{j!}\left(1+O\left(\frac{j}{d}\right)\right)}{2 d-j}=\sum_{j=0}^{i} \frac{e^{-1} \frac{1}{j!}}{2 d-j}+\sum_{j=0}^{i} \frac{O\left(\frac{1}{d}\right)+\frac{1}{j!} O\left(\frac{j}{d}\right)}{2 d-j} . \tag{4.14}
\end{equation*}
$$

Here, using that $0 \leq j \leq d^{1 / 2}$, we have

$$
\sum_{j=0}^{i} \frac{\frac{1}{j!} O\left(\frac{j^{2}}{d}\right)}{2 d-j} \leq \frac{1}{2 d-d^{1 / 2}} O\left(\frac{1}{d}\right) \sum_{j=0}^{i} \frac{j^{2}}{j!}=O\left(\frac{1}{d^{2}}\right) .
$$

Similarly,

$$
\sum_{j=0}^{i} \frac{O\left(\frac{1}{d}\right)+\frac{1}{j!} O\left(\frac{j}{d}\right)}{2 d-j} \leq \sum_{j=0}^{i} O\left(d^{-2}\right)+\sum_{j=0}^{i} \frac{j}{j!} O\left(d^{-2}\right)=O\left(i / d^{2}\right) .
$$

Putting these error bounds together with (4.13) and (4.14), we prove statement (i) of the theorem.

Let us now use that

$$
\frac{1}{2 d} e^{-1} \sum_{j=0}^{i} \frac{1}{j!} \leq \sum_{j=0}^{i} \frac{e^{-1} \frac{1}{j!}}{2 d-j} \leq \frac{1}{2 d-i} e^{-1} \sum_{j=0}^{i} \frac{1}{j!} .
$$

When $i \leq d^{1 / 2}$, and $i, d \rightarrow \infty$, we have $\frac{1}{2 d-i} \sim \frac{1}{2 d}$ and $\sum_{j=0}^{i} \frac{1}{j!} \rightarrow e$. Hence,

$$
\sum_{j=0}^{i} \frac{e^{-1} \frac{1}{j!}}{2 d-j} \sim \frac{1}{2 d}, \quad \text { as } i, d \rightarrow \infty
$$

We are left to prove statement (ii). The uniform distribution for $d^{1 / 2} \leq i \leq 2 d-1$ can be obtained from the monotonicity:

$$
p_{d}\left(d^{1 / 2}\right) \leq p_{d}(i) \leq p_{d}(2 d-1), \quad d^{1 / 2} \leq i \leq 2 d-1,
$$

if we show that $p_{d}(2 d-1)=p_{d}\left(d^{1 / 2}\right)+O\left(d^{-3 / 2}\right)$.
We write

$$
\begin{equation*}
p_{d}(2 d-1)=\sum_{j=0}^{2 d-1} \frac{q_{d}(j)}{2 d-j}=p_{d}\left(d^{1 / 2}\right)+\sum_{j=d^{1 / 2}}^{2 d-1} \frac{q_{d}(j)}{2 d-j} \leq p_{d}\left(d^{1 / 2}\right)+\sum_{j=d^{1 / 2}}^{2 d-1} q_{d}(j) \tag{4.15}
\end{equation*}
$$

Introducing the random variable

$$
X:=|\{w \sim o: o \in \pi(w)\}|,
$$

the last expression in (4.15) equals

$$
p_{d}\left(d^{1 / 2}\right)+\mathbf{P}\left[X \geq d^{1 / 2}\right] \leq p_{d}\left(d^{1 / 2}\right)+\mathbf{P}\left[X^{3} \geq d^{3 / 2}\right] \leq p_{d}\left(d^{1 / 2}\right)+\frac{\mathbf{E}\left[X^{3}\right]}{d^{3 / 2}} .
$$

Therefore, it remains to show that $\mathbf{E}\left[X^{3}\right]=O(1)$. This follows from Lemma 4.2.11, by summing over $w_{1}, \ldots, w_{3}$ (not necessarily distinct). The cases $k=1,2$ of the lemma are used to sum the contributions where one or more of the $w_{i}$ 's coincide.

### 4.3 Chapter outlook

This proof could possibly be extended to a larger class of graphs, where the degree goes to infinity. Aldous' results [1] involved a large class of graphs under symmetry assumptions; see [1, Hypothesis 7]. It would be interesting to explore further.

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## Chapter 5

## Interlacement limit of a stopped trace on a torus

## Chapter Overview

This chapter is based on a manuscript submitted for publication; see [7]. This chapter is motivated by the hashing algorithm used in Chapter 3 to study high-dimensional sandpiles. While it looks difficult to bound the running time of the algorithm directly, we can obtain a heuristic bound by making some simplifications. First, let us consider a single loop-erased walk, instead of multiple walks. Second, let us assume the simple hash function $g(x)=x(\bmod N)$, where $\bmod N$ is understood coordinatewise, from Chapter 3. As can be seen in the discussion of Section 3.2.4 and Figure 3-3, the running time is influenced by two factors, the number of random walk steps, and the multiplicity of each projected value, since the multiplicity gives how many times we must follow the links in the vertexdata table. We know that each random walk will use $O\left(L^{2}\right)$ distinct vertices. Suppose we can show that a simple random walk run up to its first exit from $(-L, L)^{d}$ and taken $\bmod N$ spreads roughly evenly over the torus $\mathbb{T}_{N}$. Suppose also that for each point on the torus $\mathbb{T}_{N}$ the expected multiplicity is $O(1)$. Then the running time and the memory used will be roughly a constant times $L^{2}$. We can expect the above is true, when $L^{2} \sim A N^{d}$.

In this chapter, we will show that the random walks spread evenly over the torus in the following sense: the probability for the translate of any finite set $K$ to be visited converges to the probability that an interlacement process visits $K$, as long as the translate of $K$ is not too close to the origin $o$ of the random walk. We will not examine
the multiplicity of visits here, but we expect that one can show using similar methods that when $K$ is a single point, this multiplicity is asymptotically a Poisson sum of independent geometric distributions, where the intensity of Poisson is the intensity of the random interlacement process.

## Statement of Authorship

| This declaration concerns the article entitled: |  |  |  |  |
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| Candidate's contribution to the paper (provide details, and also indicate as a percentage) | The candidate predo <br> Main proof ideas: 30\% looking at existing pr seeing where new m <br> Working out details of extent, this involved bounds, martingale a <br> Organization of the p between the authors | antly execu contributed methods in ods were ne <br> roofs: 60\% lying various ments <br> r: $60 \%$ - de | formulation of ideas, rature, trying to adap <br> g out most proofs in like the local CLT, G <br> on the presentation | is involved hem, and <br> tail to a large ssian <br> re shared |
| Statement from Candidate | This paper reports on Higher Degree by Re | iginal resea arch candida | nducted during the | iod of my |
| Signed | Minwei Sun |  | Date | 30-Sept-2021 |


#### Abstract

We consider a simple random walk on $\mathbb{Z}^{d}$ started at the origin and stopped on its first exit time from $(-L, L)^{d} \cap \mathbb{Z}^{d}$. Write $L$ in the form $L=m N$ with $m=m(N)$ and $N$ an integer going to infinity in such a way that $L^{2} \sim A N^{d}$ for some real constant $A>0$. Our main result is that for $d \geq 3$, the projection of the stopped trajectory to the $N$-torus locally converges, away from the origin, to an interlacement process at level $A d \sigma_{1}$, where $\sigma_{1}$ is the exit time of a Brownian motion from the unit cube $(-1,1)^{d}$ that is independent of the interlacement process. The above problem is a variation on results of Windisch (2008) and Sznitman (2009).


### 5.1 Introduction

A special case of a result of Windisch [15] - extended further in [1] — states that the trace of a simple random walk on the discrete $d$-dimensional torus $(\mathbb{Z} / N \mathbb{Z})^{d}$, for $d \geq 3$, started from stationarity and run for time $u N^{d}$ converges, in a local sense, to an interlacement process at level $u$, as $N \rightarrow \infty$. In this chapter we will be concerned with a variation on this result, for which our motivation was a heuristic analysis of an algorithm we used to simulate high-dimensional loop-erased random walk and the sandpile height distribution in Chapter 3. Let us first describe our main result and then discuss the motivating problem.

Consider a discrete-time lazy simple random walk $\left(Y_{t}\right)_{t \geq 0}$ starting at the origin $o$ in $\mathbb{Z}^{d}$. We stop the walk at the first time $T$ it exits the large box $(-L, L)^{d}$. We will take $L$ of the form $L=m N$, where $m=m(N)$ and $N$ is an integer, such that $L^{2} \sim A N^{d}$ for some $A \in(0, \infty)$, as $N \rightarrow \infty$. We consider the projection of the trajectory $\left\{Y_{t}: 0 \leq t<T\right\}$ to the $N$-torus $\mathbb{T}_{N}=[-N / 2, N / 2)^{d} \cap \mathbb{Z}^{d}$. The projection is given by the map $\varphi: \mathbb{Z}^{d} \rightarrow \mathbb{T}_{N}$, where for any $x \in \mathbb{Z}^{d}, \varphi(x)$ is the unique point of $\mathbb{T}_{N}$ such that $\varphi(x) \equiv x(\bmod N)$, where congruence $(\bmod N)$ is understood coordinate-wise.

Let $\sigma_{1}$ denote the exit time from $(-1,1)^{d}$ of a standard Brownian motion started at o. For any finite set $K^{0} \subset \mathbb{Z}^{d}$, let $\operatorname{Cap}\left(K^{0}\right)$ denote the capacity of $K^{0}$ [9]. For any $0<R<\infty$ and $x \in \mathbb{Z}^{d}$, we denote $B_{R}(x)=\left\{y \in \mathbb{Z}^{d}:|y-x|<R\right\}$, where $|\cdot|$ is the Euclidean norm. Let $\mathcal{K}_{R}$ denote the collection of all subsets of $B_{R}(o)$. Given $\mathbf{x} \in \mathbb{T}_{N}$ let $\tau_{\mathbf{x}}: \mathbb{T}_{N} \rightarrow \mathbb{T}_{N}$ denote the translation of the torus by $\mathbf{x}$. Let $g: \mathbb{N} \rightarrow(0, \infty)$ be any function satisfying $g(N) \rightarrow \infty$, as $N \rightarrow \infty$.

Theorem 5.1.1. Let $d \geq 3$. For any $0<R<\infty$, any $K^{0} \in \mathcal{K}_{R}$, and any $\mathbf{x}$ satisfying
$\tau_{\mathbf{x}} \varphi\left(B_{R}(o)\right) \cap \varphi\left(B_{g(N)}(o)\right)=\emptyset$ we have

$$
\begin{equation*}
\mathbf{P}_{o}\left[\varphi\left(Y_{t}\right) \notin \tau_{\mathbf{x}} \varphi\left(K^{0}\right), 0 \leq t<T\right]=\mathbf{E}\left[e^{-d A \sigma_{1} \operatorname{Cap}\left(K^{0}\right)}\right]+o(1) \quad \text { as } N \rightarrow \infty \tag{5.1}
\end{equation*}
$$

The error term depends on $R$ and $g$, but is uniform in $K^{0}$ and $\mathbf{x}$.
Note that the trace of the lazy simple random walk stopped at time $T$ is the same as the trace of the simple random walk stopped at the analogous exit time. We use the lazy walk for convenience of the proof.

Our result is close in spirit - but different in details - compared to a result of Sznitman [12] that is concerned with simple random walk on a discrete cylinder. The interlacement process was introduced by Sznitman in [13]. It consists of a one-parameter family $\left(\mathcal{I}^{u}\right)_{u>0}$ of random subsets of $\mathbb{Z}^{d}(d \geq 3)$, where the distribution of $\mathcal{I}^{u}$ can be characterized by the relation

$$
\begin{equation*}
\mathbf{P}\left[\mathcal{I}^{u} \cap K=\emptyset\right]=\exp (-u \operatorname{Cap}(K)) \quad \text { for any finite } \emptyset \neq K \subset \mathbb{Z}^{d} \tag{5.2}
\end{equation*}
$$

An alternative definition - that provides more insight but is more cumbersome to state - represents $\mathcal{I}^{u}$ as the trace of a Poisson cloud of bi-infinite random walk trajectories (up to time-shifts), where $u$ is an intensity parameter. We refer to [13] and the books $[3,14]$ for further details. Comparing (5.1) and (5.2) makes it clear what we mean by saying that the stopped trajectory, locally, is described by an interlacement process at the random level $u=A d \sigma_{1}$.

Our motivation to study the question in Theorem 5.1.1 was a simulation problem that arose in our numerical study of high-dimensional sandpiles in Chapter 3. We refer the interested reader to $[11,2,6]$ for background on sandpiles. In our simulations we needed to generate loop-erased random walks (LERW) from the origin $o$ to the boundary of $[-L, L]^{d}$ where $d \geq 5$. Recall that the LERW is defined by running a simple random walk from $o$ until it hits the boundary, and erasing all loops from its trajectory chronologically, as they are created. We refer to the book [9] for further background on LERW (which is not needed to understand the results in this chapter). It is known from results of Lawler [8] that in dimensions $d \geq 5$ the LERW visits on the order of $L^{2}$ vertices, the same as the simple random walk generating it. As the number of vertices visited is a lot smaller than the volume $c L^{d}$ of the box, an efficient way to store the path generating the LERW is provided by the well-known method of hashing. We refer to Chapter 3 for a discussion of this approach, and only provide a brief summary here. Assign to any $x \in[-L, L]^{d} \cap \mathbb{Z}^{d}$ an integer value $f(x) \in\left\{0,1, \ldots, C L^{2}\right\}$ that is used to
label the information relevant to position $x$, where $C$ can be a large constant or slowly growing to infinity. Thus $f$ is necessarily highly non-injective. However, we may be able to arrange that with high probability the restriction of $f$ to the simple random walk trajectory is not far from injective, and then memory use can be reduced from order $L^{d}$ to roughly $O\left(L^{2}\right)$.

A simple possible choice of the hash function $f$ can be to compose the map $\varphi:[-L, L]^{d} \cap$ $\mathbb{Z}^{d} \rightarrow \mathbb{T}_{N}$ with a linear enumeration of the vertices of $\mathbb{T}_{N}$, whose range has the required size ${ }^{1}$. The method can be expected to be effective, if the projection $\varphi(Y[0, T))$ spreads roughly evenly over the torus $\mathbb{T}_{N}$ with high probability. Our main theorem establishes a version of such a statement, as the right hand side expression in (5.1) is independent of $\mathbf{x}$.

We now make some comments on the proof of Theorem 5.1.1. We refer to [3, Theorem 3.1] for the strategy of the proof in the case when the walk is run for a fixed time $u N^{d}$. The argument presented there goes by decomposing the walk into stretches of length $\left\lfloor N^{\delta}\right\rfloor$ for some $2<\delta<d$, and then estimating the (small) probability in each stretch that $\tau_{\mathbf{x}} \varphi\left(K^{0}\right)$ is hit by the projection. We follow the same outline for the stopped lazy random walk. However, the elegant time-reversal argument given in [3] is not convenient in our setting, and we need to prove a delicate estimate on the probability that $\tau_{\mathbf{x}} \varphi\left(K^{0}\right)$ is hit, conditional on the start and end-points of the stretch. For this, we only want to consider stretches with "well-behaved" starting and end-points. We also classify stretches as "good stretch" where the total displacement is not too large, and as "bad stretch". We do this in such a way that the expected number of "bad stretches" is small and summing over the "good stretches" gives us the required behaviour.
$A$ note on constants. All constants will be positive and finite. Constants denoted $C$ or $c$ may change from line to line. If we need to refer to a constant later, it will be given an index, such as $C_{1}$.

We now describe the organization of this chapter. Section 5.1 introduces the main result and the motivating problem. In Section 5.2, we first introduce some basic notation, then we recall several useful known results on random walk and state the key propositions required for the proof of the main theorem, Theorem 5.1.1. Section 5.3 contains the proof of the main theorem, assuming the key propositions. Finally, in Section 5.4 we provide the proofs of the propositions stated in Section 5.2.

[^0]
### 5.2 Preliminaries

### 5.2.1 Some notation

We first introduce some notation used in this chapter. In section 5.1, we denoted the discrete torus $\mathbb{T}_{N}=[-N / 2, N / 2)^{d} \cap \mathbb{Z}^{d}, d \geq 3$ and the canonical projection map $\varphi: \mathbb{Z}^{d} \rightarrow \mathbb{T}_{N}$.

We write vertices and subsets of the torus in bold, i.e. $\mathbf{x} \in \mathbb{T}_{N}$ and $\mathbf{K} \subset \mathbb{T}_{N}$. In order to simplify notation, in the rest of the chapter we abbreviate $\mathbf{K}=\tau_{\mathbf{x}} \varphi\left(K^{0}\right)$, and let $K \subset \mathbb{Z}^{d}$ be a translate of $K^{0}$ with the property that $\varphi(K)=\mathbf{K}$. Let $\left(Y_{t}\right)_{t \geq 0}$ be a discrete-time lazy simple random walk on $\mathbb{Z}^{d}$, that is,

$$
\mathbf{P}\left[Y_{t+1}=y^{\prime} \mid Y_{t}=x^{\prime}\right]= \begin{cases}\frac{1}{2} & \text { when } y^{\prime}=x^{\prime} \\ \frac{1}{4 d} & \text { when }\left|y^{\prime}-x^{\prime}\right|=1\end{cases}
$$

We denote the corresponding lazy random walk on $\mathbb{T}_{N}$ by $\left(\mathbf{Y}_{t}\right)_{t \geq 0}=\left(\varphi\left(Y_{t}\right)\right)_{t \geq 0}$. Let $\mathbf{P}_{x^{\prime}}$ denote the distribution of the lazy random walk on $\mathbb{Z}^{d}$ started from $x^{\prime} \in \mathbb{Z}^{d}$, and write $\mathbf{P}_{\mathbf{x}}$ for the distribution of the lazy random walk on $\mathbb{T}_{N}$ started from $\mathbf{x}=\varphi\left(x^{\prime}\right) \in \mathbb{T}_{N}$. We write $p_{t}\left(x^{\prime}, y^{\prime}\right)=\mathbf{P}_{x^{\prime}}\left[Y_{t}=y^{\prime}\right]$ for the $t$-step transition probability. Further notation we will use:

- $L=m N$, where $L^{2} \sim A N^{d}$ as $N \rightarrow \infty$ for some constant $A \in(0, \infty)$
- $D=(-m, m)^{d}$, rescaled box, indicates which copy of the torus the walk is in
- $n=\left\lfloor N^{\delta}\right\rfloor$ for some $2<\delta<d$, long enough for the mixing property on the torus, but short compared to $L^{2}$
- $\mathbf{x}_{\mathbf{0}} \in \mathbf{K}$ is a fixed point of $\mathbf{K}$
- we write points in the original lattice $\mathbb{Z}^{d}$ with a prime, such as $y^{\prime}$, and decompose a point $y^{\prime}$ as $y N+\mathbf{y}$ with $y$ in a rescaled lattice isomorphic to $\mathbb{Z}^{d}$ and $\mathbf{y}=\varphi\left(y^{\prime}\right) \in \mathbb{T}_{N}$
- $T=\inf \left\{t \geq 0: Y_{t} \notin(-L, L)^{d}\right\}$, the first exit time from $(-L, L)^{d}$
- $S=\inf \left\{\ell \geq 0: Y_{n \ell} \notin(-L, L)^{d}\right\}$, the first multiple of $n$ when the rescaled point $Y_{n \ell} / N$ is not in $(-m, m)^{d}$

We omit the dependence on $d$ and $N$ from some notation above for simplicity.

