Journal of Pure and Applied Algebra 226 (2022) 107019

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

www.elsevier.com/locate/jpaa

Vertex F-algebra structures on the complex oriented homology of H-spaces

Jacob Gross [a], Markus Upmeier [b,*]

[a] The Mathematical Institute, Radcliffe Observatory Quarter, Woodstock Road, Oxford, OX2 6GG, UK
[b] Department of Mathematics, University of Aberdeen, Fraser Noble Building, Elphinstone Rd, Aberdeen, AB24 3UE, UK

A R T I C L E   I N F O

Article history:
Received 27 August 2021
Received in revised form 10 January 2022
Available online 20 January 2022
Communicated by C.A. Weibel

MSC:
17B69; 55N20

Keywords:
Vertex algebra
Formal group law
Generalized cohomology
H-space

A B S T R A C T

We give a topological construction of graded vertex F-algebras by generalizing Joyce's vertex algebra construction to complex-oriented homology. Given an H-space X with a $BU(1)$-action, a choice of K-theory class, and a complex oriented homology theory E, we build a graded vertex F-algebra structure on $E_*(X)$ where F is the formal group law associated with E.

Contents

* Corresponding author.
E-mail addresses: jacob.gross@maths.ox.ac.uk (J. Gross), markus.upmeier@abdn.ac.uk (M. Upmeier).

https://doi.org/10.1016/j.jpaa.2022.107019

1. Introduction and results

The algebraic topology of moduli stacks, arising for example in algebraic geometry and gauge theory, is of fundamental importance for the study of invariants. Let $\mathcal{A}$ be an additive $\mathbb{C}$-linear dg-category, whose τ-stable objects we wish to classify, for τ a stability condition. The category $\mathcal{A}$ has an associated moduli stack $\mathcal{M}_{\mathcal{A}}$ by [19]. In [8], Joyce constructs a graded vertex algebra on the ordinary homology $H_*(\mathcal{M}_{\mathcal{A}})$. Vertex algebras are algebraic structures with origins in conformal field theory which can be regarded as *singular* commutative rings whose operation $Y: V \otimes V \to V(\!(z)\!)$, the *state-to-field correspondence*, takes values in Laurent series. This profound algebraic structure is used to describe wall-crossing formulas relating the virtual fundamental classes $[\mathcal{M}_{\mathcal{A}}]^{\mathrm{virt}}_{\tau}, [\mathcal{M}_{\mathcal{A}}]^{\mathrm{virt}}_{\tau'} \in H_*(\mathcal{M}_{\mathcal{A}})$ for different stability conditions. These are powerful tools for computing invariants.

Motivated by physics, many authors currently investigate *refined invariants* such as K-theoretic Donaldson–Thomas invariants [5,6,13,18]. Here the virtual classes should be viewed in K-homology $K_*(\mathcal{M}_{\mathcal{A}})$. As a first step towards extending wall-crossing formulas to refined invariants, we here extend Joyce's construction to any generalized (complex oriented) homology theory E_* with associated formal group law $F(z, w)$. Our main result constructs a vertex F-algebra structure on $E_*(\mathcal{M}_{\mathcal{A}})$ in the sense of Li [14].

In addition, our construction of vertex F-algebra works in greater generality, namely for any topological H-space (*i.e.* abelian group up to homotopy) with an action of $B\mathrm{U}(1)$.

Let E^* be a complex oriented generalized cohomology theory with associated formal group law $F(z, w)$ over its coefficient ring R_*, see §3. As a preliminary result, we present a Laurent-polynomial version of the Conner-Floyd Chern classes (see Definition 3.4) with values in E^*.

Theorem 1.1. *For every class $\theta \in K^0(X)$ in the topological K-theory of topological space X there is an R_*-linear transformation*

$$(-) \cap C^E_z(\theta) \colon E_*(X) \longrightarrow E_*(X)[\![z]\!][z^{-1}] \qquad a \longmapsto a \cap C^E_z(\theta), \ (1.1)$$

of degree $-2r$ if θ has constant rank $r \in \mathbb{Z}$, with the following properties:

(a) (Naturality.) For continuous $f : X' \to X$, $\theta \in K^0(X)$, and $a' \in E_*(X')$

$$f_*\big(a' \cap C^E_z(f^*(\theta))\big) = f_*(a') \cap C^E_z(\theta). \tag{1.2}$$

(b) (Direct sums.) For ζ, $\theta \in K^0(X)$ and $a \in E_*(X)$ we have

$$a \cap C^E_z(\zeta + \theta) = \big[a \cap C^E_z(\zeta)\big] \cap C^E_z(\theta). \tag{1.3}$$

(c) (Normalization.) For a complex line bundle $L \to X$ and $a \in E_*(X)$ we have

$$a \cap C^E_z(L) = a \cap F(z, c^E_1(L)). \tag{1.4}$$

More generally, for any $\theta \in K^0(X)$ we have

$$a \cap C^E_z(L \otimes \theta) = i_{z,c^E_1(L)}\big(a \cap C^E_{F(z,c^E_1(L))}(\theta)\big). \tag{1.5}$$

Here, as usual, the variable z has degree -2. We prove Theorem 1.1 in §4. The notations used in (1.4) and (1.5) will be explained in Notations 3.9 & 4.1 below.

For our main result, let X be an H-space with an operation $\Phi : X \times X \to X$ that is associative, commutative, and has a unit $e \in X$ up to homotopy. Recall that the classifying space $B\mathrm{U}(1)$ for complex line bundles is an H-space with the tensor product $\mu_{B\mathrm{U}(1)}$ and trivial bundle $e_{B\mathrm{U}(1)}$. Assume there is an action Ψ of $B\mathrm{U}(1)$ on X up to homotopy, meaning $\Psi\circ(\mathrm{id}_{B\mathrm{U}(1)}\times\Psi)\simeq\Psi\circ(\mu_{B\mathrm{U}(1)}\times\mathrm{id}_X)$ and $\Psi(e_{B\mathrm{U}(1)},-)\simeq\mathrm{id}_X$. Suppose $\Psi(e,-)\simeq e$ is an h-fixed point and $\Phi\circ(\Psi\times\Psi)\circ\delta\simeq\Psi\circ(\Phi\times\mathrm{id}_{B\mathrm{U}(1)})$, where $\delta(x_1,x_2,g)=(x_1,g,x_2,g)$. The set of connected components $\pi_0(X)$ is a monoid with unit $\Omega=[e]$ and operation $\alpha+\beta=\Phi_*(\alpha\boxtimes\beta)$ and we partition $X=\coprod_{\alpha\in\pi_0(X)}X_\alpha$. Write $\Phi_{\alpha,\beta}\colon X_\alpha\times X_\beta\to X_{\alpha+\beta}$, $\Psi_\alpha\colon B\mathrm{U}(1)\times X_\alpha\to X_\alpha$ for the restrictions. Let $\theta_{\alpha,\beta}\in K^0(X_\alpha\times X_\beta)$ for all α,β.

Theorem 1.2. *Given (X,Φ,e,Ψ) as above, suppose the following identities hold for all $\alpha,\beta,\gamma\in\pi_0(X)$:*

$$(\Phi_{\alpha,\beta}\times\mathrm{id}_{X_\gamma})^*(\theta_{\alpha+\beta,\gamma})=\pi^*_{\alpha,\gamma}(\theta_{\alpha,\gamma})+\pi^*_{\beta,\gamma}(\theta_{\beta,\gamma}),\tag{1.6}$$

$$(\mathrm{id}_{X_\alpha}\times\Phi_{\beta,\gamma})^*(\theta_{\alpha,\beta+\gamma})=\pi^*_{\alpha,\beta}(\theta_{\alpha,\beta})+\pi^*_{\alpha,\gamma}(\theta_{\alpha,\gamma}),\tag{1.7}$$

$$(\Psi_\alpha\times\mathrm{id}_{X_\beta})^*(\theta_{\alpha,\beta})=\pi^*_{B\mathrm{U}(1)}(\mathcal{L})\otimes\pi^*_{\alpha,\beta}(\theta_{\alpha,\beta}),\tag{1.8}$$

$$(\mathrm{id}_{X_\alpha}\times\Psi_\beta)^*(\theta_{\alpha,\beta})=\pi^*_{B\mathrm{U}(1)}(\mathcal{L})^\vee\otimes\pi^*_{\alpha,\beta}(\theta_{\alpha,\beta}),\tag{1.9}$$

