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Mathematics and Statistics Doctoral Training Centre

On the minimisation of bending energies related to the Willmore functional under constraints on area and volume

by

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Declaration

This thesis is submitted to the University of Warwick in support of my application for the degree of Doctor of Philosophy. It has been composed by myself and has not been submitted in any previous application for any degree. The work has been carried out by myself under the supervision of Andrea Mondino.

Parts of Section 2 have been published by the author in collaboration with Andrea Mondino as *Existence and regularity of spheres minimising the Canham–Helfrich energy* in Archive for Rational Mechanics and Analysis. Parts of Section 3 have been published by the author in collaboration with Andrea Mondino as *A strict inequality for the minimization of the Willmore functional under isoperimetric constraint* in Advances in Calculus of Variations. Section 4 has been submitted for publication as *Embedded Delaunay tori and their Willmore energy*. Parts of Section 5 have been submitted for publication as *Some geometric inequalities for varifolds on Riemannian manifolds based on monotonicity identities*.

Abstract

The Willmore energy of a closed surface is defined as the integrated squared mean curvature. It appears in many areas of science and technology in current research. A slight variation, known as Canham–Helfrich functional, is obtained as a linear combination of the Willmore functional, the total mean curvature and the area. The Canham–Helfrich energy is the associated bending energy of a lipid bilayer cell membrane. Its minimisation amongst closed spherical surfaces with given fixed area and volume is referred to as the *Helfrich problem*. Minimising purely the Willmore functional in the class of surfaces with given fixed genus, while keeping the constraints on area and volume will be referred to as the *Canham problem*. By the scaling invariance of the Willmore functional, the two constraints on area and volume reduce to a single constraint on the scaling invariant isoperimetric ratio.

This thesis presents results that substantially contributed in fully solving existence of minimisers for both the Helfrich and Canham problems. Previously, Mondino–Rivière developed the notion of bubble tree which is a finite family of weak possibly branched immersions of the 2-sphere into the 3-space. Such a family of weak immersions can be parametrised by a single continuous map on the 2-sphere. They showed pre-compactness and continuity of the area functional on the class of bubble trees under weak convergence. In this thesis, we prove lower semi-continuity of the Canham–Helfrich functional as well as continuity of the volume under weak convergence of bubble trees, leading to existence of minimisers. Moreover, we show that critical bubble trees are smooth outside of finitely many branch points. In fact, the regularity result holds true for critical surfaces of any genus.

In the early 2000s, Bauer–Kuwert proved a strict inequality between the Willmore energies of two surfaces and their connected sum leading to existence of minimisers for the Willmore functional with any prescribed genus. In the proof they use bi-harmonic interpolation in order to patch an inverted surface into a huge copy of a second surface. Using their connected sum construction, we show that the same inequality remains valid in the context of isoperimetric constraints. In a first step, we determine the precise order of convergence for the isoperimetric ratio of the connected sum under scaling (up) of the second surface. Then, inspired by Huisken's volume preserving mean curvature flow, we show that the small isoperimetric deficit can be adjusted using a variational vector field supported away from the patching region. By a previous result of Keller–Mondino–Rivière, our strict inequality leads to existence of minimisers for the Canham problem, provided the minimal energy lies strictly below 8π.

In order to complete the existence part for the Canham problem in the genus one case, we construct rotationally symmetric tori consisting of two opposite signed constant mean curvature surfaces. The tori converge as varifolds to a double round sphere. Using complete elliptic integrals, we show that the resulting family can be used to obtain comparison tori of any isoperimetric ratio with Willmore energy strictly below 8π.

Additionally, we prove a general Li–Yau inequality for varifolds on Riemannian manifolds by testing the first variation identity against vector fields which are proportional to the gradient of the distance function.

1 Introduction

This thesis is about bending energies related to the Willmore functional. Given an immersed surface $f : \Sigma \to \mathbb{R}^3$, the Willmore functional $\mathcal{W}$ at f is defined by

$$\mathcal{W}(f) = \frac{1}{4} \int_\Sigma H^2 \, \mathrm{d}\mu$$

where H is the trace of the second fundamental form and μ denotes the Radon measure induced by the pull back metric of the Euclidean metric along f. Notice that there are different conventions about whether to define the mean curvature H as the arithmetic sum of the principal curvatures or the sum of the principal curvatures. Throughout the thesis we will stick to the latter convention except for Section 2. Switching the definition has the advantage that the statements can be easier put into relation with closely connected results. Typically, the Willmore functional is defined on the space of smoothly immersed surfaces. Given a non-negative integer g, we denote with $\mathcal{S}_g$ the set of smooth immersions $f : \Sigma \to \mathbb{R}^3$ where Σ is a compact smooth surface without boundary and of genus g. We will also get to know settings where the Willmore functional is defined on larger spaces; see Subsection 1.3 and Section 2 for $W^{2,2}$-immersions and Section 5 for varifolds.

