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Membership-Function-Dependent Design of L_1-Gain Output Feedback Controller for Stabilization of Positive Polynomial Fuzzy Systems

Aiwen Meng, Hak-Keung Lam, *Fellow, IEEE,* Fucai Liu, and Yingjie Yang

***Abstract*—This paper presents the L_1-gain polynomial fuzzy output feedback controller design and the stability analysis using sum-of-squares (SOS) approach for positive polynomial fuzzy-model-based (PPFMB) control systems. The polynomials, positivity and optimal L_1 performance makes some existing convex methods for general systems inapplicable. To overcome this problem, an augmented system of the positive polynomial fuzzy-model-based control system is first constructed, then by introducing some constrain conditions and mathematical techniques, the non-convex stability and positivity conditions are skillfully transformed into convex ones simultaneously. In addition, to control the systems flexibly and lower the implementation cost, the imperfect premise matching concept is taken into account for controller design. Besides, the high degree polynomial approximation method is adopted to conduct stability and positivity analysis by incorporating the information of membership functions (MFs) and the boundary information of the state variables. On the basis of the Lyapunov stability theory, the relaxed stability and positivity conditions in terms of SOS are obtained. Finally, a simulation example is presented to verify the feasibility of the theoretical results.**

***Index Terms*—Positive polynomial fuzzy-model-based (PPFMB) control systems, output feedback control, membership functions (MFs), L_1 performance, sum-of-squares (SOS).**

I. Introduction

Positive systems, whose state variables and outputs always remain in the non-negative quadrant if both of the initial conditions and input are non-negative, are often encountered in real-world applications [1]–[3]. A great deal of practical models of such systems exist in a variety of disciplines, for example, the control of the cortisol level within the hypothalamic-pituitary-adrenal gland axis in the field of biology, the human immunodeficiency virus viral mutation dynamics in the area of pharmacokinetic, the prey-predator model in the aspect of ecology, and the concentration of substances in chemical processes and so on [4]–[6]. As positive systems are closely related to our daily life, it is of great practical significance to conduct in-depth research on positive systems.

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From the perspective of system complexity, nonlinear systems are more common than linear systems in actual application, such as network systems [7], robotic systems [8], multi-agent systems [9] and nonlinear systems with stochastic impulses [10], but it also makes the study of stability analysis and control synthesis even more challenging due to the nonlinear characteristics. Fortunately, since Takagi-Sugeno (T-S) fuzzy-model-based technique provides a powerful mathematical tool to express the nonlinearities, it considerably promotes the investigation of stability analysis and control synthesis for nonlinear systems [11]–[15]. In recent years, some outstanding works on positive nonlinear systems based on T-S fuzzy models have been obtained [16]–[19]. The work in [16] mainly investigated the state estimation for positive T-S fuzzy systems. Through proposing a new algebraic algorithm and constructing a new Lyapunov function, an observer was designed for positive systems and the stability analysis results were relaxed effectively. In [17], the positive T-S fuzzy models were controlled to be asymptotically stable and positive by employing the state feedback control strategy. The stability conditions were derived in terms of the single Lyapunov-Krasovskii functional and the linear programming technique. In general, the parallel distributed compensation design concept [20] is an effective approach to reduce the conservatism of the stability analysis. But this method will cause some limitations for the design of fuzzy controllers, for instance, it will increase the complexity of controller design if the plants are complex, especially, when the number of the fuzzy rules is high or the type of membership functions is complicated. To overcome these drawbacks, imperfect premise matching concept was firstly proposed in [21] and employed in [22] to investigate the filtering problems for the positive T-S fuzzy systems. Although it could lead to conservatism, it is worth mentioning that the well-known membership-function-dependent (MFD) analysis proposed by H. K. Lam [23]–[26] provided a great help to reduce the conservatism by taking the information of membership functions (MFs) into the stability analysis. Until now, even though some MFD methods, such as, piecewise linear MFs, Taylor series MFs and approximated polynomial MFs, have been provided, this is still an open problem which is worth the effort to study further.

In recent decades, with the further study of fuzzy theory, T-S fuzzy model has been extended into polynomial fuzzy model [27]–[33] which is more effective to represent complex positive nonlinear systems. So far, plenty of researches on general polynomial fuzzy systems can be found in the literature, such as, output-feedback tracking control [30], sampled-data output-feedback tracking control [34], event-triggered networked control [35] and so on. However, the works with respect to positive polynomial fuzzy-model-based (PPFMB) systems are still not particularly fruitful, which is a strong motivation for us to carry out the task.

In the view of control synthesis of PPFMB systems, it is relatively simple to design fuzzy controllers according to the full state feedback control strategy, but in actual life, it is usually difficult to obtain the full states information of real systems. Hence, when some of the state variables are not available, this strategy does not work anymore. In this case, designing fuzzy controllers based on output feedback strategy is more effective and easier to implement because it does not dependent on the full state information but only the output state variables. Nevertheless, on the other hand, the head-scratching non-convex problem also comes with the design of the polynomial fuzzy output feedback (PFOF) controller, which make the stability analysis and control synthesis problem challenging for positive polynomial fuzzy output feedback control systems. To alleviate this challenge, we have attached much importance to designing the PFOF controllers for PPFMB systems in recent years [36], [37], but the system performance index was not taken into account in these works. Therefore, when some external disturbances are present, these existing results may no longer be applicable. After consulting the literature, it can be seen that L_1-induced performance is an effective performance index for positive systems because it expresses the sum of the values of the components [38], [39]. To our best knowledge, there is still no result corresponding to the PFOF controller design for PPFMB systems under L_1 performance. To further strengthen the research for positive nonlinear systems, it is a challenging but worthwhile task for us.

As far as the methods of stability analysis for positive systems are concerned, some of the analysis results are obtained in terms of the linear-matrix-inequality approach which is popular to be used for deriving stability and positivity conditions [40], [41]. Although this approach is successful to guarantee the stability analysis of positive nonlinear systems, there still exist some sources of conservatism because of the neglect of the characteristic of positivity. To make better use of the positivity, the linear programming method which is demonstrated to be more computationally efficient than the linear-matrix-inequality method is employed in [42]–[44]. However, for PPFMB systems, on account of the existence of polynomials in stability conditions, both of the two methods are not as effective as the sum-of-squares (SOS) method [25]. In this case, the free third-party MATLAB toolbox, i.e., SOSTOOLS [45], can be used to find feasible solutions.

On the basis of the above analysis, we need to focus on cracking the following hard nuts. Firstly, as we all known that converting non-convex stable conditions and positive conditions to convex conditions simultaneously is very difficult since most of the convex methods are just for general systems and there are no positivity constrains for general systems. When positivity constrains are taken into consideration, those convex methods for general systems may not be able to work, hence, the non-convex problem will become very tricky to be solved. In addition, the convexification is problem dependent, which means the convexificiation method from one paper may not be able to be applied to other non-convex problems, therefore, it is a hard task to find a proper method to deal with a non-convex problem. Secondly, the introduction of some constrain conditions and the absence of the information of MFs will lead to strong conservatism of the stability analysis results. How to obtain relaxed stability analysis results is still an open problem that is worth working on. Aiming at dealing with the above issues, the main contributions are made and summarized as follows:

TABLE I
DESCRIPTION OF THE ACRONYMS.

Acronyms	Explanation	Acronyms	Explanation
T-S	Takagi-Sugeno	MFD	membership-function-dependent
SOS	sum-of-squares	PPFMB	positive polynomial fuzzy-model-based
MFs	membership functions	PFOF	polynomial fuzzy output feedback

TABLE II
DESCRIPTION OF NOTATIONS.

Notation	Description	Notation	Description
$\mathbf{x}$	system state vector	$\mathbf{u}$	input vector
$\mathbf{z}$	control output	$\tilde{\mathbf{w}}$	disturbance
$\mathbf{y}$	output vector	$\mathbf{K}_j(\mathbf{y})$	static output feedback gain
$w_i(\mathbf{x})$	MFs of positive systems	$m_i(\mathbf{y})$	MFs of fuzzy controllers
ξ	augmenting vectors	λ	constant vector to be determined
γ	L_1 performance level	$\alpha_d(\mathbf{x})$	fractional function
$\eta_{ij,d}(\mathbf{x})$	approximated polynomial	$\Delta\eta_{ij,d}(\mathbf{x})$	approximation error
$\underline{\beta}_{ij,d}$	lower bound of error term	$\overline{\beta}_{ij,d}$	upper bound of error term

1) For coping with the non-convex conditions, the augmented vector method is employed to construct an augmenting system of positive L_1-gain PFOF control system. Then, through introducing some constraint conditions and mathematical skills, the non-convex stability and positivity conditions are approximated by convex ones simultaneously.

2) For reducing the conservativeness of the analysis results, a high degree polynomial approximation method is adopted to approximately express the original MFs so that the valuable information of MFs helps to derive the relaxed stability conditions. Different from other MFD methods, in our paper, the information of the boundary information of state variables is used for relaxing the stability analysis by introducing it into the high degree polynomial functions instead of by introducing a slack matrix, which help reduce the calculation burden.

The rest of this paper is formed as follows. Section II mainly introduces a PPFMB system with bounded disturbance and design a PFOF controller. Section III derives the basic stability and positivity conditions of the positive L_1-gain PFOF control system. Furthermore, a high degree polynomial approximation method is adopted to derive the relaxed stability and positivity conditions. Section IV gives a simulation example to reveal the effectiveness of the L_1-gain PFOF control schemes. Finally, a conclusion is drawn in Section V.

II. Preliminaries

In this section, some standard notations and primary preliminaries of the PPFMB system with bounded disturbance

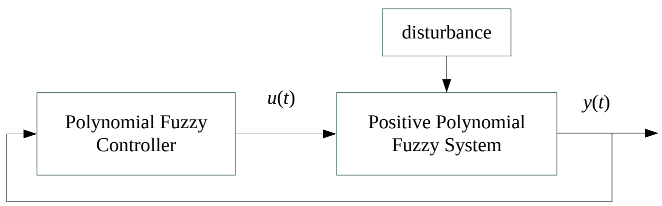

Fig. 1. The positive polynomial fuzzy closed-loop system with disturbance.

and the PFOF controller are shown. The block diagram of a polynomial fuzzy positive system with disturbance based on polynomial fuzzy output feedback control is shown in Fig. 1. The main acronyms are listed in Table I.

A. Notation

In order to facilitate the understanding, some notations used in this paper will be explained in this section [46]. $x_1^{r_1}(t), \ldots, x_n^{r_n}(t)$ represent the monomial in the vector $\mathbf{x}(t) = [x_1(t), \ldots, x_n(t)]^T$, where r_i, $i \in \{1, \ldots, n\}$, is a non-negative integer. The degree of a monomial is defined as $r = \sum_{i=1}^{n} r_i$. A polynomial $\mathbf{p}(\mathbf{x}(t))$ is indicated as the finite linear combination of monomials with real coefficients. If a polynomial $\mathbf{p}(\mathbf{x}(t))$ can be shown as $\mathbf{p}(\mathbf{x}(t)) = \sum_{j=1}^{m} \mathbf{q}_j(\mathbf{x}(t))^2$, where m is a non-zero positive integer and $\mathbf{q}_j(\mathbf{x}(t))$ is a polynomial for all j, it can be concluded that $\mathbf{p}(\mathbf{x}(t))$ is a SOS and $\mathbf{p}(\mathbf{x}(t)) \geq 0$. m_{rs} denotes the element on the r-th row and s-th column in a matrix $\mathbf{M} \in \Re^{l \times n}$. $\mathbf{M} \succeq 0$, $\mathbf{M} \succ 0$, $\mathbf{M} \preceq 0$ and $\mathbf{M} \prec 0$ represent that each element m_{rs} is non-negative, positive, non-positive and negative, respectively. $\mathbf{Q}(\mathbf{x}) = \text{diag}(x_1, \ldots, x_n)$ denotes that the matrix $\mathbf{Q}(\mathbf{x})$ is a diagonal matrix whose diagonal elements are $x_1, \ldots, x_n$. $\mathbf{I}_1 \in \Re^q$ and $\mathbf{I}_2 \in \Re^h$ are vectors with all of the elements are 1; $\mathbf{I}_n \in \Re^{n \times n}$, $\mathbf{I}_l \in \Re^{l \times l}$, $\mathbf{I}_m \in \Re^{m \times m}$ and $\mathbf{I}_h \in \Re^{h \times h}$ are the identity matrices with the specified dimensions. The other notions can be found in Table II.

B. Positive Polynomial Fuzzy Model with Disturbance

A p-rule PPFMB system with disturbance is shown:

Rule i : IF $f_1(\mathbf{x}(t))$ is M_1^i AND $\cdots$ AND $f_\Psi(\mathbf{x}(t))$ is M_Ψ^i

$$\text{THEN} \begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}_i(\mathbf{x}(t))\mathbf{x}(t) + \mathbf{B}_i(\mathbf{x}(t))\mathbf{u}(t) \\ +\mathbf{B}_{i\omega}\tilde{\mathbf{w}}(t), \\ \mathbf{z}(t) = \mathbf{D}_i(\mathbf{x}(t))\mathbf{x}(t) + \mathbf{E}_i(\mathbf{x}(t))\mathbf{u}(t) \\ +\mathbf{E}_{i\omega}\tilde{\mathbf{w}}(t), \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t), \end{cases} \tag{1}$$

where $\mathbf{x}(t) \in \Re^n$, $\mathbf{u}(t) \in \Re^m$, $\tilde{\mathbf{w}}(t) \in \Re^h$, $\mathbf{z}(t) \in \Re^q$ and $\mathbf{y}(t) \in \Re^l$ are the system state vector, the input vector, the disturbance signal, the measurement output and the controlled output, respectively; $\mathbf{A}_i(\mathbf{x}(t))$, $\mathbf{B}_i(\mathbf{x}(t))$, $\mathbf{B}_{i\omega}$, $\mathbf{D}_i(\mathbf{x}(t))$, $\mathbf{E}_i(\mathbf{x}(t))$, $\mathbf{E}_{i\omega}$, and $\mathbf{C}$ are the system matrices with appropriate dimensions for $i \in \{1, \ldots, p\}$.

The overall dynamics of the PPFMB system is introduced:

$$\begin{cases} \dot{\mathbf{x}}(t) = \sum_{i=1}^{p} w_i(\mathbf{x}(t))\big(\mathbf{A}_i(\mathbf{x}(t))\mathbf{x}(t) \\ +\mathbf{B}_i(\mathbf{x}(t))\mathbf{u}(t) + \mathbf{B}_{i\omega}\tilde{\mathbf{w}}(t)\big), \\ \mathbf{z}(t) = \sum_{i=1}^{p} w_i(\mathbf{x}(t))\big(\mathbf{D}_i(\mathbf{x}(t))\mathbf{x}(t) \\ +\mathbf{E}_i(\mathbf{x}(t))\mathbf{u}(t) + \mathbf{E}_{i\omega}\tilde{\mathbf{w}}(t)\big), \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t), \end{cases} \tag{2}$$

where $w_i(\mathbf{x}(t))$ is the normalized grade of membership with satisfying $w_i(\mathbf{x}(t)) = \frac{\prod_{\alpha=1}^{\Psi} \mu_{M_\alpha^i}(f_\alpha(\mathbf{x}(t)))}{\sum_{k=1}^{p} \prod_{\alpha=1}^{\Psi} \mu_{M_\alpha^k}(f_\alpha(\mathbf{x}(t)))}$ and $\sum_{i=1}^{p} w_i(\mathbf{x}(t)) = 1, w_i(\mathbf{x}(t)) \geq 0 \; \forall i$.

Definition 1: [42] Given any initial conditions $\mathbf{x}(0) = \mathbf{x}_0 \succeq 0$, a system can be perceived as positive if the corresponding trajectory always stays in the non-negative quadrant, i.e., $\mathbf{x}(t) \succeq 0$ for all $t \geq 0$.

Definition 2: [42] A matrix $\mathbf{M}$ is called a Metzler matrix if its off-diagonal elements are non-negative: $m_{rs} \succeq 0$, $r \neq s$.

Lemma 1: [3] System (2) is a positive system if $\mathbf{A}_i(\mathbf{x}(t))$ is a Metzler matrix, $\mathbf{B}_i(\mathbf{x}(t)) \succeq 0$, $\mathbf{B}_{i\omega} \succeq 0$, $\mathbf{D}_i(\mathbf{x}(t)) \succeq 0$, $\mathbf{E}_i(\mathbf{x}(t)) \succeq 0$, $\mathbf{E}_{i\omega} \succeq 0$, and $\mathbf{C} \succeq 0$.

Assumption 1: There exist nonlinear systems with disturbance which can be approximated by the polynomial fuzzy model (2) satisfying the conditions in Lemma 1.

Assumption 2: The disturbance signal $\tilde{\mathbf{w}}(t) \succeq 0$ is bounded.

C. Polynomial Fuzzy Output Feedback Controller

In terms of the imperfect premise matching concept [21], [24], a PFOF controller with c rules is investigated to control the PPFMB system to be asymptotically stable and positive:

Rule j : IF $h_1(\mathbf{y}(t))$ is N_1^j AND ... AND $h_\Omega(\mathbf{y}(t))$ is N_Ω^j

$$\text{THEN } \mathbf{u}(t) = \mathbf{K}_j(\mathbf{y}(t))\mathbf{y}(t), \tag{3}$$

where $\mathbf{K}_j(\mathbf{y}(t)) \in \Re^{m \times l}$ for $j \in \{1, \ldots, c\}$ is the PFOF gain of the j-th rule.

The overall PFOF controller is expressed as follows:

$$\mathbf{u}(t) = \sum_{j=1}^{c} m_j(\mathbf{y}(t))\mathbf{K}_j(\mathbf{y}(t))\mathbf{y}(t), \tag{4}$$

where $m_j(\mathbf{y}(t)) = \frac{\prod_{\beta=1}^{\Omega} \mu_{N_\beta^j}(h_\beta(\mathbf{y}(t)))}{\sum_{k=1}^{c} \prod_{\beta=1}^{\Omega} \mu_{N_\beta^k}(h_\beta(\mathbf{y}(t)))}$ and $m_j(\mathbf{y}(t))$ satisfies that $\sum_{j=1}^{c} m_j(\mathbf{y}(t)) = 1, m_j(\mathbf{y}(t)) \geq 0$, for all j.

Taking the equality $\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$ into (4), one can obtain the PFOF controller as follows:

$$\mathbf{u}(t) = \sum_{j=1}^{c} m_j(\mathbf{y}(t))\mathbf{K}_j(\mathbf{y}(t))\mathbf{C}\mathbf{x}(t). \tag{5}$$

For brevity, the t will be omitted in the following, for instance, $w_i(\mathbf{x})$, $m_j(\mathbf{y})$, $\mathbf{x}$ and $\mathbf{y}$ will replace $w_i(\mathbf{x}(t))$, $m_j(\mathbf{y}(t))$, $\mathbf{x}(t)$ and $\mathbf{y}(t)$, respectively.

D. Positive L_1-Gain Polynomial Fuzzy Control System

Based on the PPFMB system (2) and the PFOF controller (5), the positive L_1-gain PFOF control system is shown as:

$$\begin{cases} \dot{\mathbf{x}} = \sum_{i=1}^{p}\sum_{j=1}^{c} w_i(\mathbf{x})m_j(\mathbf{y})\Big(\tilde{\mathbf{A}}_{ij}(\mathbf{x},\mathbf{y})\mathbf{x} + \mathbf{B}_{i\omega}\tilde{\mathbf{w}}\Big), \\ \mathbf{z} = \sum_{i=1}^{p}\sum_{j=1}^{c} w_i(\mathbf{x})m_j(\mathbf{y})\Big(\tilde{\mathbf{D}}_{ij}(\mathbf{x},\mathbf{y})\mathbf{x} + \mathbf{E}_{i\omega}\tilde{\mathbf{w}}\Big), \\ \mathbf{y} = \mathbf{C}\mathbf{x}, \end{cases} \tag{6}$$

where $0 \le \sum_{i=1}^{p}\sum_{j=1}^{c} w_i(\mathbf{x})m_j(\mathbf{y}) \le 1$. $\tilde{\mathbf{A}}_{ij}(\mathbf{x},\mathbf{y}) = \mathbf{A}_i(\mathbf{x}) + \mathbf{B}_i(\mathbf{x})\mathbf{K}_j(\mathbf{y})\mathbf{C}$, $\tilde{\mathbf{D}}_{ij}(\mathbf{x},\mathbf{y}) = \mathbf{D}_i(\mathbf{x}) + \mathbf{E}_i(\mathbf{x})\mathbf{K}_j(\mathbf{y})\mathbf{C}$.

An augmented dynamic system of the control system (6) will be introduced below to facilitate the stability and positivity analysis. Let $\xi = [\mathbf{x}^T \quad \mathbf{y}^T]^T \in \Re^{n+l}$. According to Definition 1 and Lemma 1, we can find $\xi \succeq 0$ for all $t \ge 0$. By introducing a constant matrix $\mathbf{E} \in \Re^{(n+l)\times(n+l)}$, we have:

$$\mathbf{E}\xi(t) = \begin{bmatrix}\mathbf{I}_n & \mathbf{0} \\ \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{x} \\ \mathbf{y}\end{bmatrix} = \begin{bmatrix}\mathbf{x} \\ \mathbf{0}\end{bmatrix}, \tag{7}$$

where $\mathbf{E} = \begin{bmatrix}\mathbf{I}_n & \mathbf{0} \\ \mathbf{0} & \mathbf{0}\end{bmatrix}$, $\mathbf{I}_n \in \Re^{n\times n}$ is an identity matrix.

According to (6), (7) and $\mathbf{y} = \mathbf{C}\mathbf{x}$, the augmented positive L_1-gain PFOF control system is represented as:

$$\begin{aligned} \mathbf{E}\dot{\xi} &= \begin{bmatrix}\mathbf{I}_n & \mathbf{0} \\ \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\dot{\mathbf{x}} \\ \dot{\mathbf{y}}\end{bmatrix} = \begin{bmatrix}\dot{\mathbf{x}} \\ \mathbf{0}\end{bmatrix} \\ &= \sum_{i=1}^{p}\sum_{j=1}^{c} w_i(\mathbf{x})m_j(\mathbf{y})\Big(\bar{\mathbf{A}}_{ij}(\mathbf{x},\mathbf{y})\xi + \bar{\mathbf{B}}_{i\omega}\tilde{\mathbf{w}}\Big), \end{aligned} \tag{8}$$

where $\bar{\mathbf{A}}_{ij}(\mathbf{x},\mathbf{y}) = \begin{bmatrix}\tilde{\mathbf{A}}_{ij}(\mathbf{x},\mathbf{y}) & \mathbf{0} \\ \mathbf{C} & -\mathbf{I}_l\end{bmatrix}$, $\bar{\mathbf{B}}_{i\omega} = \begin{bmatrix}\mathbf{B}_{i\omega} \\ \mathbf{0}\end{bmatrix}$.

Remark 1: By reviewing Lemma 1, the positivity of the control system (6) can be ensured if $\tilde{\mathbf{A}}_{ij}(\mathbf{x},\mathbf{y})$ is a Metzler matrix, $\tilde{\mathbf{D}}_{ij}(\mathbf{x},\mathbf{y}) \succeq 0$, $\mathbf{B}_{i\omega} \succeq 0$, $\mathbf{E}_{i\omega} \succeq 0$, $\mathbf{C} \succeq 0$.

Definition 3: [38] The positive L_1-gain PFOF control system (6) can satisfy L_1-induced performance at the level γ, if the following inequality can be ensured with satisfying zero initial conditions

$$||\mathbf{z}||_{L_1} < \gamma||\tilde{\mathbf{w}}||_{L_1}, \tag{9}$$

where γ is the optimal level to be determined.