$$\theta|_{X_\alpha\times\{\Omega\}}=0,\quad \theta|_{\{\Omega\}\times X_\beta}=0,\tag{1.10}$$

$$\sigma^*(\theta_{\beta,\alpha})=(\theta_{\alpha,\beta})^\vee.\tag{1.11}$$

Here σ swaps the factors of $X_\alpha\times X_\beta$ and $\mathcal{L}\to B\mathrm{U}(1)$ is the universal line bundle with dual $\mathcal{L}^\vee$. With the F-shift operator $\mathcal{D}(z)$ of (3.3) below, the graded R_*-module

$$V_*=\bigoplus\nolimits_{\alpha\in\pi_0(X)}E_{*-\mathrm{rk}\,\theta_{\alpha,\alpha}}(X_\alpha)\tag{1.12}$$

is a graded nonlocal vertex F-algebra $(V_*,\mathcal{D},\Omega,Y)$ with state-to-field correspondence

$$Y(a,z)b=(\Phi_{\alpha,\beta})_*\big(\mathcal{D}_\alpha(z)\boxtimes\mathrm{id}_{E_*(X_\beta)}\big)\big[(a\boxtimes b)\cap C^E_z(\theta_{\alpha,\beta})\big].\tag{1.13}$$

Similarly, the graded R_*-module

$$\overline{V}_*=\bigoplus\nolimits_{\alpha\in\pi_0(X)}E_{*-2\,\mathrm{rk}\,\theta_{\alpha,\alpha}}(X_\alpha)\tag{1.14}$$

becomes a graded vertex F-algebra $(\overline{V}_*,\mathcal{D},\Omega,\overline{Y})$, where

$$\overline{Y}(a,z)b=(\Phi_{\alpha,\beta})_*\big(\mathcal{D}_\alpha(z)\boxtimes\mathrm{id}_{E_*(X_\beta)}\big)\big[(a\boxtimes b)\cap\overline{C}^E_z(\theta_{\alpha,\beta})\big]\tag{1.15}$$

uses the operation of degree $-4\,\mathrm{rk}\,\theta_{\alpha,\beta}$ defined by

$$c\cap\overline{C}^E_z(\theta_{\alpha,\beta})=\big[c\cap C^E_z(\theta_{\alpha,\beta})\big]\cap C^E_{\iota(z)}(\sigma^*(\theta_{\beta,\alpha})),\quad c\in E_*(X_\alpha\times X_\beta).$$

Here $\iota(z)$ is the inverse for F (see §2). The proof of Theorem 1.2 is given in §5.

As a special case, our result applies to the topological realization $X=\mathcal{M}^{\mathrm{top}}_{\mathcal{A}}$ of a moduli stack. Taking direct sums in the additive category defines Φ making $\mathcal{M}_{\mathcal{A}}$ into an H-space. Moreover, scaling morphism by $\mathrm{U}(1)$ defines an operation Ψ of the quotient stack $[*\,/\!\!/\,\mathrm{U}(1)]$, endowing $\mathcal{M}^{\mathrm{top}}_{\mathcal{A}}$ with the required action of $B\mathrm{U}(1)=[*\,/\!\!/\,\mathrm{U}(1)]^{\mathrm{top}}$. As shown in Proposition 3.3 below, this action yields an F-shift operator $\mathcal{D}(z)$. The K-theory classes $\theta_{\alpha,\beta}$ are given by the Ext-complexes in the dg-category $\mathcal{A}$, which satisfy (1.6)–(1.11). In geometric examples, one may wish to incorporate signs $\epsilon_{\alpha,\beta}$ into (1.15). These are related to orientations, see [8, §8.3]. The orientation problems were solved in the series [9–11]. For simplicity, we ignore this additional data here and set up a symmetrized construction without signs.

2. Formal groups laws and vertex F-algebras

In the section, we will keep everything general and assume the following setup. Later, the data R_* and $F(z,w)$ will arise naturally from a complex oriented cohomology theory, see §3, and V_* will be constructed from an H-space as in (1.12).

Notation 2.1.

- R_* a graded commutative ring with unit. Write R^* for the same ring with the reverse grading, $R^n = R_{-n}$, $n \in \mathbb{Z}$, and R for the ring with the grading removed
- V_* a graded module over R_*
- z, w variables of degree -2
- $F(z,w)$ a graded formal group law over R_*
- $V[\![z]\!]$ the formal power series $\sum_{i=0}^{\infty} a_i z^i$; a ring when $V = R$
- $V(\!(z)\!)$ the R_*-module of Laurent series $\sum_{i=-\infty}^{+\infty} a_i z^i$ with its partially defined product. The fact that $V(\!(z)\!)$ is *not* a ring frequently causes confusion.
- The meromorphic series $V[\![z]\!][z^{-1}]$; a ring when $V = R$.
- $i_{z,w}\colon V[\![z,w]\!][z^{-1},w^{-1},F(z,w)^{-1}] \to V(\!(z,w)\!)$ expands $F(z,w)^{-N}$, see Notation 2.4. We have $i_{z,w}(V[\![z,w]\!][F(z,w)^{-1}]) \subset V(\!(z)\!)[\![w]\!]$.
- $(-1)^a$ means $(-1)^{\mathrm{degree}(a)}$

Definition 2.2. A *graded formal group law* over R_* is a formal power series $F(z,w) = \sum_{i,j\geqslant 0} F_{ij} z^i w^j \in R[\![z,w]\!]$ with $F_{ij} \in R_{2i+2j-2}$ satisfying

$$F(z,w) = F(w,z), \qquad F(z,0) = z, \qquad F(F(z,w),v) = F(z,F(w,v)). \tag{2.1}$$

There exists a unique power series $\iota \in R[\![z]\!]$ with $F(z,\iota(z)) = 0$, the *inverse*. Note that $\iota(\iota(z)) = z$ and $\iota(F(\iota(z),w)) = F(z,\iota(w))$.

Example 2.3.

(i) The *additive formal group law* $\mathbb{G}_{\mathrm{a}}$ over $\mathbb{Z}$ (in degree zero) is defined by $F(z,w) = z + w$, and the inverse is $\iota(z) = -z$.
(ii) The *multiplicative formal group law* $\mathbb{G}_{\mathrm{m}}$ over $\mathbb{Z}$ is defined by $F(z,w) = z + w + zw$ and has $\iota(z) = (1+z)^{-1} - 1 = -z + z^2 - z^3 + \cdots$.
(iii) There is a *universal formal group law* $\mathbb{G}_{\mathrm{u}}$ over the *Lazard ring* R_L generated by variables F_{ij} subject to the relations contained in (2.1).

Notation 2.4. It follows from (2.1) that for a general formal group law

$$F(z,w) = z + w + O(zw), \qquad \iota(z) = -z + O(z^2).$$

Write $F(z,w) = z(1 + w/z + wG(z,w))$ and expand using the binomial theorem

$$i_{z,w} F(z,w)^n = \sum_{k=0}^{\infty} \binom{n}{k} z^{n-k} w^k (1 + zG(z,w))^k \in R[\![w]\!](\!(z)\!), \quad n \in \mathbb{Z}. \tag{2.2}$$

As the k-th summand has w-degree $\geqslant k$, this converges as a formal power series. Define $i_{w,z} F(z,w)^n \in R[\![z]\!](\!(w)\!)$ by expanding $F(z,w) = w(1 + z/w + zG(z,w))$ similarly. We extend $i_{z,w}$ and $i_{w,z}$ to $V[\![z,w]\!][z^{-1},w^{-1}][F(z,w)^{-1}]$ by linearity.