The first known appearance of the Willmore energy goes back to the works of POISSON [109] in 1814 and GERMAIN [47] in 1821, who studied the vibration of thin elastic plates. It appeared again in the work of KIRCHHOFF [63] in 1850 on the same subject. Notice that in particular in the presence of physical forces, one often finds the following energy per unit area

$$\frac{1}{r_1^2} + \frac{1}{r_2^2}$$

where r_1, r_2 are the radii of the osculating circles in the principal planes. Indeed, denoting with κ_1, κ_2 the principal curvatures, expanding the square and using the Gauss–Bonnet theorem, one has

$$\frac{1}{4} \int_\Sigma H^2 \, \mathrm{d}\mu = \frac{1}{4} \int_\Sigma \kappa_1^2 + \kappa_2^2 \, \mathrm{d}\mu + \pi \chi(\Sigma),$$

where $\chi(\Sigma)$ denotes the Euler characteristic of Σ. Hence, the two energies coincide up to a topological constant. In particular, they have the same minimisers. In the first half of the 20th century, the Willmore functional was studied in the context of conformal differential geometry by BLASCHKE–THOMSEN [14]. Meanwhile, the Willmore energy started to appear in many areas of science and technology. We will focus here on minimisation problems of geometric interest with applications in the study of biological cell membranes. Although some of the classical problems are fully solved, many questions remain open. Existence of minimisers for the following minimisation problems is done.

- Willmore problem [139, 136, 67, 6, 113, 115, 114, 78, 70] (see Subsection 1.2).

- Helfrich problem [28, 96, 41] (see Problem 1 and Subsection 1.4).

- Canham problem [129, 62, 97, 123, 69] (see Problem 2 and Subsection 1.5).

The results presented in this thesis substantially contribute to the solution of the Helfrich and Canham problems. While the existence part of the minimisation problems above is fully solved,

there are only partial results about the actual minimisers. Denote the minimal Willmore energy amongst surfaces with fixed genus by

$$\beta_g := \inf_{f \in \mathcal{S}_g} \mathcal{W}(f).$$

By WILLMORE [139], it is known that $\beta_0 = 4\pi$ is attained exactly by any dilation of the unit sphere. MARQUES–NEVES [85] not only solved the Willmore conjecture, that is $\beta_1 = 2\pi^2$ is attained by the Clifford torus, but they also showed that $\beta_g > \beta_1$ for all $g \geq 2$. See also RIVIÈRE [118] for a PDE based proof of the Willmore conjecture. In accordance with the results of WILLMORE [139] and MARQUES–NEVES [85], it is conjectured that β_g is attained by the stereographic projections of the minimal surfaces in the three sphere found by LAWSON [81]. Moreover, it is conjectured that β_g is strictly increasing in g. It seems also natural to conjecture that spherical solutions for the Helfrich problem as well as spherical and toroidal solutions for the Canham problem are rotationally symmetric.

Other interesting problems related to the minimisation of the Willmore functional that have been fully or partially solved are:

- Willmore conjecture [85, 118].

- Monotonicity of β [139, 67, 85, 72]: $4\pi = \beta_0 < \beta_1 < \beta_g < 8\pi$ for $g \geq 2$, $\lim_{g \to \infty} \beta_g = 8\pi$.

- Conformally constrained Willmore problem [79, 104, 30].

- Branch point analysis of Willmore surfaces [77, 11, 105].

- Willmore boundary problem [126, 29].

- Helfrich boundary problem [40].

- Willmore flow of spheres [74, 75, 76].

- Canham flow of spheres [119, 120].

- Willmore flow of tori [30].

- Willmore minimisers in Riemannian manifolds [80, 73, 94, 26, 56].

- Björling's problem for Willmore surfaces in $\mathbb{S}^3$ [18].