III. Stability Analysis

In this section, the stability and positivity analysis for positive L_1-gain PFOF control system (6) will be proved. In addition, the MFD technique is adopted so that the information of MFs can be captured and used to reduce the conservativeness of the stability and positivity analysis.

A. Basic Stability Analysis of Positive L_1-Gain PFOF Control Systems

Theorem 1: Given a PPFMB system with bounded disturbance (2), the PFOF controller (5) can ensure the positive L_1-gain PFOF control system (6) to be asymptotically stable and positive under L_1-induced performance, if there exist optimal $\gamma > 0$, $\lambda_1 \in \Re^n$, $\lambda_2 \in \Re^l$, output feedback gains $\mathbf{K}_j(\mathbf{y}) \in \Re^{m\times l}$, $j \in \{1,\ldots,c\}$ such that the following SOS-based conditions hold:

$$a_{irs}(\mathbf{x}) + \mathbf{b}_{ir}(\mathbf{x})\mathbf{K}_j(\mathbf{y})\mathbf{c}_s \text{ is SOS } \forall\, r \ne s, i, j; \tag{10}$$

$$d_{irs}(\mathbf{x}) + \mathbf{e}_{ir}(\mathbf{x})\mathbf{K}_j(\mathbf{y})\mathbf{c}_s \text{ is SOS } \forall\, r, s, i, j; \tag{11}$$

$$\rho^T\big(\mathrm{diag}(\lambda_1) - \epsilon_1\mathbf{I}_n\big)\rho \text{ is SOS}; \tag{12}$$

$$\sigma^T\big(\mathrm{diag}(\lambda_2) - \epsilon_2\mathbf{I}_l\big)\sigma \text{ is SOS}; \tag{13}$$

$$-\mathbf{k}_{jr}(\mathbf{y})\mathbf{c}_s \text{ is SOS } \forall\, r, s, j; \tag{14}$$

$$-\upsilon^T\Big(\mathrm{diag}\big(\mathbf{I}_1^T\mathbf{E}_i(\mathbf{x}) - \lambda_1^T\mathbf{B}_i(\mathbf{x})\big) + \epsilon_3(\mathbf{x})\mathbf{I}_m\Big)\upsilon \text{ is SOS } \forall\, i; \tag{15}$$

$$-\rho^T\Big(\mathrm{diag}\big(\mathbf{I}_1^T\mathbf{D}_i(\mathbf{x}) + 2\mathbf{I}_1^T\mathbf{E}_i(\mathbf{x})\mathbf{K}_j(\mathbf{y})\mathbf{C} + \lambda_1^T\mathbf{A}_i(\mathbf{x}) + \lambda_2^T\mathbf{C}\big) + \epsilon_4(\mathbf{x},\mathbf{y})\mathbf{I}_n\Big)\rho \text{ is SOS } \forall\, i, j; \tag{16}$$

$$-\mu^T\Big(\mathrm{diag}\big(\mathbf{I}_1^T\mathbf{E}_{iw} - \gamma\mathbf{I}_2^T + \lambda_1^T\mathbf{B}_{iw}\big) + \epsilon_5\mathbf{I}_h\Big)\mu \text{ is SOS } \forall\, i; \tag{17}$$

where $\epsilon_1 > 0$, $\epsilon_2 > 0$, $\epsilon_5 > 0$ are predefined scalars and $\epsilon_3(\mathbf{x}) > 0$ and $\epsilon_4(\mathbf{x},\mathbf{y}) > 0$ are predefined scalar polynomials; $\rho \in \Re^n$, $\sigma \in \Re^l$, $\upsilon \in \Re^m$ and $\mu \in \Re^h$ are arbitrary vectors independent of $\mathbf{x}$ and $\mathbf{y}$. $\mathbf{k}_{jr}(\mathbf{y}) \in R^{1\times l}$ is the r-th row of the output feedback gain $\mathbf{K}_j(\mathbf{y})$ which can be obtained if the above conditions are satisfied, for all j.

Proof : The Lyapunov function candidate [44] is chosen as:

$$V(t) = \lambda^T(\mathbf{E}\xi) = [\lambda_1^T \quad \lambda_2^T]\begin{bmatrix}\mathbf{x} \\ \mathbf{0}\end{bmatrix} = \lambda_1^T\mathbf{x}, \tag{18}$$

where $\lambda = [\lambda_1^T \quad \lambda_2^T]^T \in \Re^{n+l}$, $0 \prec \lambda_1 \in \Re^n$ and $\lambda_2 \in \Re^l$ are vectors to be determined.

Since $\mathbf{x}$ satisfies $\mathbf{x} \succeq 0$, then from (12), we have $V(t) > 0$.

Based on (8), the derivative of $V(t)$ is obtained as follows:

$$\begin{aligned} \dot{V}(t) &= \lambda^T(\mathbf{E}\dot{\xi}) = \lambda_1^T\dot{\mathbf{x}} \\ &= \sum_{i=1}^{p}\sum_{j=1}^{c} w_i(\mathbf{x})m_j(\mathbf{y})\Big(\lambda^T\bar{\mathbf{A}}_{ij}(\mathbf{x},\mathbf{y})\xi + \lambda^T\bar{\mathbf{B}}_{i\omega}\tilde{\mathbf{w}}\Big). \end{aligned}$$

When $\tilde{\mathbf{w}} = 0$, we have:

$$\dot{V}(t) = \sum_{i=1}^{p}\sum_{j=1}^{c} w_i(\mathbf{x})m_j(\mathbf{y})\Big(\lambda^T\bar{\mathbf{A}}_{ij}(\mathbf{x},\mathbf{y})\xi\Big). \tag{19}$$

If $\lambda^T\bar{\mathbf{A}}_{ij}(\mathbf{x},\mathbf{y}) \prec 0$ holds for all i, j, then $\dot{V}(t) < 0$ can be satisfied. Hence, we have:

$$\begin{aligned} \lambda^T\bar{\mathbf{A}}_{ij}(\mathbf{x},\mathbf{y}) &= [\lambda_1^T \quad \lambda_2^T]\begin{bmatrix}\tilde{\mathbf{A}}_{ij}(\mathbf{x},\mathbf{y}) & \mathbf{0} \\ \mathbf{C} & -\mathbf{I}_l\end{bmatrix} \\ &= \big[\lambda_1^T\tilde{\mathbf{A}}_{ij}(\mathbf{x},\mathbf{y}) + \lambda_2^T\mathbf{C} \quad -\lambda_2^T\big] \\ &= \big[\lambda_1^T(\mathbf{A}_i(\mathbf{x}) + \mathbf{B}_i(\mathbf{x})\mathbf{K}_j(\mathbf{y})\mathbf{C}) + \lambda_2^T\mathbf{C} \quad -\lambda_2^T\big]. \end{aligned} \tag{20}$$

From (11), we have $\mathbf{I}_1^T\mathbf{D}_i(\mathbf{x}) + \mathbf{I}_1^T\mathbf{E}_i(\mathbf{x})\mathbf{K}_j(\mathbf{y})\mathbf{C} \succeq 0$, then combining with (16), $\mathbf{I}_1^T\mathbf{E}_i(\mathbf{x})\mathbf{K}_j(\mathbf{y})\mathbf{C} + \lambda_1^T\mathbf{A}_i(\mathbf{x}) + \lambda_2^T\mathbf{C} \prec 0$ is obtained. From (14) and (15), we have:

$$\lambda_1^T\mathbf{B}_i(\mathbf{x})\mathbf{K}_j(\mathbf{y})\mathbf{C} \preceq \mathbf{I}_1^T\mathbf{E}_i(\mathbf{x})\mathbf{K}_j(\mathbf{y})\mathbf{C}. \tag{21}$$

From (21), the first term on the right hand side of (20) can be derived as follows:

$$\begin{aligned}&\lambda_1^T(\mathbf{A}_i(\mathbf{x})+\mathbf{B}_i(\mathbf{x})\mathbf{K}_j(\mathbf{y})\mathbf{C})+\lambda_2^T\mathbf{C}\\ \preceq&\mathbf{I}_1^T\mathbf{E}_i(\mathbf{x})\mathbf{K}_j(\mathbf{y})\mathbf{C}+\lambda_1^T\mathbf{A}_i(\mathbf{x})+\lambda_2^T\mathbf{C}\prec 0.\end{aligned}\tag{22}$$

Then combining with (13), $\lambda^T\bar{\mathbf{A}}_{ij}(\mathbf{x},\mathbf{y})\prec 0$ can be proved, which means $\dot{V}(t)<0$ is satisfied. Because both of $V(t)>0$ and $\dot{V}(t)<0$ are satisfied, hence, the system (6) is asymptotic stable when $\tilde{\mathbf{w}}(t)=0$.

Meanwhile, from (10) and (11), the positivity of the control system (6) is ensured when $\tilde{\mathbf{w}}(t)=0$.

When $\tilde{\mathbf{w}}(t)\neq 0$, we have:

$$\begin{aligned}&||\mathbf{z}||_{L_1}-\gamma||\tilde{\mathbf{w}}||_{L_1}+\dot{V}\\ =&\ \mathbf{I}_1^T\mathbf{z}-\gamma\mathbf{I}_2^T\tilde{\mathbf{w}}+\dot{V}\\ =&\ \mathbf{I}_1^T\Big(\sum_{i=1}^{p}\sum_{j=1}^{c}w_i(\mathbf{x})m_j(\mathbf{y})\big(\tilde{\mathbf{D}}_{ij}(\mathbf{x},\mathbf{y})\mathbf{x}+\mathbf{E}_{iw}\tilde{\mathbf{w}}\big)\Big)\\ -&\ \gamma\mathbf{I}_2^T\tilde{\mathbf{w}}+\sum_{i=1}^{p}\sum_{j=1}^{c}w_i(\mathbf{x})m_j(\mathbf{y})\lambda^T\left(\bar{\mathbf{A}}_{ij}(\mathbf{x},\mathbf{y})\xi+\bar{\mathbf{B}}_{iw}\tilde{\mathbf{w}}\right)\\ =&\ \sum_{i=1}^{p}\sum_{j=1}^{c}w_i(\mathbf{x})m_j(\mathbf{y})\Big(\big(\mathbf{I}_1^T\mathbf{E}_{iw}-\gamma\mathbf{I}_2^T+\lambda^T\bar{\mathbf{B}}_{iw}\big)\tilde{\mathbf{w}}\\ +&\ \big(\mathbf{I}_1^T\tilde{\mathbf{D}}_{ij}(\mathbf{x},\mathbf{y})[\mathbf{I}_n\quad\mathbf{0}]+\lambda^T\bar{\mathbf{A}}_{ij}(\mathbf{x},\mathbf{y})\big)\xi\Big),\end{aligned}\tag{23}$$

where $\mathbf{I}_1\in\Re^q$ and $\mathbf{I}_2\in\Re^h$ are vectors with all of the elements being 1.

Firstly, according to (17), the first term on the right hand side of (23) satisfies the following expression:

$$\mathbf{I}_1^T\mathbf{E}_{iw}-\gamma\mathbf{I}_2^T+\lambda^T\bar{\mathbf{B}}_{iw}=\mathbf{I}_1^T\mathbf{E}_{iw}-\gamma\mathbf{I}_2^T+\lambda_1^T\mathbf{B}_{iw}\prec 0.$$

Then, the second term on the right hand side of (23) can be processed as follows:

$$\begin{aligned}&\mathbf{I}_1^T\tilde{\mathbf{D}}_{ij}(\mathbf{x},\mathbf{y})[\mathbf{I}_n\quad\mathbf{0}]+\lambda^T\bar{\mathbf{A}}_{ij}(\mathbf{x},\mathbf{y})\\ =&\mathbf{I}_1^T\tilde{\mathbf{D}}_{ij}(\mathbf{x},\mathbf{y})[\mathbf{I}_n\quad\mathbf{0}]+\left[\lambda_1^T\quad\lambda_2^T\right]\begin{bmatrix}\tilde{\mathbf{A}}_{ij}(\mathbf{x},\mathbf{y}) & \mathbf{0}\\ \mathbf{C} & -\mathbf{I}_l\end{bmatrix}\\ =&\left[\mathbf{I}_1^T\tilde{\mathbf{D}}_{ij}(\mathbf{x},\mathbf{y})+\lambda_1^T\tilde{\mathbf{A}}_{ij}(\mathbf{x},\mathbf{y})+\lambda_2^T\mathbf{C}\quad -\lambda_2^T\right],\end{aligned}\tag{24}$$

where $\tilde{\mathbf{A}}_{ij}(\mathbf{x},\mathbf{y})$ and $\tilde{\mathbf{D}}_{ij}(\mathbf{x},\mathbf{y})$ can be found in (6).

Next, the first term on the right hand side of (24) can be dealt with as follows:

$$\begin{aligned}&\mathbf{I}_1^T\tilde{\mathbf{D}}_{ij}(\mathbf{x},\mathbf{y})+\lambda_1^T\tilde{\mathbf{A}}_{ij}(\mathbf{x},\mathbf{y})+\lambda_2^T\mathbf{C}\\ =&\mathbf{I}_1^T(\mathbf{D}_i(\mathbf{x})+\mathbf{E}_i(\mathbf{x})\mathbf{K}_j(\mathbf{y})\mathbf{C})\\ +&\lambda_1^T(\mathbf{A}_i(\mathbf{x})+\mathbf{B}_i(\mathbf{x})\mathbf{K}_j(\mathbf{y})\mathbf{C})+\lambda_2^T\mathbf{C}.\end{aligned}\tag{25}$$

It can be seen that $\lambda_1^T\mathbf{B}_i(\mathbf{x})\mathbf{K}_j(\mathbf{y})\mathbf{C}$ in (25) is non-convex, which is an obstacle to develop the stability and positivity analysis. Although [36], [37] offered some methods to solve this problem, unfortunately, these methods cannot be used in this paper because the term $\mathbf{I}_1^T\mathbf{E}_i(\mathbf{x})\mathbf{K}_j(\mathbf{y})\mathbf{C}$ in (25) is an additional non-convex term when these methods are adopted. Thereby, the constrain conditions (14) and (15) is introduced to approximate the non-convex term by convex one.

By introducing the constrain conditions (14) and (15), (21) is satisfied, then (25) can be derived as follows:

$$\begin{aligned}&\mathbf{I}_1^T(\mathbf{D}_i(\mathbf{x})+\mathbf{E}_i(\mathbf{x})\mathbf{K}_j(\mathbf{y})\mathbf{C})\\ +&\lambda_1^T(\mathbf{A}_i(\mathbf{x})+\mathbf{B}_i(\mathbf{x})\mathbf{K}_j(\mathbf{y})\mathbf{C})+\lambda_2^T\mathbf{C}\\ \preceq&\mathbf{I}_1^T\mathbf{D}_i(\mathbf{x})+2\mathbf{I}_1^T\mathbf{E}_i(\mathbf{x})\mathbf{K}_j(\mathbf{y})\mathbf{C}+\lambda_1^T\mathbf{A}_i(\mathbf{x})+\lambda_2^T\mathbf{C}.\end{aligned}$$

Therefore, from (16), the first term on the right hand side of (24) satisfies:

$$\mathbf{I}_1^T\tilde{\mathbf{D}}_{ij}(\mathbf{x},\mathbf{y})+\lambda_1^T\tilde{\mathbf{A}}_{ij}(\mathbf{x},\mathbf{y})+\lambda_2^T\mathbf{C}\prec 0.$$

Combining with (13), (24) satisfies:

$$\mathbf{I}_1^T\tilde{\mathbf{D}}_{ij}(\mathbf{x},\mathbf{y})[\mathbf{I}_n\quad\mathbf{0}]+\lambda^T\bar{\mathbf{A}}_{ij}(\mathbf{x},\mathbf{y})\prec 0.$$

Due to $w_i(\mathbf{x})m_j(\mathbf{y})\geq 0$, $\tilde{\mathbf{w}}\succeq 0$ and $\xi\succeq 0$, from (23), if there exist optimal $\gamma>0$, λ_2, output feedback gains $\mathbf{K}_j(\mathbf{y})$ such that the (13)-(17) are feasible, we can obtain:

$$||\mathbf{z}||_{L_1}-\gamma||\tilde{\mathbf{w}}||_{L_1}+\dot{V}\prec 0.\tag{26}$$

For any $T>0$, integrating (26) on t from 0 to T, we have

$$\begin{aligned}&\int_0^T||\mathbf{z}||_{L_1}-\gamma||\tilde{\mathbf{w}}||_{L_1}+\dot{V}d_t\\ =&\int_0^T\mathbf{I}_1^T\mathbf{z}-\gamma\mathbf{I}_2^T\tilde{\mathbf{w}}+\dot{V}d_t\\ =&\int_0^T\mathbf{I}_1^T\mathbf{z}-\gamma\mathbf{I}_2^T\tilde{\mathbf{w}}d_t+V(T)-V(0)<0.\end{aligned}\tag{27}$$

Under zero initial condition, we have $V(0)=0$. Meanwhile, $V(T)\to 0$ when $T\to\infty$. Hence, $\int_0^\infty\mathbf{I}_1^T\mathbf{z}-\gamma\mathbf{I}_2^T\tilde{\mathbf{w}}d_t<0$ is obtained, which means $\mathbf{I}_1^T\mathbf{z}<\gamma\mathbf{I}_2^T\tilde{\mathbf{w}}$.

Since the PPFMB system (2) is a positive system, therefore, $\mathbf{B}_{iw}\succeq 0$ and $\mathbf{E}_{iw}\succeq 0$. Then from (10) and (11), the positivity of the positive L_1-gain PFOF control system (6) is also ensured when $\tilde{\mathbf{w}}(t)\neq 0$.

In terms of the above analysis, the stability and positivity for the positive L_1-gain PFOF control system (6) has been proved.

Remark 2: In Theorem 1, (10)-(11) are positivity conditions which are used to guarantee the positivity of the positive L_1-gain PFOF control system (6). The rest conditions (12)-(17) are stability conditions which are used to guarantee the stability under L_1-induced performance.

Remark 3: Although the convex method given in our paper will introduce a certain degree of conservativeness, this method makes it easier to transform the non-convex stability conditions and positivity conditions into convex ones simultaneously, which means the non-convex problem can be dealt with skillfully. In order to alleviate the conservativeness, in the following, a high degree polynomial approximation method will be introduced so that the information embedded in MFs can be found and introduced into stability conditions.

B. Stability Analysis of the Positive L_1-Gain PFOF Control System via High Degree Polynomial Approximation Method

Based on (23), we can see that the cross term of MFs $w_i(\mathbf{x})m_j(\mathbf{y})$ will also influence the stability analysis, but they are ignored in the above analysis process. Thereby, the following task is to cope with the cross terms of MFs by using the high degree polynomial approximation method so that enough information of MFs can be mined.

To get a better approximation, the whole operating domain is divided into D subdomains. In each subdomain, the cross terms of MFs are approximated by high degree polynomial functions. Meanwhile, because of $\mathbf{y} = \mathbf{C}\mathbf{x}$, hence, the MFs $m_j(\mathbf{y})$ can be viewed as a function of $\mathbf{x}$. Similarly, the output feedback gain $\mathbf{K}_j(\mathbf{y})$ also can be viewed as a function of $\mathbf{x}$. Then we can get the following equality:

$$w_i(\mathbf{x})m_j(\mathbf{x}) = \eta_{ij,d}(\mathbf{x}) + \Delta\eta_{ij,d}(\mathbf{x}), \forall\, i, j, d,$$

where d means the d-th subdomain, $d \in \{1, \ldots, D\}$. $\eta_{ij,d}(\mathbf{x})$ denote the high degree approximated polynomial functions and $\Delta\eta_{ij,d}(\mathbf{x})$ denote the approximated errors with satisfying $\underline{\beta}_{ij,d} \preceq \Delta\eta_{ij,d}(\mathbf{x}) \preceq \overline{\beta}_{ij,d}$ where $\underline{\beta}_{ij,d}$ and $\overline{\beta}_{ij,d}$ are the lower bound and the upper bound of $\Delta\eta_{ij,d}(\mathbf{x})$.

In the previous work [36], we tend to increase more subdomains instead of to increase higher degrees of approximated polynomial functions to reduce the approximated errors. However, when the number of the subdomains is more, the computational burden of computers will be heavier. Hence, to overcome this difficulty, we will try to employ higher degrees of approximated polynomial functions to relax the stability conditions so that a small number of subdomains can achieve a good relaxation effect. In addition, through this method, the boundary information of the state variables can be introduced in a different way. Now, the detailed steps and explanations will be shown in the following.

For brevity, we define $w_i(\mathbf{x})m_j(\mathbf{x}) = \Xi_{ij,d}(\mathbf{x})$ for d-th subdomain. Then considering that the stability and positivity conditions should be SOS-based form, thereby, the approximated polynomial functions of the original MFs should be of even degrees so that the free third-party MATLAB toolbox, i.e., SOSTOOLS [45], can be used to find feasible solutions.

To further relax the analysis, the boundary information of the state variables will be brought into the approximated polynomial functions through the following fractional functions:

$$\alpha_d(\mathbf{x}) = \frac{f}{g(\mathbf{x} - \mathbf{x}_{d_{min}})(\mathbf{x}_{d_{max}} - \mathbf{x}) + s}, \forall\, d, \qquad (28)$$

where $f > 0$, $g > 0$ and $s > 0$ are predefined scalars. $\mathbf{x}_{d_{min}}$ and $\mathbf{x}_{d_{max}}$ are the lower bound and upper bound of the state variables $\mathbf{x}$ in each subdomain. It can be seen that $\alpha_d(\mathbf{x}) > 0$ when $\mathbf{x}$ belong to the d-th subdomain. $\alpha_{d,num} = f$ and $\alpha_{d,den}(\mathbf{x}) = g(\mathbf{x} - \mathbf{x}_{d_{min}})(\mathbf{x}_{d_{max}} - \mathbf{x}) + s$ represent the numerator and denominator of $\alpha_d(\mathbf{x})$, respectively.

Based on the above analysis, for guaranteeing the stability conditions with approximated polynomial functions to be SOS-based form, the function $\Xi^{0.5}_{ij,d}(\mathbf{x}) = \sqrt{\frac{\Xi_{ij,d}(\mathbf{x})}{\alpha_d(\mathbf{x})}}$ which is the square root of $\frac{\Xi_{ij,d}(\mathbf{x})}{\alpha_d(\mathbf{x})}$ requires to be used.

To make the design process easier to understand and follow, the detailed steps of the high degree polynomial approximation method are shown as follows:

Step 1: Calculate $\Xi^{0.5}_{ij,d}(\mathbf{x})$;

Step 2: Employ a polynomial fitting approach to get the approximation function of $\Xi^{0.5}_{ij,d}(\mathbf{x})$, which is defined as $\eta^{0.5}_{ij,d}(\mathbf{x})$;

Step 3: Obtain the approximated polynomial functions of original MFs as $\eta_{ij,d}(\mathbf{x}) = (\eta^{0.5}_{ij,d}(\mathbf{x}))^2\alpha_d(\mathbf{x})$.

Step 4: Calculate the approximated error $\Delta\eta_{ij,d}(\mathbf{x}) = \Xi_{ij,d}(\mathbf{x}) - \eta_{ij,d}(\mathbf{x})$ as well as the lower bound $\underline{\beta}_{ij,d}$ and upper bound $\overline{\beta}_{ij,d}$ of the approximated error $\Delta\eta_{ij,d}(\mathbf{x})$.

Remark 4: The information of MFs is mainly introduced into the stability condition (16) because the output feedback gain $\mathbf{K}_j(\mathbf{x})$, decision variables λ_1 and λ_2 exist in this condition simultaneously.