Note that $i_{z,w}F(z,w)^{-n} \cdot F(z,w)^n = 1$ and $i_{w,z}F(w,z)^{-n} \cdot F(z,w)^n = 1$ for all $n \geqslant 0$. For every $P(z,w) = \sum_{n \geqslant -N} a_n(z,w)F(z,w)^n \in V[\![z,w]\!][z^{-1},w^{-1}][F(z,w)^{-1}]$ we thus have

$$F(z,w)^N\big(i_{z,w}P(z,w) - i_{w,z}P(z,w)\big) = 0. \tag{2.3}$$

Definition 2.5. Let V_* be a graded R_*-module and F a graded formal group law over R_*. An *F-shift operator* is a graded R_*-linear map $\mathcal{D}(z)\colon V \to V[\![z]\!]$ with

$$\mathcal{D}(0) = \mathrm{id}_V, \qquad \mathcal{D}(z) \circ \mathcal{D}(w) = \mathcal{D}(F(z,w)). \tag{2.4}$$

Example 2.6. Let $R_* = \mathbb{Q}$, $V = \mathbb{Q}[w]$. Then $\mathcal{D}(z)(f(w)) = e^{z\frac{d}{dw}}f(w)$ defines a $\mathbb{G}_{\mathrm{a}}$-shift operator. The relation $\mathcal{D}(z)(f(w)) = f(z+w)$ motivates the terminology.

We now define vertex F-algebras. For $F = \mathbb{G}_{\mathrm{a}}$ we recover ordinary vertex algebras, see Frenkel–Ben-Zvi [3], Frenkel–Lepowsky–Meurman [4], and Kac [12].

Definition 2.7. Let $F(z,w)$ be a graded formal group law over R_*. A *graded nonlocal vertex F-algebra* is a graded R_*-module V_*, a *vacuum vector* $\Omega \in V_0$, an F-shift operator $\mathcal{D}(z)$, and a graded R_*-linear *state-to-field correspondence*

$$V \otimes_R V \longrightarrow V[\![z]\!][z^{-1}], \qquad a \otimes b \longmapsto Y(a,z)b, \tag{2.5}$$

satisfying the following axioms:

(a) *Vacuum and creation:* $Y(a,z)\Omega$ is holomorphic for all $a \in V$ and

$$Y(a,z)\Omega|_{z=0} = a, \tag{2.6}$$

$$Y(\Omega,z) = \mathrm{id}_V\,. \tag{2.7}$$

(b) *F-translation covariance:* for all $a \in V$ we have

$$Y(\mathcal{D}(w)(a),z) = i_{z,w}Y(a,F(z,w)), \tag{2.8}$$

$$\mathcal{D}(z)\Omega = \Omega. \tag{2.9}$$

(c) *Weak F-associativity:* for all $a,b,c \in V$ there exists $N \geqslant 0$ with

$$F(z,w)^N Y(Y(a,z)b,w)c = F(z,w)^N i_{z,w}Y(a,F(z,w))Y(b,w)c. \tag{2.10}$$

A graded nonlocal vertex F-algebra is a *graded vertex F-algebra* if, in addition,

$$Y(a,z)b = (-1)^{ab}\mathcal{D}(z) \circ Y(b,\iota(z))a, \qquad \text{for all } a,b \in V. \tag{2.11}$$

Remark 2.8. It is a consequence of (2.6)–(2.11) that for all $a,b,c \in V$ there exists $N \geqslant 0$ with

$$(z-w)^N Y(a,z)Y(b,w)c = (-1)^{ab}(z-w)^N Y(b,w)Y(a,z)c. \tag{2.12}$$

So our definitions agree with those given by Li [14] in the ungraded case.

3. Complex oriented cohomology and Chern classes

Let E^* be a generalized cohomology theory, see for example Rudyak [16, Ch. II, §3]. Thus, for every pair $A \subset X$ of topological spaces there is defined a graded abelian group $E^*(X,A)$. Continuous maps $f\colon (X,A) \to (X',A')$ induce homomorphisms $f^*\colon E^*(X',A') \to E^*(X,A)$ that depend only on the homotopy class of f. For a pointed space $x_0 \in X$ write $\widetilde{E}^*(X) = E^*(X,\{x_0\})$ for *reduced cohomology.* The *smash product* of (X,x_0) and (Y,y_0) is the quotient $X \wedge Y = (X \times Y)/(X \vee Y)$ with one-point union $X \vee Y = (X \times \{y_0\}) \cup (\{x_0\} \times Y)$ collapsed to become the new base-point. As part of the structure, E^* comes equipped with natural *suspension isomorphisms* $\sigma_X\colon \widetilde{E}^*(X) \to \widetilde{E}^{*+1}(X \wedge \mathcal{S}^1)$.

Suppose E^* is a multiplicative generalized cohomology theory. Then there is a bilinear *cross product* $\boxtimes\colon E^*(X,A) \otimes E^*(Y,B) \to E^*(X \times Y, X \times B \cup A \times Y)$ and *units* $1_X \in E^0(X)$, both natural. If we let $R_* = E_*(\mathrm{pt})$ be the *coefficient ring*, then $R^* = E^*(\mathrm{pt})$ for the reverse grading, which is the reason for this convention in Notation 2.1. Pulling the cross product back along the diagonal makes $E^*(X)$ a graded commutative unital R^*-algebra for the cup product '$\cup$' over R^*. Dually, there is a homological cross product that in particular makes $E_*(X)$ a graded module over R_*. There is a *cap product*

$$E_a(X) \otimes_R E^b(X) \longrightarrow E_{a-b}(X), \qquad a \otimes \varphi \mapsto a \cap \varphi$$

which is R_*-linear, unital $a \cap 1 = a$, and natural $f_*(a \cap f^*(\varphi')) = f_*(a) \cap \varphi'$, where $f\colon X \to X'$ and $\varphi' \in E^b(X')$. See Rudyak [16] for further properties.

Definition 3.1. The suspension isomorphism shows that $\widetilde{E}^*(\mathbb{CP}^1) \cong \widetilde{E}^*(\mathcal{S}^2) \cong R^{*-2}$ is a free R_*-module on a single generator. A multiplicative cohomology theory E^* is *complex orientable* if $i^*\colon \widetilde{E}^*(\mathbb{CP}^\infty) \to \widetilde{E}^*(\mathbb{CP}^1)$ is surjective, where $\mathbb{CP}^\infty \cong \operatorname{colim}_m \mathbb{CP}^m$ and $i\colon \mathbb{CP}^1 \hookrightarrow \mathbb{CP}^\infty$. A *complex orientation* is a choice of $\xi_E \in \widetilde{E}^2(\mathbb{CP}^\infty)$ such that $i^*(\xi_E)$ generates the R_*-module $\widetilde{E}^*(\mathbb{CP}^1)$.

The presence of the permanent cycle $\xi_E|_{\mathbb{CP}^m}$ implies that the Atiyah–Hirzebruch spectral sequence $H^p(\mathbb{CP}^m; E^q(\mathrm{pt})) \Longrightarrow E^{p+q}(\mathbb{CP}^m)$ collapses, see Adams [1, p. 42]. Hence we have canonical isomorphisms

$$E^*(\mathbb{CP}^m) \cong R[\xi_E]/(\xi_E^{m+1}), \qquad E^*(\mathbb{CP}^\infty) \cong \lim E^*(\mathbb{CP}^m) \cong R[\![\xi_E]\!].$$

More generally, let $P \to X$ be a bundle of projective spaces $\mathbb{CP}^m$ and suppose that $w \in E^*(P)$ restricts on every fiber P_x to generators $1_{P_x}, w|_{P_x}, \ldots, w^m|_{P_x}$ of the R_*-module $E^*(P_x)$. Then Dold's theorem implies that $E^*(P)$ is a free $E^*(X)$-module on $1_P, w, \ldots, w^m$, see [2, (7.4)]. In particular,

$$E^*(X \times \mathbb{CP}^\infty) \cong E^*(X)[\![\xi_E]\!], \quad E^*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) \cong R[\![\pi_1^*(\xi_E), \pi_2^*(\xi_E)]\!]. \tag{3.1}$$

Definition 3.2. Let ξ_E be a complex orientation of E^*. Write $\mathcal{L} \to \mathbb{CP}^\infty$ for the universal complex line bundle with $\mathcal{L}|_L = L$. Recall that $\mathbb{CP}^\infty = B\mathrm{U}(1)$ is an H-space with operation a classifying map $\mu_{\mathbb{CP}^\infty}\colon \mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty$ of the tensor product $\pi_1^*(\mathcal{L}) \otimes \pi_2^*(\mathcal{L})$ and unit t_0 the trivial line bundle. The associated *formal group law* $F = \sum_{i,j\geqslant 0} F_{ij} z^i w^j$ is defined by the expansion

$$\mu_{\mathbb{CP}^\infty}^*(\xi_E) = \sum\nolimits_{i,j\geqslant 0} F_{ij}\, \xi_E^i \boxtimes \xi_E^j, \qquad F_{ij} \in R^{2-2i-2j} = R_{2i+2j-2}. \tag{3.2}$$

As in [1, p. 42] the homology $E_*(\mathbb{CP}^\infty)$ is the free R_*-module on the dual generators t_n, $n \geqslant 0$, of degree $2n$ characterized by $\langle t_n, \xi_E^m \rangle = \delta_n^m$.