Let us stress that several of the aforementioned results have been obtained in the last ten years, and that the Willmore functional is currently a topic of high research interest in both pure and applied mathematics. Indeed, for space restrictions we did not mention several results dealing with the Willmore energy in numerical analysis (see for instance [39, 110, 38]), physics (e.g. the Hawking mass in general relativity or the Polyakov extrinsic action in string theory), or optics (e.g. bi-concave lenses). For an overview of applications in biology, see Section 1.4.

1.1 Willmore inequality

In the early 60s, WILLMORE [139] solved the minimisation problem

$$\beta_0 = \inf_{f \in \mathcal{S}_0} W(f) = 4\pi.$$

Denoting with κ_1, κ_2 the principal curvatures of an embedded sphere $f : \mathbb{S}^2 \to \mathbb{R}^3$ and denoting with K the Gauss curvature, he observed that the mean curvature H can be expressed as

$$\frac{1}{4}H^2 = K + \frac{1}{4}(\kappa_1 - \kappa_2)^2$$

and thus, by the Gauss–Bonnet theorem,

$$\mathcal{W}(f) = 4\pi + \frac{1}{4}\int_{\mathbb{S}^2}(\kappa_1 - \kappa_2)^2 \, \mathrm{d}\mu \geq 4\pi$$

with equality if and only if $\kappa_1 \equiv \kappa_2$ identically, which in turn holds true only for round spheres (in the closed case, see for instance [138]). Although the above trick via Gauss–Bonnet only works for topological spheres, the statement actually remains valid for embedded surfaces $f : \Sigma \to \mathbb{R}^3$ of any genus:

$$\frac{1}{4}\int_{\Sigma} H^2 \, \mathrm{d}\mu \geq 4\pi \tag{1.1}$$

where again, equality holds if and only f parametrises a round sphere (see [140, Theorem 7.2.2]). The inequality is also referred to as *Willmore inequality.* It was improved by LI–YAU [82, Theorem 6] for smoothly immersed closed surfaces $f : \Sigma \to \mathbb{R}^n$: If there exists $p \in \mathbb{R}^n$ with $f^{-1}(p) = \{x_1, \ldots, x_k\}$ where the x_i's are all distinct points in Σ, in other words f has a point of multiplicity k, then

$$\frac{1}{4}\int_{\Sigma} |H|^2 \, \mathrm{d}\mu \geq 4\pi k.$$

In particular, if the Willmore energy lies strictly below 8π, then f is an embedding. Because of this property, the Li–Yau inequality has become very useful for the minimisation of the Willmore functional and, more generally, for the study of immersed surfaces. In fact, it plays a crucial role in this thesis, see Remark 2.7. Due to the conformal invariance of the Willmore functional observed by BLASCHKE–THOMSEN [14] and CHEN [25] (see also WEINER [137]), Willmore's inequality has an analogue for surfaces Σ in the three sphere $\mathbb{S}^3$:

$$\frac{1}{4}\int_{\Sigma} H^2 \, \mathrm{d}\mu + |\Sigma| \geq 4\pi, \tag{1.2}$$

where $|\Sigma| := \int 1 \, \mathrm{d}\mu$ denotes the area of Σ in $\mathbb{S}^3$. Indeed, thanks to the conformal invariance, if Σ is immersed in $\mathbb{S}^3$ and $\hat{\Sigma}$ denotes its stereographic projection into $\mathbb{R}^3$, then the Willmore energies are related by

$$\int_{\Sigma}\Big(\frac{H^2}{4} + 1\Big)\mathrm{d}\mu = \frac{1}{4}\int_{\hat{\Sigma}} \hat{H}^2 \, \mathrm{d}\hat{\mu}. \tag{1.3}$$

Thus, (1.2) follows from the classical Willmore inequality (1.1). Notice also that the relation (1.3) connects Willmore surfaces in $\mathbb{R}^3$ with minimal surfaces in $\mathbb{S}^3$. Indeed, the Willmore conjecture,

corresponding to the minimisation problem

$$\beta_1 = \inf_{f \in \mathcal{S}_1} W(f) = 2\pi^2$$

was solved using minimal surface theory, see MARQUES–NEVES [85]. Similarly to Willmore's inequality in the three sphere (1.2), there is a version for surfaces Σ in the hyperbolic space $\mathbb{H}^3$:

$$\frac{1}{4} \int_{\Sigma} H^2 \, \mathrm{d}\mu - |\Sigma| \geq 4\pi. \tag{1.4}$$