Theorem 2: Given a PPFMB system with bounded disturbance (2), the PFOF controller (5) can ensure the positive L_1-gain PFOF control system (6) to be asymptotically stable and positive under L_1-induced performance, if there exist optimal $\gamma > 0$, vectors $\lambda_1 \in \Re^n$, $\lambda_2 \in \Re^l$, output feedback gains $\mathbf{K}_j(\mathbf{x}) \in \Re^{m\times l}$, and slack matrices $\mathbf{Y}_{ij,d}(\mathbf{x}) \in \Re^n$, for all $i \in \{1, \ldots, p\}$, $j \in \{1, \ldots, c\}$, $d \in \{1, \ldots, D\}$ such that the following SOS-based conditions hold:

$$a_{irs}(\mathbf{x}) + \mathbf{b}_{ir}(\mathbf{x})\mathbf{K}_j(\mathbf{x})\mathbf{c}_s \text{ is SOS } \forall\, r \neq s, i, j; \qquad (29)$$

$$d_{irs}(\mathbf{x}) + \mathbf{e}_{ir}(\mathbf{x})\mathbf{K}_j(\mathbf{x})\mathbf{c}_s \text{ is SOS } \forall\, r, s, i, j; \qquad (30)$$

$$\rho^T\big(\mathrm{diag}(\lambda_1) - \epsilon_1\mathbf{I}_n\big)\rho \text{ is SOS}; \qquad (31)$$

$$\sigma^T\big(\mathrm{diag}(\lambda_2) - \epsilon_2\mathbf{I}_l\big)\sigma \text{ is SOS}; \qquad (32)$$

$$\rho^T\Big(\mathrm{diag}\big(\mathbf{Y}_{ij,d}(\mathbf{x})\big)\Big)\rho \text{ is SOS } \forall\, i, j, d; \qquad (33)$$

$$\rho^T\Big(\mathrm{diag}\big(\mathbf{Y}_{ij,d}(\mathbf{x}) - \mathbf{H}_{ij}(\mathbf{x})\big)\Big)\rho \text{ is SOS } \forall\, i, j, d; \qquad (34)$$

$$-\,\mathbf{k}_{jr}(\mathbf{x})\mathbf{c}_s \text{ is SOS } \forall\, r, s, j; \qquad (35)$$

$$-\,\upsilon^T\Big(\mathrm{diag}\big(\mathbf{I}_1^T\mathbf{E}_i(\mathbf{x}) - \lambda_1^T\mathbf{B}_i(\mathbf{x})\big) + \epsilon_3(\mathbf{x})\mathbf{I}_m\Big)\upsilon \text{ is SOS } \forall\, i; \qquad (36)$$

$$-\,\mu^T\Big(\mathrm{diag}\big(\mathbf{I}_1^T\mathbf{E}_{iw} - \gamma\mathbf{I}_2^T + \lambda_1^T\mathbf{B}_{iw}\big) + \epsilon_4\mathbf{I}_h\Big)\mu \text{ is SOS } \forall\, i; \qquad (37)$$

$$-\,\rho^T\Big(\mathrm{diag}\Big(\sum_{i=1}^{p}\sum_{j=1}^{c}\Big(\big((\eta^{0.5}_{ij,d}(\mathbf{x}))^2\alpha_{d,num} + \underline{\beta}_{ij,d}\alpha_{d,den}(\mathbf{x})\big) \times \mathbf{H}_{ij}(\mathbf{x}) + \big(\overline{\beta}_{ij,d} - \underline{\beta}_{ij,d}\big)\alpha_{d,den}(\mathbf{x})\mathbf{Y}_{ij,d}(\mathbf{x})\Big)\Big) + \epsilon_5(\mathbf{x})\mathbf{I}_n\Big)\rho \text{ is SOS } \forall\, d; \qquad (38)$$

where $\epsilon_1 > 0$, $\epsilon_2 > 0$, $\epsilon_4 > 0$ are predefined scalars and $\epsilon_3(\mathbf{x}) > 0$ and $\epsilon_5(\mathbf{x}) > 0$ are predefined scalar polynomials; $\rho \in \Re^n$, $\sigma \in \Re^l$, $\upsilon \in \Re^m$ and $\mu \in \Re^h$ are arbitrary vectors independent of $\mathbf{x}$. $\mathbf{k}_{jr}(\mathbf{x}) \in R^{1\times l}$ is the r-th row of the output feedback gain $\mathbf{K}_j(\mathbf{x})$ which can be obtained if the above conditions are satisfied, for all j. $\mathbf{H}_{ij}(\mathbf{x}) = \mathbf{I}_1^T\mathbf{D}_i(\mathbf{x}) + 2\mathbf{I}_1^T\mathbf{E}_i(\mathbf{x})\mathbf{K}_j(\mathbf{x})\mathbf{C} + \lambda_1^T\mathbf{A}_i(\mathbf{x}) + \lambda_2^T\mathbf{C}$.

Proof : The Lyapunov function is chosen as (18). Since $\mathbf{x} \succeq 0$, then from (31), we have $V(t) > 0$.

When $\tilde{\mathbf{w}} = 0$, (19) is obtained, which means if $\sum_{i=1}^{p}\sum_{j=1}^{c} w_i(\mathbf{x})m_j(\mathbf{y})\lambda^T\bar{\mathbf{A}}_{ij}(\mathbf{x},\mathbf{y}) \prec 0$ holds for all i, j, then $\dot{V}(t) < 0$ can be satisfied. From $\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$ in (2), it can be seen that $\mathbf{y}$ is a function of $\mathbf{x}$, thereby, $\mathbf{K}_j(\mathbf{y})$ and

$m_j(\mathbf{y})$ can be also seen as the functions of $\mathbf{x}$. Then, we have:

$$\sum_{i=1}^{p}\sum_{j=1}^{c} w_i(\mathbf{x})m_j(\mathbf{x})\lambda^T\bar{\mathbf{A}}_{ij}(\mathbf{x}) = \sum_{i=1}^{p}\sum_{j=1}^{c} w_i(\mathbf{x})m_j(\mathbf{x}) \times \left[\lambda_1^T(\mathbf{A}_i(\mathbf{x})+\mathbf{B}_i(\mathbf{x})\mathbf{K}_j(\mathbf{x})\mathbf{C})+\lambda_2^T\mathbf{C} \quad -\lambda_2^T\right]. \tag{39}$$

In the following, the high degree polynomial approximation method is employed to deal with the information of membership functions. According to (33) and (34), we have:

$$\begin{aligned}
&\sum_{i=1}^{p}\sum_{j=1}^{c}\sum_{i=1}^{p}\sum_{j=1}^{c} w_i(\mathbf{x})m_j(\mathbf{x})\mathbf{H}_{ij}(\mathbf{x}) \\
&=\sum_{i=1}^{p}\sum_{j=1}^{c}\big(\eta_{ij,d}(\mathbf{x})+\Delta\eta_{ij,d}(\mathbf{x})+\underline{\beta}_{ij,d}-\underline{\beta}_{ij,d}\big)\mathbf{H}_{ij}(\mathbf{x}) \\
&=\sum_{i=1}^{p}\sum_{j=1}^{c}\Big(\big(\eta_{ij,d}(\mathbf{x})+\underline{\beta}_{ij,d}\big)\mathbf{H}_{ij}(\mathbf{x}) \\
&+\big(\Delta\eta_{ij,d}(\mathbf{x})-\underline{\beta}_{ij,d}\big)\mathbf{H}_{ij}(\mathbf{x})\Big) \\
&\preceq\sum_{i=1}^{p}\sum_{j=1}^{c}\Big(\big(\eta_{ij,d}(\mathbf{x})+\underline{\beta}_{ij,d}\big)\mathbf{H}_{ij}(\mathbf{x}) \\
&+\big(\overline{\beta}_{ij,d}-\underline{\beta}_{ij,d}\big)\mathbf{Y}_{ij,d}(\mathbf{x})\Big)\ \forall\, d,
\end{aligned} \tag{40}$$

where $\underline{\beta}_{ij,d}$ and $\overline{\beta}_{ij,d}$ are the lower and upper bound of $\Delta\eta_{ij,d}(\mathbf{x})$, respectively.

Based on the above step 3, $\eta_{ij,d}(\mathbf{x})=(\eta_{ij,d}^{0.5}(\mathbf{x}))^2\alpha_d(\mathbf{x})$ is defined, then by taking it into (40), we have:

$$\begin{aligned}
&\sum_{i=1}^{p}\sum_{j=1}^{c}\Big(\big(\eta_{ij,d}(\mathbf{x})+\underline{\beta}_{ij,d}\big)\mathbf{H}_{ij}(\mathbf{x})+\big(\overline{\beta}_{ij,d}-\underline{\beta}_{ij,d}\big)\mathbf{Y}_{ij,d}(\mathbf{x})\Big) \\
&=\sum_{i=1}^{p}\sum_{j=1}^{c}\Big(\big((\eta_{ij,d}^{0.5}(\mathbf{x}))^2\alpha_d(\mathbf{x})+\underline{\beta}_{ij,d}\big)\mathbf{H}_{ij}(\mathbf{x}) \\
&+\big(\overline{\beta}_{ij,d}-\underline{\beta}_{ij,d}\big)\mathbf{Y}_{ij,d}(\mathbf{x})\Big)\ \forall\, d.
\end{aligned} \tag{41}$$

Since $\alpha_d(\mathbf{x})$ is a fractional function with a polynomial in the denominator, it is hard to achieve feasible solutions using SOSTOOLS. To solve this problem, we first multiply both sides of (41) by the denominator $\alpha_{d,den}(\mathbf{x})$ of $\alpha_d(\mathbf{x})$. Then (41) will be derived as follows:

$$\begin{aligned}
&\sum_{i=1}^{p}\sum_{j=1}^{c}\Big(\big((\eta_{ij,d}^{0.5}(\mathbf{x}))^2\alpha_{d,num}+\underline{\beta}_{ij,d}\alpha_{d,den}(\mathbf{x})\big)\mathbf{H}_{ij}(\mathbf{x}) \\
&+\big(\overline{\beta}_{ij,d}-\underline{\beta}_{ij,d}\big)\alpha_{d,den}(\mathbf{x})\mathbf{Y}_{ij,d}(\mathbf{x})\Big)\ \forall\, d.
\end{aligned} \tag{42}$$

Thereby, from (38), $\sum_{i=1}^{p}\sum_{j=1}^{c} w_i(\mathbf{x})m_j(\mathbf{x})\mathbf{H}_{ij}(\mathbf{x})\prec 0$ is obtained. From (30), $\mathbf{I}_1^T\mathbf{D}_i(\mathbf{x})+\mathbf{I}_1^T\mathbf{E}_i(\mathbf{x})\mathbf{K}_j(\mathbf{x})\mathbf{C}\succeq 0$ is obtained. Then, in terms of the two inequalities, we have $\sum_{i=1}^{p}\sum_{j=1}^{c} w_i(\mathbf{x})m_j(\mathbf{y})\big(\mathbf{I}_1^T\mathbf{E}_i(\mathbf{x})\mathbf{K}_j(\mathbf{x})\mathbf{C}+\lambda_1^T\mathbf{A}_i(\mathbf{x})+\lambda_2^T\mathbf{C}\big)\prec 0$. By combining (35) with (36), (21) can be obtained. From (21), the inequality (22) with MFs $w_i(\mathbf{x})m_j(\mathbf{x})$ can be obtained. Hence, combining with (32), $\sum_{i=1}^{p}\sum_{j=1}^{c} w_i(\mathbf{x})m_j(\mathbf{y})\lambda^T\bar{\mathbf{A}}_{ij}(\mathbf{x},\mathbf{y})\prec 0$ can be proved, which means $\dot{V}(t)<0$. Therefore, when $\tilde{\mathbf{w}}(t)=0$, the positive L_1-gain PFOF control system (6) is asymptotic stable.

From (29) and (30), the positivity of the positive L_1-gain PFOF control system (6) is ensured when $\tilde{\mathbf{w}}(t)=0$.

When $\tilde{\mathbf{w}}(t)\neq 0$, we can obtain (23). Firstly, according to (37), $\mathbf{I}_1^T\mathbf{E}_{iw}-\gamma\mathbf{I}_2^T+\lambda^T\bar{\mathbf{B}}_{iw}\prec 0$ is ensured.

Then, $\sum_{i=1}^{p}\sum_{j=1}^{c} w_i(\mathbf{x})m_j(\mathbf{x})\big(\mathbf{I}_1^T\tilde{\mathbf{D}}_{ij}(\mathbf{x},\mathbf{y})[\mathbf{I}_n \quad \mathbf{0}]+\lambda^T\bar{\mathbf{A}}_{ij}(\mathbf{x},\mathbf{y})\big)$ in (23) can be further processed as follows:

$$\begin{aligned}
&\sum_{i=1}^{p}\sum_{j=1}^{c} w_i(\mathbf{x})m_j(\mathbf{x})\big(\mathbf{I}_1^T\tilde{\mathbf{D}}_{ij}(\mathbf{x},\mathbf{y})[\mathbf{I}_n \quad \mathbf{0}]+\lambda^T\bar{\mathbf{A}}_{ij}(\mathbf{x},\mathbf{y})\big) \\
&=\sum_{i=1}^{p}\sum_{j=1}^{c} w_i(\mathbf{x})m_j(\mathbf{x}) \\
&\times\left[\mathbf{I}_1^T\tilde{\mathbf{D}}_{ij}(\mathbf{x},\mathbf{y})+\lambda_1^T\tilde{\mathbf{A}}_{ij}(\mathbf{x},\mathbf{y})+\lambda_2^T\mathbf{C} \quad -\lambda_2^T\right],
\end{aligned} \tag{43}$$

where $\tilde{\mathbf{A}}_{ij}(\mathbf{x},\mathbf{y})$ and $\tilde{\mathbf{D}}_{ij}(\mathbf{x},\mathbf{y})$ can be found in (6).

Through using the same method in Theorem 1 to solve the non-convex problem, the first term in (43) with MFs $w_i(\mathbf{x})m_j(\mathbf{x})$ is derived as $\sum_{i=1}^{p}\sum_{j=1}^{c} w_i(\mathbf{x})m_j(\mathbf{x})\mathbf{H}_{ij}(\mathbf{x})$ which satisfies the inequalities (40)-(42).

From (38), $\sum_{i=1}^{p}\sum_{j=1}^{c} w_i(\mathbf{x})m_j(\mathbf{x})\mathbf{H}_{ij}(\mathbf{x})\prec 0$ is obtained, which implies the first term on the right hand side of (43) with MFs satisfies:

$$\sum_{i=1}^{p}\sum_{j=1}^{c} w_i(\mathbf{x})m_j(\mathbf{x})(\mathbf{I}_1^T\tilde{\mathbf{D}}_{ij}(\mathbf{x},\mathbf{y})+\lambda_1^T\tilde{\mathbf{A}}_{ij}(\mathbf{x},\mathbf{y})+\lambda_2^T\mathbf{C})\prec 0.$$

Thereby, combining (31), we have:

$$\sum_{i=1}^{p}\sum_{j=1}^{c} w_i(\mathbf{x})m_j(\mathbf{x})\big(\mathbf{I}_1^T\tilde{\mathbf{D}}_{ij}(\mathbf{x},\mathbf{y})[\mathbf{I}_n \quad \mathbf{0}]+\lambda^T\bar{\mathbf{A}}_{ij}(\mathbf{x},\mathbf{y})\big)\prec 0.$$

Due to $w_i(\mathbf{x})m_j(\mathbf{y})\geq 0$, $\tilde{\mathbf{w}}\succeq 0$ and $\xi\succeq 0$, thereby, from (23), if there exist optimal $\gamma>0$, λ_2, output feedback gains $\mathbf{K}_j(\mathbf{x})$ and slack matrix $\mathbf{Y}_{ij,d}(\mathbf{x})$ such that (32)-(38) are feasible, then (26) is obtained. The following derivation follows the same logic in Theorem 1, and is thus omitted.

As for the positivity, it also follows the same logic in Theorem 1. Hence, it is omitted.

The proof is done.

Remark 5: To reduce the number of the slack matrices, $\mathbf{Y}_{ij}(\mathbf{x})=\mathbf{Y}_{ij,d}(\mathbf{x})$ is considered, which means $\mathbf{Y}_{ij,d}(\mathbf{x})$ is the same matrix for all d.

Remark 6: In Theorem 2, (29)-(30) are positivity conditions that contribute to control the positivity of the positive L_1-gain PFOF control system (6). The rest conditions (31)-(38) are stability conditions which are used to guarantee the relaxed stability under L_1-induced performance.

To facilitate the understanding, a summary of the PFOF controller design and stability analysis is given: 1). Represent an unstable open-loop positive nonlinear system with bounded disturbance by a set of polynomial fuzzy subsystems which are combined by the MFs $w_i(\mathbf{x})$. 2). Design a set of polynomial fuzzy subcontrollers which are combined by the MFs $m_j(\mathbf{y})$. 3). Divide the entire operation domain into D subdomains and choose the fraction $\alpha_d(\mathbf{x})$ as (28). 4). In accordance of the steps 1-4 of the high degree polynomial approximation method, calculate the approximated polynomial function $\eta_{ij,d}(\mathbf{x})$ and the approximated error $\Delta\eta_{ij,d}(\mathbf{x})$ in each subdomain. Furthermore, obtain the lower bound $\underline{\beta}_{ij,d}$ and the upper bound $\overline{\beta}_{ij,d}$ of $\Delta\eta_{ij,d}(\mathbf{x})$ in each subdomain. 5). Set up the parameters, and acquire the feasible solutions through the SOSTOOLS based on the corresponding theorems.

IV. Simulation Example

In this section, the effectiveness and superiority of the theorems proposed in this paper will be verified from the following four aspects: 1) How the stability regions vary with the values of D. 2) How the stability regions change with the highest degrees of approximated polynomial function $\eta^{0.5}_{ij,d}(\mathbf{x})$. 3) How the stability regions are affected by the $\alpha_d(\mathbf{x})$. 4) Compare the stability regions obtained using the method with the ones obtained using other methods.

A. Scenario

A positive polynomial fuzzy model is shown as follows:

$$\mathbf{A}_1(x_1) = \begin{bmatrix} -0.11+0.02a & 0.34 \\ 0.95 & -1.05-x_1^2 \end{bmatrix},$$
$$\mathbf{A}_2(x_1) = \begin{bmatrix} 0.12 & 0.35+0.12x_1^2 \\ 0.88 & -1.28-x_1^2 \end{bmatrix},$$
$$\mathbf{A}_3(x_1) = \begin{bmatrix} 0.15 & 0.26+0.14x_1^2 \\ 0.68 & -1.02-x_1^2 \end{bmatrix},$$
$$\mathbf{B}_1(x_1) = \begin{bmatrix} 0.51+1.73x_1^2+0.1b \\ 1.18 \end{bmatrix},$$
$$\mathbf{B}_2(x_1) = \begin{bmatrix} 2.13+2.25x_1^2+0.1b \\ 1.24 \end{bmatrix},$$
$$\mathbf{B}_3(x_1) = \begin{bmatrix} 1.65+1.39x_1^2+0.1b \\ 1.15 \end{bmatrix},$$
$$\mathbf{D}_1(x_1) = \begin{bmatrix} 0.15+0.13x_1^2 & 0.12+0.01x_1^2 \end{bmatrix},$$
$$\mathbf{D}_2(x_1) = \begin{bmatrix} 0.16+0.12x_1^2 & 0.16+0.01x_1^2 \end{bmatrix},$$
$$\mathbf{D}_3(x_1) = \begin{bmatrix} 0.14+0.1x_1^2 & 0.12+0.01x_1^2 \end{bmatrix},$$
$$\mathbf{E}_1(x_1) = \begin{bmatrix} 0.26+0.12x_1^2 \end{bmatrix}, \mathbf{E}_2(x_1) = \begin{bmatrix} 0.28+0.11x_1^2 \end{bmatrix},$$
$$\mathbf{E}_3(x_1) = \begin{bmatrix} 0.24+0.10x_1^2 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix},$$
$$\mathbf{B}_{w1} = \begin{bmatrix} 0.01 \\ 0.01 \end{bmatrix}, \mathbf{B}_{w2} = \begin{bmatrix} 0.02 \\ 0.02 \end{bmatrix}, \mathbf{B}_{w3} = \begin{bmatrix} 0.01 \\ 0.01 \end{bmatrix},$$
$$\mathbf{E}_{w1} = \begin{bmatrix} 1.36 \end{bmatrix}, \mathbf{E}_{w2} = \begin{bmatrix} 1.29 \end{bmatrix}, \mathbf{E}_{w3} = \begin{bmatrix} 1.14 \end{bmatrix},$$
$$\mathbf{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T, \tilde{\mathbf{w}} = \kappa e^{-t}|\cos(2t)|$$

where a and b are constant scalars. The coefficients of the disturbance are chosen as $\kappa = 1, 2, 3$, respectively.

Since the matrices $\mathbf{A}_i(x_1)$, $i \in \{1, 2, 3\}$ are Metzler matrices, and the rest system matrices satisfy that each element in these matrices is non-negative, hence, based on the Lemma 1, the open-loop polynomial fuzzy system is a positive system.

The PPFMB system is with 3 fuzzy rules and the corresponding MFs are chosen as follows: $w_1(x_1) = 1-\frac{1}{1+e^{-(x_1-9)}}$, $w_2(x_1) = 1 - w_1(x_1) - w_3(x_1)$, $w_3(x_1) = \frac{1}{1+e^{-(x_1-11)}}$. According to the imperfect premise matching concept [21], the number of the fuzzy rules of the PFOF controller is chosen as 2 which is different from the number of the rules of the PPFMB system. The MFs of the PFOF controller are shown as: $m_2(x_1) = 1 - m_1(x_1)$,

$$m_1(x_1) = \begin{cases} 1, & \text{for} \quad x_1 < 2; \\ \frac{-x_1+18}{16}, & \text{for} \quad 2 \le x_1 \le 18; \\ 0, & \text{for} \quad x_1 > 18. \end{cases}$$

B. Settings when Applying Theorems

We choose the whole operating domain as $x_1 \in [0, 20]$, $\epsilon_1 = \epsilon_2 = \epsilon_3(x_1) = \epsilon_4 = \epsilon_5(x_1) = 0.001$, the parameters $a \in [3, 15]$ at the interval of 1 and $b \in [1, 9]$ at the interval of 1 as well as the highest degree of $\mathbf{Y}_{ij}(x_1)$ to be 0 for checking the stability regions.

Based on Theorem 1, the basic stability region is obtained, which is used to compare with the results based on Theorem 2. And according to Theorem 2, we mainly compare the results from four aspects: Firstly, when the number of subdomains, D, is set to different values, how the stability regions change with D. Secondly, whether different highest degrees of the approximated polynomial functions will affect the stability regions. Thirdly, how the fraction $\alpha_d(x_1)$ impact the stability regions. Finally, comparing the relaxing method given in this paper with the previous relaxing method in [36].

C. Effect of Number of Subdomains D

The numbers of subdomains are chosen as $D = 1$ and $D = 2$. Table III shows the details of the division. Recalling the steps of the high degree polynomial approximation method and in terms of the "polyfit" function, the high degree polynomial functions are obtained. Thereinto, the fraction $\alpha_d(x_1)$ is chosen as $\alpha_d(x_1) = \frac{10}{10^{-3}(x_1-x_{1,d_{min}})(x_{1,d_{max}}-x_1)+10}$, for all d. For reference, the approximated polynomial functions obtained when the highest degree of the polynomial functions is setting as 3 with the fraction $\alpha_d(x_1)$ for $D = 1$ and $D = 2$ are shown in Table IV. The corresponding approximated errors are displayed in Table V. Finally, in terms of Theorem 1 and Theorem 2, the corresponding stability regions are obtained.