Proposition 3.3. *Let (E^*, ξ_E) be a complex oriented cohomology theory with associated formal group law $F(z,w)$. Suppose $\Psi\colon B\mathrm{U}(1) \times X \to X$ satisfies the axioms for a group action of the H-space $B\mathrm{U}(1)$ on X up to homotopy. Then*

$$\mathcal{D}(z)(a) = \sum\nolimits_{k\geqslant 0} \Psi_*(t_k \boxtimes a)\, z^k, \quad a \in E_*(X), \tag{3.3}$$

defines an F-shift operator on $E_*(X)$.

Proof. Since $\Psi(t_0, x) = x$ is neutral, $\mathcal{D}(0) = \mathrm{id}_{E_*(X)}$. Define coefficients F^n_{ij} by $F(z,w)^n = \sum_{i,j\geqslant 0} F^n_{ij}\, z^i w^j$. Then $(\mu_{\mathbb{CP}^\infty})_*(t_i \boxtimes t_j) = \sum_{n\geqslant 0} F^n_{ij}\, t_n$, and so

$$\begin{aligned}\mathcal{D}(z)\circ\mathcal{D}(w)(a) &= \sum\nolimits_{i,j\geqslant 0} \Psi_*\big(t_i \boxtimes \Psi_*(t_j \boxtimes a)\big) z^i w^j \\ &= \sum\nolimits_{i,j\geqslant 0} \Psi_*\big((\mu_{\mathbb{CP}^\infty})_*(t_i \boxtimes t_j) \boxtimes a\big) z^i w^j \\ &= \sum\nolimits_{i,j,n\geqslant 0} \Psi_*(t_n \boxtimes a) F^n_{ij}\, z^i w^j = \mathcal{D}(F(z,w)). \quad \square\end{aligned}$$

Definition 3.4. Let $V \to X$ be a complex vector bundle of rank n with zero section 0_X. The bundle of projective spaces $\mathbb{P}(V) = (V \setminus 0_X)/\mathbb{C}^*$ carries a tautological line bundle $\mathcal{L}_V \to \mathbb{P}(V)$ with $\mathcal{L}_V|_L = L$. Its classifying map $f_{\mathcal{L}_V} : \mathbb{P}(V) \to \mathbb{CP}^\infty$ is unique up to homotopy. Define $w = f^*_{\mathcal{L}_V}(\xi_E)$ using the complex orientation. By the above, $E^*(\mathbb{P}(V))$ is a free $E^*(X)$-module with basis $1_{\mathbb{P}(V)}, w, \dots, w^{n-1}$. The *Conner–Floyd Chern classes* are defined by expanding w^n in this basis:

$$c^E_0(V) = 1, \qquad 0 = \sum\nolimits_{i=0}^{n} (-1)^i c^E_i(V)\cdot w^{n-i}, \qquad c^E_i(V) = 0 \ \ (\forall i > n) \tag{3.4}$$

Naturality under pullback is obvious. There is a Whitney sum formula [2, p. 47]

$$c^E_k(V \oplus W) = \sum\nolimits_{i=0}^{k} c^E_i(V) c^E_{k-i}(W). \tag{3.5}$$

For complex line bundles $\mathcal{L}_L \to \mathbb{P}(L)$ is isomorphic to $L \to X$ so $c^E_1(L) = f^*_L(\xi_E)$ for the classifying map f_L of L. In particular,

$$c^E_1(L_1 \otimes L_2) = F(c^E_1(L_1), c^E_1(L_2)). \tag{3.6}$$

Moreover, $c^E_1(\underline{\mathbb{C}}) = 0$ as ξ_E is reduced. Hence $c^E_i(\underline{\mathbb{C}}^N) = 0$ for every trivial bundle.

Example 3.5. Ordinary cohomology $E^* = H^*$ has a complex orientation ξ_H in $H^2(\mathbb{CP}^\infty) = \lim H^2(\mathbb{CP}^m)$ that is Poincaré dual to the fundamental class $[\mathbb{CP}^{m-1}] \in H_{m-2}(\mathbb{CP}^m)$ with orientation of $\mathbb{CP}^{m-1}$ fixed by the complex structure. We obtain the ordinary Chern classes, and $c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2)$ implies $F_H = \mathbb{G}_\mathrm{a}$.

Example 3.6. Topological K-theory $E^* = K^*$ on compact spaces is the group completion of isomorphism classes of complex vector bundles. Write $\mathcal{L}_m = \mathcal{L}|_{\mathbb{CP}^m}$ for the tautological complex line bundle over $\mathbb{CP}^m$, $\mathbb{C}$ for the trivial bundle, and $[\mathcal{L}_m], 1 \in K^0(\mathbb{CP}^m)$ for their classes in K-theory. The classes $[\mathcal{L}_m] - 1 \in \widetilde{K}^0(\mathbb{CP}^m)$ are compatible under restriction and define a complex orientation $\xi_K \in \widetilde{K}^2(\mathbb{CP}^\infty) = \lim \widetilde{K}^0(\mathbb{CP}^m)$. Here $F_K = \mathbb{G}_\mathrm{m}$ is the multiplicative formal group law, as

$$\mu^*([\mathcal{L}] - 1) = [\mu^*(\mathcal{L})] - 1 = [\pi_1^*(\mathcal{L}) \otimes \pi_2^*(\mathcal{L})] - 1 = \mathbb{G}_\mathrm{m}\big([\mathcal{L}] - 1, [\mathcal{L}] - 1\big).$$

For a complex vector bundle $V \to X$ of rank n one has $\pi^*(V) = \mathcal{L}_V \oplus \mathcal{L}_V^\perp$ over the projectivization $\pi\colon \mathbb{P}(V) \to X$ and $\mathcal{L}_V^\perp$. The formal power series $\Lambda_t([V]) = 1 + [V]t + [\Lambda^2 V]t^2 + \ldots \in K^0(X)[\![t]\!]$ has inverse $\Lambda_{-t}([V])$, so $\Lambda_t([V] - [W]) = \Lambda_t([V])\Lambda_{-t}([W])$. As $[\mathcal{L}_V^\perp] = \pi^*[V] - [\mathcal{L}]$ has rank $n-1$, the n-th coefficient of $\Lambda_t([\mathcal{L}_V^\perp]) = \Lambda_t([\pi^*(V)])\Lambda_{-t}(\mathcal{L})$ is $0 = [\Lambda^n(\mathcal{L}_V^\perp)] = \sum_{p=0}^n (-1)^{n-p}[\Lambda^p(V)]\cdot[\mathcal{L}]^{n-p}$. Putting $[\mathcal{L}] = w + 1$ and comparing to (3.4), $c^K_i(V) = \sum_{p=0}^i (-1)^{i+p}\binom{n-p}{n-i}[\Lambda^p(V)]$.

Example 3.7. As in Quillen [15], complex cobordism $\Omega^n_{\mathrm{U}}(X)$ for X a smooth manifold is the set of smooth maps $f\colon Z \to X$ of codimension $\dim X - \dim Z = n$ with a complex structure the stable normal bundle, modulo cobordism. The complex orientation $\xi_\Omega \in \Omega^2_{\mathrm{U}}(\mathbb{CP}^\infty) = \lim \Omega^2_{\mathrm{U}}(\mathbb{CP}^m)$ is given by $\mathbb{CP}^{m-1} \hookrightarrow \mathbb{CP}^m$, and $\mathbb{CP}^{m-1}$ is the zero set of a section of $\mathcal{L}^*_m$. So for complex line bundles $c_1^{\Omega_{\mathrm{U}}}(L)$ is represented by the zero set $s^{-1}(0)$ of a generic section $s\colon X \to L$. The formal group law is the universal law $\mathbb{G}_{\mathrm{u}}$, see Adams [1, Part I, §8].