### 5.2.2 Some auxiliary results on random walk

In this section, we collect some known results required for the proof of Theorem 5.1.1. We will rely heavily on the Local Central Limit Theorem (LCLT) [9, Chapter 2], with error term, and the Martingale maximal inequality [9, Eqn. (12.12) of Corollary 12.2.7]. We will also use the result from Equation (6.31) in [9], saying that the probability of a random walk started uniformly from the boundary of a large ball with a radius $n$ hitting a finite set $K$ before exiting the ball tends to $\operatorname{Cap}(K)$, as $n \rightarrow \infty$. In estimating some error terms in our arguments, sometimes we will use the Gaussian upper and lower bounds [5]. We also need to derive a lemma from the mixing property on the torus [10, Theorem 5.6] to show that the starting positions of different stretches are not far from uniform on the torus; see Lemma 5.2.1.

We recall the LCLT from [9, Chapter 2]. The following is a specialisation of [9, Theorem 2.3.11] to lazy simple random walk. The covariance matrix $\Gamma$ and the square root $J^{*}(x)$ of the associated quadratic form are given by

$$
\Gamma=(2 d)^{-1} I, \quad J^{*}(x)=(2 d)^{\frac{1}{2}}|x|,
$$

where $I$ is the $(d \times d)$-unit matrix.

Let $\bar{p}_{t}\left(x^{\prime}\right)$ denote the estimate of $p_{t}\left(x^{\prime}\right)$ that one obtains by the LCLT, for lazy simple random walk. We have

$$
\begin{aligned}
\bar{p}_{t}\left(x^{\prime}\right) & =\frac{1}{(2 \pi t)^{d / 2} \sqrt{\operatorname{det} \Gamma}} \exp \left(-\frac{J^{*}\left(x^{\prime}\right)^{2}}{2 t}\right) \\
& =\frac{1}{(2 \pi t)^{d / 2}(2 d)^{-d / 2}} \exp \left(-\frac{2 d\left|x^{\prime}\right|^{2}}{2 t}\right) \\
& =\frac{\bar{C}}{t^{d / 2}} \exp \left(-\frac{d\left|x^{\prime}\right|^{2}}{t}\right) .
\end{aligned}
$$

The lazy simple random walk $\left(Y_{t}\right)_{t \geq 0}$ in $\mathbb{Z}^{d}$ is aperiodic, irreducible with mean zero, finite second moment, and finite exponential moments. All the third moments of $Y_{1}$ vanish.

Theorem 5.2.1 ([9], Theorem 2.3.11). For lazy simple random walk $\left(Y_{t}\right)_{t \geq 0}$ in $\mathbb{Z}^{d}$, there exists $\rho>0$ such that for all $t \geq 1$ and all $x^{\prime} \in \mathbb{Z}^{d}$ with $\left|x^{\prime}\right|<\rho t$,

$$
p_{t}\left(x^{\prime}\right)=\bar{p}_{t}\left(x^{\prime}\right) \exp \left\{O\left(\frac{1}{t}+\frac{\left|x^{\prime}\right|^{4}}{t^{3}}\right)\right\} .
$$

The Martingale maximal inequality in [9, Eqn. (12.12) of Corollary 12.2.7] is stated as follows. Let $\left(Y_{t}^{(i)}\right)_{t \geq 0}$ denote the $i$-th coordinate of $\left(Y_{t}\right)_{t \geq 0}(1 \leq i \leq d)$. The standard deviation $\sigma$ of $Y_{1}^{(i)}$ is given by $\sigma^{2}=(2 d)^{-1}$. For all $t \geq 1$ and all $r>0$ we have

$$
\begin{equation*}
\mathbf{P}_{o}\left[\max _{0 \leq j \leq t} Y_{j}^{(i)} \geq r \sigma \sqrt{t}\right] \leq e^{-r^{2} / 2} \exp \left\{O\left(\frac{r^{3}}{\sqrt{t}}\right)\right\} . \tag{5.3}
\end{equation*}
$$

Now we state the result of [9, Eqn. (6.31)]. Recall that $B_{r}(o)$ is the discrete ball centred at $o$ with radius $r$. Let

$$
\xi_{B_{r}(o)}=\inf \left\{t \geq 1: Y_{t} \notin B_{r}(o)\right\} .
$$

Let $\partial B_{r}(o)=\left\{y^{\prime} \in \mathbb{Z}^{d} \backslash B_{r}(o): \exists x^{\prime} \in B_{r}(o)\right.$ such that $\left.\left|x^{\prime}-y^{\prime}\right|=1\right\}$. For a given finite set $K \subseteq \mathbb{Z}^{d}$, let $H_{K}$ denote the hitting time

$$
H_{K}=\inf \left\{t \geq 1: Y_{t} \in K\right\} .
$$

Then we have

$$
\begin{equation*}
\operatorname{Cap}(K)=\lim _{r \rightarrow \infty} \sum_{y^{\prime} \in \partial B_{r}(o)} \mathbf{P}_{y^{\prime}}\left[H_{K}<\xi_{B_{r}(o)}\right] . \tag{5.4}
\end{equation*}
$$

Here $\operatorname{Cap}(K)$ is the capacity of $K$; see [9, Section 6.5].

In estimating some error terms in our arguments, sometimes we will use the Gaussian upper and lower bounds [5]: there exist constants $C=C(d)$ and $c=c(d)$ such that

$$
\begin{array}{ll}
p_{t}\left(x^{\prime}, y^{\prime}\right) \leq \frac{C}{t^{d / 2}} \exp \left(-c \frac{\left|y^{\prime}-x^{\prime}\right|^{2}}{t}\right), & \text { for } x^{\prime}, y^{\prime} \in \mathbb{Z}^{d} \text { and } t \geq 1 ; \\
p_{t}\left(x^{\prime}, y^{\prime}\right) \geq \frac{c}{t^{d / 2}} \exp \left(-C \frac{\left|y^{\prime}-x^{\prime}\right|^{2}}{t}\right), \quad \text { for }\left|y^{\prime}-x^{\prime}\right| \leq t . \tag{5.5}
\end{array}
$$

Regarding mixing times, recall that lazy simple random walk on the $N$-torus mixes in time $N^{2}[10$, Theorem 5.6]. With this in mind we derive the following simple lemma.

Recall that $2<\delta<d$ and $n=\left\lfloor N^{\delta}\right\rfloor$.
Lemma 5.2.1. There exists $C=C(d)$ such that for any $N \geq 1$ and any $t \geq n$ we have

$$
\mathbf{P}_{\mathbf{o}}\left[\mathbf{Y}_{t}=\mathbf{x}\right] \leq \frac{C}{N^{d}}, \quad \mathbf{x} \in \mathbb{T}_{N} .
$$

Proof. Using the Gaussian upper bound, the left hand side can be bounded by

$$
\begin{aligned}
\sum_{x \in \mathbb{Z}^{d}} p_{t}(o, x N+\mathbf{x}) & \leq \frac{C}{t^{d / 2}} \sum_{x \in \mathbb{Z}^{d}} \exp \left(-c \frac{|x N+\mathbf{x}|^{2}}{t}\right) \leq \frac{C}{t^{d / 2}} \sum_{x \in \mathbb{Z}^{d}} \exp \left(-c \frac{|x N|^{2}}{t}\right) \\
& \leq \frac{C}{t^{d / 2}} \sum_{r=0}^{\infty}(r+1)^{d-1} \exp \left(-c \frac{r^{2} N^{2}}{t}\right) \\
& \leq \frac{C}{t^{d / 2}} \int_{0}^{\infty}(r+1)^{d-1} \exp \left(-c \frac{r^{2} N^{2}}{t}\right) d r \\
& =\frac{C}{t^{d / 2}} \int_{0}^{\infty}\left(\frac{\sqrt{t u}}{N}+1\right)^{d-1} \exp (-c u) \frac{\sqrt{t}}{2 N \sqrt{u}} d u \\
& \leq \frac{C}{N^{d}} \int_{0}^{\infty}(\sqrt{u})^{d-2} \exp (-c u) d u=\frac{C}{N^{d}} .
\end{aligned}
$$

### 5.2.3 Key propositions

In this section we state some propositions to be used in Section 5.3 to prove Theorem 5.1.1. The propositions will be proved in Section 5.4.

The strategy of the proof is to consider stretches of length $n$ of the walk, and estimate the probability in each stretch that $\mathbf{K}$ is not hit by the projection. For this, we only want to consider stretches with "well-behaved" starting and end-points, which motivates the definition of the set $\mathcal{G}_{\zeta, C_{1}}$ below. The definition involves a parameter $0<\zeta<1$, whose choice we now specify.

First, we will need $2<\delta<d$ to satisfy the inequality

$$
\begin{equation*}
2 \delta>\frac{d^{2}}{d-1} \tag{5.6}
\end{equation*}
$$

This can be satisfied if $d \geq 3$ and $\delta$ is sufficiently close to $d$, say $\delta=\frac{7}{8} d$. Since the left hand side of (5.6) equals $(\delta / d)(d-2)$, we can subsequently choose $\zeta$ such that we also have

$$
\begin{equation*}
0<\zeta<\frac{\delta}{d}, \quad \zeta(d-2)>d-\delta \tag{5.7}
\end{equation*}
$$

We now define
where $f(n)=C_{1} \sqrt{\log n}$ and recall that we write $y_{\ell}^{\prime}=y_{\ell} N+\mathbf{y}_{\ell}$ and $y_{\ell-1}^{\prime}=y_{\ell-1} N+\mathbf{y}_{\ell-1}$, and we define $y_{0}^{\prime}=o$. The time $\tau$ is corresponding to a particular value of the exiting time $S$, so $y_{\ell} \in D$ for $1 \leq \ell<\tau$ and $y_{\tau} \notin D$. The parameter $C_{1}$ will be chosen in the course of the proof. See Figure 5-1 for a visual illustration of the set $\mathcal{G}_{\zeta, C_{1}}$.


Figure 5-1: This figure explains the properties of the set $\mathcal{G}_{\zeta, C_{1}}$ (not to scale). None of the $y_{\ell}$ 's is in a shaded region.

The starting point for the proof is the following proposition that decomposes the probability we are interested in into terms involving single stretches of duration $n$.

Proposition 5.2.1. For a sufficiently large value of $C_{1}$, we have

$$
\begin{align*}
\mathbf{P}_{o} & {\left[Y_{t} \notin \varphi^{-1}(\mathbf{K}), 0 \leq t<T\right] } \\
& =\sum_{\left(\tau,\left(y_{\ell}, \mathbf{y}_{\ell}\right)_{\ell=1}^{\tau}\right) \in \mathcal{G}_{\zeta, C_{1}}} \prod_{\ell=1}^{\tau} \mathbf{P}_{y_{\ell-1}^{\prime}}\left[Y_{n}=y_{\ell}^{\prime}, Y_{t} \notin \varphi^{-1}(\mathbf{K}) \text { for } 0 \leq t<n\right]+o(1), \tag{5.9}
\end{align*}
$$

where $o(1) \rightarrow 0$ as $N \rightarrow \infty$.
Central to the proof of Theorem 5.1.1 is the following proposition, that estimates the probability of hitting a copy of $\mathbf{K}$ during a "good stretch" where the displacement $\left|y_{\ell}^{\prime}-y_{\ell-1}^{\prime}\right|$ is not too large.

Proposition 5.2.2. Let $\left(\tau,\left(y_{\ell}, \mathbf{y}_{\ell}\right)_{\ell=1}^{\tau}\right) \in \mathcal{G}_{\zeta, C_{1}}$. For a sufficiently large value of $C_{1}$,
and for all $1 \leq \ell \leq \tau$ such that $\left|y_{\ell}^{\prime}-y_{\ell-1}^{\prime}\right| \leq 10 \sqrt{n} \log \log n$ we have

$$
\begin{align*}
\mathbf{P}_{y_{\ell-1}^{\prime}} & {\left[Y_{n}=y_{\ell}^{\prime}, Y_{t} \notin \varphi^{-1}(\mathbf{K}) \text { for } 0 \leq t<n\right] } \\
& =\mathbf{P}_{y_{\ell-1}^{\prime}}\left[Y_{n}=y_{\ell}^{\prime}\right]\left(1-\frac{1}{2} \frac{\operatorname{Cap}(\mathbf{K}) n}{N^{d}}(1+o(1))\right) . \tag{5.10}
\end{align*}
$$

In addition to this proposition (that we prove in Section 5.4.3), we will need a weaker version under less restriction on the distance $\left|y_{\ell}^{\prime}-y_{\ell-1}^{\prime}\right|$. This will be needed to handle some error terms and it will be useful to demonstrate some of our proof ideas. It will be proved in Section 5.4.1.

Proposition 5.2.3. Let $\left(\tau,\left(y_{\ell}, \mathbf{y}_{\ell}\right)_{\ell=1}^{\tau}\right) \in \mathcal{G}_{\zeta, C_{1}}$. For all $2 \leq \ell \leq \tau$ we have

$$
\begin{equation*}
\mathbf{P}_{y_{\ell-1}^{\prime}}\left[Y_{n}=y_{\ell}^{\prime}, Y_{t} \notin \varphi^{-1}(\mathbf{K}) \text { for all } 0 \leq t<n\right]=\mathbf{P}_{y_{\ell-1}^{\prime}}\left[Y_{n}=y_{\ell}^{\prime}\right]\left(1-O\left(\frac{n}{N^{d}}\right)\right) . \tag{5.11}
\end{equation*}
$$

and for the first stretch we have

$$
\begin{equation*}
\mathbf{P}_{o}\left[Y_{n}=y_{1}^{\prime}, Y_{t} \notin \varphi^{-1}(\mathbf{K}) \text { for all } 0 \leq t<n\right]=\mathbf{P}_{o}\left[Y_{n}=y_{1}^{\prime}\right](1-o(1)) . \tag{5.12}
\end{equation*}
$$

Our final proposition is needed to estimate the number of stretches that are 'bad'.
Proposition 5.2.4. We have

$$
\begin{align*}
\mathbf{P}[\# & \left.\left\{1 \leq \ell \leq \frac{N^{d}}{n} C_{1} \log N:\left|Y_{n \ell}-Y_{n(\ell-1)}\right|>10 \sqrt{n} \log \log n\right\} \geq \frac{N^{d}}{n} \frac{1}{\log \log n}\right] \\
& \rightarrow 0, \tag{5.13}
\end{align*}
$$

as $N \rightarrow \infty$.

### 5.3 Proof of the main theorem assuming the key propositions

This section is the proof of the main theorem, Theorem 5.1.1.

Proof of Theorem 5.1.1 assuming Propositions 5.2.1-5.2.4.
We first denote $\mathcal{L}=\left\{\ell:\left|y_{\ell}^{\prime}-y_{\ell-1}^{\prime}\right| \leq 10 \sqrt{n} \log \log n\right\}$, then we have $\mathcal{L}^{c}=\{\ell:$ $\left.\left|y_{\ell}^{\prime}-y_{\ell-1}^{\prime}\right|>10 \sqrt{n} \log \log n\right\}$.

By Propositions 5.2.1, 5.2.2 and 5.2.3, we have

$$
\begin{align*}
& \mathbf{P}_{o}\left[Y_{t} \notin \varphi^{-1}(\mathbf{K}), 0 \leq t<T\right] \\
& =o(1)+\sum_{\left(\tau,\left(y_{\ell}, \mathbf{y} \ell\right)_{\ell=1}^{\tau}\right) \in \mathcal{G}_{\zeta, C_{1}}} \prod_{\ell=1}^{\tau} \mathbf{P}_{y_{\ell-1}^{\prime}}\left[Y_{n}=y_{\ell}^{\prime}\right]  \tag{5.14}\\
& \quad \times(1-o(1)) \prod_{\substack{2 \leq \ell \leq \tau \\
\ell \in \mathcal{L}}}\left(1-\frac{1}{2} \operatorname{Cap}(\mathbf{K}) \frac{n}{N^{d}}(1+o(1))\right) \prod_{\substack{2 \leq \ell \leq \tau \\
\bar{\ell} \in \mathcal{L}^{c}}}\left(1-O\left(\frac{n}{N^{d}}\right)\right) .
\end{align*}
$$

By Proposition 5.2.4, we have

$$
\begin{equation*}
\left|\mathcal{L}^{c}\right| \leq \frac{N^{d}}{n} \frac{1}{\log \log n} \quad \text { with probability going to } 1 \text {, as } N \rightarrow \infty \tag{5.15}
\end{equation*}
$$

By (5.15), we can lower bound the last product in (5.14) by

$$
\exp \left(-O\left(\frac{n}{N^{d}}\right) \frac{N^{d}}{n} \frac{1}{\log \log n}\right)=e^{o(1)}=(1+o(1)) .
$$

Since the product is also at most 1 , it equals $1+o(1)$.