In order to show the influence of the number of subdomains D for stability regions, we mainly compare the results in the following cases. Case 1: the highest degree of the polynomial functions is set as 3 without the fraction $\alpha_d(x_1)$, then comparing the stability regions for $D = 1$ and $D = 2$. Case 2: the highest degree of the polynomial functions is set as 5 without the fraction $\alpha_d(x_1)$, then comparing the stability regions for $D = 1$ and $D = 2$. Case 3: the highest degree of the polynomial functions is set as 3 with the fraction $\alpha_d(x_1)$, then comparing the stability regions for $D = 1$ and $D = 2$. Case 4: the highest degree of the polynomial functions is set as 5 with the fraction $\alpha_d(x_1)$, then comparing the stability regions for $D = 1$ and $D = 2$.

From Figs. 2 to 5, it can be seen that when the fraction $\alpha_d(x_1)$ is removed/considered and the highest degrees of the polynomial functions keep the same, the stability regions based on $D = 2$ are larger than the ones based on $D = 1$. Taking the stability regions in Fig. 5 for example, the two stability regions belong to Case 4. It is distinct that the stability region for $D = 1$ ("×") is smaller than the one for $D = 2$ ("+"). From what has been discussed above, we may safely draw the conclusion that the number of the subdomains is larger, the stability region is more extensive.

Besides, the stability region obtained by Theorem 1 also is shown in Fig. 2 ("○"). From the comparison of these stability regions in Fig. 2, it comes to a conclusion that the stability regions obtained by Theorem 2 are larger than the one obtained by Theorem 1. The results confirm that the information of MFs is greatly helpful to relax the stability analysis.

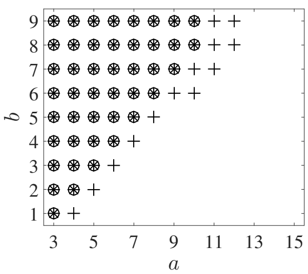

Fig. 2. Stability region given by Theorem 2 when the highest degree of polynomial functions is setting as 3 without $\alpha_d(x_1)$ for $D = 1$ ("×") and $D = 2$ ("+"). And the stability region given by Theorem 1 ("○").

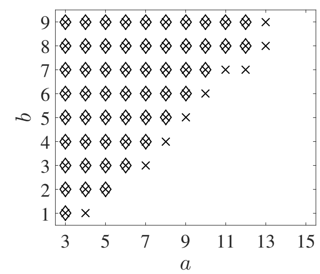

Fig. 3. Stability region given by Theorem 2 when the highest degree of polynomial functions is setting as 5 without $\alpha_d(x_1)$ for $D = 1$ ("◇") and $D = 2$ ("×").

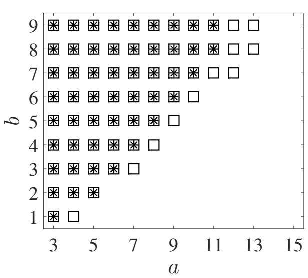

Fig. 4. Stability region given by Theorem 2 when the highest degree of polynomial functions is setting as 3 with $\alpha_d(x_1)$ for $D = 1$ ("∗") and $D = 2$ ("□").

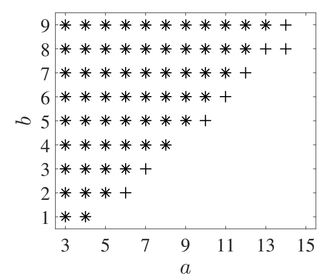

Fig. 5. Stability region given by Theorem 2 when the highest degree of polynomial functions is setting as 5 with $\alpha_d(x_1)$ for $D = 1$ ("×") and $D = 2$ ("+").

D. Effect of the Degrees of Polynomial Functions

For showing how the highest degrees of polynomial functions affect the relaxed stability regions, the following cases are considered. Case 1: the number of subdomains is chosen as $D = 1$, when the fraction $\alpha_d(x_1)$ is removed, comparing the stability region for the highest degree of the polynomial functions being 3 with the one for the highest degree of the polynomial functions being 5. Case 2: the number of subdomains is chosen as $D = 2$, when the fraction $\alpha_d(x_1)$ is removed, comparing the stability region for the highest degree of the polynomial functions being 3 with the one for the highest degree of the polynomial functions being 5. Case 3: the number of subdomains is chosen as $D = 1$, when the fraction $\alpha_d(x_1)$ is considered, comparing the stability region for the highest degree of the polynomial functions being 3 with the one for the highest degree of the polynomial functions being 5. Case 4: the number of subdomains is chosen as $D = 2$, when the fraction $\alpha_d(x_1)$ is considered, comparing the stability region for the highest degree of the polynomial functions being 3 with the one for the highest degree of the polynomial functions being 5.

Through comparing the stability regions in Fig. 2 with the stability regions in Fig. 3, or selecting the stability regions in Fig. 4 and the stability regions in Fig. 5 for comparison, it serves to show that when the fraction $\alpha_d(x_1)$ is removed/considered and the number of subdomains D remains the same, the stability regions obtained by setting the highest degree of polynomial functions as 3 are smaller than the ones obtained by setting the highest degree of polynomial functions as 5. For instance, picking out the stability region ("□") in Fig. 4 and the stability region ("+") in Fig. 5 to satisfy Case 4. By comparing the two stability regions, we can see that the stability region ("□") is smaller than the stability region ("+"), which means the higher degrees of polynomial functions will lead to more relaxed results.

E. Effect of the Fraction $\alpha_d(x_1)$

In the following, we will analyze the influence of the fraction $\alpha_d(x_1)$ for stability regions in four cases. Case 1: the number of the subdomains is $D = 1$ and the highest degree of the polynomial functions is set as 3, then comparing the stability region obtained without considering the fraction $\alpha_d(x_1)$ with the one obtained with considering the fraction $\alpha_d(x_1)$. Case 2: the number of the subdomains is $D = 1$ and the highest degree of the polynomial functions is set as 5, then comparing the stability region obtained without considering the fraction $\alpha_d(x_1)$ with the one obtained with considering the fraction $\alpha_d(x_1)$. Case 3: the number of the subdomains is $D = 2$ and the highest degree of the polynomial functions is set as 3, then comparing the stability region obtained without considering the fraction $\alpha_d(x_1)$ with the one obtained with considering the fraction $\alpha_d(x_1)$. Case 4: the number of the subdomains is $D = 2$ and the highest degree of the polynomial functions is set as 5, then comparing the stability region obtained without considering the fraction $\alpha_d(x_1)$ with the one

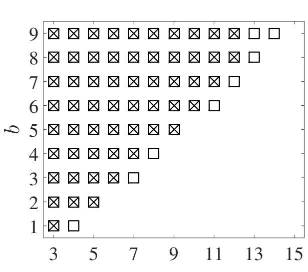

Fig. 6. Stability regions given when the highest degree of polynomial functions is setting as 2 without the boundary information of the state variables for $D = 4$ by using the relaxed method in [36] ("×") and by using Theorem 2 in this paper ("□").

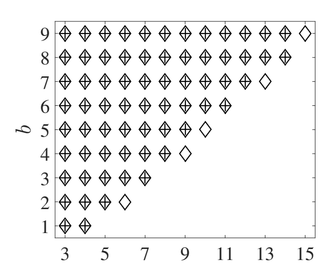

Fig. 7. Stability region given when the highest degree of polynomial functions is setting as 2 with the boundary information of the state variables for $D = 4$ ("+") by using the relaxed method in [36] and by using Theorem 2 in this paper ("◇").

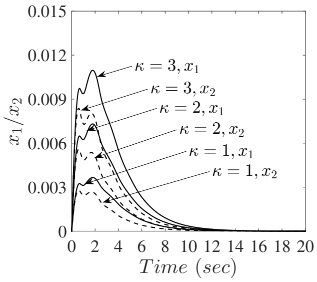

Fig. 8. Time responses of the states x_1 and x_2 for stability point $(10, 8)$ represented as ("×") in Fig. 1 with different κ.

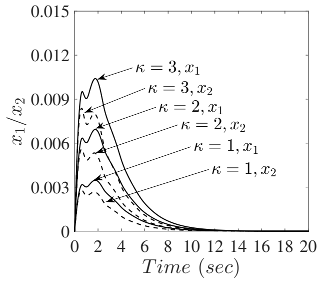

Fig. 9. Time responses of the states x_1 and x_2 for stability point $(4, 7)$ represented as ("+") in Fig. 1 with different κ.

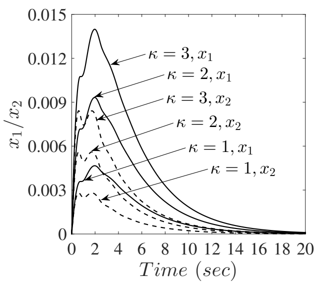

Fig. 10. Time responses of the states x_1 and x_2 for stability point $(5, 2)$ represented as ("⋄") in Fig. 2 with different κ.

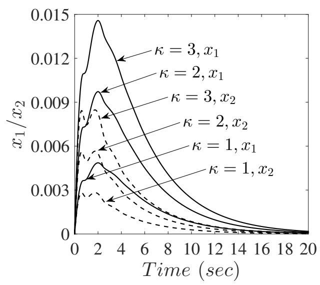

Fig. 11. Time responses of the states x_1 and x_2 for stability point $(4, 1)$ represented as ("×") in Fig. 2 with different κ.

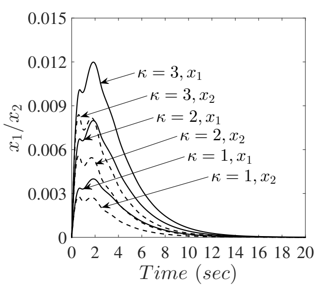

Fig. 12. Time responses of the states x_1 and x_2 for stability point $(9, 6)$ represented as ("∗") in Fig. 3 with different κ.

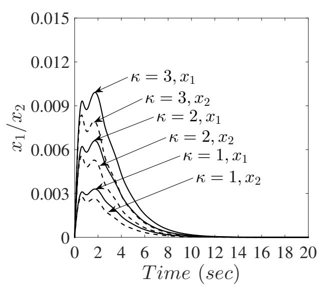

Fig. 13. Time responses of the states x_1 and x_2 for stability point $(7, 9)$ represented as ("□") in Fig. 3 with different κ..

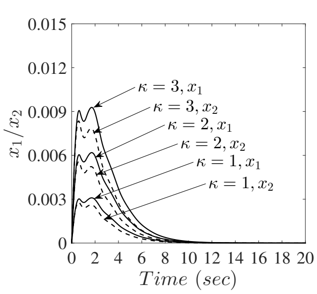

Fig. 14. Time responses of the states x_1 and x_2 for stability point $(3, 9)$ represented as ("×") in Fig. 4 with different κ.

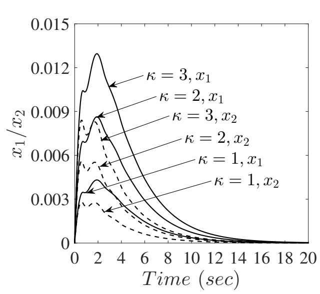

Fig. 15. Time responses of the states x_1 and x_2 for stability point $(10, 5)$ represented as ("+") in Fig. 4 with different κ.

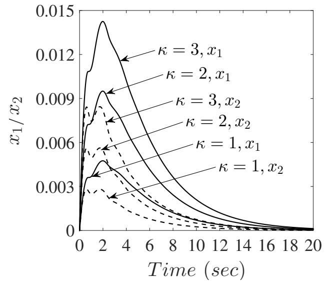

Fig. 16. Time responses of the states x_1 and x_2 for stability point $(3, 1)$ represented as ("×") in Fig. 5 with different κ.

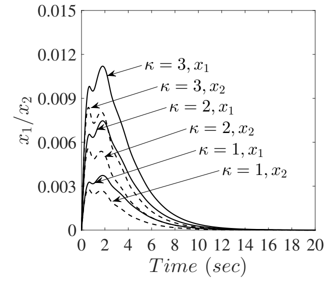

Fig. 17. Time responses of the states x_1 and x_2 for stability point $(14, 9)$ represented as ("□") in Fig. 5 with different κ.

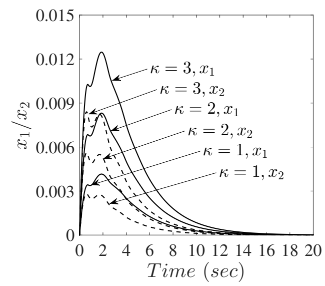

Fig. 18. Time responses of the states x_1 and x_2 for stability point $(11, 6)$ represented as ("+") in Fig. 6 with different κ.

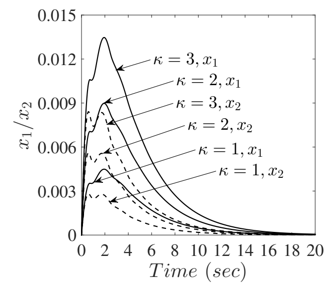

Fig. 19. Time responses of the states x_1 and x_2 for stability point $(9, 4)$ represented as ("⋄") in Fig. 6 with different κ.

obtained with considering the fraction $\alpha_d(x_1)$.

Based on the four cases, it reveals that when highest degrees of polynomial functions are set as the same and the numbers of subdomains D remain the same, the stability regions obtained without considering $\alpha_d(x_1)$ are smaller than the ones obtained with considering $\alpha_d(x_1)$. For example, the stability region ("×") in Fig. 3 and the stability region ("+") in Fig. 5 are chosen for comparison and the results show that the stability region ("×") is smaller than the stability region ("+"), which means $\alpha_d(x_1)$ is in favour of extending the stability regions.

F. Comparison with Other Results

For comparing with the approximated method in [36], we will discuss the results in two cases. Case 1: setting the number of subdomains as $D = 4$ and the highest degree of the approximated polynomial functions as 2 without considering the boundary information of the state variables, comparing the stability region obtained by using the approximated method in this paper with the one obtained by using the approximated method in [36]. Case 2: setting the number of subdomains as $D = 4$ and the highest degree of the approximated polynomial functions as 2 with considering the boundary information of the state variables, comparing the stability region obtained by using the approximated method in this paper with the one obtained by using the approximated method in [36].

For case 1, the stability regions are shown in Fig. 6. For case 2, the stability regions are shown in Fig. 7. Through comparing the two sets of stability regions, it arrives at a conclusion that the method in this paper can generate better relaxation effect than the method in [36] no matter the boundary information of the state variables is taken into account or not.

It is worth mentioning that both of the bounded disturbance and the L_1-induced performance were not considered in [36], whereas, both of the them are considered in this paper. Thereby, only the approximated method but not the whole Theorem 2 in [36] is used for comparison.

G. Time Response of the System State Variables

For different stability points in different stability regions, the corresponding time responses of the system state variables x_1 and x_2 are checked in this section. Considering the limitation of the length of the article, only one stability point (a, b) in each stability region is picked out for testing. In Fig. 2, the chosen stability point are $a = 10, b = 8$ ("$\times$") and $a = 4, b = 7$ ("+"), respectively. In Fig. 3, the chosen stability point are $a = 5, b = 2$ ("$\diamond$") and $a = 4, b = 1$ ("$\times$"), respectively. In Fig. 4, the chosen stability point are $a = 9, b = 6$ ("$*$") and $a = 7, b = 9$ ("$\square$"), respectively. In Fig. 5, the chosen stability point are $a = 3, b = 9$ ("$\times$") and $a = 10, b = 5$ ("+"), respectively. In Fig. 6, the chosen stability point are $a = 3, b = 1$ ("$\times$") and $a = 14, b = 9$ ("$\square$"). In Fig. 7, the chosen stability point are $a = 11, b = 6$ ("+") and $a = 9, b = 4$ ("$\diamond$"). The Corresponding feedback gain matrices $\mathbf{K}_j$ of these stability points are obtained and shown in Table VI. In addition, for the sake of verifying the influence of the bounded disturbance on the stability of the positive L_1-gain PFOF control system, we set the coefficients of the disturbance as $\kappa = 1, 2, 3$, respectively. The time responses have been shown from Fig. 8 to Fig. 19.

According to the time responses from Fig. 8 to Fig. 19, it demonstrates that the positive L_1-gain PFOF control system can be stabilized succesfully, meanwhile, the positivity and the optimal L_1 performance index can be guaranteed as well. Furthermore, the smaller the parameter κ of the disturbance, the faster the time responses converge to 0. Hence, the control effect is better with smaller parameter κ of the disturbance.

V. Conclusion

This paper has investigated the PFOF controller design and the stability and positivity analysis of the positive L_1-gain PFOF control system using SOS-based method. For dealing with the non-convex problem, some constraint conditions and mathematical skills have been employed. Although the introduction of the constraint conditions may lead to a certain degree of conservativeness, this method makes it easier to approximate the non-convex stability conditions and positivity conditions by convex ones simultaneously. Besides, in order to reduce the conservatism of the stability analysis, the high degree polynomial approximation method has been adopted to extract the information embedded in MFs. A simulation example has been given to illustrate the feasibility and effectiveness of the analysis results.

TABLE III
Different Operating Subdomains of x_1.

Region d	$D = 1$	$D = 2$	$D = 4$
Region 1	$0 \leq x_1 \leq 20$	$0 \leq x_1 < 10$	$0 \leq x_1 < 5$
Region 2	–	$10 \leq x_1 \leq 20$	$5 \leq x_1 < 10$
Region 3	–	–	$10 \leq x_1 < 15$
Region 4	–	–	$15 \leq x_1 \leq 20$

TABLE IV
The Approximated Polynomial Functions $\eta^{0.5}_{ij,d}(x_1)$ for $D = 1$ and $D = 2$ for the Highest Degree of Polynomial Functions Being 3 with $\alpha_d(x_1)$.

$D = 1$	$d = 1$	$\eta^{0.5}_{11,1}(x_1)$	$6.3130 \times 10^{-4}x_1^3 - 1.7641 \times 10^{-2}x_1^2 + 5.7551 \times 10^{-2}x_1^1 + 9.6546 \times 10^{-1}$
		$\eta^{0.5}_{21,1}(x_1)$	$1.1378 \times 10^{-4}x_1^3 - 7.7796 \times 10^{-3}x_1^2 + 1.1164 \times 10^{-1}x_1^1 - 1.5089 \times 10^{-1}$
		$\eta^{0.5}_{31,1}(x_1)$	$-7.0663 \times 10^{-4}x_1^3 + 1.6805 \times 10^{-2}x_1^2 - 6.7333 \times 10^{-2}x_1^1 + 5.5717 \times 10^{-2}$
		$\eta^{0.5}_{12,1}(x_1)$	$7.0663 \times 10^{-4}x_1^3 - 2.5593 \times 10^{-2}x_1^2 + 2.4310 \times 10^{-1}x_1^1 - 2.2204 \times 10^{-1}$
		$\eta^{0.5}_{22,1}(x_1)$	$-1.1378 \times 10^{-4}x_1^3 - 9.5264 \times 10^{-4}x_1^2 + 6.3003 \times 10^{-2}x_1^1 - 1.1963 \times 10^{-1}$
		$\eta^{0.5}_{32,1}(x_1)$	$-6.3130 \times 10^{-4}x_1^3 + 2.0237 \times 10^{-2}x_1^2 - 1.0948 \times 10^{-1}x_1^1 + 1.1059 \times 10^{-1}$
$D = 2$	$d = 1$	$\eta^{0.5}_{11,1}(x_1)$	$-7.6875 \times 10^{-4}x_1^3 + 2.9775 \times 10^{-3}x_1^2 - 1.7858 \times 10^{-2}x_1^1 + 1.0145 \times 10^{-0}$
		$\eta^{0.5}_{21,1}(x_1)$	$-3.3708 \times 10^{-4}x_1^3 + 1.1525 \times 10^{-2}x_1^2 - 3.0840 \times 10^{-2}x_1^1 + 3.3940 \times 10^{-2}$
		$\eta^{0.5}_{31,1}(x_1)$	$6.2894 \times 10^{-4}x_1^3 - 3.8023 \times 10^{-3}x_1^2 + 1.2038 \times 10^{-2}x_1^1 - 8.2119 \times 10^{-4}$
		$\eta^{0.5}_{12,1}(x_1)$	$-1.8757 \times 10^{-3}x_1^3 + 1.6844 \times 10^{-2}x_1^2 + 5.9328 \times 10^{-2}x_1^1 - 6.0911 \times 10^{-2}$
		$\eta^{0.5}_{22,1}(x_1)$	$3.4780 \times 10^{-4}x_1^3 + 3.1928 \times 10^{-3}x_1^2 - 1.4631 \times 10^{-2}x_1^1 + 1.0548 \times 10^{-2}$
		$\eta^{0.5}_{32,1}(x_1)$	$9.1223 \times 10^{-4}x_1^3 - 7.4345 \times 10^{-3}x_1^2 + 1.9898 \times 10^{-2}x_1^1 - 1.1168 \times 10^{-2}$
	$d = 2$	$\eta^{0.5}_{11,2}(x_1)$	$-9.1223 \times 10^{-4}x_1^3 + 4.7299 \times 10^{-2}x_1^2 - 8.1719 \times 10^{-1}x_1^1 + 4.7108 \times 10^{-0}$
		$\eta^{0.5}_{21,2}(x_1)$	$-3.4780 \times 10^{-4}x_1^3 + 2.4061 \times 10^{-2}x_1^2 - 5.3044 \times 10^{-1}x_1^1 + 3.7774 \times 10^{-0}$
		$\eta^{0.5}_{31,2}(x_1)$	$1.8757 \times 10^{-3}x_1^3 - 9.5698 \times 10^{-2}x_1^2 + 1.5177 \times 10^{-0}x_1^1 - 7.1422 \times 10^{-0}$
		$\eta^{0.5}_{12,2}(x_1)$	$-6.2894 \times 10^{-4}x_1^3 + 3.3934 \times 10^{-2}x_1^2 - 6.1467 \times 10^{-1}x_1^1 + 3.7505 \times 10^{-0}$
		$\eta^{0.5}_{22,2}(x_1)$	$3.3708 \times 10^{-4}x_1^3 - 8.6997 \times 10^{-3}x_1^2 - 2.5667 \times 10^{-2}x_1^1 + 1.3305 \times 10^{-0}$
		$\eta^{0.5}_{32,2}(x_1)$	$7.6875 \times 10^{-4}x_1^3 - 4.3147 \times 10^{-2}x_1^2 + 8.2126 \times 10^{-1}x_1^1 - 4.3016 \times 10^{-0}$

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TABLE V
THE UPPER BOUND $\overline{\beta}_{ij,d}$ AND THE LOWER BOUND $\underline{\beta}_{ij,d}$ OF THE APPROXIMATION ERROR $\Delta\eta_{ij,d}(x_1)$ FOR $D = 1$ AND $D = 2$ FOR THE HIGHEST DEGREE OF POLYNOMIAL FUNCTIONS BEING 3 WITH $\alpha_d(x_1)$.