Lemma 3.8. *Let $V \to X$ be a complex vector bundle over a finite CW complex. Then each of the Conner–Floyd Chern classes $c_i^E(V)$ is nilpotent.*

Proof. There is a finite open cover $X = \bigcup_{\lambda=1}^N U_\lambda$ with U_λ contractible and $V|_{U_\lambda}$ trivial. From the long exact sequence of the pair (X, U_λ) we see that we may lift $c_i^E(V)$ along $j_\lambda^*\colon E^{2i}(X, U_\lambda) \to E^{2i}(X)$ to a class $x_\lambda \in E^{2i}(X, U_\lambda)$. The diagram

$$\begin{array}{ccc} \prod_{\lambda=1}^N E^{2i}(X, U_\lambda) & \xrightarrow{\ \cup\ } & E^{2iN}\big(X, \bigcup_{\lambda=1}^N U_\lambda\big) = E^{2iN}(X,X) = \{0\} \\ \Big\downarrow{\scriptstyle \prod_{\lambda=1}^N j_\lambda^*} & & \Big\downarrow{\scriptstyle j^*} \\ \prod_{\lambda=1}^N E^{2i}(X) & \xrightarrow{\ \cup\ } & E^{2iN}(X) \end{array}$$

commutes by naturality of '∪', so $c_i^E(V)^N = \prod_{\lambda=1}^N j_\lambda^*(x_\lambda) = j^*\big(\prod_{\lambda=1}^N x_\lambda\big) = 0$. □

Notation 3.9. When X is a finite CW complex, it follows that we may substitute w by $c_1^E(L)$ in the formal group law $F(z, w)$. To define the right hand side of (1.4) also for infinite CW complexes X, let $\{X_i \mid i \in I\}$ be the direct system of finite subcomplexes $X_i \subset X$ ordered by inclusion. The *pro-group E-cohomology* is the inverse limit $\hat{E}^*(X) = \lim E^*(X_i)$. The family of all restrictions $F(z, c_1^E(L|_{X_i}))$ determines an element we write $F(z, c_1^E(L)) \in \hat{E}^*(X)[\![z]\!]$. As homology and direct limits commute, see [17, Prop. 7.53], we have $E_*(X) = \operatorname{colim} E_*(X_i)$ and therefore a well-defined cap product $E_*(X) \otimes \hat{E}^*(X) \to E_*(X)$. This defines (1.4) in general.

4. Proof of Theorem 1.1

Step 1: Vector bundles over finite CW complexes. For a complex line bundle $L \to X$ over a finite CW complex X define $C_z^E(L) = F(z, c_1^E(L))$. For $V \to X$ a rank n complex vector bundle we proceed by the splitting principle. As in Definition 3.4 over the projectivization $p\colon \mathbb{P}(V) \to X$ we can split off a line bundle from $p^*(V)$ and $p^*\colon E^*(Y) \to E^*(X)$ is injective. Iterating, we find $q\colon Y \to X$ and line bundles $L_1, \ldots, L_n \to Y$ with $L_1 \oplus \cdots \oplus L_n = q^*(V)$ and $q^*\colon E^*(Y) \to E^*(X)$ is injective. By (3.5), the class $q^*(c_k^E(V))$ is the k-th elementary symmetric polynomial in the *Chern roots* $c_1^E(L_1), \ldots, c_1^E(L_1)$. As the expression

$$F(z, c_1^E(L_1)) \cup \cdots \cup F(z, c_1^E(L_n)) = q^*(C_z^E(V)) \tag{4.1}$$

is a symmetric polynomial in the Chern roots, the fundamental theorem of symmetric polynomials implies it has a (unique) preimage $C_z^E(V)$ in $E^*(X)[\![z]\!]$. The map (1.1) is obtained by combining the class $C_z^E(V)$ with the cap product

$$\cap\colon E_*(X) \otimes E^*(X)[\![z]\!] \to E_*(X)[\![z]\!].$$

(a) For naturality, let $f\colon X' \to X$ and use the pullback $Q\colon Y' = X' \times_X Y \to X'$ with its canonical map $F\colon Y' \to Y$ to split $V' = f^*(V)$ as $Q^*(V') \cong F^* q^*(V) \cong F^*(L_1) \oplus \cdots \oplus F^*(L_n)$. Naturality of the

Conner–Floyd Chern classes implies that the pullback $F^*q^*(C_z^E(V)) = Q^*f^*(C_z^E(V))$ of (4.1) along F is $Q^*C_z^E(V')$. Thus,

$$C_z^E(f^*(V)) = f^*(C_z^E(V)). \tag{4.2}$$

(b) Let $V, W \to X$ be vector bundles. Pick $q\colon Y \to X$ such that both $q^*(V) = L_1 \oplus \cdots \oplus L_n$ and $q^*(W) = S_1 \oplus \cdots \oplus S_m$ split into line bundles with q^* injective. Then $q^*C_z^E(V)$ equals (4.1), $q^*C_z^E(W) = F(z, c_1^E(S_1)) \cup \cdots \cup F(z, c_1^E(S_m))$, and

$$q^*C_z^E(V \oplus W) = F(z, c_1^E(L_1)) \cup \cdots \cup F(z, c_1^E(S_m)) = q^*C_z^E(V) \cup q^*C_z^E(W).$$

Hence

$$C_z^E(V \oplus W) = C_z^E(V) \cup C_z^E(W). \tag{4.3}$$

This proves that cap product with $C_z^E(V)$ satisfies Theorem 1.1(a)&(b). Part (c) holds by construction. For (d), in the case of line bundles the operation $(-) \cap F(z, c_1^E(L)) = \sum_{i,j\geqslant 0} F_{ij} z^i [(-) \cap c_1^E(L)^j]$ has degree -2, as $F_{ij} \in R_{2i+2j-2}$. It then follows from (4.1) that in general $(-) \cap C_z^E(V)$ has degree $-2\,\mathrm{rk}(V)$.

(e) Let $V \to X$ be a vector bundle, $L \to X$ a complex line bundle, and suppose $q^*(V)$ splits as above. Then $q^*(L \otimes V) = (q^*(L) \otimes L_1) \oplus \cdots \oplus (q^*(L) \otimes L_n)$ and so

$$\begin{aligned}
q^*C_z^E(L \otimes V) &= F\big(z, c_1^E(q^*(L) \otimes L_1)\big) \cup \cdots \cup F\big(z, c_1^E(q^*(L) \otimes L_n)\big) \\
&\overset{(3.6)}{=} F\big(z, F(q^*c_1^E(L), c_1^E(L_1))\big) \cup \cdots \cup F\big(z, F(q^*c_1^E(L), c_1^E(L_n))\big) \\
&\overset{(2.1)}{=} F\big(F(z, q^*c_1^E(L)), c_1^E(L_1)\big) \cup \cdots \cup F\big(F(z, q^*c_1^E(L)), c_1^E(L_n)\big) \\
&= q^*C^E_{F(z,c_1^E(L))}(V).
\end{aligned}$$

Hence

$$C_z^E(L \otimes V) = C^E_{F(z,c_1^E(L))}(V). \tag{4.4}$$

Step 2: Extension to K-theory of finite CW complexes. So far, we have constructed a homomorphism $C_z^E\colon (\mathrm{Vect}(X), \oplus) \to (E^*(X)[\![z]\!], \cup)$ on the monoid of complex vector bundles $V \to X$ up to isomorphism over a finite CW complex. We claim that every class $C_z^E(V)$ is invertible in the larger ring $E^*(X)[\![z]\!][z^{-1}]$. Indeed, there exists a vector bundle $W \to X$ with $V \oplus W \cong \underline{\mathbb{C}}^N$ trivial and therefore $C_z^E(V) \cup C_z^E(W) = C_z^E(\underline{\mathbb{C}}^N) = F(z, c_1^E(\underline{\mathbb{C}}))^N = z^N$. As X is a finite CW complex, its topological K-theory is the group completion of $(\mathrm{Vect}(X), \oplus)$ whose universal property allows us to uniquely extend the homomorphism to $C_z^E\colon K^0(X) \to (E^*(X)[\![z]\!][z^{-1}], \cup)$. It is easy to check that properties (a)–(d) continue to hold.