We will soon see in our Theorem 1 that both inequalities (1.2) and (1.4) can be generalised for surfaces in Riemannian manifolds with an upper bound on the sectional curvature. Previous work in this direction was done by KLEINER [64] who showed

$$\frac{1}{4} \int_{\Sigma_0} H^2 \, \mathrm{d}\mu + b|\Sigma_0| \geq 4\pi \tag{1.5}$$

for minimisers Σ_0 of the isoperimetric profile in a complete one-connected 3-dimensional Riemannian manifold without boundary and with sectional curvatures bounded above by $b \leq 0$. Subsequently, RITORÉ [112] showed that (1.4) remains valid for all $C^{1,1}$ surfaces in a 3-dimensional Cartan–Hadamard manifold with sectional curvatures bounded above by -1. SCHULZE [127] showed that the classical Willmore inequality (1.1) holds true for integral 2-varifolds in 3-dimensional Cartan–Hadamard manifolds, see [127, Lemma 6.7]. Then, he showed that (1.5) remains valid for integral 2-varifolds in n-dimensional Cartan–Hadamard manifolds, see [128, Theorem 1.4]. Finally, CHAI [23] upgraded (1.4) to a Li–Yau inequality. That is, given a smoothly immersed surface $f : \Sigma \to \mathbb{H}^n$ in the hyperbolic space $\mathbb{H}^n$ which has a point of multiplicity k, then

$$\frac{1}{4} \int_{\Sigma} |H|^2 \, \mathrm{d}\mu - |\Sigma| \geq 4\pi k.$$

Notice also the recent generalisation of the Willmore inequality to higher dimensional submanifolds by AGOSTINIANI–FOGAGNOLO–MAZZIERI [3]. They showed that for closed codimension 1 submanifolds M in a non-compact n-dimensional Riemannian manifold (N, g) with non-negative Ricci curvature, there holds

$$\int_M \left| \frac{H}{n-1} \right|^{n-1} \, \mathrm{d}\mu \geq \mathrm{AVG}(g) |\mathbb{S}^{n-1}|$$

where $\mathrm{AVG}(g)$ denotes the asymptotic volume ratio of (N, g). See also CHEN [24] for the earlier Euclidean version.

In the following theorem, we generalise the results of CHAI [23] to Riemannian manifolds with an upper bound on the sectional curvature.

Theorem 1 (See Corollary 5.15 and SCHARRER [124, Theorem 1.7]). *Suppose $n \geq 3$ is an integer, (N, g) is an n-dimensional Riemannian manifold, Σ is a smooth closed surface, $f : \Sigma \to N$ is a smooth immersion, $p \in N$, $f^{-1}\{p\} = \{x_1, \ldots x_k\}$ where the x_i's are distinct points in Σ, $b \in \mathbb{R}$, and the sectional curvature K of N satisfies $K \leq b$ on the image of f. Let H be the trace of the second fundamental form of the immersion f, μ be the Radon measure on Σ induced by the*

pull-back metric of g along f, and $|\Sigma| := \int_\Sigma 1 \, d\mu$ be the area of Σ in N. Then, the following two statements hold.

1. *If $b > 0$, and the image of f is contained in a geodesic ball around p (see Definition 5.1) of radius strictly less than $\frac{\pi}{\sqrt{b}}$, then*

$$\frac{1}{4} \int_\Sigma |H|_g^2 \, d\mu + b \int_\Sigma \cos(\sqrt{b}r) \, d\mu + b|\Sigma| \geq 4\pi k,$$

   where $r = d(p, \cdot)$ is the distance to p in N.

2. *If $b \leq 0$ and the image of f is contained in a geodesically star-shaped open neighbourhood of p (see Definition 5.1), then*

$$\frac{1}{4} \int_\Sigma |H|_g^2 \, d\mu + b|\Sigma| \geq 4\pi k.$$

In particular, if the left hand side is strictly smaller than 8π, then f is an embedding.

Notice that the inequality in (1) leads to

$$\frac{1}{4} \int_\Sigma |H|_g^2 \, d\mu + 2b|\Sigma| \geq 4\pi k$$

which up to a larger constant recovers the spherical version (1.2). In Section 5 we will prove Theorem 1 for varifolds with boundary leading to the most general Li–Yau inequality up to now, see Theorem 5.13. The statement follows from monotonicity inequalities obtained by testing the first variation identity of the area with the vector field $\sin(\sqrt{b}r)\nabla r$ if $b > 0$ and $\sinh(\sqrt{|b|}r)\nabla r$ if $b < 0$, where again $r = d(p, \cdot)$ is the distance to p in N, and p is the point of multiplicity k.