$\underline{\beta}_{ij,d}$ / $\overline{\beta}_{ij,d}$		$D = 1$	$D = 2$	
		$d = 1$	$d = 1$	$d = 2$
$i = 1$	$\underline{\beta}_{ij,d}$	-5.2596×10^{-2}	-2.9310×10^{-2}	-1.9219×10^{-3}
$j = 1$	$\overline{\beta}_{ij,d}$	9.2399×10^{-2}	3.1331×10^{-2}	7.3161×10^{-3}
$i = 1$	$\underline{\beta}_{ij,d}$	-4.9301×10^{-2}	-1.2106×10^{-2}	-1.1509×10^{-3}
$j = 2$	$\overline{\beta}_{ij,d}$	7.6625×10^{-2}	1.9406×10^{-2}	1.1027×10^{-3}
$i = 2$	$\underline{\beta}_{ij,d}$	-4.0680×10^{-2}	-6.1588×10^{-2}	-5.1238×10^{-2}
$j = 1$	$\overline{\beta}_{ij,d}$	1.4416×10^{-1}	2.1705×10^{-2}	1.5072×10^{-2}
$i = 2$	$\underline{\beta}_{ij,d}$	-4.0680×10^{-2}	-5.1238×10^{-2}	-6.1588×10^{-2}
$j = 2$	$\overline{\beta}_{ij,d}$	1.4416×10^{-1}	1.5072×10^{-2}	2.1705×10^{-2}
$i = 3$	$\underline{\beta}_{ij,d}$	-4.9301×10^{-2}	-1.1509×10^{-3}	-1.2106×10^{-2}
$j = 1$	$\overline{\beta}_{ij,d}$	7.6625×10^{-2}	1.1027×10^{-3}	1.9406×10^{-2}
$i = 3$	$\underline{\beta}_{ij,d}$	-5.2596×10^{-2}	-1.9219×10^{-3}	-2.9310×10^{-2}
$j = 2$	$\overline{\beta}_{ij,d}$	9.2399×10^{-2}	7.3161×10^{-3}	3.1331×10^{-2}

TABLE VI
THE OUTPUT FEEDBACK GAIN K_j FOR CHOSEN STABILITY POINTS.

(a, b)	K_1, K_2	(a, b)	K_1, K_2
$a = 10$	$K_1 = -5.7114 \times 10^{-1}$	$a = 4$	$K_1 = -5.6770 \times 10^{-1}$
$b = 8$	$K_2 = -5.5996 \times 10^{-1}$	$b = 7$	$K_2 = -5.5938 \times 10^{-1}$
$a = 5$	$K_1 = -5.7143 \times 10^{-1}$	$a = 4$	$K_1 = -5.7141 \times 10^{-1}$
$b = 2$	$K_2 = -5.5247 \times 10^{-1}$	$b = 1$	$K_2 = -5.6905 \times 10^{-1}$
$a = 9$	$K_1 = -5.7143 \times 10^{-1}$	$a = 7$	$K_1 = -5.6854 \times 10^{-1}$
$b = 6$	$K_2 = -5.6503 \times 10^{-1}$	$b = 9$	$K_2 = -5.5649 \times 10^{-1}$
$a = 3$	$K_1 = -5.6794 \times 10^{-1}$	$a = 10$	$K_1 = -5.7127 \times 10^{-1}$
$b = 9$	$K_2 = -5.5827 \times 10^{-1}$	$b = 5$	$K_2 = -5.6459 \times 10^{-1}$
$a = 3$	$K_1 = -5.7143 \times 10^{-1}$	$a = 14$	$K_1 = -5.7139 \times 10^{-1}$
$b = 1$	$K_2 = -5.6959 \times 10^{-1}$	$b = 9$	$K_2 = -5.6878 \times 10^{-1}$
$a = 11$	$K_1 = -5.7143 \times 10^{-1}$	$a = 9$	$K_1 = -5.7143 \times 10^{-1}$
$b = 6$	$K_2 = -5.6961 \times 10^{-1}$	$b = 4$	$K_2 = -5.7121 \times 10^{-1}$

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January 4, 2021

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Dear Prof. Jonathan Garibaldi, Editor-in-Chief,

Re: Resubmission of manuscript with reference TFS-2020-0730

I hereby resubmit the manuscript entitled "***Membership-Function-Dependent Design of L_1-Gain Output Feedback Controller for Stabilization of Positive Polynomial Fuzzy Systems***" for your consideration of publication in the *IEEE Transactions on Fuzzy Systems*. We would like to express our sincere gratitude to you, the Associate Editor and the reviewers for the comments and questions. Our responses to these comments and questions, and the details about the revision are attached. For any further information, please feel free to contact me. My personal particulars are as follows:

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Yours Sincerely,

Aiwen Meng

Responses to the **Associate Editor's** comments and questions for the paper entitled "***Membership-Function-Dependent Design of L_1-Gain Output Feedback Controller for Stabilization of Positive Polynomial Fuzzy Systems***" with **Paper No.: TFS-2020-0730**.

We would like to express our sincere gratitude to you for valuable and constructive comments. Our responses are as follows.

Comment 1

Unfortunately, the manuscript can not be accepted in its current form. In case you decide to submit it again please take into account each and every comment provided by the reviewers.

Response 1

The authors would like to thank the Associate Editor for the comment. The authors have carefully addressed the reviewers' concern and prepared a revised manuscript for resubmission.

Responses to the **Reviewer 1's** comments and questions for the paper entitled "***Membership-Function-Dependent Design of L_1-Gain Output Feedback Controller for Stabilization of Positive Polynomial Fuzzy Systems***" with **Paper No.: TFS-2020-0730**.

We would like to express our sincere gratitude to you for valuable and constructive comments. Our responses are as follows.

Comment 1
This paper mainly aims at positive polynomial fuzzy systems and the polynomials functions are of n-degree. If we consider the function $\mathbf{x}(t) = [x_0(t)...x_{n-1}(t)]$ in $f_1(\mathbf{x}(t))$, and $\mathbf{x}(t) = [x_n(t)...x_{2n-1}(t)]$ in $f_2(\mathbf{x}(t))$, the same conclusion in the part of the fuzzy controller design can be drawn?

Response 1
Yes, the same conclusion in the part of the fuzzy controller can be obtained. If we use the function as the reviewer mentioned, the dimension of the state variables will be extended, but the expression of the polynomial fuzzy model is invariant, in other words, the analysis does not limit the degrees of the fuzzy controller and which state variables it depends on. Therefore, we can draw the same conclusion.

Comment 2
Please state the assumptions in this paper. They must be clear in order to make the readers understand this paper better.

Response 2
The authors would like to thank the reviewer for this comment. The following assumptions have been added to the revised paper.

Assumption 1 *There exists nonlinear systems with disturbance which can be approximated by the polynomial fuzzy model (2) satisfying the conditions in Lemma 1.*

Assumption 2 *The disturbance signal* $\tilde{\mathbf{w}}(t) \succeq 0$ *is bounded.*

The above assumptions are added on the right column, lines 25-28, page 3 of the revised paper.

Comment 3
How fuzzy logic is helpful here?

Response 3
In this paper, we assume that the polynomial fuzzy model of a complex non-

linear positive system has been obtained based on the fuzzy logic. Thereby, we directly use the polynomial fuzzy model of a complex nonlinear positive system in this paper. Because the focus of us is not mainly on how to use fuzzy logic to deal with a nonlinear system but on how to realize the stability and positivity analysis based on fuzzy models. Therefore, how to use the fuzzy logic to obtain positive polynomial fuzzy models and to design fuzzy controllers is not considered.

Comment 4
What about the conservatism of the analysis results? Please discuss it.

Response 4
The authors would like to thank the reviewer for the comment. we have discussed the conservatism in the Remark 3 in the revised paper, the details are shown as follows:

Remark 3: Although the convex method given in our paper will introduce a certain degree of conservativeness, this method makes it easier to transform the non-convex stability conditions and positivity conditions into convex ones simultaneously, which means the non-convex problem can be dealt with skillfully. In order to alleviate the conservativeness, in the following, a high degree polynomial approximation method will be introduced so that the information embedded in MFs can be found and introduced into stability conditions.

The changes can be found on the right column, lines 44-52, page 5 of the revised paper.

Comment 5
In my opinion, for $(31), (32), (33)$, (43), they should be just for any "d" instead of any "i, j, d" because there are two sets of summation symbols for i and j.

Response 5
The authors would like to thank the reviewer for the comment. The expressions (31), (32), (33) and (43) in previous version have been changed as (40), (41), (42) and (38), respectively, in the revised paper. As suggested by this reviewer, we have revised "$\forall i, j, d$" as "$\forall d$" in these expressions.

Comment 6
What is the whole operating domain in the simulation section? Please give the scope of the whole operating domain in the simulation.

Response 6
The authors would like to thank the reviewer for the comment. We have

added the sentence "We choose the whole operating domain as $x_1 \in [0, 20]$," into the simulation so that the whole operating domain is clear for readers.

The change can be found on the right column, line 11, page 8 of the revised paper.

Comment 7
The following reference should be updated to enrich the background, such as 10.1109/TFUZZ.2020.2999746, 10.1109/TFUZ Z.2020.2982618.

Response 7
The authors would like to thank the reviewer for the comment. We have cited these papers as reference [7] and [8] in the revised paper.

The changes can be found in the right column, line 16, page 1 of the revised paper.

Comment 8
Please add the advantages and limitations of proposed approach in the conclusion.

Response 8
The authors would like to thank the reviewer for the comment. We have added advantages and limitations of the proposed approach in the conclusion. The details are shown as follows:

This paper has investigated the PFOF controller design and the stability and positivity analysis of the positive L_1-gain PFOF control system using SOS-based method. For dealing with the non-convex problem, some constraint conditions and mathematical skills have been employed. Although the introduction of the constraint conditions may lead to a certain degree of conservativeness, this method makes it easier to approximate the non-convex stability conditions and positivity conditions by convex ones simultaneously. Besides, in order to reduce the conservatism of the stability analysis, the high degree polynomial approximation method has been adopted to extract the information embedded in MFs. A simulation example has been given to illustrate the feasibility and effectiveness of the analysis results.

The changes can be found in the left column lines 42-51, page 11 of the revised paper.

Responses to the **Reviewer 2's** comments and questions for the paper entitled "***Membership-Function-Dependent Design of L_1-Gain Output Feedback Controller for Stabilization of Positive Polynomial Fuzzy Systems***" with **Paper No.: TFS-2020-0730**.

We would like to express our sincere gratitude to you for valuable and constructive comments. Our responses are as follows.

Comment 1
Compared with the authors' previous work, the non-convex problem is transformed into a convex one via the inequality (16). While the conservatism of such a simple inequality cannot be evaluated.

Response 1
The authors would like to thank the reviewer for the comment. We agree with you that the introduction of the constraint conditions will lead to a certain degree of conservativeness, but as we all known that converting non-convex stable conditions and positive conditions to convex conditions simultaneously is very difficult since most of the convex methods are just for general systems and there are no positivity constrains for general systems. When positivity constrains are taken into consideration, those convex methods for general systems may not be able to work. In addition, the convexification is problem dependent, which means the convexificiation method from one paper may not be able to be applied to other non-convex problems, therefore, it is a hard task to find a proper method to deal with the non-convex problem so that the stability conditions and positivity conditions can be ensured to be convex simultaneously. In our paper, the first task is to find a proper method to transform the non-convex conditions into convex ones. After solving this tricky problem, the second task is to reduce the conservativeness by employing membership-function-dependent techniques. Therefore, although the convex method in our paper will bring some conservatism, the high degree polynomial approximation method can effectively relax the conservativeness. To clarify the concern, we have discussed it in Remark 3. The details are shown as follows:

Remark 3: Although the convex method given in our paper will introduce a certain degree of conservativeness, this method makes it easier to transform the non-convex stability conditions and positivity conditions into convex ones simultaneously, which means the non-convex problem can be dealt with skillfully. In order to reduce the conservativeness, in the following, a high degree polynomial approximation method will be introduced so that the information embedded in MFs can be found and introduced into stability conditions.

The changes can be found on the right column, lines 44-52, page 5 of the revised paper.

Comment 2

As for the so-called second difficulty, the High Degree Polynomial Approximation Method has also been used in the existing results, please clarify the contribution of the paper.

Response 2

The authors would like to thank the reviewer for the comment. We agree with you that the High Degree Polynomial Approximation Method has been used in the existing results, but it was proposed in recent years, which means this approach is still not very mature and there is still a lot of room for improvement. Therefore, based on this idea, we firstly divide the whole operate domain into several sub-domains, and then the cross terms of $w_i(\mathbf{x})m_j(\mathbf{x})$ in each sub-domain are approximated by high degree approximation polynomials. Different from the existed membership function-dependent methods, in our paper, the information of state variables are used for relaxing the stability analysis by introducing it into the high degree polynomial functions instead of through a slack matrix to introduce it into the stability analysis, which is help to reduce the calculation burden. This idea can be found in the equality (28) in the revised paper, which also is shown as follows:

$$\alpha_d(\mathbf{x}) = \frac{f}{g(\mathbf{x} - \mathbf{x}_{d_{min}})(\mathbf{x}_{d_{max}} - \mathbf{x}) + s}, \forall\, d,$$

where $f > 0$, $g > 0$ and $s > 0$ are predefined scalars. $\mathbf{x}_{d_{min}}$ and $\mathbf{x}_{d_{max}}$ are the lower bound and upper bound of the state variables $\mathbf{x}$ in each subdomain. It can be seen that $\alpha_d(\mathbf{x}) > 0$ when the state variables $\mathbf{x}$ belong to the d-th subdomain.

To verify the effect of the boundary information of the state variables, we have analyzed the influence of the fraction $\alpha_d(\mathbf{x})$ for relaxing stability regions in four cases in Section IV-E. The details are shown as follows:

Case 1: the number of the subdomains is $D = 1$ and the highest degree of the polynomial functions is set as 3, then comparing the stability region obtained without considering the fraction $\alpha_d(x_1)$ with the one obtained with considering the fraction $\alpha_d(x_1)$. Case 2: the number of the subdomains is $D = 1$ and the highest degree of the polynomial functions is set as 5, then comparing the stability region obtained without considering the fraction $\alpha_d(x_1)$ with the one obtained with considering the fraction $\alpha_d(x_1)$. Case 3: the number of the subdomains is $D = 2$ and the highest degree of the polynomial functions is set as 3, then comparing the stability region obtained without considering the fraction $\alpha_d(x_1)$ with the one obtained with considering the fraction $\alpha_d(x_1)$. Case 4: the number of the subdomains is $D = 2$ and the highest degree of the polynomial functions is set as 5, then comparing the stability region obtained without considering the fraction $\alpha_d(x_1)$ with the one obtained with considering the fraction $\alpha_d(x_1)$.

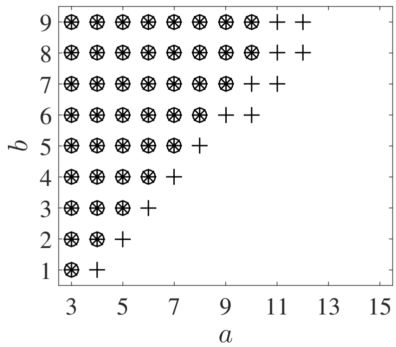

Fig 1. Stability region given by Theorem 2 when the highest degree of polynomial functions is setting as 3 without $\alpha_d(x_1)$ for $D = 1$ ("×") and $D = 2$ ("+"). And the stability region given by Theorem 1 ("○").

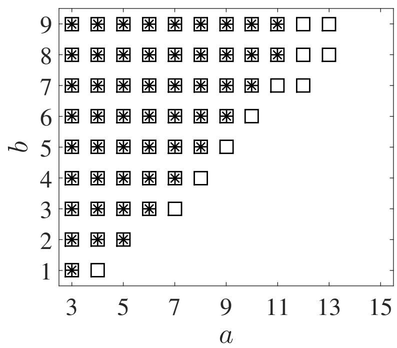

Fig 2. Stability region given by Theorem 2 when the highest degree of polynomial functions is setting as 3 with $\alpha_d(x_1)$ for $D = 1$ ("$*$") and $D = 2$ ("□").

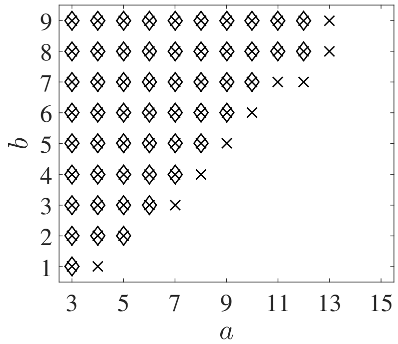

Fig 3. Stability region given by Theorem 2 when the highest degree of polynomial functions is setting as 5 without $\alpha_d(x_1)$ for $D = 1$ ("$\diamond$") and $D = 2$ ("$\times$").

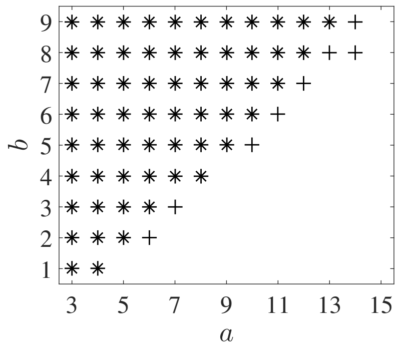

Fig 4. Stability region given by Theorem 2 when the highest degree of polynomial functions is setting as 5 with $\alpha_d(x_1)$ for $D = 1$ ("$\times$") and $D = 2$ ("$+$").

Based on the obtained stability regions, it reveals that when highest degrees of polynomial functions are set as the same and the numbers of subdomains D remain the same, the stability regions obtained without considering $\alpha_d(x_1)$ are smaller than the ones obtained with considering $\alpha_d(x_1)$. For example, the stability region ("$\times$") in Fig. 3 and the stability region ("$+$") in Fig. 4 are chosen for comparison and the results show that the stability region ("$\times$") is smaller than the stability region ("$+$"), which means $\alpha_d(x_1)$ is in favour of extending the stability regions.

Besides, to the best of our knowledge, this approach has not been employed to study the relaxation of the stability analysis for positive polynomial fuzzy systems, which encourages us to carry out this work.

Finally, we also add some content in the Introduction to clarify the main contributions of this paper. The details are shown as follows:

"Aiming at dealing with the above issues, the main contributions are made and summarized as follows:

1) For dealing with the non-convex conditions, the augmented vector method is employed to construct an augmenting system of positive L_1-gain PFOF control system. Then, through introducing some constraint conditions and using some mathematical skills, the non-convex stability and positivity conditions are approximated by convex ones simultaneously.

2) For reducing the conservativeness of the analysis results, a high degree polynomial approximation method is adopted to approximately express the original MFs so that the valuable information of MFs helps to derive the relaxed stability conditions. Different from other MFD methods, in our paper, the information of the boundary information of state variables is used for relaxing the stability analysis by introducing it into the high degree polynomial functions instead of by introducing a slack matrix, which help reduce the calculation burden."

The changes can be found on the right column, lines 28-45, page 2 of the revised paper.

Comment 3
So many acronyms are used in the paper, which reduce the readability.

Response 3
The authors would like to thank the reviewer for the comment. As suggested by the reviewer, we have deleted some acronyms and summarized the rest acronyms in Table I to improve the readability of this paper. The details are shown as follows:

1. "LMI" is replaced by "linear-matrix-inequality".

2. "IMP" is replaced by " imperfect premise matching".

3. "LP" is replaced by " linear programming".

4. "LKF" is replaced by "Lyapunov-Krasovskii functional".

Table I Description of the Acronyms

Acronyms	Explanation	Acronyms	Explanation
T-S	Takagi-Sugeno	MFD	membership-function-dependent
SOS	sum-of-squares	PPFMB	positive polynomial fuzzy-model-based
MFs	membership functions	PFOF	polynomial fuzzy output feedback

These changes can be found on the pages of 1-3.

Comment 4
Some related references are missing.

Response 4
The authors would like to thank the reviewer for the comment. As suggested by the reviewer, we have cited more references in the revised paper. For example, the reference [1], [2], [7], [8], [9], [10] in the revised paper.

[1] D. G. Luenberger, Introduction to dynamic systems: Theory, models and applications. John Wiley and Sons, 1979.

[2] V. D. Hof and M. J., Positive linear observers for linear compartmental systems, Siam Journal on Control and Optimization, vol. 36, no. 2, pp. 590C608, 1998.

[7] H. Liang, X. Guo, Y. Pan, and T. Huang, Event-triggered fuzzy bipartite tracking control for network systems based on distributed reduced-order observers(revised manuscript of tfs-2019-1049), IEEE Transactions on Fuzzy Systems, pp.1-1, 2020.

[8] Y. Pan, P. Du, H. Xue, and H.-K. Lam, Singularity-free fixed-time fuzzy control for robotic systems with user-defined performance, IEEE Transactions on Fuzzy Systems, pp.1-1, 2020.

[9] M. Xue, Y. Tang, W. Ren, and F. Qian, "Practical output synchronization for asynchronously switched multi-agent systems with adaption to fast-switching perturbations, Automatica, vol. 116, no. 108917, Jun. 2020.

[10] Y. Tang, X.Wu, P.Shi, and F. Qian, "Input-to-state stability for nonlinear systems with stochastic impulses, Automatica, vol. 113, p. 108766, 2020.

Responses to the **Reviewer 3's** comments and questions for the paper entitled "***Membership-Function-Dependent Design of L_1-Gain Output Feedback Controller for Stabilization of Positive Polynomial Fuzzy Systems***" with **Paper No.: TFS-2020-0730**.

We would like to express our sincere gratitude to you for valuable and constructive comments. Our responses are as follows.

Comment 1
However, there are a number of results about the issues of positive systems, polynomial fuzzy systems, output feedback control, L_1-gain analysis, SOS methods, and etc. This paper is a simple permutation and combination of these issues and considered in a general without a profound contribution. Therefore, it might not be suitable for the journal.

Response 1
The authors would like to thank the reviewer for the comment. Although there are a number of results about the issues of positive systems, polynomial fuzzy systems, output feedback control, L_1-gain analysis, SOS methods, but when considering these topics together, the difficulty will be vastly increased, the main reasons are as follows:

Firstly, the positivity constrains of positive systems will make the non-convex problem harder to be solved because in this case, we not only need to ensure the stability conditions to be convex conditions, but also need to ensure the positivity conditions to be convex conditions simultaneously, which will directly lead to the fact that many existing convex methods for general systems are not applicable to positive systems. Besides, convexification is problem dependent, which means a convex method from one paper may not be able to be applied to other non-convex problems.

Secondly, since polynomial fuzzy models can express more complexity nonlinear systems than T-S fuzzy models, hence, we study the positive nonlinear systems through the polynomial fuzzy models in our paper. But the existence of the polynomials in the polynomial fuzzy systems will make the stability analysis and controller design very challenging, especially, when solving the non-convex problem, if the inverse matrices are used in a convex method to facilitate the stability analysis, then such a method may not be able to cope with the non-convex problem for polynomial fuzzy systems due to the existence of the polynomials in matrices. In addition, L_1-gain output feedback control will further make the stability and positivity analysis difficult because the number of non-convex terms will be increased. The more non-convex terms there are, the more complex the problem becomes.

Thirdly, although there are some membership-function-dependent methods to relax the conservatism of the stability analysis for general systems, few researchers apply these methods to the stability analysis of positive polynomial fuzzy systems. Moreover, different from these existed membership-function-dependent methods, in our paper, the information of state variables are used for relaxing the stability analysis by introducing it into the high degree polynomial functions instead of by introducing a slack matrix, which is help to reduce the calculation burden.

Based on the above analysis, the work in our paper is not just a simple permutation and combination of these topics. We need to take all of these problems into account when this paper is investigated.

Comment 2
This is a well-written conference paper, and in this shape even if rejected from another IEEE journal, can be resubmitted to the conference series of adequate scope.