Also due to (5.15), we have

$$
\tau-\frac{N^{d}}{n} \frac{1}{\log \log n} \leq|\mathcal{L}| \leq \tau
$$

Since $\tau \geq \frac{N^{d}}{n}(\sqrt{\log \log n})^{-1}$, we have $|\mathcal{L}|=(1+o(1)) \tau$. This implies that the penultimate product in (5.14) equals

$$
\begin{equation*}
\left(1-\frac{1}{2} \operatorname{Cap}(\mathbf{K}) \frac{n}{N^{d}}(1+o(1))\right)^{(1+o(1)) \tau}=\exp \left(-\frac{1}{2} \operatorname{Cap}(\mathbf{K}) \frac{n}{N^{d}} \tau(1+o(1))\right) . \tag{5.16}
\end{equation*}
$$

By summming over $\left(y_{\ell}, \mathbf{y}_{\ell}\right)_{\ell=1}^{\tau}$ we get that (5.14) equals

$$
\begin{equation*}
o(1)+\sum_{\tau}^{\prime} \mathbf{E}\left[\mathbf{1}_{S=\tau} \exp \left(-\frac{1}{2} \operatorname{Cap}(\mathbf{K}) \frac{n}{N^{d}} \tau(1+o(1))\right)\right], \tag{5.17}
\end{equation*}
$$

where the primed summation denotes restriction to $\frac{N^{d}}{n}(\sqrt{\log \log n})^{-1} \leq \tau \leq(\log N) \frac{N^{d}}{n}$. Since $S$ satisfies the bounds on $\tau$ with probability going to 1 , the latter expression equals

$$
\begin{equation*}
o(1)+\mathbf{E}\left[e^{-\frac{1}{2} \operatorname{Cap}(\mathbf{K}) \frac{n}{N d} S}\right] . \tag{5.18}
\end{equation*}
$$

Let $\Gamma_{n}$ denote the covariance matrix for $Y_{n}$, so that $\Gamma_{n}=\frac{n}{2 d} I$. Let $Z_{1}=\sqrt{\frac{2 d}{n}} Y_{n}$ in distribution, with the covariance matrix $\Gamma_{Z}=I$. Let $Z_{\ell}=\sqrt{\frac{2 d}{n}} Y_{n \ell}$ in distribution for $\ell \geq 0$.

Recall that $S=\inf \left\{\ell \geq 0: Y_{n \ell} \notin(-L, L)^{d}\right\}$. Since $L^{2} \sim A N^{d}$, the event $\left\{Y_{n \ell} \notin\right.$ $\left.(-L, L)^{d}\right\}$ is the same as $\left\{Y_{n \ell} \notin\left(-(1+o(1)) \sqrt{A} N^{d / 2},(1+o(1)) \sqrt{A} N^{d / 2}\right)^{d}\right\}$. Converting to events in terms of $Z$ we have

$$
Z_{\ell} \notin\left(-\sqrt{2 d A(1+o(1))}\left(N^{d} / n\right)^{1 / 2}, \sqrt{2 d A(1+o(1))}\left(N^{d} / n\right)^{1 / 2}\right)^{d}
$$

Now we can write $S$ as

$$
S=\inf \left\{\ell \geq 0: Z_{\ell} \notin\left(-\sqrt{2 d A(1+o(1))}\left(N^{d} / n\right)^{1 / 2}, \sqrt{2 d A(1+o(1))}\left(N^{d} / n\right)^{1 / 2}\right)^{d}\right\}
$$

Let $\sigma_{1}=\inf \left\{t>0: B_{t} \notin(-1,1)^{d}\right\}$ be the exit time of Brownian motion from $(-1,1)^{d}$. By Donsker's Theorem [4, Theorem 8.1.5] we have

$$
\mathbf{P}\left[S \leq 2 d A(1+o(1)) \frac{N^{d}}{n} t\right] \rightarrow \mathbf{P}\left[\sigma_{1} \leq t\right]
$$

Then we have that $\frac{n}{N^{d}} S$ converges in distribution to $c \sigma_{1}$, with $c=2 d A$. This completes the proof.

### 5.4 Proofs of the key propositions

### 5.4.1 Proof of Proposition 5.2.3

In the proof of the proposition we will need the following lemma that bounds the probability of hitting some copy of $\mathbf{K}$ in terms of the Green's function of the random walk. Recall that the Green's function is defined by

$$
G\left(x^{\prime}, y^{\prime}\right)=\sum_{t=0}^{\infty} p_{t}\left(x^{\prime}, y^{\prime}\right)
$$

and in all $d \geq 3$ satisfies the bound [9]

$$
G\left(x^{\prime}, y^{\prime}\right) \leq \frac{C_{G}}{\left|y^{\prime}-x^{\prime}\right|^{d-2}}
$$

for a constant $C_{G}=C_{G}(d)$. For part (ii) of the lemma recall that $\mathbf{K} \cap \varphi\left(B_{g(N)}(o)\right)=\emptyset$.

Lemma 5.4.1. Let $d \geq 3$.
(i) If $y^{\prime} \in \mathbb{Z}^{d}$ satisfies $\varphi\left(y^{\prime}\right) \notin B\left(\mathbf{x}_{0}, N^{\zeta}\right)$, then for all sufficiently small $\varepsilon>0$ we have

$$
\begin{equation*}
\sum_{t=0}^{N^{2+6 \varepsilon}} \sum_{x \in \mathbb{Z}^{d}} \sum_{x^{\prime} \in \mathbf{K}+x N} p_{t}\left(y^{\prime}, x^{\prime}\right) \leq \frac{C}{N^{\zeta(d-2)}} . \tag{5.19}
\end{equation*}
$$

(ii) If $g(N) \leq N^{\zeta}$, then for all sufficiently small $\varepsilon>0$ we have

$$
\begin{equation*}
\sum_{t=0}^{N^{2+6 \varepsilon}} \sum_{x \in \mathbb{Z}^{d}} \sum_{x^{\prime} \in \mathbf{K}+x N} p_{t}\left(o, x^{\prime}\right) \leq \frac{C}{g(N)^{(d-2)}} . \tag{5.20}
\end{equation*}
$$

(iii) If $y^{\prime} \in \mathbb{Z}^{d}$ satisfies $\varphi\left(y^{\prime}\right) \notin B\left(\mathbf{x}_{0}, N^{\zeta}\right)$, then for all sufficiently small $\varepsilon>0$ we have

$$
\begin{equation*}
\sum_{\substack{x \in \mathbb{Z}^{d}\\
}}^{\substack{\begin{subarray}{c}{x^{\prime} \in \mathbf{K}+x N \\
\left|x^{\prime}-y^{\prime}\right| \leq n^{\frac{1}{2}+\varepsilon}} }}\end{subarray}} G\left(y^{\prime}, x^{\prime}\right) \leq \frac{C}{N^{d-\delta-2 \delta \varepsilon}} . \tag{5.21}
\end{equation*}
$$

Proof. (i) Due to (5.3), the probability that $\left|Y_{t}-y^{\prime}\right|>N^{1+4 \varepsilon}$ for some $0 \leq t \leq N^{2+6 \varepsilon}$ is stretched-exponentially small in $N$. Excluding this event, we have the upper bound

$$
\sum_{\substack{x^{\prime} \in \varphi^{-1}(\mathbf{K}) \\\left|x^{\prime}-y^{\prime}\right| \leq N^{1+4 \varepsilon}}} G\left(y^{\prime}, x^{\prime}\right) .
$$

Let $Q(k N)$ be the cube with radius $k N$ centred at $o$ and then $y^{\prime}+(Q(k N) \backslash Q((k-1) N))$ are disjoint annuli for $k=1,2, \ldots$. We decompose the sum over $x^{\prime}$ according to which annulus $x^{\prime}$ falls into. For $k \geq 2$ we have

$$
\sum_{\substack{x^{\prime} \in \varphi^{-1}(\mathbf{K}) \\ x^{\prime}-y^{\prime} \in Q(k N) \backslash Q((k-1) N)}} \frac{C_{G}}{\left|y^{\prime}-x^{\prime}\right|^{d-2}} \leq|\mathbf{K}| C k^{d-1} C_{G}(N k)^{2-d} \leq|\mathbf{K}| C k N^{2-d},
$$

where $C_{G}$ is the Green's function constant. The contribution from a copy of $\mathbf{K}$ in $y^{\prime}+Q(N)$ will be of order $N^{2-d}$ if its distance from $y^{\prime}$ is at least $N / 3$, say. Note that there is at most one copy of $\mathbf{K}$ within distance $N / 3$ of $y^{\prime}$, which may have a distance as small as $N^{\zeta}$.

We have to sum over the following values of $k$ :

$$
k=1, \ldots, \frac{N^{1+4 \varepsilon}}{N}=N^{4 \varepsilon}
$$

Since $x^{\prime} \in \varphi^{-1}(\mathbf{K})$ and $y^{\prime} \notin \varphi^{-1}\left(B\left(\mathbf{x}_{0}, N^{\zeta}\right)\right)$ for $\mathbf{x}_{0} \in \mathbf{K}$, the distance between $x^{\prime}$ and $y^{\prime}$ is at least $N^{\zeta}$. Therefore, we get the upper bound as follows:

$$
\begin{aligned}
\sum_{\substack{x^{\prime} \in \varphi^{-1}(\mathbf{K}) \\
\left|x^{\prime}-y^{\prime}\right| \leq N^{1+4 \varepsilon}}} G\left(y^{\prime}, x^{\prime}\right) & \leq|\mathbf{K}| N^{\zeta(2-d)}+\sum_{k=1}^{N^{4 \varepsilon}}|\mathbf{K}| C k N^{2-d} \\
& \leq|\mathbf{K}| N^{\zeta(2-d)}+C|\mathbf{K}| N^{2-d} \times N^{8 \varepsilon} \leq C|\mathbf{K}| N^{\zeta(2-d)} .
\end{aligned}
$$

Here the last inequality follows from the choice of $\zeta$, (5.7), for sufficiently small $\varepsilon>0$.
(ii) The proof is essentially the same, except for the contribution of the "nearest" copy of $\mathbf{K}$, which is now $C|\mathbf{K}| g(N)^{2-d}$.
(iii) The proof is very similar to that in part (i). Recall that $n=\left\lfloor N^{\delta}\right\rfloor$. This time we need to sum over $k=1, \ldots, n^{\frac{1}{2}+\varepsilon} / N$, which results in the bound

$$
C|\mathbf{K}| N^{-\zeta(d-2)}+C|\mathbf{K}| N^{2-d} \times N^{\delta+2 \delta \varepsilon-2}=C|\mathbf{K}|\left[N^{-\zeta(d-2)}+N^{\delta-d+2 \delta \varepsilon}\right] .
$$

Here, for $\varepsilon>0$ for small enough, the second term dominates due to the choice of $\zeta$; see (5.7).

Proof of Proposition 5.2.3. Since

$$
\begin{aligned}
& \mathbf{P}_{y_{\ell-1}^{\prime}}\left[Y_{n}=y_{\ell}^{\prime}, Y_{t} \notin \varphi^{-1}(\mathbf{K}) \text { for } 0 \leq t<n\right] \\
& \quad=\mathbf{P}_{y_{\ell-1}^{\prime}}\left[Y_{n}=y_{\ell}^{\prime}\right]-\mathbf{P}_{y_{\ell-1}^{\prime}}\left[Y_{n}=y_{\ell}^{\prime}, Y_{t} \in \varphi^{-1}(\mathbf{K}) \text { for some } 0 \leq t<n\right]
\end{aligned}
$$

we need to show that

$$
\mathbf{P}_{y_{\ell-1}^{\prime}}\left[Y_{n}=y_{\ell}^{\prime}, Y_{t} \in \varphi^{-1}(\mathbf{K}) \text { for some } 0 \leq t<n\right]=O\left(\frac{n}{N^{d}}\right) \mathbf{P}_{y_{\ell-1}^{\prime}}\left[Y_{n}=y_{\ell}^{\prime}\right] .
$$

Define $A(x)=\left\{Y_{n}=y_{\ell}^{\prime}, Y_{t} \in x N+\mathbf{K}\right.$ for some $\left.0 \leq t<n\right\}$, so that

$$
\begin{equation*}
\mathbf{P}_{y_{\ell-1}^{\prime}}\left[Y_{n}=y_{\ell}^{\prime}, Y_{t} \in \varphi^{-1}(\mathbf{K}) \text { for some } 0 \leq t<n\right] \leq \sum_{x \in \mathbb{Z}^{d}} \mathbf{P}_{y_{\ell-1}^{\prime}}[A(x)] . \tag{5.22}
\end{equation*}
$$

We have

$$
\begin{equation*}
\mathbf{P}_{y_{\ell-1}^{\prime}}[A(x)] \leq \sum_{n_{1}+n_{2}=n} \sum_{x^{\prime} \in \mathbf{K}+x N} p_{n_{1}}\left(y_{\ell-1}^{\prime}, x^{\prime}\right) p_{n_{2}}\left(x^{\prime}, y_{\ell}^{\prime}\right) . \tag{5.23}
\end{equation*}
$$

We bound this by splitting up the sum into different contributions. Let $\varepsilon>0$ that will
be chosen sufficiently small in the course of the proof.
Case 1. $n_{1}, n_{2} \geq N^{2+6 \varepsilon}$ and $\left|y_{\ell-1}^{\prime}-x^{\prime}\right| \leq n_{1}^{\frac{1}{2}+\varepsilon},\left|x^{\prime}-y_{\ell}^{\prime}\right| \leq n_{2}^{\frac{1}{2}+\varepsilon}$. By the LCLT we have that

$$
\begin{aligned}
p_{n_{1}}\left(y_{\ell-1}^{\prime}, x^{\prime}\right) & \leq C p_{n_{1}}\left(y_{\ell-1}^{\prime}, u^{\prime}\right) \quad \text { for any } u^{\prime} \in \mathbb{T}_{N}+x N, \\
p_{n_{2}}\left(x^{\prime}, y_{\ell}^{\prime}\right) & \leq C p_{n_{2}}\left(u^{\prime}, y_{\ell}^{\prime}\right) \quad \text { for any } u^{\prime} \in \mathbb{T}_{N}+x N .
\end{aligned}
$$

For this note that we have

$$
\begin{aligned}
\left|\frac{d\left|y_{\ell-1}^{\prime}-x^{\prime}\right|^{2}}{n_{1}}-\frac{d\left|y_{\ell-1}^{\prime}-u^{\prime}\right|^{2}}{n_{1}}\right| & \leq \frac{d\left|x^{\prime}-u^{\prime}\right|^{2}}{n_{1}}+\frac{2 d\left|\left\langle x^{\prime}-u^{\prime}, y_{\ell-1}^{\prime}-x^{\prime}\right\rangle\right|}{n_{1}} \\
& \leq C \frac{N^{2}}{n_{1}}+\frac{C N \cdot n_{1}^{\frac{1}{2}+\varepsilon}}{n_{1}},
\end{aligned}
$$

where the first term tends to 0 and the rest equals

$$
C N n_{1}^{-\frac{1}{2}+\varepsilon} \leq C N \cdot N^{(2+6 \varepsilon)\left(-\frac{1}{2}+\varepsilon\right)}=C N^{-\varepsilon+6 \varepsilon^{2}} \rightarrow 0, \quad \text { as } N \rightarrow \infty .
$$

A similar observation shows the estimate for $p_{n_{2}}\left(x^{\prime}, y_{\ell}^{\prime}\right)$.
It follows that the contribution of the values of $n_{1}, n_{2}$ and $x$ in Case 1 to the right hand side of (5.22) is at most

$$
\frac{C}{N^{d}} \sum_{n_{1}+n_{2}=n} \sum_{u^{\prime} \in \mathbb{Z}^{d}} p_{n_{1}}\left(y_{\ell-1}^{\prime}, u^{\prime}\right) p_{n_{2}}\left(u^{\prime}, y_{\ell}^{\prime}\right)=\frac{C}{N^{d}} \sum_{n_{1}+n_{2}=n} p_{n}\left(y_{\ell-1}^{\prime}, y_{\ell}^{\prime}\right) \leq \frac{C n}{N^{d}} p_{n}\left(y_{\ell-1}^{\prime}, y_{\ell}^{\prime}\right),
$$

where $1 / N^{d}$ comes from that $u^{\prime}$ is uniformly chosen from $\mathbb{T}_{N}+x N$.
Case 2a. $n_{1}, n_{2} \geq N^{2+6 \varepsilon}$ but $\left|x^{\prime}-y_{\ell-1}^{\prime}\right|>n_{1}^{\frac{1}{2}+\varepsilon}$. In this case we bound $p_{n_{2}}\left(x^{\prime}, y_{\ell}^{\prime}\right) \leq 1$ and have that the contribution of this case to the right hand side of (5.22) is at most

$$
\begin{aligned}
\sum_{\substack{n_{1}+n_{2}=n \\
n_{1}, n_{2} \geq N^{2+6 \varepsilon}}} \mathbf{P}_{y_{\ell-1}^{\prime}}\left[\left|Y_{n_{1}}-y_{\ell-1}^{\prime}\right|>n_{1}^{1 / 2+\varepsilon}\right] & \leq \sum_{\substack{n_{1}+n_{2}=n \\
n_{1}, n_{2} \geq N^{2+6 \varepsilon}}} C \exp \left(-c n_{1}^{2 \varepsilon}\right) \\
& \leq C n \exp \left(-c N^{4 \varepsilon}\right)=o\left(p_{n}\left(y_{\ell-1}^{\prime}, y_{\ell}^{\prime}\right)\right),
\end{aligned}
$$

where in the first step we used (5.3) and in the last step we used the Gaussian lower bound (5.5) for $p_{n}$. Indeed, by the Gaussian lower bound for $p_{n}$, we have

$$
\begin{equation*}
p_{n}\left(y_{\ell-1}^{\prime}, y_{\ell}^{\prime}\right) \geq \frac{c}{n^{d / 2}} \exp \left(-\frac{C\left|y_{\ell}^{\prime}-y_{\ell-1}^{\prime}\right|^{2}}{n}\right) \geq \frac{c}{n^{d / 2}} \exp (-C \log n) \tag{5.24}
\end{equation*}
$$

Then we have

$$
\begin{aligned}
0 & \leq \frac{C n \exp \left(-c N^{4 \varepsilon}\right)}{c n^{-d / 2} \exp (-C \log n)} \\
& \leq C n^{1+d / 2} \exp \left(-c N^{4 \varepsilon}+C \log n\right) \rightarrow 0, \text { as } N \rightarrow \infty
\end{aligned}
$$

Case 2b. $n_{1}, n_{2} \geq N^{2+6 \varepsilon}$ but $\left|y_{\ell}^{\prime}-x^{\prime}\right|>n_{2}^{1 / 2+\varepsilon}$. This case can be handled very similarly to Case 2a.