1.2 The classical Willmore problem

In the previous section we have seen that the minimal Willmore energy amongst all surfaces immersed into $\mathbb{R}^3$ is attained exactly by any dilation of the unit sphere. In particular, given any immersion $f : \Sigma \to \mathbb{R}^3$ of a surface Σ with genus $g \geq 1$, there holds $\mathcal{W}(f) > \mathcal{W}(\mathbb{S}^2) = 4\pi$. This leads to the question whether or not the infimum

$$\beta_g = \inf_{f \in \mathcal{S}_g} \mathcal{W}(f) \tag{1.6}$$

is attained and, if so, what are the minimisers. We will refer to this question as the *Willmore problem*. A first result on existence of Willmore minimisers was contributed by SIMON [136]. He proved that the infimum in (1.6) is attained provided that

$$\beta_g < \min\{8\pi, \omega_g\}, \tag{1.7}$$

where

$$\omega_g := \min\left\{4\pi + \sum_{i=1}^p (\beta_{g_i} - 4\pi) : g = \sum_{i=1}^p g_i, \, 1 \leq g_i < g\right\}. \tag{1.8}$$

5

SIMON'S [136] result was proven using what is now known as the *ambient approach*. That is, convergence of surfaces is given by convergence of the induced Radon measures. To be more precise, let $\Sigma_1, \Sigma_2, \ldots$ be a sequence of genus g surfaces with unit area embedded into $\mathbb{R}^3$ such that $\mathcal{W}(\Sigma_k) \to \boldsymbol{\beta}_g$ as $k \to \infty$. In other words, Σ_k is a minimising sequence of (1.6). Then, by classical functional analysis (see Theorem 2 of Section 1.9 in [44]), after passing to a subsequence, there exists a Radon measure μ on $\mathbb{R}^3$ such that

$$\lim_{k \to \infty} \int_{\Sigma_k} f \, \mathrm{d}\mathscr{H}^2 = \int_{\mathbb{R}^3} f \, \mathrm{d}\mu \qquad \text{whenever } f \in C(\mathbb{R}^3, \mathbb{R}) \tag{1.9}$$

where $\mathscr{H}^2$ is the 2-dimensional Hausdorff measure. SIMON [136] then showed that $\mu = \mathscr{H}^2 \llcorner \Sigma$ for a smoothly embedded surface Σ which attains $\boldsymbol{\beta}_g$. Locally, inside a small given ball, each of the surfaces Σ_k can be approximated by a layer of Lipschitz graphs with small Lipschitz constants. The assumption $\boldsymbol{\beta}_g < 8\pi$ implies that this layer consist of actually only one sheet (compare this with the Li–Yau inequality discussed in Section 1.1). Hence, as a starting point of the regularity theory, one can employ the Arzela–Ascoli theorem to deduce that outside of finitely many points, the limit μ is locally given by the graph of a Lipschitz function. Investigating regularity around the finitely many points where energy might have concentrated once again relies on the assumption that $\boldsymbol{\beta}_g < 8\pi$, compare also Remark 2.7 and Proposition 2.1 in this thesis. Finally, the assumption $\boldsymbol{\beta}_g < \boldsymbol{\omega}_g$ is needed to deduce that the genus of Σ indeed equals g as required.

Notice that the weaker the notion of convergence of the sequence Σ_k, the harder it is to prove lower semi-continuity of the Willmore functional. Indeed, SIMON [136] did not prove lower semi-continuity of the Willmore functional under the convergence in (1.9). That was done later by SCHÄTZLE [125]. Instead, in order to show that Σ actually attains $\boldsymbol{\beta}_g$, SIMON [136] used a comparison argument by locally solving the Dirichlet–Neumann problem for biharmonic functions.