Response 2
The authors would like to thank the reviewer for the comment. In order to clarify the difficulties and contributions of this paper, we have added the following content into the revised paper. The details are shown as follows:

"On the basis of the above analysis, we need to focus on cracking the following hard nuts. Firstly, as we all known that converting non-convex stable conditions and positive conditions to convex conditions simultaneously is very difficult since most of the convex methods are just for general systems and there are no positivity constrains for general systems. When positivity constrains are taken into consideration, those convex methods for general systems may not be able to work, hence, the non-convex problem will become very tricky to be solved. In addition, the convexification is problem dependent, which means the convexificiation method from one paper may not be able to be applied to other non-convex problems, therefore, it is a hard task to find a proper method to deal with a non-convex problem. Secondly, the introduction of some constrain conditions and the absence of the information of MFs will lead to strong conservatism of the stability analysis results. How to obtain relaxed stability analysis results is still an open problem that is worth working on. Aiming at dealing with the above issues, the main contributions are made and summarized as follows:

1) For coping with the non-convex conditions, the augmented vector method is employed to construct an augmenting system of positive L_1-gain PFOF control system. Then, through introducing some constraint conditions and

mathematical skills, the non-convex stability and positivity conditions are approximated by convex ones simultaneously.

2) For reducing the conservativeness of the analysis results, a high degree polynomial approximation method is adopted to approximately express the original MFs so that the valuable information of MFs helps to derive the relaxed stability conditions. Different from other MFD methods, in our paper, the information of the boundary information of state variables is used for relaxing the stability analysis by introducing it into the high degree polynomial functions instead of by introducing a slack matrix, which help reduce the calculation burden."

The changes can be found on the left column, lines 51-59 and the right column, lines 17-45, page 2 of the revised paper.

To verify the effectiveness and superiority of the theorems proposed in this paper, a numerical simulation has been done and the results have been discussed from the following four aspects: 1) How the stability regions vary with the values of D. 2) How the stability regions change with the highest degrees of approximated polynomial function $\eta^{0.5}_{ij,d}(\mathbf{x})$. 3) How the stability regions are affected by the $\alpha_d(\mathbf{x})$. 4) Compare the stability regions obtained using the method with the ones obtained using other methods. In addition, for each of the aspects, we analyze and compare the simulation results from different cases. To further check the correctness of the results, the time responses of some stability points have been tested as well. The results indicates that the asymptotic stability and positivity of the positive L_1-gain PFOF control system can be guaranteed, meanwhile, the optimal L_1 performance index can be satisfied as well.

Therefore, considering the results and contributions, we would like to have a try to resubmit this paper to the IEEE transaction on Fuzzy Systems. After seriously revising and improving, we hope this revision now meets the expectation of the reviewer, and is suitable for publication in IEEE Transactions on Fuzzy Systems.

Comment 3

The abstract is organized not well and the writing idea is also not clear, such that the readers face difficulty in understanding the presented points. Thus, the abstract can be reorganized carefully.

Response 3

As suggested by the reviewer, we have revised the abstract in the revised paper. The details are shown as follows:

This paper presents the L_1-gain polynomial fuzzy output feedback controller design and the stability analysis using sum-of-squares (SOS) approach for positive polynomial fuzzy-model-based (PPFMB) control systems. The polynomials, positivity and optimal L_1 performance makes some existing convex methods for general systems inapplicable. To overcome this problem, an augmented system of the positive polynomial fuzzy-model-based control system is first constructed, then by introducing some constrain conditions and mathematical techniques, the non-convex stability and positivity conditions are skillfully transformed into convex ones simultaneously. In addition, to control the systems flexibly and lower the implementation cost, the imperfect premise matching concept is taken into account for controller design. Besides, the high degree polynomial approximation method is adopted to conduct stability and positivity analysis by incorporating the information of membership functions (MFs) and the boundary information of the state variables. On the basis of the Lyapunov stability theory, the relaxed stability and positivity conditions in terms of SOS are obtained. Finally, a simulation example is presented to verify the feasibility of the theoretical results.

The changes can be found on the left column, lines 14-40, page 1 of the revised paper.

Comment 4
As mentioned above, the research points and methods studied in this paper have basically been solved and the contribution is relatively weak. The main results obtained are standard and not novel in essential.

Response 4
The authors would like to thank the reviewer for the comment. To our best knowledge, most of the work related to the L_1-gain control are for positive linear systems and positive switched systems, such as the references [R1]-[R3], but there are still no papers considering the L_1-gain control for positive polynomial fuzzy systems with disturbance. Besides, since the polynomial fuzzy-model-based technique is still on the start-up stage, hence, the relevant results on positive nonlinear systems in terms of polynomial fuzzy models are relatively few, moreover, most of the results are from our team. As for the L_1-gain static output control for positive polynomial fuzzy systems with disturbance, the related researches results are fewer.

In addition, as we all know that converting non-convex stable conditions and positive conditions to convex conditions simultaneously is very challenging since most of the convex methods are just for general systems and there are no positivity constrains for general systems. When positivity constrains are taken into consideration, those convex methods for general systems may be inapplicable. In addition, the convexification is problem dependent, which means the convex method from one paper may not be able to be applied to other non-convex problems, therefore, it is a hard task to find a proper method to deal with a non-convex problem. In our paper, by introducing some constrain conditions and mathematical techniques, the non-convex stability conditions and positivity conditions have been approximated into convex ones simultaneously, which means the non-convex problem have been dealt with skillfully.

Furthermore, in order to reduce the conservativeness, in our paper, a high degree polynomial approximation method is introduced so that the information embedded in MFs can be found and introduced into stability conditions. Based on this idea, we not only approximate the cross terms of $w_i(\mathbf{x})m_j(\mathbf{x})$ by high degree approximation polynomials but also try to introduce the boundary information of the state variables into the high degree polynomials at the same time. This idea can be found in the equality (28) in the revised paper, which also is shown as follows:

$$\alpha_d(\mathbf{x}) = \frac{f}{g(\mathbf{x} - \mathbf{x}_{d_{min}})(\mathbf{x}_{d_{max}} - \mathbf{x}) + s}, \forall\ d,$$

where $f > 0$, $g > 0$ and $s > 0$ are predefined scalars. $\mathbf{x}_{d_{min}}$ and $\mathbf{x}_{d_{max}}$ are the lower bound and upper bound of the state variables $\mathbf{x}$ in each subdomain. It can be seen that $\alpha_d(\mathbf{x}) > 0$ when the state variables $\mathbf{x}$ belong to the d-th subdomain.

To verify the effect of the boundary information of the state variables, we have analyzed the influence of the fraction $\alpha_d(\mathbf{x})$ for relaxing stability regions in four cases, please refer to the details in Section IV-E in the revised paper.

Finally, we also add some content in the Introduction to clarify the main contributions of this paper. The details are shown as follows:

"Aiming at dealing with the above issues, the main contributions are made and summarized as follows:

1) For dealing with the non-convex conditions, the augmented vector method is employed to construct an augmenting system of positive L_1-gain PFOF control system. Then, through introducing some constraint conditions and using some mathematical skills, the non-convex stability and positivity conditions are approximated by convex ones simultaneously.

2) For reducing the conservativeness of the analysis results, a high degree

polynomial approximation method is adopted to approximately express the original MFs so that the valuable information of MFs helps to derive the relaxed stability conditions. Different from other MFD methods, in our paper, the information of the boundary information of state variables is used for relaxing the stability analysis by introducing it into the high degree polynomial functions instead of by introducing a slack matrix, which help reduce the calculation burden."

The changes can be found on the right column, lines 28-45, page 2 of the revised paper.

[R1] M Xiang, Z Xiang. Stability. L_1-gain and control synthesis for positive switched systems with time-varying delay, Nonlinear Analysis Hybrid Systems, 2013, 9:9-17.

[R2] Y Ebihara, D Peaucelle, and D Arzelier. Optimal L_1-controller synthesis for positive systems and its robustness properties, American Control Conference (ACC), 2012. IEEE, 2012.

[R3] J Zhang, Z Han, and F Zhu. L_1-gain analysis and control synthesis of positive switched systems. Taylor and Francis, 2015, 46(12):2111-2121.

Comment 5
The difficulties of handling the problem of fuzzy L_1-gain output feedback control for positive polynomial fuzzy systems should be highlighted. Are these difficulties essential?

Response 5
As suggested by the reviewer, we have highlighted the difficulties and the contributions in the Introduction of the revised paper. The details have been shown in the response to Comment 2, please refer to it.

In addition, the reasons for why these difficulties are essential are listed as follows: Firstly, in our paper, the obtained initial stability conditions are non-convex, which cannot be solved by using SOSTOOLS directly. Therefore, it is essential to transform the non-convex conditions into convex ones. Moreover, not only does stability for positive systems need to be guaranteed, but also positivity needs to be guaranteed, so when we deal with non-convex stability conditions, the convexification of positivity conditions should be taken into account as well, which makes the non-convex problem more tricky.

Secondly, when stability conditions are derived without considering membership functions, the obtained feasible solutions are suitable for any type of membership functions. However, for a specific positive system, it only needs to find the feasible solutions under specific membership functions. So the lack of the information of membership functions will lead to conservative results. But introducing the original membership functions into the stability analysis directly not only complicates the analysis process, but also makes it impossible for the SOSTOOLS to deal with this kind of stability conditions because original membership functions generally are not the Sum-of-Squares form. Therefore, how to capture the information of membership functions and introduce it into the stability analysis is also essential and meaningful.

Comment 6

What is the practical significance of studying the positive polynomial fuzzy systems in this paper? The authors can combine the actual application examples for specific analysis in the Simulation Examples.

Response 6

The authors would like to thank the reviewer for the comment. As mentioned in our paper, many practical models of positive systems exist in a variety of disciplines, for example, the field of biology, the area of pharmacokinetic, the aspect of ecology, and so on. In order to highlight the practical significance of studying positive systems, we add some relevant content in the Introduction in the revised paper and give an example with a real system in the following.

"Positive systems, whose state variables and outputs always remain in the non-negative quadrant if both of the initial conditions and input are non-negative, are often encountered in real-world applications [1]-[3]. A great deal of practical models of such systems exist in a variety of disciplines, for example, the control of the cortisol level within the hypothalamic-pituitary-adrenal gland axis in the field of biology, the human immunodeficiency virus viral mutation dynamics in the area of pharmacokinetic, the prey-predator model in the aspect of ecology, and the concentration of substances in chemical processes and so on [4]-[6]. As positive systems are closely related to our daily life, it is of great practical significance to conduct in-depth research on positive systems."

The changes can be found on the left column, lines 34-46, page 1 of the revised paper.

Based on [R4], a biological system model is chosen and applied to check the effective of Theorem 2, but due to the limitation of the pages, the corresponding results are just shown in this reply letter for checking by the reviewers. In the following, a single species with a stage structure model is given

$$\dot{x}_1(t) = \alpha x_2(t) - \gamma_1 x_1(t) - \beta x_1(t) - \eta x_1^2(t) + \delta \tilde{w}(t) + \xi u(t),$$

$$\dot{x}_2(t) = \beta x_1(t) - \gamma_2 x_2(t),$$

where $x_1(t)$ is the density of immature population of the species and $x_2(t)$ denotes the density of mature population of the species. α is the birth rate of the immature population; for the immature population, γ_1 and β are called the death rate and transformation rate of immature, respectively; η is called the density restriction for the immature population, γ_2 is called the death rate of the mature population; δ is the immigrant rates of immature population of the species from other areas to this area; $\tilde{w}(t)$ is the disturbance, which denotes the density of the immigrant immature population of this species; ξ is the control rate, and $u(t)$ is the control input, which denotes the controlled density of the immature population of this species.

Next, for simplify, $x_1(t)$ and $x_2(t)$ will be denoted by x_1 and x_2, respectively.

Defining the density of the immature species as $x_1 \in [0, 20]$, then we have $f(x_1) = x_1 = \mu_{11}(x_1) f_{max} + \mu_{12}(x_1) f_{min}$, where $\mu_{11}(x_1) = \frac{f(x_1) - f_{min}}{f_{max} - f_{min}}$, $\mu_{11}(x_1) = 1 - \mu_{12}(x_1)$, $f_{max} = 20$, $f_{min} = 0$. By employing the sector nonlinearity technique, we get the fuzzy model of this system, which is shown as follows:

$$\begin{aligned}
\text{Rule 1 : } & \text{IF } x_1 \text{ is LARGE} \\
& \text{THEN } \dot{\mathbf{x}} = \mathbf{A}_1(\mathbf{x})\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{B}_{1\omega}\tilde{\mathbf{w}}, \\
\text{Rule 2 : } & \text{IF } x_1 \text{ is SMALL} \\
& \text{THEN } \dot{\mathbf{x}} = \mathbf{A}_2(\mathbf{x})\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{B}_{1\omega}\tilde{\mathbf{w}},
\end{aligned}$$

Then the overall fuzzy model of this system is shown as follows:

$$\begin{aligned}
\dot{\mathbf{x}} &= \sum_{i=1}^{2} w_i(\mathbf{x}) \big(\mathbf{A}_i(\mathbf{x})\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{B}_{1\omega}\tilde{\mathbf{w}} \big), \\
\mathbf{y} &= \mathbf{C}\mathbf{x},
\end{aligned}$$

where $w_i(\mathbf{x})$, $i \in 1, 2$ is the membership function of the systems.

$$\begin{aligned}
& \mathbf{A}_1 = \begin{bmatrix} -f_{max}\eta - \gamma_1 - \beta & \alpha \\ \beta & -\gamma_2 \end{bmatrix}, \mathbf{A}_2 = \begin{bmatrix} -f_{min}\eta - \gamma_1 - \beta & \alpha \\ \beta & -\gamma_2 \end{bmatrix}, \\
& \mathbf{B} = \begin{bmatrix} \xi & 0 \end{bmatrix}^T, \mathbf{B}_{1\omega} = \begin{bmatrix} \delta & 0 \end{bmatrix}^T, \mathbf{x} = [\ x_1 \quad x_2\ ]^T,
\end{aligned}$$

where α, β, η, γ, k, β_1, γ_1, γ_2, η_1, δ and ξ are positive constant scalars.

The membership functions of the system are $w_1(x_1) = \frac{x_1}{20}$ and $w_2(x_1) = 1 - w_1(x_1)$. Then we assume that the output is $y = x_1$, so the output matrix is $\mathbf{C} = [1 \quad 0]$. The controlled output z and the disturbance $\tilde{w}$ are chosen as $z = 1.5x_1 + 1.1u$ and $\tilde{w} = 5e^{-t}|\cos(2t)|$, respectively.

In the following, a polynomial fuzzy controller is designed as follows:

$$\begin{aligned} \text{Rule 1} : &\text{ IF } \mathbf{y} \text{ is LARGE} \\ &\text{ THEN } \mathbf{u} = \mathbf{K}_1(\mathbf{y})\mathbf{y}, \\ \text{Rule 2} : &\text{ IF } \mathbf{y} \text{ is SMALL} \\ &\text{ THEN } \mathbf{u} = \mathbf{K}_2(\mathbf{y})\mathbf{y}, \end{aligned}$$

where $\mathbf{K}_1(\mathbf{y})$ and $\mathbf{K}_2(\mathbf{y})$ are the output feedback gains.

Then the polynomial fuzzy controller is designed as follows:

$$\mathbf{u} = \sum_{j=1}^{2} m_j(\mathbf{y})\mathbf{K}_j(\mathbf{y})\mathbf{C}\mathbf{x},$$

where $m_j(\mathbf{y})$, $j \in 1, 2$ is the membership function of the controller.

Because $y = x_1$ and the ranges of y is assumed as $y \in [0, 20]$, hence, m_j is related to x_1 and is chosen as follows:

$$m_1(x_1) = \begin{cases} 1, & \text{for} \quad x_1 < 2 \\ \frac{-x_1+18}{16}, & \text{for} \quad 2 \le x_1 \le 18 \\ 0. & \text{for} \quad x_1 > 18 \end{cases}$$

When we choose $\alpha = 0.3; \gamma_1 = 0.1; \beta = 0.18; \eta = 0.01; \gamma_2 = 0.1$, $\delta = 0.5$ and $\xi = 0.5$, the system matric $\mathbf{A}_1$ and $\mathbf{A}_2$ are Metzler, $\mathbf{B} \succeq 0$ and $\mathbf{C} \succeq 0$, which means the real system is a positive system. Then the whole operating region is divided into 2 sub-regions and the highest degree of the approximation polynomial functions is chosen as 3. The degrees of $\mathbf{K}_j(\mathbf{x})$ and $\mathbf{Y}_{ij,d}(\mathbf{x})$ are chosen as 0, respectively. And by setting $\epsilon_1 = \epsilon_2(\mathbf{x}) = 0.001$, the decision variables $\lambda_1, \lambda_2, \mathbf{K}_j(\mathbf{x})$ and $\mathbf{Y}_{ij,d}(\mathbf{x})$ are obtained as follows:

$$\begin{aligned} &\gamma = 1.1020, \lambda_1 = [\ 2.2020 \quad 9.1600\ ]^T, \lambda_2 = 4.3095 \times 10^{-2}, \mathbf{K}_1 = -1.2989, \\ &\mathbf{K}_2 = -1.2905, \mathbf{Y}_{ij,d}(:,:,1,1) = \left[\ 1.3534 \quad 1.0218\ \right]^T, \\ &\mathbf{Y}_{ij,d}(:,:,1,2) = \left[\ 1.7592 \quad 1.2711\ \right]^T, \mathbf{Y}_{ij,d}(:,:,2,1) = \left[\ 1.6862 \quad 1.3124\ \right]^T, \\ &\mathbf{Y}_{ij,d}(:,:,2,2) = \left[\ 1.3291 \quad 9.4836 \times 10^{-1}\ \right]^T, \end{aligned}$$

By choosing $\mathbf{x}(0) = [1 \quad 2]^T$, $\mathbf{x}(0) = [10 \quad 10]^T$ and $\mathbf{x}(0) = [15 \quad 18]^T$, respectively, the corresponding time responses are shown in the following figures:

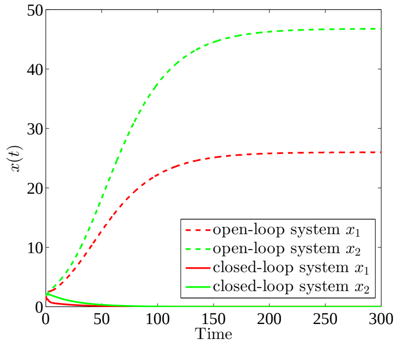

Fig 1. Time response of the states x_1 and x_2 with initial condition $\mathbf{x}(0) = [1 \quad 2]^T$ and time span of 300s for the open-loop and the closed-loop system, respectively.

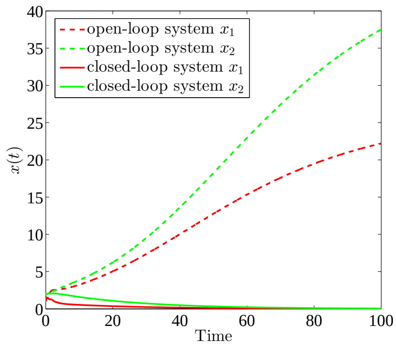

Fig 2. Time response of the states x_1 and x_2 with initial condition $\mathbf{x}(0) = [1 \quad 2]^T$ and time span of 100s for the open-loop and the closed-loop system, respectively.

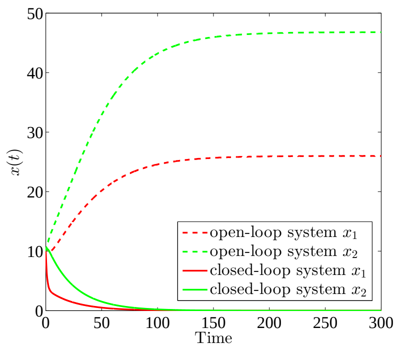

Fig 3. Time response of the states x_1 and x_2 with initial condition $\mathbf{x}(0) = [10 \quad 10]^T$ and time span of 300s for the open-loop and the closed-loop system, respectively.

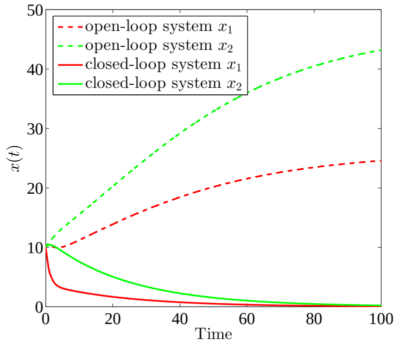

Fig 4. Time response of the states x_1 and x_2 with initial condition $\mathbf{x}(0) = [10 \quad 10]^T$ and time span of 100s for the open-loop and the closed-loop system, respectively.

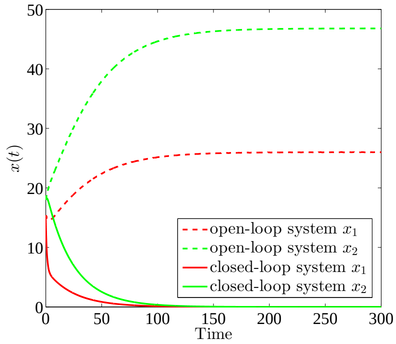

Fig 5. Time response of the states x_1 and x_2 with initial condition $\mathbf{x}(0) = [15 \quad 18]^T$ and time span of 300s for the open-loop and the closed-loop system, respectively.

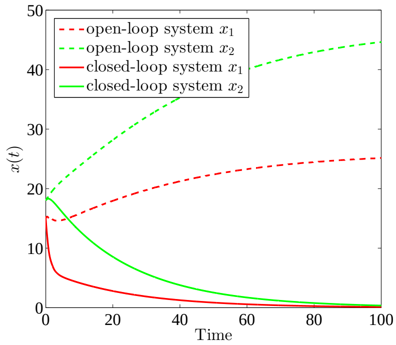

Fig 6. Time response of the states x_1 and x_2 with initial condition $\mathbf{x}(0) = [15 \quad 18]^T$ and time span of 100s for the open-loop and the closed-loop system, respectively.

According to the time responses of the open-loop system, we can see that the original systems are reasonable, because the density of immature population and the mature polulation of the species are limited, which caters for the fact that the species should not keep growing practically.

It can be seen from the time responses of the closed-loop system that the controller can drive the system to be asymptotically stable. Meanwhile, we can see that the positivity of the closed-loop system also can be ensured because all of the state variables always stay in the positive quadrant. Therefore, the validity and reliability of the theory in our paper are verified.

[R4] Zhang, Q., Liu, C., Zhang, X. "Complexity, analysis and control of singular biological systems." (Springer, London, 2012).

Comment 7
Is there a difference in the problem presented in this paper compared with the problem for general systems (non-positive systems)? Please the authors compare the proposed results with the existing related results for general linear systems (non-positive systems).

Response 7
Yes, there are many differences between the work in our paper and the work for general linear systems.

Firstly, for a positive system, the system matrix $\mathbf{A}_i(\mathbf{x})$ should be a Metzler matrix which means the non-diagonal elements are non-negative. Meanwhile the elements of the rest matrices such as $\mathbf{B}_i(\mathbf{x})$ and $\mathbf{C}$ should be non-negative. While for a general system, there is no the above characteristic for system matrices. As far as a closed-loop positive system is concerned, both of the convex stability and the positivity require to be ensured at the same time, hence, transferring the non-convex stability and positivity conditions into convex ones simultaneously will make the stability analysis process become relatively complex. While for a closed-loop general system, there is no requirement for positivity, which means the stability analysis process for general systems is simpler than for positive systems.