Notation 4.1. As X is a finite CW complex, we may write $\theta = [V] - [\underline{\mathbb{C}}^\ell]$. Expand $C_z^E(V) = \sum_{n\geqslant 0}^\infty C_n(V) z^n$. Then

$$C_z^E(\theta) = \sum\nolimits_{n\geqslant 0} C_n(V) z^{n-\ell}. \tag{4.5}$$

In Notation 2.1 we have defined $i_{z,w}\big(F(z,w)^{-\ell} \sum_{n\geqslant -\ell}^\infty C_n(V) F(z,w)^n\big)$ as a holomorphic series in w which we can substitute by the nilpotent $c_1^E(L)$, see Lemma 3.8. This defines $i_{z,c_1^E(L)} C^E_{F(z,c_1^E(L))}(\theta) \in E^*(X)[\![z]\!][z^{-1}]$ for finite X. When X is infinite, the classes for the restrictions of θ to all finite subcomplexes $X_i \subset X$ define $i_{z,c_1^E(L)} C^E_{F(z,c_1^E(L))}(\theta) \in \hat{E}(X)(\!(z)\!)$ in pro-group E-cohomology, see Notation 3.9.

We prove (e). As just seen, $C_z^E(L) = F(z, c_1^E(L))$ is invertible in $E^*(X)[\![z]\!][z^{-1}]$. Therefore $i_{z,w}F(z, c_1^E(L))^n = F(z, c_1^E(L))^n$ for all $n \in \mathbb{Z}$. Using Notation 4.1, we have

$$\begin{aligned} C_z^E(L \otimes \theta) &\overset{(4.3)}{=} C_z(L \otimes V) C_z(L)^{-\ell} \\ &\overset{(4.4)}{=} C_{F(z,c_1^E(L))}(V) F(z, c_1^E(L))^{-\ell} \\ &= \sum\nolimits_{n \geqslant 0} C_n(V) F(z, c_1^E(L))^{n-\ell} = i_{z, c_1^E(L)} C^E_{F(z,c_1^E(L))}(\theta). \end{aligned}$$

Step 3: Infinite complexes. Let $\{X_i \mid i \in I\}$ be the direct system of finite subcomplexes of a CW complex X ordered by inclusion. Write $\iota(i)\colon X_i \subset X$ and $\iota(i,j)\colon X_i \subset X_j$ for the inclusions. For $\theta \in K^0(X)$, Step 2 yields for each $i \in I$ a map

$$E_*(X_i) \xrightarrow{\cap C_z(\iota(i)^*\theta)} E_*(X_i)[\![z]\!][z^{-1}] \xrightarrow{\iota(i)_*} E_*(X)[\![z]\!][z^{-1}]. \tag{4.6}$$

By naturality, $\iota(i,j)_*(a) \cap C_z^E(\iota(j)^*\theta) = \iota(i,j)_*(a \cap C_z^E(\iota(i)^*\theta))$ so the maps (4.6) determine a homomorphism $E_*(X) \cong \operatorname{colim} E_*(X_i) \to E_*(X)[\![z]\!][z^{-1}]$ on the colimit, using that homology and direct limits commute, see [17, Prop. 7.53]. Equivalently, the restrictions $C_z^E(\theta|_{X_i})$ define a class $C_z^E(\theta) \in \hat{E}^*(X)(\!(z)\!)$ in pro-group E-cohomology. Using the cap product $E_*(X) \otimes \hat{E}^*(X)(\!(z)\!) \to E_*(X)(\!(z)\!)$ we can define $(-) \cap C_z^E(\theta)\colon E_*(X) \to E_*(X)(\!(z)\!)$ which, a priori, has a larger codomain.

Finally, properties (a)–(e) pass to the limit.

Step 5: General topological spaces. By the CW approximation theorem, there is a CW complex X' with a weak homotopy equivalence $f\colon X' \to X$. Then

$$a \cap C_z(\theta) = f_*(f_*^{-1}(a) \cap C_z(f^*\theta))$$

is well-defined, since this equation holds for a homotopy equivalence $f\colon X' \to X'$ by (1.2). With this definition, the properties (a)–(e) carry over to X. □

5. Proof of Theorem 1.2

We verify Definition 2.7(a)–(c) for the graded module $V_* = \bigoplus E_{*-\operatorname{rk}\theta_{\alpha,\alpha}}(X_\alpha)$, vacuum vector $\Omega = e_*(1)$, F-shift operator (3.3), and state-to-field correspondence (1.13). Here, $e\colon \mathrm{pt} \to X_0$ is the H-space unit and $1 \in E_0(\mathrm{pt}) = R^0$.

Writing $|a|_V = |a| + \operatorname{rk}\theta_{\alpha,\alpha}$ for the shifted degree, we have

$$|Y(a,z)b|_V = |Y(a,z)b)| + \operatorname{rk}\theta_{\alpha+\beta,\alpha+\beta} = (|a| - \operatorname{rk}\theta_{\alpha,\alpha})(|b| - \operatorname{rk}\theta_{\beta,\beta}) = |a|_V \cdot |b|_V,$$

for $a \in E_{*-\operatorname{rk}\theta_{\alpha,\alpha}}(X_\alpha)$, $b \in E_{*-\operatorname{rk}\theta_{\beta,\beta}}(X_\beta)$, so that Y preserves the grading of V_*.

(a) Let $a \in E_*(X_\alpha)$, $b \in E_*(X_\beta)$. As e is a fixed point, $\Psi_*(t_k \boxtimes \Omega) = 0$ for $k > 0$ and $\Psi_*(t_0 \boxtimes \Omega) = \Omega$. Hence $\mathcal{D}(z)\Omega = \Omega$. Let $\varphi = (e, \mathrm{id}_{X_\beta})\colon X_\beta \to X_\Omega \times X_\beta$. Then

$$\begin{aligned} (\Omega \boxtimes b) \cap C_z^E(\theta_{\Omega,\beta}) &= \varphi_*(b) \cap C_z^E(\theta_{\Omega,\beta}) \\ &= \varphi_*\big(b \cap \varphi^* C_z^E(\theta_{\Omega,\beta})\big) \overset{(1.10)}{=} \varphi_*(b \cap 1) = \Omega \boxtimes b, \end{aligned}$$

and so $Y(\Omega, z)b = (\Phi_{\Omega,\beta})_*(\mathcal{D}(z)\Omega \boxtimes b) = b$, proving (2.7). Similarly,

$$Y(a,z)\Omega = (\Phi_{\alpha,\Omega})_*(\mathcal{D}(z) \boxtimes \mathrm{id}_{X_\Omega})(a \boxtimes \Omega) = \mathcal{D}(z)(a)$$

is holomorphic with $\mathcal{D}(0)(a) = a$ for $z = 0$, proving (2.6).

(b) We have already shown $\mathcal{D}(z)\Omega = \Omega$. To prove (2.8), we first need a lemma.

Lemma 5.1. *For the universal complex line bundle $\mathcal{L} \to \mathbb{CP}^\infty$ and $n \in \mathbb{Z}$*

$$\sum_{k\geqslant 0} t_k \cap i_{z,c_1^E(\mathcal{L})} F(z, c_1^E(\mathcal{L}))^n w^k = \sum_{\ell\geqslant 0} t_\ell \, i_{z,w} F(z,w)^n w^\ell. \tag{5.1}$$

Moreover, for all $a \in E_*(X_\alpha)$, $b \in E_*(X_\beta)$ we have

$$(\mathcal{D}_\alpha(w)a \boxtimes b) \cap C_z^E(\theta_{\alpha,\beta}) = (\mathcal{D}_\alpha(w) \times \mathrm{id}_{X_\beta})\big[(a \boxtimes b) \cap i_{z,w} C^E_{F(z,w)}(\theta_{\alpha,\beta})\big], \tag{5.2}$$

$$(a \boxtimes \mathcal{D}_\beta(w)b) \cap C_z^E(\theta_{\alpha,\beta}) = (\mathrm{id}_{X_\alpha} \times \mathcal{D}_\beta(w))\big[(a \boxtimes b) \cap i_{z,w} C^E_{F(z,\iota(w))}(\theta_{\alpha,\beta})\big]. \tag{5.3}$$