Case 3a. $n_{1}<N^{2+6 \varepsilon}$ and $\left|x^{\prime}-y_{\ell-1}^{\prime}\right| \leq N^{\frac{\delta}{2}-\varepsilon}$. By the LCLT we have

$$
\begin{aligned}
p_{n_{2}}\left(x^{\prime}, y_{\ell}^{\prime}\right) & =\frac{C}{n_{2}^{d / 2}} \exp \left(-\frac{d\left|y_{\ell}^{\prime}-x^{\prime}\right|^{2}}{n_{2}}\right)(1+o(1)) \\
p_{n}\left(y_{\ell-1}^{\prime}, y_{\ell}^{\prime}\right) & =\frac{C}{n^{d / 2}} \exp \left(-\frac{d\left|y_{\ell}^{\prime}-y_{\ell-1}^{\prime}\right|^{2}}{n}\right)(1+o(1))
\end{aligned}
$$

We claim that

$$
\begin{equation*}
p_{n_{2}}\left(x^{\prime}, y_{\ell}^{\prime}\right) \leq C p_{n}\left(y_{\ell-1}^{\prime}, y_{\ell}^{\prime}\right) \tag{5.25}
\end{equation*}
$$

We first note that $n_{2}=n-n_{1}=n(1+o(1))$, then we deduce that $n_{2}^{-d / 2}=O\left(n^{-d / 2}\right)$ and

$$
\exp \left(-\frac{d\left|y_{\ell}^{\prime}-y_{\ell-1}^{\prime}\right|^{2}}{n}\right) \geq \exp \left(-\frac{d\left|y_{\ell}^{\prime}-y_{\ell-1}^{\prime}\right|^{2}}{n_{2}}\right)
$$

Since we have $\left|x^{\prime}-y_{\ell-1}^{\prime}\right| \leq N^{\frac{\delta}{2}-\varepsilon}$ in the exponent, we have, as $N \rightarrow \infty$,

$$
\frac{\left|x^{\prime}-y_{\ell-1}^{\prime}\right|^{2}}{n_{2}} \leq \frac{N^{\delta-2 \varepsilon}}{n_{2}} \rightarrow 0
$$

and

$$
\frac{\left|y_{\ell}^{\prime}-y_{\ell-1}^{\prime}\right|\left|x^{\prime}-y_{\ell-1}^{\prime}\right|}{n_{2}} \leq \frac{n^{\frac{1}{2}} C_{1} \sqrt{\log n} N^{\frac{\delta}{2}-\varepsilon}}{n_{2}} \rightarrow 0
$$

These imply that

$$
\begin{aligned}
\left|\frac{\left|y_{\ell}^{\prime}-y_{\ell-1}^{\prime}\right|^{2}-\left|y_{\ell}^{\prime}-x^{\prime}\right|^{2}}{n_{2}}\right| & \leq\left|\frac{\left|y_{\ell}^{\prime}-y_{\ell-1}^{\prime}\right|^{2}-\left|\left(y_{\ell}^{\prime}-y_{\ell-1}^{\prime}\right)+\left(y_{\ell-1}^{\prime}-x^{\prime}\right)\right|^{2}}{n_{2}}\right| \\
& \leq \frac{\left|x^{\prime}-y_{\ell-1}^{\prime}\right|^{2}}{n_{2}}+\frac{2\left|y_{\ell}^{\prime}-y_{\ell-1}^{\prime}\right|\left|x^{\prime}-y_{\ell-1}^{\prime}\right|}{n_{2}} \rightarrow 0
\end{aligned}
$$

Thus (5.25) follows from comparing the LCLT approximations of the two sides.
We now have that the contribution of this case to the right hand side of (5.22) is at
most

$$
\begin{aligned}
C p_{n}\left(y_{\ell-1}^{\prime}, y_{\ell}^{\prime}\right) \sum_{n_{1}<N^{2+6 \varepsilon}} \sum_{x \in \mathbb{Z}^{d}} \sum_{x^{\prime} \in \mathbf{K}+x N} p_{n_{1}}\left(y_{\ell-1}^{\prime}, x^{\prime}\right) & \leq \frac{C}{N^{\zeta(d-2)}} p_{n}\left(y_{\ell-1}^{\prime}, y_{\ell}^{\prime}\right) \\
& \leq C \frac{n}{N^{d}} p_{n}\left(y_{\ell-1}^{\prime}, y_{\ell}^{\prime}\right),
\end{aligned}
$$

where in the first step we used Lemma 5.4.1(i) and the last step holds for the value of $\zeta$ we chose; cf. (5.7).

Case 3b. $n_{1}<N^{2+6 \varepsilon}$ but $\left|x^{\prime}-y_{\ell-1}^{\prime}\right|>N^{\frac{\delta}{2}-\varepsilon}$. Bounding the $p_{n_{2}}$ term by 1 and using the Gaussian upper bound (5.5) for $p_{n_{1}}$, we get

$$
\begin{aligned}
& \sum_{\substack{n_{1}+n_{2}=n \\
n_{1}<N^{2+6 \varepsilon}}} \sum_{x \in \mathbb{Z}^{d}} \sum_{x^{\prime} \in \mathbf{K}+x N} p_{n_{1}}\left(y_{\ell-1}^{\prime}, x^{\prime}\right) p_{n_{2}}\left(x^{\prime}, y_{\ell}^{\prime}\right) \leq \sum_{n_{1}<N^{2+6 \varepsilon}} \frac{C}{n_{1}^{d / 2}} \exp \left(-\frac{N^{\delta-2 \varepsilon}}{N^{2+6 \varepsilon}}\right) \\
& \quad \leq C N^{O(1)} \exp \left(-N^{\delta-2-8 \varepsilon}\right)=o(1) p_{n}\left(y_{\ell-1}^{\prime}, y_{\ell}^{\prime}\right), \quad \text { as } N \rightarrow \infty
\end{aligned}
$$

In the last step we used a Gaussian lower bound for $p_{n}$; cf. (5.24).
Case 4 a. $n_{2}<N^{2+6 \varepsilon}$ and $\left|y_{\ell}^{\prime}-x^{\prime}\right| \leq N^{\frac{\delta}{2}-\varepsilon}$. This case can be handled very similarly to Case 3a.

Case 4b. $n_{2}<N^{2+6 \varepsilon}$ and $\left|y_{\ell}^{\prime}-x^{\prime}\right|>N^{\frac{\delta}{2}-\varepsilon}$. This case can be handled very similarly to Case 3b.

Therefore, we discussed all possible cases and proved statement (5.11) of the proposition as required.

The proof of (5.12) is similar to the first part with only a few modifications. In this part we have to show that

$$
\mathbf{P}_{o}\left[Y_{n}=y_{1}^{\prime}, Y_{t} \in \varphi^{-1}(\mathbf{K}) \text { for some } 0 \leq t<n\right]=o(1) \mathbf{P}_{o}\left[Y_{n}=y_{1}^{\prime}\right]
$$

Define $A_{0}(x)=\left\{Y_{n}=y_{1}^{\prime}, Y_{t} \in x N+\mathbf{K}\right.$ for some $\left.0 \leq t<n\right\}$, so that

$$
\begin{equation*}
\mathbf{P}_{o}\left[Y_{n}=y_{1}^{\prime}, Y_{t} \in \varphi^{-1}(\mathbf{K}) \text { for some } 0 \leq t<n\right] \leq \sum_{x \in \mathbb{Z}^{d}} \mathbf{P}_{o}\left[A_{0}(x)\right] \tag{5.26}
\end{equation*}
$$

We have

$$
\begin{equation*}
\mathbf{P}_{o}\left[A_{0}(x)\right] \leq \sum_{n_{1}+n_{2}=n} \sum_{x^{\prime} \in \mathbf{K}+x N} p_{n_{1}}\left(o, x^{\prime}\right) p_{n_{2}}\left(x^{\prime}, y_{1}^{\prime}\right) \tag{5.27}
\end{equation*}
$$

We bound the term above by splitting up the sum into the same cases as in the proof of (5.11). The different cases can be handled very similarly to the first part. The difference is only in Case 3a while applying the Green's function bound Lemma 5.4.1.

In Case 3a, by the LCLT, we can deduce that

$$
p_{n_{2}}\left(x^{\prime}, y_{1}^{\prime}\right) \leq C p_{n}\left(o, y_{1}^{\prime}\right) .
$$

If $g(N)>N^{\zeta}$, the bound of Lemma 5.4.1(i) can be used as before. If $g(N) \leq N^{\zeta}$, by Lemma 5.4.1(ii), we have that the contribution of this case to the right hand side of (5.26) is at most

$$
C p_{n}\left(o, y_{1}^{\prime}\right) \sum_{n_{1}<N^{2+6 \varepsilon}} \sum_{x \in \mathbb{Z}^{d}} \sum_{x^{\prime} \in \mathbf{K}+x N} p_{n_{1}}\left(o, x^{\prime}\right) \leq \frac{C}{g(N)^{d-2}} p_{n}\left(o, y_{1}^{\prime}\right)=o(1) p_{n}\left(o, y_{1}^{\prime}\right) .
$$

Here we used that $g(N) \rightarrow \infty$.

Note that Case 4a can be handled in the same way as in the proof of (5.11), since the distance between $y_{1}^{\prime}$ and any copy of $\mathbf{K}$ is at least $N^{\zeta}$.

Therefore, we discussed all possible cases and proved (5.12) as required.

### 5.4.2 Proof of Proposition 5.2.1

Proof of Proposition 5.2.1. We denote the error term in (5.9) as $E$, which we claim to satisfy $|E| \leq E_{1}+E_{2}+E_{3}+E_{4}$, with

$$
\begin{aligned}
& E_{1}=\mathbf{P}\left[\frac{S n}{N^{d}}<(\sqrt{\log \log n})^{-1}\right]+\mathbf{P}\left[\frac{S n}{N^{d}}>\log N\right] \\
& E_{2}=\mathbf{P}\left[\exists \ell: 1 \leq \ell \leq \log N \frac{N^{d}}{n} \text { such that } Y_{\ell n} \in \varphi^{-1}\left(B\left(\mathbf{x}_{\mathbf{0}}, N^{\zeta}\right)\right)\right] \\
& E_{3}=\mathbf{P}\left[\exists t: T \leq t<S n \text { such that } Y_{t} \in \varphi^{-1}(\mathbf{K})\right] . \\
& E_{4}=\mathbf{P}\left[\exists \ell: 1 \leq \ell \leq \log N \frac{N^{d}}{n} \text { such that }\left|Y_{\ell n}-Y_{(\ell-1) n}\right|>f(n) n^{\frac{1}{2}}\right] .
\end{aligned}
$$

Since $T \leq S n$, we have

$$
\mathbf{P}_{o}\left[Y_{t} \notin \varphi^{-1}(\mathbf{K}), 0 \leq t<T\right]-\mathbf{P}_{o}\left[Y_{t} \notin \varphi^{-1}(\mathbf{K}), 0 \leq t<S n\right] \leq E_{3} .
$$

By the Markov property, for $\left(\tau,\left(y_{\ell}, \mathbf{y}_{\ell}\right)_{\ell=1}^{\tau}\right) \in \mathcal{G}_{\zeta, C_{1}}$,

$$
\begin{aligned}
& \prod_{\ell=1}^{\tau} \mathbf{P}_{y_{\ell-1}^{\prime}}\left[Y_{n}=y_{\ell}^{\prime}, Y_{t} \notin \varphi^{-1}(\mathbf{K}) \text { for } 0 \leq t<n\right] \\
& =\mathbf{P}_{o}\left[\begin{array}{l}
Y_{n \ell}=y_{\ell}^{\prime} \text { for } 0 \leq \ell \leq \tau ; Y_{t} \notin \varphi^{-1}(\mathbf{K}) \text { for } 0 \leq t<\tau n ; \\
Y_{n \ell} \notin \varphi^{-1}\left(B\left(\mathbf{x}_{\mathbf{0}}, N^{\zeta}\right)\right) \text { for } 0<\ell \leq \tau
\end{array}\right] .
\end{aligned}
$$

We denote the probability on the right hand side by $p\left(\tau,\left(y_{\ell}, \mathbf{y}_{\ell}\right)_{\ell=1}^{\tau}\right)$. On the event of the right hand side, since $y_{\tau} \in D^{c}$, we have $S=\tau$ and the events in the definitions of $E_{1}, E_{2}$ and $E_{4}$ do not occur. Hence

$$
\mid \sum_{\left(\tau,\left(y_{\ell}, \mathbf{y}_{\ell}\right)_{\ell=1}^{\tau}\right) \in \mathcal{G}_{\zeta, C_{1}}} p\left(\tau,\left(y_{\ell}, \mathbf{y}_{\ell}\right)_{\ell=1}^{\tau}\right)-\mathbf{P}_{o}\left[Y_{t} \notin \varphi^{-1}(\mathbf{K}) \text { for } 0 \leq t<S n\right] \mid \leq E_{1}+E_{2}+E_{4} .
$$

The proof follows since $E_{j} \rightarrow 0$, for $j=1,2,3,4$, as is shown below.

We bound $E_{1}, E_{2}, E_{3}$ and $E_{4}$ in the following lemmas.
Lemma 5.4.2. We have $E_{1} \rightarrow 0$ as $N \rightarrow \infty$.

Proof. By the definitions of $S$ and $T$, we first notice that

$$
\begin{aligned}
\mathbf{P} & {\left[S<(\sqrt{\log \log n})^{-1} \frac{N^{d}}{n}\right] } \\
& \leq \mathbf{P}\left[T<(\sqrt{\log \log n})^{-1} N^{d}\right] \\
& \leq \sum_{1 \leq i \leq d}\left(\mathbf{P}\left[\max _{0 \leq j \leq(\sqrt{\log \log n})^{-1} N^{d}} Y_{j}^{(i)} \geq L\right]+\mathbf{P}\left[\max _{0 \leq j \leq(\sqrt{\log \log n})^{-1} N^{d}}-Y_{j}^{(i)} \geq L+1\right]\right),
\end{aligned}
$$

where $Y^{(i)}$ denotes the $i$-th coordinate of the $d$-dimensional lazy random walk.
We are going to use (5.3). Setting $t=(\sqrt{\log \log n})^{-1} N^{d}$ and $r \sigma \sqrt{t}=L$, we can evaluate each term (similarly for the event with $-Y_{j}^{(i)}$ ) in the sum

$$
\begin{aligned}
& \mathbf{P}\left[\max _{0 \leq j \leq(\sqrt{\log \log n})^{-1} N^{d}} Y_{j}^{(i)} \geq L\right] \\
& \quad \leq \exp \left\{-\frac{1}{2} \frac{L^{2}}{\sigma^{2}(\sqrt{\log \log n})^{-1} N^{d}}+O\left(\frac{L^{3}}{\sigma^{3}(\sqrt{\log \log n})^{-2} N^{2 d}}\right)\right\} .
\end{aligned}
$$

Recall that $L^{2} \sim A N^{d}$ and $\sigma^{2}=1 / 2 d$, we have the upper bound

$$
\begin{aligned}
\exp \left(-(1+o(1)) \frac{1}{2} \frac{2 d \cdot A N^{d}}{(\sqrt{\log \log n})^{-1} N^{d}}\right) & =\exp (-(1+o(1)) A d \sqrt{\log \log n}) \\
& \rightarrow 0, \quad \text { as } N \rightarrow \infty
\end{aligned}
$$

and

$$
\begin{aligned}
\exp \left\{O\left(\frac{\left(A N^{d}\right)^{3 / 2}}{\sigma^{3}(\sqrt{\log \log n})^{-2} N^{2 d}}\right)\right\} & =\exp \left\{O\left(N^{-d / 2}(\log \log n)\right)\right\} \\
& =(\log n)^{O\left(N^{-d / 2}\right)}=O(1), \quad \text { as } N \rightarrow \infty
\end{aligned}
$$

Now we use the Central Limit Theorem (CLT) and induction to estimate the probability $\mathbf{P}\left[\frac{S n}{N^{d}}>\log N\right]$. By the CLT, if we have a lazy random walk starting from the origin $o$ at time 0 , the walk exits $(-L, L)^{d}$ in $N^{d}$ steps with at least positive probability $c_{0}$ independent of $N$, for sufficient large $N$.

$$
\mathbf{P}_{o}\left[S>\frac{N^{d}}{n}\right] \leq \mathbf{P}_{o}\left[Y_{N^{d}} \in[-L, L)^{d}\right]=1-\mathbf{P}_{o}\left[Y_{N^{d}} \notin[-L, L)^{d}\right] \leq 1-c_{0} .
$$

Assume that

$$
\mathbf{P}_{o}\left[S>k \frac{N^{d}}{n}\right] \leq\left(1-c_{0}\right)\left(1-c_{0}^{\prime}\right)^{k-1} \leq e^{-c_{0}^{\prime} k}
$$

where $c_{0}$ and $c_{0}^{\prime}$ comes from the CLT with $c_{0}^{\prime} \leq c_{0}$; see below for details.
By the Markov property,

$$
\begin{aligned}
\mathbf{P}_{o}\left[S>(k+1) \frac{N^{d}}{n}\right] & =\mathbf{P}_{o}\left[S>k \frac{N^{d}}{n}\right] \mathbf{P}_{o}\left[\left.S>(k+1) \frac{N^{d}}{n} \right\rvert\, S>k \frac{N^{d}}{n}\right] \\
& \leq \mathbf{P}_{o}\left[S>k \frac{N^{d}}{n}\right] \max _{z \in(-L, L)^{d}} \mathbf{P}_{z}\left[Y_{N^{d}} \in(-L, L)^{d}\right] \\
& \leq \mathbf{P}_{o}\left[S>k \frac{N^{d}}{n}\right] \max _{z \in(-L, L)^{d}} \mathbf{P}_{z}\left[Y_{N^{d}} \in z+(-2 L, 2 L)^{d}\right] \\
& \leq \mathbf{P}_{o}\left[S>k \frac{N^{d}}{n}\right] \mathbf{P}_{o}\left[Y_{N^{d}} \in(-2 L, 2 L)^{d}\right] \\
& \leq\left(1-c_{0}\right)\left(1-c_{0}^{\prime}\right)^{k-1}\left(1-c_{0}^{\prime}\right)=\left(1-c_{0}\right)\left(1-c_{0}^{\prime}\right)^{k} \leq e^{-c_{0}^{\prime}(k+1)}
\end{aligned}
$$

where $c_{0}$ and $c_{0}^{\prime}$ comes from the CLT with $c_{0}^{\prime} \leq c_{0}$.
Coming to the second term in $E_{1}$, note that since the rescaled exit time $S$ is of the scale
$N^{d} / n$, we have shown above that for some $c$ we have $\mathbf{P}_{o}\left[S>k \frac{N^{d}}{n}\right] \leq e^{-c k}$ for all $k \geq 0$. Applying this with $k=\log N, \mathbf{P}_{o}\left[S>\log N \frac{N^{d}}{n}\right] \leq e^{-c \log N} \rightarrow 0$ as required.