Secondly, in terms of complexity of positive systems, we mainly aim at the positive nonlinear systems instead of positive linear systems. As we all know, a positive nonlinear system is more complex than a positive linear system due to the nonlinearities. For this reason, most of the existing results are for positive linear systems, although some researchers have tried to start studying positive nonlinear systems through T-S fuzzy models, this kind of fuzzy model can only express relatively simple positive nonlinear systems. In our paper, we mainly employ the advanced polynomial fuzzy model to express a positive nonlinear system, which makes the results suitable for more complicated positive systems. However, it is worth mentioning that the polynomials existing in the subsystems and membership functions will make the stability and positivity analysis challenging because many matrix processing techniques cannot be used directly, especially when the inverse matrix of a polynomial matrix is used in the method. That is also the reason why not many researchers have adopted the polynomial fuzzy model to investigate the control synthesis and stability analysis for positive nonlinear systems. While for a general linear system, it is simpler to design a controller and analyze the stability because it is not constrained by the positive conditions and it is not affected by polynomials.

Thirdly, considering the unique positivity, a novel Lyapunov function (linear copositive Lyapunov function) $V(t) = \lambda^T \mathbf{x}(t)$ is used for the stability analysis of positive polynomial fuzzy systems. This kind of Lyapunov function not only can capture the unique positivity but also can reduce the difficulty of analysis and calculation. While for a general linear system, in general, the quadratic Lyapunov function $V(t) = \mathbf{x}^T(t)\mathbf{P}(x)(t)$ is used to analyze the stability. By comparing the two kind of Lyapunov functions, it can be concluded that the former is simpler and more helpful to facilitate the stability and positivity analysis.

Comment 8

The significance of the results is scarce as well since they are extremely specific: a special class of the membership functions is considered, which guarantees stability and positivity not in general, but only provided that some SOS conditions are satisfied.

Response 8

The authors would like to thank the reviewer for the comment. It is well known that the stability conditions are valid for any arbitrary membership functions when the information of membership functions are ignored, which leads to the results in conservativeness. As the stability conditions only need to be valid under the specific membership functions used in the investigated fuzzy plant and fuzzy controller, therefore, bringing the information of membership functions into the stability analysis contributes to reduce the conservatism of the analysis results.

In our paper, the information of membership functions are not considered into the stability analysis when the Theorem 1 is derived, therefore, the Theorem 1 is conservative. In order to relax the conservatism, Theorem 2 is derived by taking the information of membership functions into account. But we just use some mathematical symbols to represent the detailed information of membership functions in Theorem 2, such as $\eta_{ij,d}(\mathbf{x})$ (the approximation polynomials of the original membership functions), $\overline{\beta}_{ij,d}$ and $\underline{\beta}_{ij,d}$ (the upper bound and lower boundary of the approximated errors), which means these information can be obtained based on any classes of membership functions instead of a special class of membership functions.

However, in the Simulation Example, for verifying the reliability of the Theorem 2, we need to obtain these detailed information of membership functions based on specific membership functions so that these mathematical symbols can be assigned values and used for simulating. And that is the reason why we have to choose specific membership functions in the Simulation Example.

To sum up, in the process of derivation, the positive polynomial fuzzy systems, the fuzzy controller and the corresponding membership functions are universal. But when we try to verify the effective of Theorems by a numeral example, we have to choose specific membership functions because the stability conditions in Theorem 2 contain the information of membership functions. If we still use the mathematical symbols instead of the detailed values of the information of special membership functions, the specific feasible solutions of these decision variables in Theorem 2 cannot be obtained by SOSTOOLS.

Comment 9
In Eq(12), the L_1 performance index has been applied. Firstly, more specific derivations should be given more clearly. Besides, why $V(\infty)$ and $V(0)$ are equal to 0 for $T \to \infty$? This issue should be explained more clearly.

Response 9
The authors would like to thank the reviewer for the comment. As suggested by this reviewer, in order to make the derivations more clear and formal, we have shown the formal proofs for both Theorem 1 and Theorem 2 in the revised paper. The changes can be found from pages 4-7.

As for the question: why $V(\infty)$ and $V(0)$ are equal to 0 for $T \to \infty$? I think it may since our expression was not very clear in the previous version, which led to the misunderstanding of the reviewer. In fact, we want to express that under zero initial condition, $V(0)$ satisfies $V(0) = 0$. And when $T \to \infty$, $V(\infty)$ satisfies $V(\infty) \to 0$. In the revised paper, we have shown the derivation processes more formally so that readers can better understand the theorems in our paper.

Comment 10

The authors approximate the non-convex term into convex one by introducing the Eq (17). However, the Eq (17) will bring more conservativeness for the main results, which will against the purpose of reducing conservativeness.

Response 10

The authors would like to thank the reviewer for the comment. We agree with you that the introduction of the constraint conditions will lead to a certain degree of conservativeness, but as we all known that converting non-convex stable conditions and positive conditions to convex conditions simultaneously is very difficult since most of the convex methods are just for general systems and there are no positivity constrains for general systems. When positivity constrains are taken into consideration, those convex methods for general systems may not be able to work. In addition, the convexification is problem dependent, which means the convexificiation method from one paper may not be able to be applied to other non-convex problems, therefore, it is a hard task to find a proper method to deal with the non-convex problem so that the stability conditions and positivity conditions can be ensured to be convex simultaneously. In our paper, the first task is to find a proper method to transform the non-convex conditions into convex ones. After solving this tricky problem, the second task is to reduce the conservativeness by employing membership-function-dependent techniques. Therefore, although the convex method in our paper will bring some conservatism, the high degree polynomial approximation method can effectively relax the conservativeness. To clarify the concern, we have discussed it in Remark 3 in the revised paper:

Remark 3: Although the convex method given in our paper will introduce a certain degree of conservativeness, this method makes it easier to transform the non-convex stability conditions and positivity conditions into convex ones simultaneously, which means the non-convex problem can be dealt with skillfully. In order to reduce the conservativeness, in the following, a high degree polynomial approximation method will be introduced so that the information embedded in MFs can be found and introduced into stability conditions.

The changes can be found on the right column, lines 44-52, page 5 of the revised paper.

Responses to the **Reviewer 4's** comments and questions for the paper entitled "***Membership-Function-Dependent Design of L_1-Gain Output Feedback Controller for Stabilization of Positive Polynomial Fuzzy Systems***" with **Paper No.: TFS-2020-0730**.

We would like to express our sincere gratitude to you for valuable and constructive comments. Our responses are as follows.

Comment 1
What do they call "non-convex stability conditions"?

Response 1
A convex condition requires each term is linear, if there exist nonlinear terms in stability conditions, they belong to non-convex stability conditions. In general, there are two different cases that may lead to non-convex stability conditions. Firstly, non-convex conditions may be caused by coupling of decision variables. For example, the (25) in the revised paper is non-convex since the decision variables λ_1 and the $\mathbf{K}_j(\mathbf{y})$ is coupled in this condition.

$$\begin{aligned}&\mathbf{I}_1^T\tilde{\mathbf{D}}_{ij}(\mathbf{x},\mathbf{y})+\lambda_1^T\tilde{\mathbf{A}}_{ij}(\mathbf{x},\mathbf{y})+\lambda_2^T\mathbf{C}\\=&\mathbf{I}_1^T(\mathbf{D}_i(\mathbf{x})+\mathbf{E}_i(\mathbf{x})\mathbf{K}_j(\mathbf{y})\mathbf{C})+\lambda_1^T(\mathbf{A}_i(\mathbf{x})+\mathbf{B}_i(\mathbf{x})\mathbf{K}_j(\mathbf{y})\mathbf{C})+\lambda_2^T\mathbf{C}\prec 0.\end{aligned}$$

To solve this non-convex condition, (14) and (15) are introduced, we have:

$$\begin{aligned}&\mathbf{I}_1^T(\mathbf{D}_i(\mathbf{x})+\mathbf{E}_i(\mathbf{x})\mathbf{K}_j(\mathbf{y})\mathbf{C})+\lambda_1^T(\mathbf{A}_i(\mathbf{x})+\mathbf{B}_i(\mathbf{x})\mathbf{K}_j(\mathbf{y})\mathbf{C})+\lambda_2^T\mathbf{C}\\&\preceq\mathbf{I}_1^T\mathbf{D}_i(\mathbf{x})+2\mathbf{I}_1^T\mathbf{E}_i(\mathbf{x})\mathbf{K}_j(\mathbf{y})\mathbf{C}+\lambda_1^T\mathbf{A}_i(\mathbf{x})+\lambda_2^T\mathbf{C}.\end{aligned}$$

Secondly, non-convex conditions may be caused by the existence of other nonlinear terms such as $sin(\mathbf{x}), cos(\mathbf{x}), 1/f(\mathbf{x})$. For example, in the revised paper, the condition (41) is non-convex as well because $\alpha_d(\mathbf{x})$ which belongs to the nonlinear term $1/f(\mathbf{x})$ exists in this condition.

$$\begin{aligned}&\sum_{i=1}^{p}\sum_{j=1}^{c}\Big((\eta_{ij,d}(\mathbf{x})+\underline{\beta}_{ij,d})\mathbf{H}_{ij}(\mathbf{x})+(\overline{\beta}_{ij,d}-\underline{\beta}_{ij,d})\mathbf{Y}_{ij,d}(\mathbf{x})\Big)\\&=\sum_{i=1}^{p}\sum_{j=1}^{c}\Big(((\eta_{ij,d}^{0.5}(\mathbf{x}))^2\alpha_d(\mathbf{x})+\underline{\beta}_{ij,d})\mathbf{H}_{ij}(\mathbf{x})+(\overline{\beta}_{ij,d}-\underline{\beta}_{ij,d})\mathbf{Y}_{ij,d}(\mathbf{x})\Big)\preceq 0,\ \forall\ d.\end{aligned}$$

Then by multiplying both sides of (41) by the denominator of $\alpha_d(\mathbf{x})$, this non-convex condition can be approximated by the following convex condition:

$$\sum_{i=1}^{p}\sum_{j=1}^{c}\Big(\big((\eta_{ij,d}^{0.5}(\mathbf{x}))^2\alpha_{d,num}+\underline{\beta}_{ij,d}\alpha_{d,den}(\mathbf{x})\big)\mathbf{H}_{ij}(\mathbf{x})$$
$$+\big(\overline{\beta}_{ij,d}-\underline{\beta}_{ij,d}\big)\alpha_{d,den}(\mathbf{x})\mathbf{Y}_{ij,d}(\mathbf{x})\Big)\preceq 0,\ \forall\, d.$$

Comment 2

Why in equation (1), specifically, in the equation for controlled output, $\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$, does the matrix $\mathbf{C}$ not depend on the rule? that is, why not write $\mathbf{C}_i$, unlike the other matrices?

Response 2

Yes, we mainly aim at the case that the output $\mathbf{y}(t)$ is independent on the fuzzy rules, therefore, the output matrix $\mathbf{C}$ does not written as $\mathbf{C}_i$ in our paper. When the output is considered to be dependent on fuzzy rules, it will be written as $\mathbf{y} = \sum_{i=1}^{p} w_i(\mathbf{x})\mathbf{C}_i\mathbf{x}$, which will lead to the stability and positivity analysis more challenging and more effort needs to be made to deal with the non-convex problem. Therefore, in this paper, we consider the case that the output is independent on the fuzzy rules, that is, $\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$.

Comment 3

It should be noted that the fuzzy system is Type 1. Show each rule of the fuzzy rule base.

Response 3

Yes, this fuzzy system belongs to Type 1. We totally agree with you that showing each rule of the fuzzy rule base will be more intuitive and helpful for readers to better understand the fuzzy systems, however, in order to keep this paper concise, we prefer to show the fuzzy rules of fuzzy model and the fuzzy rules of fuzzy controller as (1) and (3) in this paper, respectively.

Actually, the (1) can represent each fuzzy rule of the fuzzy model because it shows the i-th fuzzy rule and $i \in \{1, 2, \ldots, p\}$ where p is where c is the number of the fuzzy rules of the fuzzy controller.

Comment 4

It would be nice to add diagram of the system.

Response 4

The authors would like to thank the reviewer for the comment. As suggested by this reviewer, we have added the diagram of system in the revised paper.

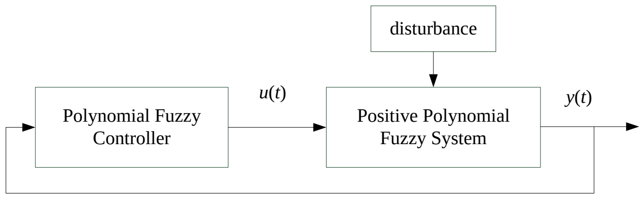

Fig. 1 The positive polynomial fuzzy closed-loop system with disturbance.

Comment 5

Indicate whether the elements of the disturbance signal $\tilde{\mathbf{w}}$ are bounded.

Response 5

The authors would like to thank the reviewer for the comment. As suggested by the reviewer, we have added an assumption in the revised paper to indicate that the elements of the disturbance signal should be bounded.

Assumption 2 : The disturbance signal $\tilde{\mathbf{w}}(t) \succeq 0$ is bounded.

The changes can be found on the right column, lines 26, page 3 of the revised paper.

Comment 6

The proof or any comment about the proof of theorems 1 and 2 is missing. Furthermore, there is no explicit way to obtain the feedback control gains matrix of the output $\mathbf{K}_j(\mathbf{y})$. Nothing is stated about the membership functions for the plant in Example 1, only for the controller.

Response 6

The authors would like to thank the reviewer for the comment. As suggested by this reviewer, the formal proofs for both of Theorem 1 and Theorem 2 have been shown in the revised paper, which can be found from pages 4-7.

To make it clear that how to obtain the feedback control gains matrix of the output $\mathbf{K}_j(\mathbf{y})$, we add the following sentence in the revised paper: "$\mathbf{k}_{jr}(\mathbf{y}) \in R^{1\times l}$ is the r-th row of the output feedback gain $\mathbf{K}_j(\mathbf{y})$ which can be obtained directly if the above conditions are satisfied, for all j."

The changes can be found on the right column, lines 23-25, page 4 and the right column, lines 48-50, page 6 of the revised paper.

As suggested by this reviewer, we also add the statement for the membership functions of the PPFMB system in the revised paper.

The PPFMB system is with 3 fuzzy rules and the corresponding MFs are chosen as follows: $w_1(x_1) = 1 - \frac{1}{1+e^{-(x_1-9)}}$, $w_2(x_1) = 1 - w_1(x_1) - w_3(x_1)$,

$w_3(x_1) = \frac{1}{1+e^{-(x_1-11)}}$. According to the imperfect premise matching concept [37], the number of the fuzzy rules of the PFOF controller is chosen as 2 which is different from the number of the rules of the PPFMB system. The MFs of the PFOF controller are shown as: $m_2(x_1) = 1 - m_1(x_1)$, $m_2(x_1) = 1 - m_1(x_1)$,

$$m_1(x_1) = \begin{cases} 1, & \text{for} \quad x_1 < 2; \\ \frac{-x_1+18}{16}, & \text{for} \quad 2 \leq x_1 \leq 18; \\ 0, & \text{for} \quad x_1 > 18. \end{cases}$$

The changes can be found on the left column, lines 54-59, page 8 and on the right column, lines 2-7, page 8 of the revised paper.

Comment 7

Show in detailed manner the application examples in order to let the reader to reproduce the experiments. It would have been desirable to include a second example with some real dynamics system.

Response 7

The authors would like to thank the reviewer for the comment. In order to highlight the practical significance of studying positive systems, we add some relevant content in the Introduction in the revised paper and give an example with a real system in the following.

"Positive systems, whose state variables and outputs always remain in the non-negative quadrant if both of the initial conditions and input are non-negative, are often encountered in real-world applications [1]-[3]. A great deal of practical models of such systems exist in a variety of disciplines, for example, the control of the cortisol level within the hypothalamic-pituitary-adrenal gland axis in the field of biology, the human immunodeficiency virus viral mutation dynamics in the area of pharmacokinetic, the prey-predator model in the aspect of ecology, and the concentration of substances in chemical processes and so on [4]-[6]. As positive systems are closely related to our daily life, it is of great practical significance to conduct in-depth research on positive systems."

The changes can be found on the left column, lines 34-46, page 1 of the revised paper.

Based on [R1], a biological system model is chosen and applied to check the effective of Theorem 2, but due to the limitation of the pages, the corresponding results are just shown in this reply letter for checking by the reviewers.

In the following, a single species with a stage structure model is given

$$\dot{x}_1(t) = \alpha x_2(t) - \gamma_1 x_1(t) - \beta x_1(t) - \eta x_1^2(t) + \delta \tilde{w}(t) + \xi u(t),$$
$$\dot{x}_2(t) = \beta x_1(t) - \gamma_2 x_2(t),$$

where $x_1(t)$ is the density of immature population of the species and $x_2(t)$ denotes the density of mature population of the species. α is the birth rate of the immature population; for the immature population, γ_1 and β are called the death rate and transformation rate of immature, respectively; η is called the density restriction for the immature population, γ_2 is called the death rate of the mature population; δ is the immigrant rates of immature population of the species from other areas to this area; $\tilde{w}(t)$ is the disturbance, which denotes the density of the immigrant immature population of this species; ξ is the control rate, and $u(t)$ is the control input, which denotes the controlled density of the immature population of this species.

Next, for simplify, $x_1(t)$ and $x_2(t)$ will be denoted by x_1 and x_2, respectively.

Defining the density of the immature species as $x_1 \in [0, 20]$, then we have $f(x_1) = x_1 = \mu_{11}(x_1) f_{max} + \mu_{12}(x_1) f_{min}$, where $\mu_{11}(x_1) = \frac{f(x_1) - f_{min}}{f_{max} - f_{min}}$, $\mu_{11}(x_1) = 1 - \mu_{12}(x_1)$, $f_{max} = 20$, $f_{min} = 0$. By employing the sector nonlinearity technique, we get the fuzzy model of this system, which is shown as follows:

$$\begin{aligned} \text{Rule 1 : } & \text{IF } x_1 \text{ is LARGE} \\ & \text{THEN } \dot{\mathbf{x}} = \mathbf{A}_1(\mathbf{x})\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{B}_{1\omega}\tilde{\mathbf{w}}, \\ \text{Rule 2 : } & \text{IF } x_1 \text{ is SMALL} \\ & \text{THEN } \dot{\mathbf{x}} = \mathbf{A}_2(\mathbf{x})\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{B}_{1\omega}\tilde{\mathbf{w}}, \end{aligned}$$

Then the overall fuzzy model of this system is shown as follows:

$$\dot{\mathbf{x}} = \sum_{i=1}^{2} w_i(\mathbf{x}) \big(\mathbf{A}_i(\mathbf{x})\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{B}_{1\omega}\tilde{\mathbf{w}} \big),$$
$$\mathbf{y} = \mathbf{C}\mathbf{x},$$

where $w_i(\mathbf{x})$, $i \in 1, 2$ is the membership function of the systems.

$$\mathbf{A}_1 = \begin{bmatrix} -f_{max}\eta - \gamma_1 - \beta & \alpha \\ \beta & -\gamma_2 \end{bmatrix}, \mathbf{A}_2 = \begin{bmatrix} -f_{min}\eta - \gamma_1 - \beta & \alpha \\ \beta & -\gamma_2 \end{bmatrix},$$
$$\mathbf{B} = \begin{bmatrix} \xi & 0 \end{bmatrix}^T, \mathbf{B}_{1\omega} = \begin{bmatrix} \delta & 0 \end{bmatrix}^T, \mathbf{x} = [\, x_1 \quad x_2 \,]^T,$$

where α, β, η, γ, k, β_1, γ_1, γ_2, η_1, δ and ξ are positive constant scalars.

The membership functions of the system are $w_1(x_1) = \frac{x_1}{20}$ and $w_2(x_1) = 1 - w_1(x_1)$. Then we assume that the output is $y = x_1$, so the output matrix is $\mathbf{C} = [1 \quad 0]$. The controlled output z and the disturbance $\tilde{w}$ are chosen as $z = 1.5x_1 + 1.1u$ and $\tilde{w} = 5e^{-t}|\cos(2t)|$, respectively.

In the following, a polynomial fuzzy controller is designed as follows:

$$\begin{aligned}\text{Rule } 1 : &\text{ IF } \mathbf{y} \text{ is LARGE}\\ &\text{ THEN } \mathbf{u} = \mathbf{K}_1(\mathbf{y})\mathbf{y},\\ \text{Rule } 2 : &\text{ IF } \mathbf{y} \text{ is SMALL}\\ &\text{ THEN } \mathbf{u} = \mathbf{K}_2(\mathbf{y})\mathbf{y},\end{aligned}$$

where $\mathbf{K}_1(\mathbf{y})$ and $\mathbf{K}_2(\mathbf{y})$ are the output feedback gains.

Then the polynomial fuzzy controller is designed as follows:

$$\mathbf{u} = \sum_{j=1}^{2} m_j(\mathbf{y})\mathbf{K}_j(\mathbf{y})\mathbf{C}\mathbf{x},$$

where $m_j(\mathbf{y})$, $j \in 1, 2$ is the membership function of the controller.

Because $y = x_1$ and the ranges of y is assumed as $y \in [0, 20]$, hence, m_j is related to x_1 and is chosen as follows:

$$m_1(x_1) = \begin{cases} 1, & \text{for} \quad x_1 < 2 \\ \frac{-x_1+18}{16}, & \text{for} \quad 2 \leq x_1 \leq 18 \\ 0. & \text{for} \quad x_1 > 18 \end{cases}$$

When we choose $\alpha = 0.3; \gamma_1 = 0.1; \beta = 0.18; \eta = 0.01; \gamma_2 = 0.1$, $\delta = 0.5$ and $\xi = 0.5$, the system matric $\mathbf{A}_1$ and $\mathbf{A}_2$ are Metzler, $\mathbf{B} \succeq 0$ and $\mathbf{C} \succeq 0$, which means the real system is a positive system. Then the whole operating region is divided into 2 sub-regions and the highest degree of the approximation polynomial functions is chosen as 3. The degrees of $\mathbf{K}_j(\mathbf{x})$ and $\mathbf{Y}_{ij,d}(\mathbf{x})$ are chosen as 0, respectively. And by setting $\epsilon_1 = \epsilon_2(\mathbf{x}) = 0.001$, the decision variables $\lambda_1, \lambda_2, \mathbf{K}_j(\mathbf{x})$ and $\mathbf{Y}_{ij,d}(\mathbf{x})$ are obtained as follows:

$$\begin{aligned}&\gamma = 1.1020, \lambda_1 = [\ 2.2020 \quad 9.1600\ ]^T, \lambda_2 = 4.3095 \times 10^{-2}, \mathbf{K}_1 = -1.2989,\\ &\mathbf{K}_2 = -1.2905, \mathbf{Y}_{ij,d}(:,:,1,1) = \left[\ 1.3534 \quad 1.0218\ \right]^T,\\ &\mathbf{Y}_{ij,d}(:,:,1,2) = \left[\ 1.7592 \quad 1.2711\ \right]^T, \mathbf{Y}_{ij,d}(:,:,2,1) = \left[\ 1.6862 \quad 1.3124\ \right]^T,\\ &\mathbf{Y}_{ij,d}(:,:,2,2) = \left[\ 1.3291 \quad 9.4836 \times 10^{-1}\ \right]^T,\end{aligned}$$

By choosing $\mathbf{x}(0) = [1 \quad 2]^T$, $\mathbf{x}(0) = [10 \quad 10]^T$ and $\mathbf{x}(0) = [15 \quad 18]^T$, respectively, the corresponding time responses are shown in the following figures:

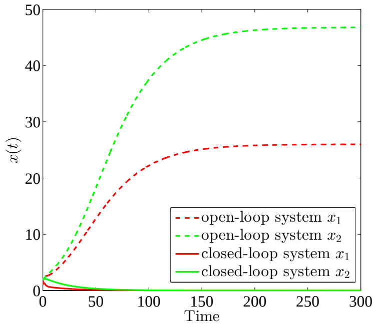

Fig 1. Time response of the states x_1 and x_2 with initial condition $\mathbf{x}(0) = [1 \quad 2]^T$ and time span of 300s for the open-loop and the closed-loop system, respectively.