Proof. Introduce the expansion $i_{z,w}F(z,w)^n = \sum_{i\in\mathbb{Z}, j\geqslant 0} F^n_{ij}\, z^i w^j$. Then

$$t_k \cap i_{z,c_1^E(\mathcal{L})} F(z, c_1^E(\mathcal{L}))^n = t_k \cap \sum_{\substack{i\in\mathbb{Z}\\ j\geqslant 0}} F^n_{ij}\, z^i c_1^E(\mathcal{L})^j = \sum_{\substack{i\in\mathbb{Z}\\ j\geqslant 0}} F^n_{ij}\, z^i t_{k-j}, \tag{5.4}$$

where $t_k = 0$ for $k < 0$. Summing (5.4) over all k, the summands with $k < j$ vanish, so we may restrict the sum to $k \geqslant j$ and reindexing by $\ell = k - j$ gives (5.1):

$$\sum_{\substack{i\in\mathbb{Z}\\ j\geqslant 0}} \sum_{\ell\geqslant 0} F^n_{ij}\, z^i w^j t_\ell w^\ell = \sum_{\ell\geqslant 0} t_\ell \, i_{z,w} F(z,w)^n w^\ell$$

For (5.2) we compute

$$\begin{aligned}
(\mathcal{D}_\alpha(w)a \boxtimes b) \cap C_z^E(\theta_{\alpha,\beta}) &\overset{(3.3)}{=} \sum_{k\geqslant 0} (\Psi_\alpha \times \mathrm{id}_{X_\beta})_*(t_k \boxtimes a \boxtimes b) \cap C_z^E(\theta_{\alpha,\beta}) w^k \\
&= (\Psi_\alpha \times \mathrm{id}_{X_\beta})_* \sum_{k\geqslant 0} (t_k \boxtimes a \boxtimes b) \cap (\Psi_\alpha \times \mathrm{id}_{X_\beta})^* C_z^E(\theta_{\alpha,\beta}) w^k \\
&\overset{(1.8)}{=} (\Psi_\alpha \times \mathrm{id}_{X_\beta})_* \sum_{k\geqslant 0} (t_k \boxtimes a \boxtimes b) \cap C_z^E(\mathcal{L} \boxtimes \theta_{\alpha,\beta}) w^k \\
&\overset{(1.5)}{=} (\Psi_\alpha \times \mathrm{id}_{X_\beta})_* \sum_{k\geqslant 0} (t_k \boxtimes a \boxtimes b) \cap i_{z,c_1^E(\mathcal{L})} C^E_{F(z,c_1^E(\mathcal{L}))}(\theta_{\alpha,\beta}) w^k \\
&\overset{(5.1)}{=} (\Psi_\alpha \times \mathrm{id}_{X_\beta})_* \sum_{\ell\geqslant 0} (t_\ell \boxtimes a \boxtimes b) \cap i_{z,w} C^E_{F(z,w)}(\theta_{\alpha,\beta}) w^\ell \\
&= (\mathcal{D}_\alpha(w) \times \mathrm{id}_{X_\beta})\big[(a \boxtimes b) \cap i_{z,w} C^E_{F(z,w)}(\theta_{\alpha,\beta})\big].
\end{aligned}$$

For (5.3) we similarly use (1.9) which replaces $c_1^E(\mathcal{L})$ by its formal inverse $\iota(c_1^E(\mathcal{L}))$ above, so the same argument with $F(z,\iota(w))$ in place of $F(z,w)$ gives (5.3). □

It is now easy to verify (2.8): Let $a \in E_*(X_\alpha)$, $b \in E_*(X_\beta)$. Then

$$\begin{aligned}
Y(\mathcal{D}_\alpha(w)a, z)b &\overset{(1.13)}{=} (\Phi_{\alpha,\beta})_*(\mathcal{D}_\alpha(z) \boxtimes \mathrm{id}_\beta)[(\mathcal{D}_\alpha(w)a \boxtimes b) \cap C_z^E(\theta_{\alpha,\beta})] \\
&\overset{(5.2)}{=} (\Phi_{\alpha,\beta})_*(\mathcal{D}_\alpha(z)\mathcal{D}_\alpha(w) \boxtimes \mathrm{id}_\beta)\big[(a \boxtimes b) \cap i_{z,w} C^E_{F(z,w)}(\theta_{\alpha,\beta})\big] \\
&\overset{(2.4)}{=} i_{z,w} Y(a, F(z,w))b.
\end{aligned}$$

(c) Firstly, $\Phi \circ (\Psi \times \Psi) \circ \delta \simeq \Psi \circ (\Phi \times \mathrm{id}_{BU(1)})$ and $\Delta_*(t_k) = \sum_{i+j=k} t_i \boxtimes t_j$ imply

$$\mathcal{D}_{\alpha+\beta}(z)(\Phi_{\alpha,\beta})_* = (\Phi_{\alpha,\beta})_*(\mathcal{D}_\alpha(z) \boxtimes \mathcal{D}_\beta(z)). \tag{5.5}$$

Let $a \in E_*(X_\alpha)$, $b \in E_*(X_\beta)$, $c \in E_*(X_\gamma)$. On the one hand

$$\begin{aligned}
Y(Y(a,z)b,w)c &= (\Phi_{\alpha+\beta,\gamma})_*(\mathcal{D}_{\alpha+\beta}(w)\boxtimes \mathrm{id}_\gamma)\\
&\quad \big[(\Phi_{\alpha,\beta})_*(\mathcal{D}_\alpha(z)\boxtimes \mathrm{id}_\beta)[(a\boxtimes b)\cap C_z^E(\theta_{\alpha,\beta})]\boxtimes c\cap C_w^E(\theta_{\alpha+\beta,\gamma})\big]\\
&\overset{(5.5)}{=} (\Phi_{\alpha+\beta,\gamma})_*(\Phi_{\alpha,\beta})_*(\mathcal{D}_\alpha(w)\boxtimes \mathcal{D}_\beta(w)\boxtimes \mathrm{id}_\gamma)\\
&\quad \big[(\mathcal{D}_\alpha(z)\boxtimes \mathrm{id}_\beta\boxtimes \mathrm{id}_\gamma)\big((a\boxtimes b\boxtimes c)\cap C_z^E(\theta_{\alpha,\beta})\cap(\Phi_{\alpha,\beta}\times \mathrm{id}_\gamma)^* C_w^E(\theta_{\alpha+\beta,\gamma})\big)\big]\\
&\overset{(2.4),(5.2)}{=} (\Phi_{\alpha+\beta,\gamma})_*(\Phi_{\alpha,\beta})_*(\mathcal{D}_\alpha(w)\mathcal{D}_\alpha(z)\boxtimes \mathcal{D}_\beta(w)\boxtimes \mathrm{id}_\gamma)\\
&\quad \big[(a\boxtimes b\boxtimes c)\cap C_z^E(\theta_{\alpha,\beta})\cap i_{w,z}C_{F(w,z)}^E(\theta_{\alpha,\gamma})\cap C_w^E(\theta_{\beta,\gamma})\big],
\end{aligned}$$

and on the other hand

$$\begin{aligned}
i_{z,w}Y(a,F(z,w))Y(b,w)c &= i_{z,w}(\Phi_{\alpha,\beta+\gamma})_*(\mathcal{D}_\alpha(F(z,w))\boxtimes \mathrm{id}_{\beta+\gamma})\\
&\quad \big[\big(a\boxtimes(\Phi_{\beta,\gamma})_*(\mathcal{D}_\beta(w)\boxtimes \mathrm{id}_\gamma)\big[(b\boxtimes c)\cap C_w^E(\theta_{\beta,\gamma})\big]\big)\cap C_{F(z,w)}^E(\theta_{\alpha,\beta+\gamma})\big]\\
&\overset{(5.5),(1.7)}{=} i_{z,w}(\Phi_{\alpha,\beta+\gamma})_*(\mathrm{id}_\alpha\boxtimes\Phi_{\beta,\gamma})_*(\mathcal{D}_\alpha(F(z,w))\boxtimes \mathrm{id}_\beta\boxtimes \mathrm{id}_\gamma)\\
&\quad \big[(\mathrm{id}_\alpha\boxtimes\mathcal{D}_\beta(w)\boxtimes \mathrm{id}_\gamma)\big[(a\boxtimes b\boxtimes c)\cap C_w^E(\theta_{\beta,\gamma})\big]\cap C_{F(z,w)}^E(\theta_{\alpha,\beta})\cap C_{F(z,w)}^E(\theta_{\alpha,\gamma})\big]\\
&\overset{(2.4),(5.3)}{=} (\Phi_{\alpha,\beta+\gamma})_*(\mathrm{id}_\alpha\boxtimes\Phi_{\beta,\gamma})_*(\mathcal{D}_\alpha(w)\mathcal{D}_\alpha(z)\boxtimes\mathcal{D}_\beta(w)\boxtimes \mathrm{id}_\gamma)\\
&\quad \big[(a\boxtimes b\boxtimes c)\cap C_w^E(\theta_{\beta,\gamma})\cap C_z^E(\theta_{\alpha,\beta})\cap i_{z,w}C_{F(z,w)}^E(\theta_{\alpha,\gamma})\big].
\end{aligned}$$

As $Y(Y(a,z)b,w)c$ and $Y(a,F(z,w))Y(b,w)c$ are both expansions in negative powers of $F(z,w)$ of the same series in different variables, there exist some $N \gg 0$ with $F(z,w)^N Y(Y(a,z)b,w)c = F(z,w)^N Y(a,F(z,w))Y(b,w)c$, see (2.3).