Lemma 5.4.3. We have $E_{2} \leq C \log N \frac{N^{d \zeta}}{n}$ for some $C$. Consequently, $E_{2} \rightarrow 0$.

Proof. Since the number of points in $B\left(\mathbf{x}_{\mathbf{0}}, N^{\zeta}\right)$ is $O\left(N^{d \zeta}\right)$, and since we are considering times after the first stretch, the random walk is well mixed, so the probability to visit any point in the torus is $O\left(1 / N^{d}\right)$. Using a union bound we have

$$
E_{2} \leq C \log N \frac{N^{d}}{n} N^{d \zeta} \frac{1}{N^{d}}=C \log N \frac{N^{d \zeta}}{n}
$$

Since $\zeta<\delta / d$, we have

$$
E_{2} \rightarrow 0, \quad \text { as } N \rightarrow \infty
$$

Before we bound the error term $E_{3}$, we first introduce the following lemma. Let $T_{0}^{(i)}=$ $\inf \left\{t \geq 0: Y_{t}^{(i)}=0\right\}$, where recall that $Y^{(i)}$ denotes the $i$-th coordinate of the $d$ dimensional lazy random walk. We will denote by $t_{0}$ an instance of $T_{0}^{(i)}$.

Lemma 5.4.4. For all $1 \leq i \leq d$, for any $-n \leq y<0$ and $0<t_{0} \leq n$ we have

$$
\begin{aligned}
& \mathbf{E}_{y}\left[Y_{n}^{(i)} \mid T_{0}^{(i)}\right.\left.=t_{0}, Y_{n}^{(i)}>0\right] \\
& \leq C n^{\frac{1}{2}} \\
& \mathbf{E}_{y}\left[\left(Y_{n}^{(i)}\right)^{2} \mid T_{0}^{(i)}\right.\left.=t_{0}, Y_{n}^{(i)}>0\right]
\end{aligned}
$$

Proof. Using the Markov property at time $t_{0}$, we get

$$
\begin{aligned}
\mathbf{E}_{y}\left[Y_{n}^{(i)} \mid T_{0}^{(i)}=t_{0}, Y_{n}^{(i)}>0\right] & =\mathbf{E}_{0}\left[Y_{n-t_{0}}^{(i)} \mid Y_{n-t_{0}}^{(i)}>0\right]=\frac{\mathbf{E}_{0}\left[Y_{n-t_{0}}^{(i)} \mathbf{1}_{\left\{Y_{n-t_{0}}^{(i)}>0\right\}}\right]}{\mathbf{P}_{0}\left[Y_{n-t_{0}}^{(i)}>0\right]} \\
& \leq C_{0}\left(\mathbf{E}_{0}\left[\left(Y_{n-t_{0}}^{(i)}\right)^{2}\right]\right)^{\frac{1}{2}} \leq C\left(n-t_{0}\right)^{\frac{1}{2}} \leq C n^{\frac{1}{2}}
\end{aligned}
$$

where the third step is due to Jensen's inequality and $\mathbf{P}_{0}\left[Y_{n-t_{0}}^{(i)}>0\right] \geq c_{0}$ for some $c_{0}>0$, and the second to last step is due to $\mathbf{E}_{0}\left[\left(Y_{n-t_{0}}^{(i)}\right)^{2}\right]=\left(n-t_{0}\right) / 2 d$.

We can similarly bound the conditional expectation of $\left(Y_{n}^{(i)}\right)^{2}$ as follows:

$$
\begin{aligned}
& \mathbf{E}_{y}\left[\left(Y_{n}^{(i)}\right)^{2} \mid T_{0}^{(i)}=t_{0}, Y_{n}^{(i)}>0\right]=\mathbf{E}_{0}\left[\left(Y_{n-t_{0}}^{(i)}\right)^{2} \mid Y_{n-t_{0}}^{(i)}>0\right] \\
&=\frac{\mathbf{E}_{0}\left[\left(Y_{n-t_{0}}^{(i)}\right)^{2} \mathbf{1}_{\left\{Y_{n-t_{0}}^{(i)}>0\right\}}\right]}{\mathbf{P}_{0}\left[Y_{n-t_{0}}^{(i)}>0\right]} \leq C_{0}\left(\mathbf{E}_{0}\left[\left(Y_{n-t_{0}}^{(i)}\right)^{2}\right]\right) \leq C\left(n-t_{0}\right) \leq C n .
\end{aligned}
$$

Lemma 5.4.5. We have $E_{3} \rightarrow 0$ as $N \rightarrow \infty$.

Proof. First we are going to bound the time difference between $T$ and $S n$. We are going to consider separately the cases when $Y_{T}$ is in each face of the cube $[-L, L)^{d}$. Assume that we have $Y_{T}^{(i)}=L$ for some $1 \leq i \leq d$. (The arguments needed are very similar when $Y_{T}^{(i)}=-L-1$ for some $1 \leq i \leq d$, and this will not be given.)

Let us consider the lazy random walk $\left(Y_{t}\right)_{t \geq 0}$ in multiples of $n$ steps. Let

$$
s_{1}=\min \{\ell n: \ell n \geq T\}-T,
$$

and similarly, let

$$
s_{r+1}=r n+\min \{\ell n: \ell n \geq T\}-T, \quad r \geq 1
$$

We let $M_{0}=L-Y_{T+s_{1}}^{(i)}$ and $M_{r}=L-Y_{T+s_{r+1}}^{(i)}$ for $r \geq 1$. We have that $\left(M_{r}\right)_{r \geq 0}$ is a martingale. Let $\tilde{S}=\inf \left\{r \geq 0: M_{r} \leq 0\right\}$, and we are going to bound $\mathbf{P}\left[\tilde{S}>N^{\varepsilon_{1}}\right]$ for some small $\varepsilon_{1}$ that we are going to choose in the course of the proof. We are going to adapt an argument in [10, Proposition 17.19] to this purpose.

Define

$$
T_{h}=\inf \left\{r \geq 0: M_{r} \leq 0 \text { or } M_{r} \geq h\right\},
$$

where we set $h=\sqrt{n} \sqrt{N^{\varepsilon_{1}}}$. Let $\left(\mathcal{F}_{r}\right)_{r \geq 0}$ denote the filtration generated by $\left(M_{r}\right)_{r \geq 0}$. We have

$$
\begin{equation*}
\operatorname{Var}\left(M_{r+1} \mid \mathcal{F}_{r}\right)=n \sigma^{2} \quad \text { for all } r \geq 0, \tag{5.28}
\end{equation*}
$$

where recall that $\sigma^{2}$ is the variance of $Y_{1}^{(i)}$.
We first estimate $\mathbf{E}\left[M_{0} \mid \tilde{S}>0\right]$. Since $0 \leq s_{1}<n$, by the same argument as in Lemma
5.4.4 we have that

$$
\mathbf{E}\left[M_{0} \mid Y_{T+s_{1}}^{(i)}<L\right]=\mathbf{E}\left[L-Y_{T+s_{1}}^{(i)} \mid L-Y_{T+s_{1}}^{(i)}>0\right] \leq C n^{\frac{1}{2}}
$$

We first bound $\mathbf{P}\left[M_{T_{h}} \geq h \mid M_{0}\right]$. Since ( $M_{r \wedge T_{h}}$ ) is bounded, by the Optional Stopping Theorem, we have

$$
\begin{aligned}
M_{0} & =\mathbf{E} M_{T_{h}}=\mathbf{E}\left[M_{T_{h}} \mathbf{1}_{\left\{M_{T_{h}} \leq 0\right\}}\right]+\mathbf{E}\left[M_{T_{h}} \mathbf{1}_{\left\{M_{T_{h}} \geq h\right\}}\right] \\
& =-m_{-}(h)+\mathbf{E}\left[M_{T_{h}} \mathbf{1}_{\left\{M_{T_{h}} \geq h\right\}}\right] \\
& \geq-m_{-}(h)+h \mathbf{P}\left[M_{T_{h}} \geq h\right],
\end{aligned}
$$

where we denote $\mathbf{E}\left[M_{T_{h}} \mathbf{1}_{\left\{M_{T_{h}} \leq 0\right\}}\right]$ by $-m_{-}(h) \leq 0$ and the last step is due to Markov's inequality. Hence, we have

$$
M_{0}+m_{-}(h) \geq h \mathbf{P}\left[M_{T_{h}} \geq h\right] .
$$

We bound $m_{-}(h)$ using Lemma 5.4.4

$$
m_{-}(h) \leq \max _{y \leq L} \mathbf{E}_{y}\left[Y_{n}^{(i)}-L \mid Y_{n}^{(i)}>L\right] \leq C n^{\frac{1}{2}}
$$

Hence, we have

$$
\mathbf{P}\left[M_{T_{h}} \geq h \mid M_{0}\right] \leq \frac{M_{0}}{h}+\frac{C n^{\frac{1}{2}}}{h}
$$

We now estimate $\mathbf{P}\left[T_{h} \geq r \mid M_{0}\right]$. Let $G_{r}=M_{r}^{2}-h M_{r}-\sigma^{2} n r$. The sequence $\left(G_{r}\right)$ is a martingale by (5.28).

Since $\operatorname{Var}\left(M_{r+1} \mid \mathcal{F}_{r}\right)=\mathbf{E}\left[M_{r+1}^{2} \mid \mathcal{F}_{r}\right]-M_{r}^{2}=n \sigma^{2}$

$$
\begin{aligned}
\mathbf{E}\left[G_{r+1} \mid \mathcal{F}_{r}\right] & =\mathbf{E}\left[M_{r+1}^{2} \mid \mathcal{F}_{r}\right]-h \mathbf{E}\left[M_{r+1} \mid \mathcal{F}_{r}\right]-n \sigma^{2}(r+1) \\
& =\mathbf{E}\left[M_{r+1}^{2} \mid \mathcal{F}_{r}\right]-n \sigma^{2}-h M_{r}-n \sigma^{2} r=M_{r}^{2}-h M_{r}-n \sigma^{2} r=G_{r} .
\end{aligned}
$$

We can bound both the 'overshoot' above $h$ and the 'undershoot' below 0 by Lemma 5.4.4. For the 'undershoot' below 0 we have
$\mathbf{E}\left[\left(M_{T_{h}}-h\right) M_{T_{h}} \mid M_{T_{h}} \leq 0\right]=\mathbf{E}\left[M_{T_{h}}^{2} \mid M_{T_{h}} \leq 0\right]+\mathbf{E}\left[-h M_{T_{h}} \mid M_{T_{h}} \leq 0\right] \leq C n+C h n^{1 / 2}$.

For the 'overshoot' above $h$, write $M_{T_{h}}=: N_{T_{h}}+h$, then we have

$$
\left(M_{T_{h}}-h\right) M_{T_{h}}=N_{T_{h}}\left(h+N_{T_{h}}\right),
$$

Hence

$$
\mathbf{E}\left[\left(M_{T_{h}}-h\right) M_{T_{h}} \mid M_{T_{h}} \geq h\right]=\mathbf{E}\left[h N_{T_{h}} \mid N_{T_{h}} \geq 0\right]+\mathbf{E}\left[N_{T_{h}}^{2} \mid N_{T_{h}} \geq 0\right] \leq C h n^{1 / 2}+C n .
$$

For $r<T_{h}$, we have $\left(M_{r}-h\right) M_{r}<0$ Therefore, we have

$$
\mathbf{E}\left[M_{r \wedge T_{h}}^{2}-h M_{r \wedge T_{h}}\right] \leq C h n^{1 / 2}+C n .
$$

Since $\left(G_{r \wedge T_{h}}\right)$ is a martingale

$$
\begin{aligned}
-h M_{0} \leq G_{0} \leq \mathbf{E} G_{r \wedge T_{h}} & =\mathbf{E}\left[M_{r \wedge T_{h}}^{2}-h M_{r \wedge T_{h}}\right]-\sigma^{2} n \mathbf{E}\left[r \wedge T_{h}\right] \\
& \leq C n^{\frac{1}{2}} h+C n-\sigma^{2} n \mathbf{E}\left[r \wedge T_{h}\right] .
\end{aligned}
$$

We conclude that $\mathbf{E}\left[r \wedge T_{h} \mid M_{0}\right] \leq \frac{h\left(M_{0}+C n^{\frac{1}{2}}\right)+C n}{\sigma^{2} n}$. Letting $r \rightarrow \infty$, by the Monotone Convergence Theorem,

$$
\mathbf{E}\left[T_{h} \mid M_{0}\right] \leq \frac{h\left(M_{0}+C n^{\frac{1}{2}}\right)+C n}{\sigma^{2} n}
$$

where $h=\sqrt{n} \sqrt{N^{\varepsilon_{1}}}$. This gives

$$
\mathbf{P}\left[T_{h}>N^{\varepsilon_{1}} \mid M_{0}\right] \leq \frac{1}{N^{\varepsilon_{1}}}\left[\frac{\sqrt{n} \sqrt{N^{\varepsilon_{1}}} M_{0}+C n \sqrt{N^{\varepsilon_{1}}}+C n}{\sigma^{2} n}\right]
$$

Taking expectations of both sides, we have

$$
\begin{aligned}
\mathbf{P}\left[T_{h}>N^{\varepsilon_{1}}\right] & \leq \frac{1}{N^{\varepsilon_{1}}}\left[\frac{\sqrt{n} \sqrt{N^{\varepsilon_{1}}} \mathbf{E} M_{0}+C n \sqrt{N^{\varepsilon_{1}}}+C n}{\sigma^{2} n}\right] \\
& =\frac{\mathbf{E} M_{0}}{\sigma^{2} \sqrt{n} \sqrt{N^{\varepsilon_{1}}}}+\frac{C}{\sigma^{2} \sqrt{N^{\varepsilon_{1}}}}+\frac{C}{\sigma^{2} N^{\varepsilon_{1}}} \leq \frac{C}{\sqrt{N^{\varepsilon_{1}}}} .
\end{aligned}
$$

Combining the above bounds, we get

$$
\mathbf{P}\left[\tilde{S}>N^{\varepsilon_{1}}\right] \leq \mathbf{P}\left[M_{T_{h}} \geq h\right]+\mathbf{P}\left[T_{h}>N^{\varepsilon_{1}}\right] \leq \frac{\mathbf{E}\left[M_{0}\right]}{h}+\frac{C n^{\frac{1}{2}}}{h}+\frac{C}{\sqrt{N^{\varepsilon_{1}}}} \leq \frac{C}{\sqrt{N^{\varepsilon_{1}}}}
$$

We now bound the probability that a copy of $\mathbf{K}$ is hit between times $T$ and $s_{N^{\varepsilon_{1}}}$.
We first show that the probability that the lazy random walk on the torus is in the ball $B\left(\mathrm{x}_{0}, N^{\zeta}\right)$ at time $T$ goes to 0 . Indeed, we have

$$
\begin{gathered}
\mathbf{P}_{o}\left[Y_{T} \in \varphi^{-1}\left(B\left(\mathbf{x}_{0}, N^{\zeta}\right)\right)\right]=\sum_{y^{\prime} \in \partial(-L, L)^{d} \cap \varphi^{-1}\left(B\left(\mathbf{x}_{0}, N \zeta\right)\right)} \mathbf{P}_{o}\left[Y_{T}=y^{\prime}\right] \\
\leq C N^{\zeta(d-1)} \frac{L^{d-1}}{N^{d-1}} \frac{C}{L^{d-1}}=C N^{(\zeta-1)(d-1)},
\end{gathered}
$$

where we have $\zeta<\delta / d<1$, so the last expression goes to 0 . Here we used that $\mathbf{P}_{o}\left[Y_{T}=y^{\prime}\right] \leq C / L^{d-1}$, for example using a half-space Poisson kernel estimate $[9$, Theorem 8.1.2].

Condition on the location $y^{\prime}$ of the walk at the exit time $T$. For $y^{\prime} \notin \varphi^{-1}\left(B\left(\mathbf{x}_{0}, N^{\zeta}\right)\right)$ we first bound the probability of hitting $\mathbf{K}$ between the times between 0 and $s_{2}$. After time $s_{2}$, the random walk is well mixed, and we can apply a simpler union bound.

We thus have the upper bound

$$
\sum_{t=0}^{s_{2}} \sum_{x^{\prime} \in \varphi^{-1}(\mathbf{K})} p_{t}\left(y^{\prime}, x^{\prime}\right) \leq \mathbf{P}\left[\max _{0 \leq t \leq s_{2}}\left|Y_{t}^{(i)}-y^{\prime}\right| \geq n^{\frac{1}{2}+\varepsilon}\right]+\sum_{\substack{x^{\prime} \in \varphi^{-1}(\mathbf{K}) \\\left|x^{\prime}-y^{\prime}\right| \leq n^{\frac{1}{2}}+\varepsilon}} G\left(y^{\prime}, x^{\prime}\right)
$$

The first term is stretched-exponentially small due to the Martingale maximal inequality (5.3). The Green's function term is bounded by Lemma 5.4.1(iii).