WILLMORE [139] computed the energy now bearing his name of rotationally symmetric tori with radii $0 < r < R$. He showed that the energy is minimal if $\frac{r}{R} = \frac{1}{\sqrt{2}}$ which results in the so-called Clifford torus $\mathbb{T}_{1/\sqrt{2}}$ and $\mathcal{W}(\mathbb{T}_{1/\sqrt{2}}) = 2\pi^2$. In particular, $\boldsymbol{\beta}_1 \leq 2\pi^2 < 8\pi$. Moreover, by definition, $\boldsymbol{\omega}_1 = \infty$ which means the strict inequality in (1.7) is satisfied for $g = 1$. Thus, SIMON [136] in particular proved existence of Willmore tori (i.e. genus $g = 1$ minimisers). Later, MARQUES–NEVES [85] proved that indeed $\boldsymbol{\beta}_1 = 2\pi^2$ which solved the Willmore problem for $g = 1$. For $g \geq 2$, KUSNER [67] showed that $\boldsymbol{\beta}_g < 8\pi$ by estimating the area of minimal surfaces in the three sphere $\mathbb{S}^3$ found by LAWSON [81] (recall (1.3)). Hence, in order to prove existence of Willmore minimisers with prescribed genus $g \geq 2$, the missing step was to show that

$$\boldsymbol{\beta}_g < \boldsymbol{\omega}_g. \tag{1.10}$$

There were some suggestions on that inequality before it was finally proven. Namely SIMON [136] conjectured that $\boldsymbol{\beta}_g \geq 6\pi$ for all $g \geq 1$ which would imply $\boldsymbol{\omega}_g > 8\pi$, reducing the compactness assumption in (1.7) to the 8π-bound proven by KUSNER [67]. As we shall see shortly, SIMON'S conjecture is now known to be true. Moreover, SIMON [136] explained that the non-strict

inequality

$$\boldsymbol{\beta}_g \leq \boldsymbol{\omega}_g \tag{1.11}$$

is indeed always true. To see this, he suggested to choose p surfaces $f_1 \in \mathcal{S}_{g_1}, \ldots, f_p \in \mathcal{S}_{g_p}$ with Willmore energies close to $\boldsymbol{\beta}_{g_1}, \ldots, \boldsymbol{\beta}_{g_p}$, respectively. Then, to each surface f_i, apply a sphere inversion (also referred to as *Möbius map*)

$$I_{a_i} : \mathbb{R}^3 \setminus \{a_i\} \to \mathbb{R}^3, \qquad I_{a_i}(x) = \frac{x - a_i}{|x - a_i|^2} \tag{1.12}$$

for some point $a_i \in \operatorname{im} f_i$ of multiplicity one, turning the surface f_i into an unbounded surface $I_{a_i} \circ f_i$ with planar end and Willmore energy $\mathcal{W}(I_{a_i} \circ f_i) = \mathcal{W}(f_i) - 4\pi$. This procedure is often described as *inverting the surface f_i at a_i*. (In fact, SIMON [136] suggested to choose a_i close to the image of f_i which results in a surface that already looks like a round sphere; the final construction however will look the same). Then, focus on the part of $I_{a_i} \circ f_i$ that carries the genus of f_i, cut away the planar end, and glue the part with the genus into a large round sphere. The glueing can be done at small cost in terms of Willmore energy in such a way that the resulting surface looks like a round sphere with a cap of g_i handles, having the same genus as f_i and Willmore energy close to the sum $\mathcal{W}(\mathbb{S}^2) + \mathcal{W}(I_{a_i} \circ f_i)$. Glueing suitable sphere inversions of the surfaces $f_1, \ldots, f_p$ all into the same large sphere, results in a surface f with $\operatorname{genus}(f) = \operatorname{genus}(f_1) + \ldots + \operatorname{genus}(f_p)$ that looks like a round sphere with p caps and Willmore energy

$$\mathcal{W}(f) \approx \mathcal{W}(\mathbb{S}^2) + \sum_{i=1}^{p} \mathcal{W}(I_{a_i} \circ f_i) \approx 4\pi + \sum_{i=1}^{p} (\boldsymbol{\beta}_{g_i} - 4\pi)$$

which indeed implies the non-strict inequality (1.11). In fact, in order to prove either of the inequalities (1.10) or (1.11) one might assume that $p = 2$ in the definition of $\boldsymbol{\omega}_g$ (see Equation (1.8)). The general case then follows by induction. KUSNER [67] developed the *conformal connected-sum $M\#N$* of two given immersed surfaces M and N in $\mathbb{R}^3$ satisfying

$$\mathcal{W}(M\#N) = \mathcal{W}(M) + \mathcal{W}(N) - 4\pi,$$

which also implies the non-strict inequality (1.11). This kind of equation can be found in many mathematical concepts. Notable for instance is that the same equation holds true for the Euler characteristic χ of the connected sum $M\#N$ of two n-manifolds M and N:

$$\chi(M\#N) = \chi(M) + \chi(N) - \chi(\mathbb{S}^n).$$

Later, KUSNER [68] suggested to invert the two surfaces at nonumbilic points after which the planar end of each surface is asymptotic to the graph of a biharmonic function with higher order terms decaying at least as fast as $1/r$. Therefore, one can weld together two such inverted surfaces along a line in their planar ends and estimate the saved energy in terms of the energy of a biharmonic graph. Inspired by this idea, BAUER–KUWERT [6] finally found a proof for the strict inequality (1.10) and thus completed the existence part for the Willmore problem. Given two smoothly immersed surfaces $f_i : \Sigma_i \to \mathbb{R}^3$ with $i = 1, 2$ neither of which is a round sphere, they constructed an immersed surface $f : \Sigma \to \mathbb{R}^3$ with topological type of the connected sum

$\Sigma_1 \# \Sigma_2$ by inverting the first surface f_1 at a nonumbilic point and glueing the inverted surface directly into a large copy of the second surface, again at a nonumbilic point. The glueing was done by the graph of a biharmonic function. Thereby they inferred

$$\mathcal{W}(f) < \mathcal{W}(f_1) + \mathcal{W}(f_2) - 4\pi \tag{1.13}$$

which implies (1.10).

An alternative way to prove the strict inequality (1.10) for the high genus case follows from a result of KUWERT–LI–SCHÄTZLE [72]. They proved that $\lim_{g \to \infty} \beta_g = 8\pi$ which implies $\lim_{g \to \infty} \omega_g > 8\pi$. Moreover, the inequality $\beta_g \geq 2\pi^2$ for all $g \geq 1$ proven by MARQUES–NEVES [85] implies in particular $\beta_g \geq 6\pi$ as conjectured by SIMON [136] and thus $\omega_g \geq 8\pi$ leading to an alternative proof of (1.10).

It remains to mention that SIMON's [136] results were actually proven not only for surfaces embedded into $\mathbb{R}^3$ but more generally, for surfaces embedded into $\mathbb{R}^n$ with $n \geq 3$. This results in constants β_g^n that are non-increasing in n. The proof of MARQUES–NEVES [85] however only holds for $n = 3$. In this thesis we are interested in minimisation problems with constraints on the volume. Thus, we will restrict ourselves to the case $n = 3$.

1.3 The parametric approach

In the previous section we have seen how the existence part of Willmore problem was solved by the ambient approach. We will now present the mathematical framework of an alternative way developed later. Namely, in the *parametric approach*, the direct methods in calculus of variations for the Willlmore functional are formalised into the language of Sobolev mappings and functional analysis. Thereby, as we shall see in this thesis, the techniques are made accessible not only to the classical Willmore problem but to various related minimisation problems. The compactness part in this setting was developed by KUWERT–SCHÄTZLE [78] and KUWERT–LI [70] using previous results by MÜLLER–ŠVERÁK [100], and independently by RIVIÈRE [115, 114] using HÉLEIN's moving frames technique [51]. The regularity part for the Willmore problem in this setting was established by RIVIÈRE [113].

Let Σ be a closed surface endowed with a smooth metric g_0. A map $\vec{\Phi} : \Sigma \to \mathbb{R}^3$ is called a *weak branched conformal immersion with finite total curvature* if and only if there exists a positive integer N, finitely many points $b_1, \ldots, b_N \in \Sigma$ such that

$$\vec{\Phi} \in W^{1,\infty}(\Sigma, \mathbb{R}^3) \cap W_{\text{loc}}^{2,2}(\Sigma \setminus \{b_1, \cdots, b_N\}, \mathbb{R}^3), \tag{1.14}$$

there holds

$$\begin{cases} |\partial_{x^1} \vec{\Phi}| = |\partial_{x^2} \vec{\Phi}| \\ \partial_{x^1} \vec{\Phi} \cdot \partial_{x^2} \vec{\Phi} = 0 \end{cases} \tag{1.15}$$

almost everywhere for any conformal chart x of Σ,

$$\log |\mathrm{d}\vec{\Phi}| \in L_{\text{loc}}^\infty(\Sigma \setminus \{b_1, \ldots, b_N\}),$$

and the Gauss map $\vec{n}$ defined by

$$\vec{n} := \frac{\partial_{x^1}\vec{\Phi} \times \partial_{x^2}\vec{\Phi}}{|\partial_{x^1}\vec{\Phi} \times \partial_{x^2}\vec{\Phi}|}$$