The same calculations show that (1.15) is a nonlocal vertex F-algebra and that the state-to-field correspondence $\overline{Y}(a,z)b$ preserves the degree shifted by $2\chi(\alpha,\alpha)$. It remains to prove $(-1)^{ab}\overline{Y}(a,z)b = \mathcal{D}_{\alpha+\beta}(z)\overline{Y}(b,\iota(z))a$. Notice $\sigma^*(\overline{C}_z^E(\theta_{\alpha,\beta})) = \overline{C}_{\iota(z)}^E(\theta_{\beta,\alpha})$ for the swap $\sigma\colon X_\beta\times X_\alpha \to X_\alpha\times X_\beta$. Using $\Phi_{\beta,\alpha}\simeq\Phi_{\alpha,\beta}\circ\sigma$ we find

$$\begin{aligned}
\mathcal{D}_{\alpha+\beta}(z)\overline{Y}(b,\iota(z))a &= \mathcal{D}_{\alpha+\beta}(z)(\Phi_{\beta,\alpha})_*(\mathcal{D}_\beta(\iota(z))\boxtimes \mathrm{id}_\alpha)\big[(b\boxtimes a)\cap\overline{C}_{\iota(z)}^E(\theta_{\beta,\alpha})\big]\\
&= \mathcal{D}_{\alpha+\beta}(z)(\Phi_{\alpha,\beta})_*(\mathrm{id}_\alpha\boxtimes\mathcal{D}_\beta(\iota(z)))\sigma_*\big[(b\boxtimes a)\cap\sigma^*\overline{C}_z^E(\theta_{\alpha,\beta})\big]\\
&\overset{(5.5)}{=} (\Phi_{\alpha,\beta})_*(\mathcal{D}_\alpha(z)\boxtimes \mathrm{id}_\beta)\big[\sigma_*(b\boxtimes a)\cap\overline{C}_z^E(\theta_{\alpha,\beta})\big]\\
&= (\Phi_{\alpha,\beta})_*(\mathcal{D}_\alpha(z)\boxtimes \mathrm{id}_\beta)\big[(-1)^{ab}(a\boxtimes b)\cap\overline{C}_z^E(\theta_{\alpha,\beta})\big] = (-1)^{ab}\overline{Y}(a,z)b. \quad\square
\end{aligned}$$

Remark 5.2. For the additive formal group law $\mathbb{G}_a$ and ordinary homology, this was shown by Joyce [8, Thm. 3.14]. When X is the derived category of a finite quiver or of certain smooth projective complex varieties, then taking $F(X,Y) = X+Y$ in (1.15) gives a (super) lattice vertex algebra [7, Thm. 5.7] [8, Thm. 5.19].

Remark 5.3. A similar construction applies to H-spaces X with $BO(1)$-actions, the classifying space for *real* line bundles, and homology with $\mathbb{Z}_2$-coefficients. Since $H^*(BO(1)) = \mathbb{Z}_2[\![\xi]\!]$ there is a shift operator $\mathcal{D}(u)\colon H_*(X;\mathbb{Z}_2)\to H_*(X;\mathbb{Z}_2)[\![u]\!]$ for u a variable of degree -1. One can then build, just as in Theorem 1.1, an operator $(-)\cap W_u(\theta)$ of degree $-\operatorname{rk}\theta_{\alpha,\beta}$, where $\theta_{\alpha,\beta}\in KO(X_\alpha\times X_\beta)$, with normalization $a\cap W_u(L) = a\cap(u+w_1(L))$ for the first Stiefel–Whitney class of a real line bundle $L\to X$. Then $Y(a,z)b = (\Phi_{\alpha,\beta})_*(\mathcal{D}_\alpha(u)\boxtimes \mathrm{id}_\beta)\big[(a\otimes b)\cap W_u(\theta_{\alpha,\beta})\big]$ makes $V = H_*(X;\mathbb{Z}_2)$ into a vertex algebra over $\mathbb{Z}_2$.

Acknowledgements

The authors thank Dominic Joyce for many discussions and suggestions. They also thank Mikhail Kapranov, Kobi Kremnitzer, Sven Meinhardt, and Konrad Voelkel for helpful conversations.

References

[1] J.F. Adams, Stable Homotopy and Generalised Homology, Chicago Lectures in Math., University of Chicago Press, Chicago, Ill.-London, 1974.
[2] P.E. Conner, E.E. Floyd, The Relation of Cobordism to K-Theories, Lecture Notes in Math., vol. 28, Springer-Verlag, Berlin-New York, 1966.
[3] E. Frenkel, D. Ben-Zvi, Vertex Algebras and Algebraic Curves, Math. Surveys and Monographs, vol. 88, A.M.S., Providence, RI, 2004.
[4] I. Frenkel, J. Lepowsky, A. Meurman, Vertex Operator Algebras and the Monster, Pure and Applied Mathematics, vol. 134, Academic Press, Inc., Boston, MA, 1988.
[5] L. Göttsche, H. Nakajima, K. Yoshioka, K-theoretic Donaldson invariants via instanton counting, Pure Appl. Math. Q. 5 (2009) 1029–1111, special issue in honor of F. Hirzebruch.
[6] L. Göttsche, M. Kool, R.A. Williams, Verlinde formulae on complex surfaces I: K-theoretic invariants, arXiv:1903.03869, 2019.
[7] J. Gross, The homology of moduli stacks of complexes, arXiv:1907.03269, 2019.
[8] D. Joyce, Ringel–Hall style vertex algebra and Lie algebra structures on the homology of moduli spaces, http://people.maths.ox.ac.uk/~joyce/hall.pdf.
[9] D. Joyce, Y. Tanaka, M. Upmeier, On orientations for gauge-theoretic moduli spaces, Adv. Math. 362 (2020), arXiv:1811.01096.
[10] D. Joyce, M. Upmeier, Canonical orientations for moduli spaces of G_2-instantons with gauge group $\mathrm{SU}(m)$ or $\mathrm{U}(m)$, J. Differ. Geom. (2022), in press.
[11] D. Joyce, M. Upmeier, On spin structures and orientations for gauge-theoretic moduli spaces, Adv. Math. 381 (2021).
[12] V.G. Kac, Vertex Algebras for Beginners, Univ. Lect. Series, vol. 10, A.M.S., Providence, RI, 1998.
[13] Y.-H. Kiem, M. Savvas, K-theoretic generalized Donaldson–Thomas invariants, Int. Math. Res. Not. (2020), https://doi.org/10.1093/imrn/rnaa097.
[14] H.-S. Li, Vertex F-algebras and their ϕ-coordinated modules, J. Pure Appl. Algebra 215 (2011) 1645–1662.
[15] D. Quillen, Elementary proofs of some results of cobordism theory using Steenrod operations, Adv. Math. 7 (1971) 29–56.
[16] Y.B. Rudyak, On Thom Spectra, Orientability, and Cobordism, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 1998.
[17] R.M. Switzer, Algebraic Topology—Homotopy and Homology, Die Grundlehren der Math. Wiss., vol. 212, Springer-Verlag, New York-Heidelberg, 1975.
[18] R.P. Thomas, Equivariant K-theory and refined Vafa–Witten invariants, Commun. Math. Phys. 378 (2020) 1451–1500.
[19] B. Toën, M. Vaquié, Moduli of objects in dg-categories, Ann. Sci. Éc. Norm. Supér. 40 (2007) 387–444.