After time $s_{2}$, by the mixing property, we have that

$$
\sum_{t=s_{2}}^{s_{N^{\varepsilon_{1}}}} \mathbf{P}_{y}\left[Y_{t} \in \varphi^{-1}(K)\right] \leq n \cdot N^{\varepsilon_{1}}|\mathbf{K}| \frac{C}{N^{d}}=C N^{\delta+\varepsilon_{1}-d}
$$

Therefore, combining the above upper bounds, we have the required result.

$$
E_{3} \leq C \cdot N^{-\frac{\varepsilon_{1}}{2}}+C \cdot N^{\delta-d+2 \delta \varepsilon}+C \cdot N^{\delta-d+\varepsilon_{1}} \rightarrow 0, \quad \text { as } N \rightarrow \infty
$$

if $\varepsilon$ and $\varepsilon_{1}$ are sufficiently small.

Lemma 5.4.6. We have $E_{4} \leq C e^{-c f(n)^{2}} \frac{N^{d} \log N}{n}$ for some $C$. There exists $C_{1}$ such that if $f(n) \geq C_{1} \sqrt{\log N}$, then $E_{4} \rightarrow 0$.

Proof. By the Martingale maximal inequality (5.3), we have that

$$
E_{4} \leq C e^{-c f(n)^{2}} \frac{N^{d} \log N}{n} .
$$

There exists $C_{1}$ such that if $f(n) \geq C_{1} \sqrt{\log N}$, we have

$$
E_{4} \rightarrow 0, \quad \text { as } N \rightarrow \infty .
$$

### 5.4.3 Proof of Proposition 5.2.2

In this section we need $C_{1}$ large enough so that we have

$$
\begin{equation*}
e^{-d f(n)^{2}} N^{d} n \rightarrow 0 \tag{5.29}
\end{equation*}
$$

We have

$$
\mathbf{P}_{y_{\ell-1}^{\prime}}\left[Y_{n}=y_{\ell}^{\prime}, Y_{t} \in \varphi^{-1}(\mathbf{K}) \text { for some } 0 \leq t<n\right]=\mathbf{P}\left[\cup_{x \in \mathbb{Z}^{d}} A(x)\right],
$$

where

$$
A(x)=\left\{Y_{n}=y_{\ell}^{\prime}, Y_{t} \in x N+\mathbf{K} \text { for some } 0 \leq t<n\right\} .
$$

The strategy is to estimate the probability via the Bonferroni inequalities:

$$
\begin{align*}
\sum_{x} \mathbf{P}_{y_{\ell-1}^{\prime}}[A(x)]-\sum_{x_{1} \neq x_{2}} \mathbf{P}_{y_{\ell-1}^{\prime}}\left[A\left(x_{1}\right) \cap A\left(x_{2}\right)\right] & \leq \mathbf{P}\left[\cup_{x \in \mathbb{Z}^{d}} A(x)\right]  \tag{5.30}\\
& \leq \sum_{x} \mathbf{P}_{y_{\ell-1}^{\prime}}[A(x)] .
\end{align*}
$$

We are going to use a parameter $A_{n}$ that we choose as $A_{n}=10 \log \log n$ so that in particular $A_{n} \rightarrow \infty$.

Case (I): $\left|y_{\ell}^{\prime}-y_{\ell-1}^{\prime}\right| \leq A_{n} \sqrt{n}$.
We will show that the main contribution in (5.30) comes from $x$ in the set:

$$
G=\left\{x \in \mathbb{Z}^{d}:\left|y_{\ell-1}^{\prime}-x N\right| \leq A_{n}^{2} \sqrt{n},\left|x N-y_{\ell}^{\prime}\right| \leq A_{n}^{2} \sqrt{n}\right\} .
$$

We first examine $\mathbf{P}_{y_{\ell-1}^{\prime}}[A(x)]$ for $x \in G$. Putting $B_{0, x}=B\left(\mathbf{x}_{0}+x N, N^{\zeta}\right)$, let $n_{1}$ be the


Figure 5-2: The decomposition of a path hitting a copy of $\mathbf{K}$ into three subpaths (not to scale).
time of the last visit to $\partial B_{0, x}$ before hitting $\mathbf{K}+x N$, let $n_{1}+n_{2}$ be the time of the first hit of $\mathbf{K}+x N$, and let $n_{3}=n-n_{1}-n_{2}$. See Figure 5-2 for an illustration of this decomposition. Then we can write:

$$
\begin{align*}
\mathbf{P}_{y_{\ell-1}^{\prime}}[A(x)]= & \sum_{n_{1}+n_{2}+n_{3}=n} \sum_{z^{\prime} \in \partial B_{0, x}} \sum_{x^{\prime} \in \mathbf{K}+x N} \widetilde{p}_{n_{1}}^{(x)}\left(y_{\ell-1}^{\prime}, z^{\prime}\right)  \tag{5.31}\\
& \times \mathbf{P}_{z^{\prime}}\left[H_{\mathbf{K}+x N}=n_{2}<\xi_{B_{0, x}}, Y_{H_{\mathbf{K}+x N}}=x^{\prime}\right] p_{n_{3}}\left(x^{\prime}, y_{\ell}^{\prime}\right)
\end{align*}
$$

where

$$
\widetilde{p}_{n_{1}}^{(x)}\left(y_{\ell-1}^{\prime}, z^{\prime}\right)=\mathbf{P}_{y_{\ell-1}^{\prime}}\left[Y_{n_{1}}=z^{\prime}, Y_{t} \notin \mathbf{K}+x N \text { for } 0 \leq t \leq n_{1}\right]
$$

We are going to use another parameter $\varepsilon_{n}$ that will need to go to 0 slowly. We choose it as $\varepsilon_{n}=(10 \log \log n)^{-1} \rightarrow 0$. The main contribution to (5.31) will be when $n_{1} \geq \varepsilon_{n} n$, $n_{3} \geq \varepsilon_{n} n$ and $n_{2} \leq N^{2 \delta / d} \sim n^{2 / d}$. Therefore, we split the sum over $n_{1}, n_{2}, n_{3}$ in (5.31) into the main contribution $I(x)$ and an error term $I I(x)$ according to:

$$
\begin{equation*}
\mathbf{P}_{y_{\ell-1}^{\prime}}[A(x)]=I(x)+I I(x):=\sum_{\substack{n_{1}+n_{2}+n_{3}=n \\ n_{1}, n_{3} \geq \varepsilon_{n} n, n_{2} \leq N^{2 \delta / d}}}+\sum_{\substack{n_{1}+n_{2}+n_{3}=n \\ n_{1}<\varepsilon_{n} n \text { or } n_{3}<\varepsilon_{n} n \\ \text { or } n_{2}>N^{2 \delta / d}}} \tag{5.32}
\end{equation*}
$$

Lemma 5.4.7. When $x \in G$ and $n_{3} \geq \varepsilon_{n} n$, we have

$$
p_{n_{3}}\left(x^{\prime}, y_{\ell}^{\prime}\right)=(1+o(1)) p_{n_{3}}\left(u^{\prime}, y_{\ell}^{\prime}\right) \quad \text { for all } x^{\prime} \in \mathbf{K}+x N \text { and all } u^{\prime} \in \mathbb{T}_{N}+x N
$$

with the o(1) term uniform in $x^{\prime}$ and $u^{\prime}$.

Proof. By the LCLT, we have

$$
\begin{aligned}
& p_{n_{3}}\left(x^{\prime}, y_{\ell}^{\prime}\right)=\frac{C}{n_{3}^{d / 2}} \exp \left(-\frac{d\left|y_{\ell}^{\prime}-x^{\prime}\right|^{2}}{n_{3}}\right)(1+o(1)), \\
& p_{n_{3}}\left(u^{\prime}, y_{\ell}^{\prime}\right)=\frac{C}{n_{3}^{d / 2}} \exp \left(-\frac{d\left|y_{\ell}^{\prime}-u^{\prime}\right|^{2}}{n_{3}}\right)(1+o(1)) .
\end{aligned}
$$

We compare the exponents

$$
\begin{aligned}
\left|\frac{d\left|y_{\ell}^{\prime}-x^{\prime}\right|^{2}}{n_{3}}-\frac{d\left|y_{\ell}^{\prime}-u^{\prime}\right|^{2}}{n_{3}}\right| & \leq \frac{d\left|x^{\prime}-u^{\prime}\right|^{2}}{n_{3}}+\frac{2 d\left|\left\langle x^{\prime}-u^{\prime}, y_{\ell}^{\prime}-x^{\prime}\right\rangle\right|}{n_{3}} \\
& \leq C \frac{N^{2}}{n_{3}}+\frac{C N \cdot A_{n}^{2} \sqrt{n}}{n_{3}} \rightarrow 0,
\end{aligned}
$$

as $N \rightarrow \infty$.

Lemma 5.4.8. When $x \in G$ and $n_{1} \geq \varepsilon_{n} n$, we have

$$
\widetilde{p}_{n_{1}}^{(x)}\left(y_{\ell-1}^{\prime}, z^{\prime}\right)=(1+o(1)) p_{n_{1}}\left(y_{\ell-1}^{\prime}, u^{\prime}\right) \quad \text { for all } z^{\prime} \in \partial B_{0, x} \text { and all } u^{\prime} \in \mathbb{T}_{N}+x N,
$$

with the o(1) term uniform in $z^{\prime}$ and $u^{\prime}$.

Proof. The statement boils down to showing the following claim:

$$
\mathbf{P}_{y_{\ell-1}^{\prime}}\left[Y_{n_{1}}=z^{\prime}, Y_{t} \in \mathbf{K}+x N \text { for some } 0 \leq t \leq n_{1}\right]=o(1) p_{n_{1}}\left(y_{\ell-1}^{\prime}, z^{\prime}\right) .
$$

For this, observe that by (5.5) we have

$$
\begin{align*}
p_{n_{1}}\left(y_{\ell-1}^{\prime}, z^{\prime}\right) & \geq \frac{c}{n_{1}^{d / 2}} \exp \left(-C \frac{\left|z^{\prime}-y_{\ell-1}^{\prime}\right|^{2}}{n_{1}}\right) \\
& \geq \frac{c}{n^{d / 2}} \exp \left(-c \frac{A_{n}^{2} n+N^{\zeta}}{\varepsilon_{n} n}\right)  \tag{5.33}\\
& \geq \frac{c}{n^{d / 2}} \exp \left(-C(\log \log n)^{O(1)}\right) \\
& =n^{-d / 2+o(1)} .
\end{align*}
$$

On the other hand, using the Markov property, (5.5), and the fact that for $x^{\prime} \in \mathbf{K}+x N$
we have $\left|y_{\ell-1}^{\prime}-x^{\prime}\right| \geq c N^{\zeta}$ and $\left|x^{\prime}-z^{\prime}\right| \geq c N^{\zeta}$, we get

$$
\begin{align*}
\mathbf{P}_{y_{\ell-1}^{\prime}} & {\left[Y_{n_{1}}=z^{\prime}, Y_{t} \in \mathbf{K}+x N \text { for some } 0 \leq t \leq n_{1}\right] } \\
& \leq \sum_{1 \leq m \leq n_{1}-1} \sum_{x^{\prime} \in \mathbf{K}+x N} p_{m}\left(y_{\ell-1}^{\prime}, x^{\prime}\right) p_{n_{1}-m}\left(x^{\prime}, z^{\prime}\right) \\
& \leq C \sum_{1 \leq m \leq n_{1}-1} \frac{1}{m^{d / 2}} \frac{1}{\left(n_{1}-m\right)^{d / 2}} \exp \left(-c \frac{N^{2 \zeta}}{m}\right) \exp \left(-c \frac{N^{2 \zeta}}{n_{1}-m}\right)  \tag{5.34}\\
& =C\left[\sum_{1 \leq m \leq n_{1} / 2}+\sum_{n_{1} / 2 \leq m \leq n_{1}-1}\right] .
\end{align*}
$$

Due to symmetry, it is enough to bound the sum over $1 \leq m \leq n_{1} / 2$. This gives

$$
\begin{align*}
\frac{C}{n_{1}^{d / 2}} & \sum_{1 \leq m \leq n_{1} / 2} \frac{1}{m^{d / 2}} \exp \left(-c \frac{N^{2 \zeta}}{m}\right) \\
& =\frac{C}{n_{1}^{d / 2}}\left[\sum_{1 \leq m \leq N^{2 \zeta}}+\sum_{N^{2 \zeta}<m \leq n_{1} / 2}\right] . \tag{5.35}
\end{align*}
$$

In the second sum we can bound the exponential by 1 , and get the upper bound

$$
\frac{C}{n_{1}^{d / 2}} N^{\zeta(2-d)}=o\left(n^{-d / 2+o(1)}\right) .
$$

In the first sum, we group terms on dyadic scales $k$ so that $2^{k} \leq N^{2 \zeta} / m \leq 2^{k+1}$, $k=0, \ldots \log _{2} N^{2 \zeta}$. This gives the bound

$$
\frac{C}{n_{1}^{d / 2}} \sum_{k=0}^{\log _{2} N^{2 \zeta}} \frac{\left(2^{k}\right)^{d / 2-1}}{\left(N^{2 \zeta}\right)^{d / 2-1}} \exp \left(-c 2^{k}\right) \leq \frac{C}{n_{1}^{d / 2}} \frac{1}{N^{\zeta(d-2)}}
$$

which is of the same order as the other term.

The previous two lemmas allow us to write, for $x \in G$, the main term $I(x)$ as

$$
\begin{align*}
I(x)= & \frac{1+o(1)}{N^{d}} \sum_{\substack{u^{\prime} \in \mathbb{T}_{N}+x N}} \sum_{\substack{n_{1}+n_{2}+n_{3}=n \\
n_{1}, n_{3} \geq \varepsilon_{n} n \\
n_{2} \leq N^{2 \delta / d}}} p_{n_{1}}\left(y_{\ell-1}^{\prime}, u^{\prime}\right) p_{n_{3}}\left(u^{\prime}, y_{\ell}^{\prime}\right) \\
& \sum_{z^{\prime} \in \partial B_{0, x}} \mathbf{P}_{z^{\prime}}\left[H_{\mathbf{K}+x N}=n_{2}<\xi_{B_{0, x}}\right] .
\end{align*}
$$

Lemma 5.4.9. Assume that $n_{1}, n_{3} \geq \varepsilon_{n} n$ and $n_{2} \leq N^{2 \delta / d}$.
(i) We have

$$
p_{n_{1}+n_{3}}\left(y_{\ell-1}^{\prime}, y_{\ell}^{\prime}\right)=(1+o(1)) p_{n}\left(y_{\ell-1}^{\prime}, y_{\ell}^{\prime}\right) .
$$

(ii) We have

$$
\begin{equation*}
\sum_{x \in G} \sum_{u^{\prime} \in \mathbb{T}_{N}+x N} p_{n_{1}}\left(y_{\ell-1}^{\prime}, u^{\prime}\right) p_{n_{3}}\left(u^{\prime}, y_{\ell}^{\prime}\right)=(1+o(1)) p_{n_{1}+n_{3}}\left(y_{\ell-1}^{\prime}, y_{\ell}^{\prime}\right) . \tag{5.37}
\end{equation*}
$$

Proof. (i) When $n_{2} \leq N^{2 \delta / d}=n^{2 / d}$, we have

$$
n_{1}+n_{3}=n\left(1-O\left(n^{-1+2 / d}\right)\right) .
$$

Hence the exponential term in the LCLT for $p_{n_{1}+n_{3}}\left(y_{\ell-1}^{\prime}, y_{\ell}^{\prime}\right)$ is

$$
\exp \left(-\frac{\left|y_{\ell}^{\prime}-y_{\ell-1}^{\prime}\right|^{2}}{n}\left(1+O\left(n^{-1+2 / d}\right)\right)\right)=(1+o(1)) \exp \left(-\frac{\left|y_{\ell}^{\prime}-y_{\ell-1}^{\prime}\right|^{2}}{n}\right)
$$

where we used that $\left|y_{\ell}^{\prime}-y_{\ell-1}^{\prime}\right|^{2} \leq A_{n}^{2} n=n o\left(n^{1-2 / d}\right)$.
(ii) If we summed over all $x \in \mathbb{Z}^{d}$, we would get exactly $p_{n_{1}+n_{3}}\left(y_{\ell-1}^{\prime}, y_{\ell}^{\prime}\right)$. Thus the claim amounts to showing that

$$
\begin{equation*}
\sum_{x \in \mathbb{Z}^{d} \backslash G} \sum_{u^{\prime} \in \mathbb{T}_{N}+x N} p_{n_{1}}\left(y_{\ell-1}^{\prime}, u^{\prime}\right) p_{n_{3}}\left(u^{\prime}, y_{\ell}^{\prime}\right)=o(1) p_{n_{1}+n_{3}}\left(y_{\ell-1}^{\prime}, y_{\ell}^{\prime}\right) . \tag{5.38}
\end{equation*}
$$

First, note that from the Local CLT we have

$$
p_{n_{1}+n_{3}}\left(y_{\ell-1}^{\prime}, y_{\ell}^{\prime}\right)=(1+o(1)) \bar{p}_{n_{1}+n_{3}}\left(y_{\ell-1}^{\prime}, y_{\ell}^{\prime}\right) .
$$

In order to estimate the left hand side of (5.38), using (5.5), the contribution of $\{x \in$ $\left.\mathbb{Z}^{d} \backslash G: \max \left\{\left|y_{\ell-1}^{\prime}-x N\right|,\left|x N-y_{\ell-1}^{\prime}\right|\right\}>A_{n}^{2} \sqrt{n}\right\}$ can be estimated as follows. First, we have

$$
p_{n_{1}+n_{3}}\left(y_{\ell-1}^{\prime}, y_{\ell}^{\prime}\right) \geq \frac{c}{n^{d / 2}} \exp \left(-A_{n}^{2}+o(1)\right) \geq \frac{c}{n^{d / 2}} \exp \left(-100(\log \log n)^{2}\right)
$$

On the other hand, note that either $n_{1} \geq n / 3$ or $n_{3} \geq n / 3$. Without loss of generality, assume that $n_{3} \geq n / 3$. Then the contribution to the left hand side of (5.38), using (5.5), and by summing in dyadic shells with radii $2^{k} A_{n}^{2} \sqrt{n}, k=0,1,2, \ldots$ we get the
bound

$$
\begin{align*}
\sum_{k=0}^{\infty} & C\left(A_{n}^{2} \sqrt{n}\right)^{d} 2^{d k} \frac{C}{n_{1}^{d / 2}} \exp \left(-c 2^{2 k} A_{n}^{4} n / n_{1}\right) \frac{1}{n_{3}^{d / 2}} \exp \left(-c 2^{2 k} A_{n}^{4} n / n_{3}\right) \\
& \leq \sum_{k=0}^{\infty} C A_{n}^{2 d} 2^{d k} \frac{1}{\varepsilon_{n}^{d / 2}} \exp \left(-c 2^{2 k} A_{n}^{4}\right) \frac{1}{n^{d / 2}} \exp \left(-c 2^{2 k} A_{n}^{4}\right)  \tag{5.39}\\
& \leq \frac{C}{n^{d / 2}} \frac{A_{n}^{2 d}}{\varepsilon_{n}^{d / 2}} \sum_{k=0}^{\infty} \exp \left(-c 2^{2 k}(\log \log n)^{4}+d k \log 2\right) \\
& =\frac{C}{n^{d / 2}} o\left(\exp \left(-100(\log \log n)^{2}\right)\right) .
\end{align*}
$$

The above lemma allows us to write

$$
\begin{align*}
\sum_{x \in G} I(x) & =\frac{1+o(1)}{N^{d}} p_{n}\left(y_{\ell-1}^{\prime}, y_{\ell}^{\prime}\right) \sum_{\substack{n_{1}+n_{2}+n_{3}=n \\
n_{1}, n_{3} \geq z_{n} n \\
n_{2} \leq \mathcal{Z}^{2 \delta /}}} \sum_{z^{\prime} \in \partial B_{0, x}} \mathbf{P}_{z^{\prime}}\left[H_{\mathbf{K}+x N}=n_{2}<\xi_{B_{0, x}}\right]  \tag{5.40}\\
& =\frac{(1+o(1)) n}{N^{d}} p_{n}\left(y_{\ell-1}^{\prime}, y_{\ell}^{\prime}\right) \sum_{\substack{ \\
n_{2} \leq N^{2 \delta / d}}} \sum_{z^{\prime} \in \partial B_{0, x}} \mathbf{P}_{z^{\prime}}\left[H_{\mathbf{K}+x N}=n_{2}<\xi_{B_{0, x}}\right] .
\end{align*}
$$

The next lemma will help us extract the $\operatorname{Cap}(K)$ contribution from the right hand side of (5.31).