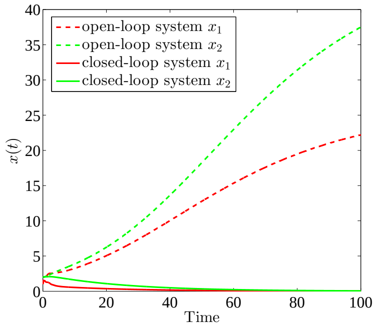

Fig 2. Time response of the states x_1 and x_2 with initial condition $\mathbf{x}(0) = [1 \quad 2]^T$ and time span of 100s for the open-loop and the closed-loop system, respectively.

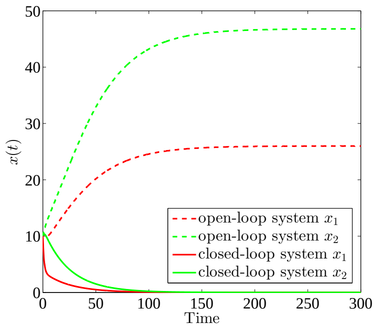

Fig 3. Time response of the states x_1 and x_2 with initial condition $\mathbf{x}(0) = [10 \quad 10]^T$ and time span of 300s for the open-loop and the closed-loop system, respectively.

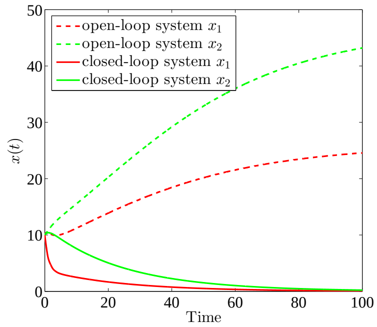

Fig 4. Time response of the states x_1 and x_2 with initial condition $\mathbf{x}(0) = [10 \quad 10]^T$ and time span of 100s for the open-loop and the closed-loop system, respectively.

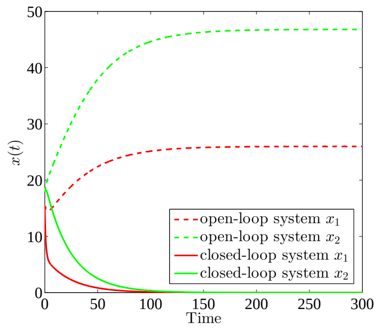

Fig 5. Time response of the states x_1 and x_2 with initial condition $\mathbf{x}(0) = [15 \quad 18]^T$ and time span of 300s for the open-loop and the closed-loop system, respectively.

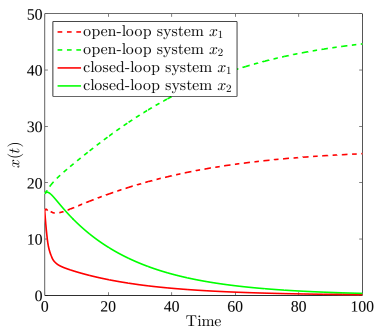

Fig 6. Time response of the states x_1 and x_2 with initial condition $\mathbf{x}(0) = [15 \quad 18]^T$ and time span of 100s for the open-loop and the closed-loop system, respectively.

According to the time responses of the open-loop system, we can see that the original systems are reasonable, because the density of immature population and the mature polulation of the species are limited, which caters for the fact that the species should not keep growing practically.

It can be seen from the time responses of the closed-loop system that the controller can drive the system to be asymptotically stable. Meanwhile, we can see that the positivity of the closed-loop system also can be ensured because all of the state variables always stay in the positive quadrant. Therefore, the validity and reliability of the theory in our paper are verified.

[R1] Zhang, Q., Liu, C., Zhang, X. "Complexity, analysis and control of singular biological systems." (Springer, London, 2012).

Comment 8
To extract the information from the original membership functions, high-degree polynomial approximations were used, however, goodness of fit with r or r^2 is not reported.

Response 8
The authors would like to thank the reviewer for the comment. As suggested by this reviewer, we have obtained the goodness of fit with r^2 for the degrees of approximation polynomials being 3 and 5 when $D = 1$. The details are shown in the following:

Table I Goodness of Fit with r^2

(i,j)	degree of polynomials being 3	degree of polynomials being 5
$(1,1)$	9.9146×10^{-1}	9.9696×10^{-1}
$(1,2)$	9.1942×10^{-1}	9.9247×10^{-1}
$(2,1)$	5.4276×10^{-1}	8.6451×10^{-1}
$(2,2)$	5.4276×10^{-1}	8.6451×10^{-1}
$(3,1)$	9.1942×10^{-1}	9.9247×10^{-1}
$(3,2)$	9.9146×10^{-1}	9.9696×10^{-1}

From the above Table, we can see that when the degree of the approximation polynomials is 3, the cross terms of membership functions $w_1(\mathbf{x})m_2(\mathbf{x})$ and $w_2(\mathbf{x})m_1(\mathbf{x})$ cannot be fitted very well, but when the degree of the approximation polynomials is 5, the cross terms of membership functions $w_1(\mathbf{x})m_2(\mathbf{x})$ and $w_2(\mathbf{x})m_1(\mathbf{x})$ can be fitted better. It is reasonable because when the degree of the approximation polynomials is higher, the obtained approximation effect is better.

Comment 9
It is concluded that using a single example ... that a broader region of stability is obtained. However, this is mathematically inaccurate.

Response 9
The authors would like to thank the reviewer for the comment. Although we

only show a simulation example for verifying the reliability and effectiveness of the theorems in our paper, we analyze this example from different aspects. Besides, for each of the aspects, we analyze and compare the simulation results based not only on one case but on multiple cases. Therefore, the obtained conclusions aiming at each of the aspects are convincing. The details are shown as follows:

1) How the stability regions vary with the values of D. In order to show the influence of the number of subdomains D for stability regions, we mainly compare the results in the following cases.

Case 1: the highest degree of the polynomial functions is set as 3 without the fraction $\alpha_d(x_1)$, then comparing the stability regions for $D = 1$ and $D = 2$. Please compare the stability region ("×") with the stability region ("+") in Fig. 2 in the revised paper.

Case 2: the highest degree of the polynomial functions is set as 5 without the fraction $\alpha_d(x_1)$, then comparing the stability regions for $D = 1$ and $D = 2$. Please compare the stability region ("⋄") with the stability region ("×") in Fig. 3 in the revised paper.

Case 3: the highest degree of the polynomial functions is set as 3 with the fraction $\alpha_d(x_1)$, then comparing the stability regions for $D = 1$ and $D = 2$. Please compare the stability region ("∗") with the stability region ("□") in Fig. 4 in the revised paper.

Case 4: the highest degree of the polynomial functions is set as 5 with the fraction $\alpha_d(x_1)$, then comparing the stability regions for $D = 1$ and $D = 2$. Please compare the stability region ("×") with the stability region ("+") in Fig. 5 in the revised paper.

By comparing and analyzing the simulation results in the above four cases, we can safely draw a conclusion that the number of the subdomains is larger, the stability region is more extensive.

2) How the stability regions change with the highest degrees of approximated polynomial function. For showing relaxation effect of the highest degrees of approximated polynomial function, the following four cases are considered.

Case 1: the number of subdomains is chosen as $D = 1$, when the fraction $\alpha_d(x_1)$ is removed, comparing the stability region for the highest degree of the polynomial functions being 3 with the one for the highest degree of the polynomial functions being 5. Please compare the stability region ("×") in Fig. 2 with the stability region ("⋄") in Fig. 3 in the revised paper.

Case 2: the number of subdomains is chosen as $D = 2$, when the fraction $\alpha_d(x_1)$ is removed, comparing the stability region for the highest degree of the polynomial functions being 3 with the one for the highest degree of the polynomial functions being 5. Please compare the stability region (“+”) in Fig. 2 with the stability region (“×”) in Fig. 3 in the revised paper.

Case 3: the number of subdomains is chosen as $D = 1$, when the fraction $\alpha_d(x_1)$ is considered, comparing the stability region for the highest degree of the polynomial functions being 3 with the one for the highest degree of the polynomial functions being 5. Please compare the stability region (“∗”) in Fig. 4 with the stability region (“×”) in Fig. 5 in the revised paper.

Case 4: the number of subdomains is chosen as $D = 2$, when the fraction $\alpha_d(x_1)$ is considered, comparing the stability region for the highest degree of the polynomial functions being 3 with the one for the highest degree of the polynomial functions being 5. Please compare the stability region (“□”) in Fig. 4 with the stability region (“+”) in Fig. 5 in the revised paper.

By comparing and analyzing the simulation results in the above four cases, we can safely draw a conclusion that the higher degrees of polynomial functions will lead to more relaxed results.

3) How the stability regions are affected by the $\alpha_d(x_1)$. We analyze the influence of the fraction $\alpha_d(x_1)$ for stability regions in four cases.

Case 1: the number of subdomains is $D = 1$ and the highest degree of the polynomial functions is set as 3, then comparing the stability region obtained without considering the fraction $\alpha_d(x_1)$ with the one obtained with considering the fraction $\alpha_d(x_1)$. Please compare the stability region (“×”) in Fig. 2 with the stability region (“∗”) in Fig. 4 in the revised paper.

Case 2: the number of subdomains is $D = 1$ and the highest degree of the polynomial functions is set as 5, then comparing the stability region obtained without considering the fraction $\alpha_d(x_1)$ with the one obtained with considering the fraction $\alpha_d(x_1)$. Please compare the stability region (“⋄”) in Fig. 3 with the stability region (“×”) in Fig. 5 in the revised paper.

Case 3: the number of subdomains is $D = 2$ and the highest degree of the polynomial functions is set as 3, then comparing the stability region obtained without considering the fraction $\alpha_d(x_1)$ with the one obtained with considering the fraction $\alpha_d(x_1)$. Please compare the stability region (“+”) in Fig. 2 with the stability region (“□”) in Fig. 4 in the revised paper.

Case 4: the number of subdomains is $D = 2$ and the highest degree of the polynomial functions is set as 5, then comparing the stability region obtained without considering the fraction $\alpha_d(x_1)$ with the one obtained with considering the fraction $\alpha_d(x_1)$. Please compare the stability region ("×") in Fig. 3 with the stability region ("+") in Fig. 5 in the revised paper.

By comparing and analyzing the simulation results in the above four cases, we can safely draw a conclusion that $\alpha_d(x_1)$ is in favour of extending the stability regions.

4) Compare the stability regions obtained using the method with the ones obtained using other methods. For comparing with the approximated method in [36] in the revised paper, we will discuss the results in two cases.

Case 1: setting the number of subdomains as $D = 4$ and the highest degree of the approximated polynomial functions as 2 without considering the boundary information of the state variables, comparing the stability region obtained by using the approximated method in this paper with the one obtained by using the approximated method in [36] in the revised paper. Please compare the stability region ("×") with the stability region ("□") in Fig. 6.

Case 2: setting the number of subdomains as $D = 4$ and the highest degree of the approximated polynomial functions as 2 with considering the boundary information of the state variables, comparing the stability region obtained by using the approximated method in this paper with the one obtained by using the approximated method in [36] in the revised paper. Please compare the stability region ("+") in Fig. 2 with the stability region ("⋄") in Fig. 7.

By comparing and analyzing the simulation results in the above two cases, it arrives at a conclusion that the method in our paper can generate better relaxation effect than the method in [36] no matter the boundary information of the state variables is taken into account or not.

Comment 10
Define each acronym, each variable, constant, symbol used in each equation.

Response 10
The authors would like to thank the reviewer for the comment. As suggested by the reviewer, we have deleted some acronyms which only occur once or twice. The acronyms that still remain in use have been summarised in Table I in the revised paper. Meanwhile, we have defined the constant, symbol used in each equation in Table II. The details are shown as follows:

1. "LMI" is replaced by "linear-matrix-inequality".

2. "IMP" is replaced by "imperfect premise matching".

3. "LP" is replaced by " linear programming".

4. "LKF" is replaced by "Lyapunov-Krasovskii functional".

Table I Description of the Acronyms

Acronyms	Explanation	Acronyms	Explanation
T-S	Takagi-Sugeno	MFD	membership-function-dependent
SOS	sum-of-squares	PPFMB	positive polynomial fuzzy-model-based
MFs	membership functions	PFOF	polynomial fuzzy output feedback

Table II Description of Notations

Notation	Description	Notation	Description
$\mathbf{x}$	system state vector	$\mathbf{u}$	input vector
$\mathbf{z}$	control output	$\tilde{\mathbf{w}}$	disturbance
$\mathbf{y}$	output vector	$\mathbf{K}_j(\mathbf{y})$	static output feedback gain
$w_i(\mathbf{x})$	MFs of positive systems	$m_i(\mathbf{y})$	MFs of fuzzy controllers
ξ	augmenting vectors	λ	constant vector to be determined
γ	L_1 performance level	$\alpha_d(\mathbf{x})$	fractional function
$\eta_{ij,d}(\mathbf{x})$	approximated polynomial	$\Delta\eta_{ij,d}(\mathbf{x})$	approximation error
$\underline{\beta}_{ij,d}$	lower bound of error term	$\overline{\beta}_{ij,d}$	upper bound of error term

The changes can be found on the right column, lines 2-18, page 2.

Responses to the **Reviewer 5's** comments and questions for the paper entitled "***Membership-Function-Dependent Design of L_1-Gain Output Feedback Controller for Stabilization of Positive Polynomial Fuzzy Systems***" with **Paper No.: TFS-2020-0730**.

We would like to express our sincere gratitude to you for valuable and constructive comments. Our responses are as follows.

Comment 1
Please clarify the motivation of this work, since nearly every paragraph seems to contain a motivation as its last sentence, but they just turn out to be some claims of adding up complexity or reducing conservativeness. Is there any practical sense in studying such a complicated system as well as using the mentioned control strategies.

Response 1
The authors would like to thank the reviewer for this comment. As mentioned in the first paragraph in the Introduction, a great deal of practical models of positive systems exist in a variety of disciplines, for example, the field of biology, the area of pharmacokinetic, the aspect of ecology, and so on. In order to highlight the practical significance of studying positive systems, we add some relevant content in the first paragraph in the Introduction. The details are shown as follows:

"Positive systems, whose state variables and outputs always remain in the non-negative quadrant if both of the initial conditions and input are non-negative, are often encountered in real-world applications [1]-[3]. A great deal of practical models of such systems exist in a variety of disciplines, for example, the control of the cortisol level within the hypothalamic-pituitary-adrenal gland axis in the field of biology, the human immunodeficiency virus viral mutation dynamics in the area of pharmacokinetic, the prey-predator model in the aspect of ecology, and the concentration of substances in chemical processes and so on [4]-[6]. As positive systems are closely related to our daily life, it is of great practical significance to conduct in-depth research on positive systems. "

The changes can be found on the left column, lines 42-49, page 1.

In addition, in the view of the control strategies, it is relatively simple to design a fuzzy controller based on the full states feedback control strategy for positive polynomial fuzzy systems, but in actual life, it is usually difficult to obtain the full states information of real systems. In this case, designing output feedback fuzzy controllers is more effective and easier to implement because it only depends on output variables instead of the full states information. Therefore, investigating such a complicate systems and using this kind of control strategy have important practical significance.

In order to highlight the practical significance of using the output feedback control strategy, we add some relevant content in the Introduction. The details are shown as follows:

"In the view of control synthesis of PPFMB systems, it is relatively simple to design fuzzy controllers according to the full state feedback control strategy, but in actual life, it is usually difficult to obtain the full states information of real systems. Hence, when some of the state variables are not available, this strategy does not work anymore. In this case, designing fuzzy controllers based on output feedback strategy is more effective and easier to implement because it does not dependent on the full state information but only the output state variables."

The changes can be found on the left column, lines 10-16, page 2.

Comment 2
The paper is stuffed with acronyms which impede a fluent understanding of the content. It is suggested to make a clear table for summarizing these acronyms.

Response 2
The authors would like to thank the reviewer for this comment. To enhance the readability of this paper, we delete some acronyms which only occur once or twice. After processing, the acronyms that still remain in use have been summarised in Table I in the revised paper. The details are shown as follows:

1. "LMI" is replaced by "linear-matrix-inequality".

2. "IMP" is replaced by " imperfect premise matching".

3. "LP" is replaced by " linear programming".

4. "LKF" is replaced by "Lyapunov-Krasovskii functional".

Table I Description of the Acronyms

Acronyms	Explanation	Acronyms	Explanation
T-S	Takagi-Sugeno	MFD	membership-function-dependent
SOS	sum-of-squares	PPFMB	positive polynomial fuzzy-model-based
MFs	membership functions	PFOF	polynomial fuzzy output feedback

The changes can be found on the right column, lines 2-7, page 2.

Comment 3

It appears that the studied fuzzy models are formulated based on the type-1 fuzzy set. Why not use the more general type-2 fuzzy set to model such a system? Moreover, please compare this work with [R1], since the latter has considered the PFMB with type-2 fuzzy sets and also used the SOS tool to ensure stability. Then what is the difficulty in considering type-2 fuzzy set for this system in the presence of positive dynamics?

[R1] B. Xiao, H.-K. Lam, and H. Li, Stabilization of Interval Type-2 Polynomial-Fuzzy-Model-Based Control Systems, IEEE Transactions on Fuzzy Systems, vol. 25, no. 1, pp. 205-217, 2017.

Response 3

Yes, the studied fuzzy models are formulated based on the Type-1 fuzzy set instead of Type-2 fuzzy set. There are some differences from the Type-1 and Type-2. Firstly, both of the Type-1 fuzzy set and Type-2 fuzzy set can cope with the nonlinearities in control systems, but Type-2 fuzzy set has the capability to handle the uncertainties in nonlinear systems, while Type-1 fuzzy set cannot deal with the uncertainties because the membership functions in Type-1 fuzzy set do not contain any uncertain information. Secondly, comparing with the Type-2 fuzzy set, using Type-1 fuzzy set to analyze positive nonlinear systems is relatively simpler because there are only nonlinearities and positivity to be considered, but no uncertainties require to be considered. Therefore, at present, most of studied results on positive nonlinear systems are in terms of Type-1 fuzzy set instead of Type-2 fuzzy set.

In our paper, the existence of polynomials, positivity constrain and disturbance have made the stability analysis very complex, especially, how to transform the non-convex positivity conditions and stability conditions into convex ones simultaneously has become head-scratching, therefore, the more complex Type-2 fuzzy set is not taken into consideration in this paper.

Although the authors in [R1] have considered the PFMB with type-2 fuzzy sets and also used the SOS tool to ensure stability, the work in our paper and the work in [R1] cannot be compared with each other directly. We will give the reasons from the following several aspects.

Firstly, from the perspective of the type of control systems, in our paper, the positive polynomial fuzzy systems with disturbance based on Type-1 fuzzy set are investigated, which means the unique positivity and the disturbance of positive systems are considered but the uncertainties are not considered. However in [R1], the general polynomial fuzzy systems based on Type-2 fuzzy set are studied, which means the authors mainly coped with the uncertainties for general systems but not for positive systems with disturbance in [R1].

Secondly, from the view of the control strategy, in our paper, considering the case where only some of the state variables are measurable, we investigate the static output feedback control strategy, while in [R1], the authors mainly study the state feedback control strategy. Comparing with state feedback control strategy, the static output feedback control strategy is more practical because in many real systems, it is hard to obtain full state variables of the systems. But it is worth mentioning that static output control strategy will cause non-convex problem or make non-convex problem more difficult to deal with, which is the reason why relatively few study results on static output feedback control for positive nonlinear systems have been existed.

Thirdly, from the angle of the stability analysis, in our paper, the linear co-positive Lyapunov function is employed to analyze the stability of the closed-loop systems, which not only can well capture the elegant positivity of positive systems, but also can reduce the difficulty of the analysis process. However, in [R1], the quadratic polynomial Lyapunov function is employed, although this kind of Lyapunov function also can be used for positive systems in theory, it cannot well capture the unique positivity of positive systems, therefore, when quadratic Lyapunov function is employed for positive systems, it will lead to relatively conservative results.

Based on above analysis, we can see that there are many different aspects from the the work in our paper and the work in [R1].

In our opinion, the difficulties in considering Type-2 fuzzy set for positive nonlinear systems mainly include the following several aspects: (1) The non-convex problem will become more difficult to be solved when the Type-2 fuzzy set is used to investigate the positive nonlinear systems because the uncertainties will be introduced into the stability analysis for Interval Type-2 positive polynomial fuzzy systems. Hence, the uncertainties, positivity and disturbance will make the stability analysis very challenging. (2) Novel membership function approximation methods require to be proposed when Type-2 fuzzy set is used because most of the existing membership function-dependent techniques are for membership functions without uncertain information.

Comment 4

The organization of Section III needs adjustment, since the content between Section III and Section III-A seems still the part of formulating the system and giving preliminaries, which is supposed to be presented in Section II.

Response 4
The authors would like to thank the reviewer for this comment. As suggested by this reviewer, we have adjusted the organization of Section III. And the Section III-A has been removed into the Section II-D.

Comment 5
The content of Theorem 1 and Theorem 2 are excessively long. Moreover, why the last parts of both Theorems 1 and 2 look like summaries of the notation usage? Since they took up too much space of the theorem presentation, moving them into the notation part (Section II-A) could be better. Not every formula needs to be numbered if you do not actually mention it in the text.

Response 5
The authors would like to thank the reviewer for this comment. As suggested by this reviewer, some of the notations in the last parts of both Theorems 1 and 2 have been moved into the notation part (Section II-A). Furthermore, the numbers of some formula which are not mentioned in the text have been removed.

The changes can be found on the left column, lines 27-30, page 3.

Comment 6
Why are there no formal proofs for both Theorem 1 and Theorem 2? It is understandable that some analysis has been made before each theorem presentation, but it is quite unclear if this analysis can be deemed the strict proof for the corresponding theorem. It is suggested to more formally present your results.

Response 6
The authors would like to thank the reviewer for this comment. As suggested by this reviewer, we have revised this paper and the formal proofs for both Theorem 1 and Theorem 2 have been presented.

The changes can be found from pages 4-7.

Comment 7
Please carefully refine the language, including correcting some typos and grammar errors, such as "In recent..." (page 1, third paragraph), the title of Section IV-F "compare with other results", etc.

Response 7
The authors would like to thank the reviewer for this comment. After a double check, we have tried our best to revise these typos and grammar

errors, for example

1. " In recent," has been revised as "In recent decades". This change can be found on the right column, line 52, page 1.

2." imperfectly matched premises" has been revised as "imperfect premise matching". This change can be found on the right column, line 43, page 1.

3." Compare with other results" has been revised as " Comparison with other results". This change can be found on the right column, line 52, page 10.

4." that is known " has been revised as ", which is well known". This change can be found on the left column, line 35, page 2.

Comment 8

Regarding the fuzzy systems, is it possible to make some extensions to hybrid fuzzy systems, see e. g.,

Mengqi Xue, et al., Practical output synchronization for asynchronously switched multi-agent systems with adaption to fast-switching perturbations, Automatica, vol. 116, art. No. 108917,Jun. 2020. (Regular Paper)

Input-to-State Stability for Nonlinear Systems with Stochastic Impulses, Automatica, vol. 113, no. 3, art no. 108766, Mar. 2020.

Response 8

The authors would like to thank the reviewer for this comment. We have studied the above papers and cited them in our paper as the references [9] and [10]. After learning about the hybrid fuzzy systems, in our opinion, it is possible to combine the positive switch systems with the switched multi-agent systems, but we have not study further. In the future, we would like to have a try to make some extensions to hybrid fuzzy systems.