Lemma 5.4.10. We have

$$
\begin{align*}
& \sum_{n_{2}=0}^{N^{28 / d}} \sum_{z^{\prime} \in \partial B_{0, x}} \mathbf{P}_{z^{\prime}}\left[H_{\mathbf{K}+x N}=n_{2}<\xi_{B_{0, x}}\right]  \tag{5.41}\\
& \quad=\frac{1}{2} \operatorname{Cap}(K)(1+o(1))
\end{align*}
$$

Proof. Performing the sum over $n_{2}$ and using time-reversal allows us to write the left hand side of (5.41) as

$$
\begin{align*}
& \left(\sum_{z^{\prime} \in \partial B_{0, x}} \mathbf{P}_{z^{\prime}}\left[H_{\mathbf{K}+x N}<\xi_{B_{0, x}}\right]\right)-\left(\sum_{z^{\prime} \in \partial B_{0, x}} \mathbf{P}_{z^{\prime}}\left[N^{2 \delta / d}<H_{\mathbf{K}+x N}<\xi_{B_{0, x}}\right]\right)  \tag{5.42}\\
& \quad=\frac{1}{2} \operatorname{Cap}(K)-\sum_{\mathbf{x} \in \mathbf{K}} \mathbf{P}_{\mathbf{x}+x N}\left[N^{2 \delta / d}<\xi_{B_{0, x}}<H_{\mathbf{K}+x N}\right]
\end{align*}
$$

The subtracted term in the right hand side of (5.42) is at most

$$
|\mathbf{K}| \max _{\mathbf{x} \in \mathbf{K}} \mathbf{P}_{\mathbf{x}+x N}\left[\xi_{B_{0, x}}>N^{2 \delta / d}\right] .
$$

Since $\zeta<\delta / d$, this expression is $o(1)$.

From the above lemma we get that the main contribution equals

$$
\begin{equation*}
\sum_{x \in G} I(x)=(1+o(1)) \frac{n}{N^{d}} \frac{1}{2} \operatorname{Cap}(K) p_{n}\left(y_{\ell-1}^{\prime}, y_{\ell}^{\prime}\right) . \tag{5.43}
\end{equation*}
$$

It is left to estimate all the error terms.
Lemma 5.4.11. We have

$$
\sum_{x \in G} I I(x)=o(1) \frac{n}{N^{d}} p_{n}\left(y_{\ell-1}^{\prime}, y_{\ell}^{\prime}\right) .
$$

Proof. We split the estimates according to which condition is violated in the sum.
Case 1. $n_{2}>N^{2 \delta / d}$. We claim that

$$
\mathbf{P}_{z^{\prime}}\left[H_{\mathbf{K}+x N}=n_{2}<\xi_{B_{0, x},}, Y_{H_{\mathbf{K}+x N}}=x^{\prime}\right] \leq C \exp \left(-N^{\varepsilon / 2}\right) p_{n_{2}}\left(z^{\prime}, x^{\prime}\right) .
$$

By (5.3), we have

$$
\mathbf{P}_{z^{\prime}}\left[H_{\mathbf{K}+x N}=n_{2}<\xi_{B_{0, x}}, Y_{H_{\mathbf{K}+x N}}=x^{\prime}\right] \leq C \exp \left(-N^{\varepsilon}\right)
$$

By (5.5) on $p_{n_{2}}$ and since $\zeta<\delta / d$

$$
p_{n_{2}}\left(z^{\prime}, x^{\prime}\right) \geq \frac{C}{n_{2}^{d / 2}} \exp \left(-\frac{N^{2 \zeta}}{n_{2}}\right) \geq C \exp \left(-N^{\varepsilon / 2}\right)
$$

We also have the bound

$$
\widetilde{p}_{n_{1}}^{(x)}\left(y_{\ell-1}^{\prime}, z^{\prime}\right) \leq p_{n_{1}}\left(y_{\ell-1}^{\prime}, z^{\prime}\right) .
$$

We then get (summing over $z^{\prime}$ and $x^{\prime}$ ) that the contribution to $\sum_{x \in \mathbb{Z}^{d}} I I(x)$ from Case 1 is at most

$$
\begin{aligned}
C \exp \left(-N^{\varepsilon / 2}\right) N^{(d-1) \zeta} \sum_{n_{1}+n_{2}+n_{3}=n} p_{n}\left(y_{\ell-1}^{\prime}, y_{\ell}^{\prime}\right) & \leq C n^{2} \exp \left(-N^{\varepsilon / 2}\right) p_{n}\left(y_{\ell-1}^{\prime}, y_{\ell}^{\prime}\right) \\
& =o(1) \frac{n}{N^{d}} p_{n}\left(y_{\ell-1}^{\prime}, y_{\ell}^{\prime}\right) .
\end{aligned}
$$

Case 2. $n_{2} \leq N^{28 / d}$ and $n_{1}<\varepsilon_{n} n$. Note that since $n_{2} \leq N^{2} \leq \varepsilon_{n} n$, if we put $n_{1}^{\prime}=n_{1}+n_{2}$ and $n_{3}^{\prime}=n_{3}$, we can upper bound the contribution of this case by

$$
\sum_{\substack{n_{1}^{\prime}+n_{3}^{\prime}=n \\ n_{1}^{\prime} \leq 2 \varepsilon_{n} n}} \sum_{x \in G} \sum_{x^{\prime} \in \mathbf{K}+x N} p_{n_{1}^{\prime}}\left(y_{\ell-1}^{\prime}, x^{\prime}\right) p_{n_{3}^{\prime}}\left(x^{\prime}, y_{\ell}^{\prime}\right) .
$$

Now we can model the bound on the argument for Proposition 5.2.3 as follows.
Case 2-(i). $N^{2+6 \varepsilon} \leq n_{1}^{\prime} \leq 2 \varepsilon_{n} n$ and $\left|y_{\ell-1}^{\prime}-x^{\prime}\right| \leq\left(n_{1}^{\prime}\right)^{\frac{1}{2}+\varepsilon}$ and $\left|x^{\prime}-y_{\ell}^{\prime}\right| \leq n^{\frac{1}{2}+\varepsilon}$. The LCLT allows us to replace $x^{\prime}$ by $u^{\prime} \in \mathbb{T}+x N$ both in $p_{n_{1}^{\prime}}$ and $p_{n_{3}^{\prime}}$, yielding the upper bound

$$
\begin{aligned}
\frac{C}{N^{d}} \sum_{\substack{n_{1}^{\prime}+n_{3}^{\prime}=n \\
n_{1}^{\prime} \leq 2 \varepsilon_{n} n}} \sum_{u^{\prime} \in \mathbb{Z}^{d}} p_{n_{1}^{\prime}}\left(y_{\ell-1}^{\prime}, u^{\prime}\right) p_{n_{3}^{\prime}}\left(u^{\prime}, y_{\ell}^{\prime}\right) & =\frac{C}{N^{d}} \sum_{\substack{n_{1}^{\prime}+n_{3}^{\prime}=n \\
n_{1}^{\prime} \leq 2 \varepsilon_{n} n}} p_{n}\left(y_{\ell-1}^{\prime}, y_{\ell}^{\prime}\right)=C \frac{2 \varepsilon_{n} n}{N^{d}} p_{n}\left(y_{\ell-1}^{\prime}, y_{\ell}^{\prime}\right) \\
& =o(1) \frac{n}{N^{d}} p_{n}\left(y_{\ell-1}^{\prime}, y_{\ell}^{\prime}\right) .
\end{aligned}
$$

Case 2-(ii). $N^{2+6 \varepsilon} \leq n_{1}^{\prime} \leq 2 \varepsilon_{n} n$ but either $\left|y_{\ell-1}^{\prime}-x^{\prime}\right|>\left(n_{1}^{\prime}\right)^{\frac{1}{2}+\varepsilon}$ or $\left|x^{\prime}-y_{\ell}^{\prime}\right|>$ $n^{\frac{1}{2}+\varepsilon}$. This case gives a stretched-exponentially small contribution as in the proof of Proposition 5.2.3.

Case 2-(iii). $n_{1}^{\prime}<N^{2+6 \varepsilon}$. This case can be handled exactly as Cases 3 a and 3 b of Proposition 5.2.3.

Case 3. $n_{2} \leq N^{28 / d}$ and $n_{3}<\varepsilon_{n} n$. This case can be handled very similarly to Case 2.

Lemma 5.4.12. We have

$$
\sum_{x \in \mathbb{Z}^{d} \backslash G} \mathbf{P}[A(x)]=o(1) \frac{n}{N^{d}} p_{n}\left(y_{\ell-1}^{\prime}, y_{\ell}^{\prime}\right) .
$$

Proof. By the same arguments as in Lemma 5.4.9(ii), we have

$$
p_{n}\left(y_{\ell-1}^{\prime}, y_{\ell}^{\prime}\right) \geq \frac{C}{n^{d / 2}} \exp \left(-100(\log \log n)^{2}\right)
$$

For $x \in \mathbb{Z}^{d} \backslash G$, let $k$ be the dyadic scale that satisfies

$$
2^{k} A_{n}^{2} \sqrt{n} \leq\left|x^{\prime}-y_{\ell-1}^{\prime}\right|<2^{k+1} A_{n}^{2} \sqrt{n}
$$

Then we have

$$
\begin{aligned}
\mathbf{P}[A(x)] & \leq \sum_{1 \leq m \leq n-1} \sum_{x^{\prime} \in \mathbf{K}+x N} p_{m}\left(y_{\ell-1}^{\prime}, x^{\prime}\right) p_{n-m}\left(x^{\prime}, y_{\ell}^{\prime}\right) \\
& \leq C|\mathbf{K}| \sum_{1 \leq m \leq n-1} \frac{1}{m^{d / 2}} \frac{1}{(n-m)^{d / 2}} \exp \left(-c \frac{2^{2 k} A_{n}^{4} n}{m}\right) \exp \left(-c \frac{2^{2 k} A_{n}^{4} n}{n-m}\right)
\end{aligned}
$$

Due to symmetry of the right hand side, it is enough to consider the contribution of $1 \leq m \leq n / 2$, which is bounded by

$$
\begin{aligned}
\frac{C}{n^{d / 2}} & \exp \left(-c 2^{2 k} A_{n}^{4}\right) \sum_{1 \leq m \leq n / 2} \frac{1}{m^{d / 2}} \exp \left(-c \frac{2^{2 k} A_{n}^{4} n}{m}\right) \\
& \leq \frac{C}{n^{d / 2}} \exp \left(-c 2^{2 k} A_{n}^{4}\right) \sum_{k^{\prime}=1}^{\log _{2} n} \sum_{m: 2^{k^{\prime}} \leq n / m<2^{k^{\prime}+1}} \frac{2^{k^{\prime} d / 2}}{n^{d / 2}} \exp \left(-c 2^{2 k} A_{n}^{4} 2^{k^{\prime}}\right) \\
& \leq \frac{C}{n^{d}} \exp \left(-c 2^{2 k} A_{n}^{4}\right) \sum_{k^{\prime}=1}^{\infty} \frac{n}{2^{k^{\prime}}} \exp \left(-c 2^{2 k} A_{n}^{4} 2^{k^{\prime}}+k^{\prime} d / 2 \log 2\right) \\
& \leq \frac{C n}{n^{d}} \exp \left(-c 2^{2 k} A_{n}^{4}\right)
\end{aligned}
$$

Now summing over $x \in \mathbb{Z}^{d} \backslash G$ we have that

$$
\begin{aligned}
\sum_{x \in \mathbb{Z}^{d} \backslash G} \mathbf{P}[A(x)] & \leq \frac{C n}{n^{d}} \sum_{k=0}^{\infty} \frac{1}{N^{d}}\left(2^{k} A_{n}^{2} \sqrt{n}\right)^{d} \exp \left(-c 2^{2 k} A_{n}^{4}\right) \\
& \leq \frac{C}{n^{d / 2}} \frac{n}{N^{d}} \sum_{k=0}^{\infty} \exp \left(-c 2^{2 k} A_{n}^{4}+k d \log 2+2 d \log A_{n}\right) \\
& =o(1) \frac{1}{n^{d / 2}} \frac{n}{N^{d}} \exp \left(-100(\log \log n)^{2}\right)
\end{aligned}
$$

Lemma 5.4.13. We have

$$
\sum_{x_{1} \neq x_{2} \in \mathbb{Z}^{d}} \mathbf{P}\left[A\left(x_{1}\right) \cap A\left(x_{2}\right)\right]=o(1) \frac{n}{N^{d}} p_{n}\left(y_{\ell-1}^{\prime}, y_{\ell}^{\prime}\right)
$$

Proof. The summation on the left hand side is bounded above by

$$
\begin{aligned}
\mathbf{P}\left[A\left(x_{1}\right) \cap A\left(x_{2}\right)\right] \leq \sum_{m_{1}+m_{2}+m_{3}=n} \sum_{\substack{x_{1}^{\prime} \in \mathbf{K}+x_{1} N \\
x_{2}^{\prime} \in \mathbf{K}+x_{2} N}} & {\left[p_{m_{1}}\left(y_{\ell-1}^{\prime}, x_{1}^{\prime}\right) p_{m_{2}}\left(x_{1}^{\prime}, x_{2}^{\prime}\right) p_{m_{3}}\left(x_{2}^{\prime}, y_{\ell}^{\prime}\right)\right.} \\
& \left.+p_{m_{1}}\left(y_{\ell-1}^{\prime}, x_{2}^{\prime}\right) p_{m_{2}}\left(x_{2}^{\prime}, x_{1}^{\prime}\right) p_{m_{3}}\left(x_{1}^{\prime}, y_{\ell}^{\prime}\right)\right] .
\end{aligned}
$$

Due to symmetry it is enough to consider the first term inside the summation. The estimates are again modelled on the proof of Proposition 5.2.3.

Case 1. $m_{1}+m_{2} \geq n / 2$ and $\left|x_{2}^{\prime}-y_{\ell-1}^{\prime}\right| \leq 2 C_{1} \sqrt{n} \sqrt{\log n}$. In this case we can use the calculations of Proposition 5.2.3 with $x_{2}^{\prime}$ playing the role of $y_{\ell}^{\prime}$ to perform the summation over $x_{1}^{\prime}$ and $x_{1}$ and get the upper bound:

$$
\begin{equation*}
C \frac{n}{N^{d}} \sum_{m_{1}^{\prime}+m_{2}^{\prime}=n} \sum_{x_{2} \in \mathbb{Z}^{d}} \sum_{x_{2}^{\prime} \in \mathbf{K}+x_{2} N} p_{m_{1}^{\prime}}\left(y_{\ell-1}^{\prime}, x_{2}^{\prime}\right) p_{m_{2}^{\prime}}\left(x_{2}^{\prime}, y_{\ell}^{\prime}\right), \tag{5.44}
\end{equation*}
$$

where we have written $m_{1}^{\prime}=m_{1}+m_{2}$ and $m_{2}^{\prime}=m_{3}$. Using again the calculations in Proposition 5.2.3 yields the upper bound

$$
\begin{equation*}
C\left(\frac{n}{N^{d}}\right)^{2} p_{n}\left(y_{\ell-1}^{\prime}, y_{\ell}^{\prime}\right)=o(1) \frac{n}{N^{d}} p_{n}\left(y_{\ell-1}^{\prime}, y_{\ell}^{\prime}\right) \tag{5.45}
\end{equation*}
$$

Case 2. $m_{1}+m_{2} \geq n / 2$ and $2 C_{1} \sqrt{n} \sqrt{\log n}<\left|x_{2}^{\prime}-y_{\ell-1}^{\prime}\right| \leq n^{\frac{1}{2}+\varepsilon}$. First sum over all $x_{1}^{\prime} \in \mathbb{Z}^{d}$ to get the upper bound

$$
\begin{equation*}
C n \sum_{m_{1}^{\prime}+m_{2}^{\prime}=n} \sum_{x_{2} \in \mathbb{Z}^{d}} \sum_{x_{2}^{\prime} \in \mathbf{K}+x_{2} N}^{\prime} p_{m_{1}^{\prime}}\left(y_{\ell-1}^{\prime}, x_{2}^{\prime}\right) p_{m_{2}^{\prime}}\left(x_{2}^{\prime}, y_{\ell}^{\prime}\right), \tag{5.46}
\end{equation*}
$$

where the primed summation denotes the restriction $2 C_{1} \sqrt{n} \sqrt{\log n}<\left|x_{2}^{\prime}-y_{\ell-1}^{\prime}\right| \leq$ $n^{\frac{1}{2}+\varepsilon}$. The choice of $C_{1}$ (recall (5.29)) implies that $p_{m_{1}^{\prime}}$ is $o\left(1 / n N^{d}\right)$. Due to the triangle inequality we also have $\left|y_{\ell}^{\prime}-x_{2}^{\prime}\right|>C_{1} \sqrt{n} \sqrt{\log n}$. Using the LCLT for $p_{m_{2}^{\prime}}$ we get that

$$
\begin{equation*}
p_{m_{2}^{\prime}}\left(x_{2}^{\prime}, y_{\ell}^{\prime}\right) \leq \frac{C}{\left(m_{2}^{\prime}\right)^{d / 2}} \exp \left(-d C_{1}^{2} n \log n / m_{2}^{\prime}\right) \leq \frac{C}{n^{d / 2}} \exp \left(-d C_{1}^{2} \log n\right) \leq C p_{n}\left(y_{\ell-1}^{\prime}, y_{\ell}^{\prime}\right) \tag{5.47}
\end{equation*}
$$

Case 3. $m_{1}+m_{2} \geq n / 2$ and $\left|x_{2}^{\prime}-y_{\ell-1}^{\prime}\right|>n^{\frac{1}{2}+\varepsilon}$ Summing over all $x_{1}^{\prime} \in \mathbb{Z}^{d}$, we get the transition probability $p_{m_{1}+m_{2}}\left(y_{\ell-1}^{\prime}, x_{2}^{\prime}\right)$. This is stretched-exponentially small, and hence this case satisfies the required bound.

Case 4. $m_{2}+m_{3} \geq n / 2$. Due to symmetry, this case can be handled analogously to

Cases 1-3.

### 5.4.4 Proof of Proposition 5.2.4

Proof. By Martingale maximal inequality (5.3) used in the last step,

$$
\begin{aligned}
\mathbf{P}_{y_{\ell-1}^{\prime}}\left[\left|Y_{n}-y_{\ell-1}^{\prime}\right|>\sqrt{n}(10 \log \log n)\right] & =\mathbf{P}_{0}\left[\left|Y_{n}\right|>\sqrt{n}(10 \log \log n)\right] \\
& \leq \exp \left(-c 100(\log \log n)^{2}\right)
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
& \mathbf{E}\left[\#\left\{1 \leq \ell \leq \frac{N^{d}}{n} C_{1} \log N:\left|Y_{n \ell}-Y_{n(\ell-1)}\right|>10 \sqrt{n} \log \log n\right\}\right] \\
& \quad \leq \frac{N^{d}}{n} C_{1} \log N \exp \left(-c(\log \log n)^{2}\right) \leq \frac{N^{d}}{n} C \exp \left(-(c / 2)(\log \log n)^{2}\right)
\end{aligned}
$$

By Markov's inequality, it follows that

$$
\begin{gathered}
\mathbf{P}\left[\#\left\{1 \leq \ell \leq \frac{N^{d}}{n} C_{1} \log N:\left|Y_{n \ell}-Y_{n(\ell-1)}\right|>10 \sqrt{n} \log \log n\right\} \geq \frac{N^{d}}{n}(\log \log n)^{-1}\right] \\
\quad \leq \frac{\frac{N^{d}}{n} C \exp \left(-(c / 2)(\log \log n)^{2}\right)}{\frac{N^{d}}{n}(\log \log n)^{-1}} \leq C \frac{\exp \left(-(c / 2)(\log \log n)^{2}\right)}{(\log \log n)^{-1}} \rightarrow 0,
\end{gathered}
$$

as $N \rightarrow \infty$.

### 5.5 Chapter outlook

The following are some possible generalisations regarding the proof of the main theorem:
(1) It is not essential that we restrict to the simple random walk: any random walk for which the results in Section 5.2, hold (such as finite range symmetric walks) would work equally well.
(2) The paper [15] considers several distant sets $K^{1}, \ldots, K^{r}$, and we believe this would also be possible here, but would lead to further technicalities in the presentation.
(3) It is also not essential that the rescaled domain be $(-1,1)^{d}$, and we believe it could be replaced by any other domain with sufficient regularity of the boundary.

In this chapter, we considered a single simple random walk starting from a fixed point (the origin $o$ ) to hit a finite set on $\mathbb{Z}^{d}$. We could extend this result with some modification to a finite number of independent random walks, say $i$ independent random walks. In this case, the probability that $i$ independent random walks do not hit $\varphi^{-1}(\mathbf{K})$ converges to

$$
\mathbf{E}\left[e^{-A d \operatorname{Cap}(\mathbf{K})\left(\sigma_{1}+\sigma_{2}+\cdots+\sigma_{i}\right)}\right]+o(1), \quad \text { as } N \rightarrow \infty
$$

where $\sigma_{1}, \sigma_{2}, \ldots, \sigma_{i}$ are the exit time of independent Brownian motions from the unit cubic $(-1,1)^{d}$.

The potential implications of this connection are for the numerical methods about the height probability and the whole avalanche: From Chapter 3, we know that we need $2 d+1$ random walks to generate the height probability in $d$ dimensions. In this case, we could give a heuristic upper bound $O\left(L^{2}\right)$ for the average running time for generating the height probability. A rigorous upper bound on the running time would require further study of the nearby random walks. In terms of the whole avalanche, the memory use is $O\left(L^{6}\right)$.

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## Chapter 6

## Conclusion

In this thesis, I worked on critical exponents in sandpiles both analytically and numerically. My research involved: numerical simulation of sandpile critical exponents as well as some theoretical questions inspired by these simulations that involve detailed random walk estimates and random interlacements. The following sections are future open questions and long term goals of my current research in sandpiles. Some of the open questions are also inspired by the simulation results in Chapter 3 .

### 6.1 Future open questions

### 6.1.1 Sandpiles on a 4-dimensional hierarchical lattice

In Abelian sandpiles, the exact behaviour of the toppling probability in 4 dimensions is not known, but Járai, Redig and Saada [35] expected that $\mathbf{P}[x$ topples] $=$ $|x|^{d-2} \log ^{-\alpha}|x|$ with $\alpha \geq 0$. Detecting the exact exponent numerically is hard, since even when the size of the box $[-L, L]^{4}$ is very $\operatorname{large}, \log L$ is still very small. However, we could modify our previous simulation method in Chapter 3 to check at least the leading order of the behaviour in 4 dimensions, i.e. $|x|^{-2}$. We would like to further study the exact behaviour theoretically, which is a challenging problem. As a simplification, we plan to study the exact behaviour on a hierarchical model by adapting the renormalization group method in Bauerschmidt, Brydges and Slade's book [4]. This method is novel in the context of the sandpile model, which has not been done in the literature.

The reasons why we choose to study the exact behaviour in 4 dimensional hierarchical lattice is the following two. First, we have studied numerically in 2 and 3 dimensions.

For $d \geq 5$, the critical exponent of the toppling probability in sandpile model is well understood. Second, we plan to start with the hierarchical lattice, because it is easier than in $\mathbb{Z}^{4}$.

Renormalization approach requires the understanding of the continuous height sandpile model. We have only studied the discrete sandpile model in this thesis so far. Let us now briefly introduce the continuous height sandpile model.

A continuous model with deterministic additions, called the Abelian avalanche model, has been introduced on finite graphs by Gabrielov [17]. Let $V$ be a finite set of $N$ elements (sites), and let $\Delta$ be a $N \times N$ real matrix with indices in $V$, such as

$$
\begin{array}{ll}
\Delta_{i i}>0, & \text { for all } i \\
\Delta_{i j} \leq 0, & \text { for all } i \neq j
\end{array}
$$

and $s_{i}=\sum_{j} \Delta_{i j} \geq 0$ for all $i$. The value $s_{i}$ is called the dissipation at a site $i$.
We define a positive real value $h_{i}$ as the height at every site $i$. The set $h=\left\{h_{i}\right\}$ is called the configuration of the system. For every site $i$, a threshold $H_{i}$ is defined. The configurations $h$ with $h_{i}<H_{i}$ for all $i$ are called stable. For every stable configuration, the height $h_{i}$ increases in time with a constant rate $\nu_{i} \geq 0$ until it is greater than a threshold $H_{i}$ at a site $i$. Then the site $i$ topples, and the heights are redistributed as follows:

$$
h_{j} \rightarrow h_{j}-\Delta_{i j}, \quad \text { for all } j .
$$

Similarly, as in the discrete case, if after the redistribution, any heights are greater than thresholds at some other sites, these sites also topples according to the above redistribution law, and so on, until we arrive at a stable configuration. The sequence of topplings is called an avalanche. This model has the Abelian property, i.e. the stable configuration after an avalanche, and the number of topplings at any site during the avalanche, do not depend on the order of topplings during the avalanche. Hence, we call this model an Abelian avalanche model.

Járai, Redig and Saada [35] later studied a continuous height sandpile model, which a special case of the model introduced by Gabrielov [17]. Let $\Lambda \subset \mathbb{Z}^{d}$ be finite. A continuous configuration on $\Lambda$ is a collection of heights occupying the sites in $\Lambda$. We write a configuration as a map $\eta: \Lambda \rightarrow[0, \infty)$. Let $\gamma \geq 0$ be a real parameter. We say that a configuration $\eta$ is $\gamma$-stable if $\eta_{x} \leq 2 d+\gamma$ for all $x \in \Lambda$. We say that $x$ is allowed to $\eta$-topple if $\eta_{x} \geq 2 d+\gamma$. A $\gamma$-toppling at $x$ means that height 1 is sent along each edge incident to $x$ in $\mathbb{Z}^{d}$, and height $\gamma$ is lost, thereby decreasing the height at


Figure 6-1: Blocks in $\mathcal{B}_{j}$ for $j=0,1,2$ when $d=2, N=2, L=2$.
$x$ by $2 d+\gamma$ and increasing the height at each neighbour by 1 . This model has the Abelian property as well, i.e. any configuration has a unique $\gamma$-stabilization arrived at by carrying out all possible $\gamma$-topplings. The order of topplings during the avalanche does not affect the stable configuration after an avalanche. They also called this model the Abelian avalanche model in the paper [35]. In this model, the infinite volume limit when $\gamma=0$ is the same as the discrete sandpile model, which is the limit of the infinite volume model with dissipation $\gamma$ when $\gamma \downarrow 0$ [35].

We would like to study a variant of a model introduced by Gabrielov [17] , which adapts the renormalization group method. Let us now briefly introduce the hierarchical model in [4, Chapter 4].

For $d \geq 2$, although in our case $d=4$, we introduce a general version here. Throughout our discussion and analysis of the hierarchical model, let $L>1$ be the side length of a 1-block and $N \in \mathbb{N}_{+}$be the maximum level of blocks. Set the hypercube $\Lambda_{N}=$ $\left[0, L^{N}-1\right]^{d} \cap \mathbb{Z}^{d}$. As in Fig 6-1, we partition $\Lambda_{N}$ into disjoint blocks of side length $L^{j}$ (number of vertices), with $j=0,1,2, \ldots, N$. For $j=0,1,2, \ldots, N$, we denote $\mathcal{B}_{j}$ as the set of disjoint blocks $B$ of side length $L^{j}$ so that we get a partition $\Lambda_{N}=\cup_{B \in \mathcal{B}_{j}} B$. An element $B \in \mathcal{B}_{j}$ is a called a $j$-block.

For a sandpile model in hierarchical lattice, let $\eta$ denote the configuration on $\Lambda_{N}$, which is a map $\eta: \Lambda_{N} \rightarrow[0, \infty)$. The toppling matrix $\Delta^{H, N}$ should be the hierarchical Laplacian such that

$$
\begin{aligned}
& \Delta_{x, x}^{H, N}=M, \quad \text { for all } x \\
& \Delta_{x, y}^{H, N}=-m^{j, N}, \quad \text { for all } x \neq y,
\end{aligned}
$$

where the constants $m^{j, N}$ depend on the least level $j$ at which $x$ and $y$ are in the same block; see [4] for details.

Let $N(x, y ; h)$ be the number of times $y$ toppled during stabilization if amount of $h$ is added at $x$. Define the addition operators acting on $\eta$ as

$$
A_{x}^{h}(\eta)_{y}:=\eta_{y}+h \delta_{x y}-\sum_{z \in \Lambda_{N}} \Delta_{z, y}^{H, N} N(x, z ; h), \quad \text { for all } y \in \Lambda_{N}
$$

i.e. add amount of sand $h$ at $x$ and stabilize.

Gabrielov [17] showed that there exists such stationary measure $\mu_{N}$ and set $\mathcal{R}_{\Lambda_{N}}$ of recurrent sandpiles invariant under the addition operators $A_{x}^{h}$. We would be interested in computing these more specifically for the hierarchical model.

Similarly, as in the discrete sandpile model, we can prove the Dhar's formula, which states that the expected number of times $y$ toppled when we add height $h$ at $x$ is given by the Green's function.

From the definition of the addition operators $A_{x}^{h}$, we have for all $y \in \Lambda_{N}$

$$
\mathbf{E}_{\mu_{N}}\left[\eta_{y}\right]=\mathbf{E}_{\mu_{N}}\left[\eta_{y}\right]+h \delta_{x y}-\sum_{z \in \Lambda_{N}} \Delta_{z, y}^{H, N} \mathbf{E}_{\mu_{N}}[N(x, z ; h)] .
$$

Rearranging, we get

$$
\sum_{z \in \Lambda_{N}} \mathbf{E}_{\mu_{N}}[N(x, z ; h)] \Delta_{z, y}^{H, N}=h \delta_{x y} .
$$

Finally, we have the Dhar's formula as follows:

$$
\mathbf{E}^{N}[N(x, y ; h)]=h G_{N}(x, y)=h\left(\Delta^{H, N}\right)_{x, y}^{-1},
$$

where $G_{N}(x, y)$ is the Green's function on $\Lambda_{N}$,
As in the next step, we need to compute $\operatorname{Var}_{\mu_{N}}[N(o, x ; h)]$. It is helpful to start with some simple examples regarding the renormalization method to discuss the support of the stationary measure $\mu_{N}$. First, we check the case when $N=1$, this might be trivial with product measure only. Then, we could move on to look at the case when $N=2$, $L=2$, and so on.

### 6.1.2 Rotational invariance in 2D

From our simulation of the toppling probability in 2 dimensions, we found similar behaviour in different radial directions; see Figure 3-4 and Figure 3-5 in Chapter 3. It appears that asymptotically, the toppling probability only depends on the Euclidean
distance from the origin (in the infinite volume limit). Lawler, Schramm and Werner [49] proved that the scaling limit of a loop-erased random walk is the radial SLE ${ }_{2}$ path, which is conformally invariant. We hope to use this result to prove the probability that the vertex toppled in the last wave of an avalanche is rotationally symmetric on the full plane. In the infinite volume limit case, the main difficulty is to bridge the gap between the behaviour of the loop-erased paths near the origin and their large scale behaviour. In a bounded domain, we expect that a conformal covariance statement holds. This would be very interesting to explore.

The reason for looking at the last wave is a simplification, as it is simpler than looking at the toppling probability itself. In the case of last wave, there is a bijection between the last wave and the two-component spanning forest of the domain under the condition that the origin is not connected to the sink; see [6] for details.

### 6.1.3 Other possible future work

It is also useful to study other dynamical properties in Abelian sandpiles. Bhupatiraju, Hanson and Járai [6] proved bounds on the behaviour of various avalanche characteristics such as the probability that a given vertex topples, the radius of the avalanche cluster, and the number of vertices toppled. Their results yield rigorous inequalities for the relevant critical exponents in $d \geq 2$. Hutchcroft [31] deduced that the critical exponents describing the diameter and total number of topplings in an avalanche on a large class of graphs in $d \geq 5$. In the case of $\mathbb{Z}^{d}, d \geq 5$, some of the results regarding critical exponents recover earlier results in [6]. We would be interested in study these quantities in the case of $\mathbb{Z}^{d}, d=2,3$.

### 6.2 Research goals in Abelian sandpiles

The final goal of my current research in Abelian sandpiles is to provide a rigorous mathematical analysis of the results from the simulations. In particular, it will consider two aspects connected to the simulation. It is expected that the first aspect will provide a rigorous analysis of the algorithm designed and prove an upper bound on its average running time. To start with this problem, we have shown for a single random walk starting from a fixed point (the origin $o$ ) to hit a finite set is a random interlacement by looking into stretches of random walks in Chapter 5. We could extend this result to a finite number of (almost) independent random walks with slight modification by considering the random walks starting at the origin and its neighbours. In addition to lending rigorous support to the use of the algorithm, we expect that it will shed new
light on the mean-field values of critical exponents in sandpiles.
As in Figure 3-25, there is a drop in the toppling probability near the boundary of the box $[-L, L]^{d}$ when $d=5$ and $L=32$, which is clearly not in the $|x|^{-(d-2)}$ behaviour. Further extending our analysis to growing number of random walks in a torus, we would need the hashtable size $N^{d}$, where $A N^{d} \sim L^{6}$, so there is no sample discarded when using the hashing method. Hutchcroft [31] showed that the probability that the avalanche reaches radius $L / 2$ is $L^{-2}$. The avalanche size reaches $L^{4}$ is with the probability of the same order, because the avalanche size exponent is $1 / 2$ in dimensions 5 and higher [31]. We can heuristically expect that the size of largest avalanche in the box $[-L, L]^{d}$ is of size $L^{4}$. This requires $L^{2} \times L^{4}$ steps to simulate the whole avalanche. Hence, we require the hashtable size as stated above.

In further work, we hope to prove rigorous upper bounds for the values of the exponent we estimated. This is a much more challenging open question for which only lower bounds are known at the moment. Bhupatiraju, Hanson and Járai [6] proved lower bounds for these values of exponents by considering the last wave. We know that the first moment of topplings is the Green's function, i.e. the expected number of visits at $x$, but the second moment is unknown. If we could bound the second moment, we can either improve the lower bound, or even prove the upper bound.

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[^0]:    ${ }^{1}$ This is slightly different from what was used for the simulations in Chapter 3 , but easier to analyse.