in any local positive chart x of Σ satisfies

$$\vec{n} \in W^{1,2}(\Sigma, \mathbb{R}^3). \tag{1.16}$$

The space of weak branched conformal immersions with finite total curvature is denoted by $\mathcal{F}_\Sigma$. We define the L^∞-metric g pointwise for almost every $p \in \Sigma$ by

$$g_p(X, Y) := \mathrm{d}\vec{\Phi}_p(X) \cdot \mathrm{d}\vec{\Phi}_p(Y)$$

for elements X, Y of the tangent space $T_p\Sigma$. In the usual way, g induces a Radon measure μ_g on Σ. The conditions (1.14) and (1.15) imply that

$$g(\cdot, \cdot) \leq C_1 g_0(\cdot, \cdot) \tag{1.17}$$

almost everywhere for some finite $C_1 > 0$. In particular, μ_g is absolutely continuous with respect to the Radon measure μ_{g_0} induced by the reference metric and condition (1.16) implies

$$|\mathrm{d}\vec{n}|_g \in L^2(\Sigma, \mu_g). \tag{1.18}$$

If in addition to (1.17) there holds

$$C_0 g_0(\cdot, \cdot) \leq g(\cdot, \cdot)$$

for some $C_0 > 0$, then $\vec{\Phi}$ is called a *Lipschitz immersion.* The space of Lipschitz immersions is denoted by $\mathcal{E}_\Sigma$. The subtle but important difference of the two spaces $\mathcal{E}_\Sigma$ and $\mathcal{F}_\Sigma$ is subject of Remark 2.7. We define the second fundamental form $\vec{\mathbb{I}}$ pointwise for almost every $p \in \Sigma$ by

$$\vec{\mathbb{I}}_p : T_p\Sigma \times T_p\Sigma \to \mathbb{R}^3, \qquad \vec{\mathbb{I}}_p(X, Y) := -[\mathrm{d}\vec{n}_p(X) \cdot \mathrm{d}\vec{\Phi}_p(Y)]\vec{n}.$$

The mean curvature vector $\vec{H}$ and the scalar mean curvature H are given by

$$\vec{H} := \mathrm{tr}\,\vec{\mathbb{I}}, \qquad H := \vec{n} \cdot \vec{H}.$$

Note that (1.18) ensures

$$H \in L^2(\Sigma, \mu_g).$$

In particular, we can define the Willmore functional on the space $\mathcal{F}_\Sigma$:

$$\mathcal{W}(\vec{\Phi}) = \frac{1}{4} \int_\Sigma H^2 \, \mathrm{d}\mu_g.$$

For more properties of weakly conformal immersions, see Section 2.1.

Each member $\vec{\Phi}$ of the space $\mathcal{E}_\Sigma$ induces a conformal structure on Σ which we will denote by $c_{\vec{\Phi}}$ (see [117, Corollary 4.4]). Given any finite number $\Lambda > 0$ and a compact subset K of the

moduli space $\mathcal{M}(\Sigma)$ of Σ, after passing to a subsequence, composing with Möbius transformations and reparametrisations, each sequence in

$$\{\vec{\Phi} \in \mathcal{E}_\Sigma : \mathcal{W}(\vec{\Phi}) \leq \Lambda,\, c_{\vec{\Phi}} \in K\}$$

converges weakly in the pre-Hilbert space $W_{\mathrm{loc}}^{2,2}(\Sigma \setminus \{b_1, \ldots, b_N\}, \mathbb{R}^3)$ for some $b_1, \ldots, b_N \in \Sigma$ to a limit $\vec{\Phi}_\infty \in \mathcal{F}_\Sigma$ (see [70, Theorem 4.1] or [115]). If $\mathcal{W}(\vec{\Phi}_\infty) < 8\pi$, then $\vec{\Phi}_\infty \in \mathcal{E}_\Sigma$. Moreover, for any $\delta > 0$, it holds that

$$\left\{ c_{\vec{\Phi}} : \vec{\Phi} \in \mathcal{E}_\Sigma,\, \mathcal{W}(\vec{\Phi}) \leq \min\{8\pi, \boldsymbol{\omega}_{\mathrm{genus}(\Sigma)}\} - \delta \right\}$$

is a compact subset of $\mathcal{M}(\Sigma)$ (see [70, Theorem 5.3] or [114]). Therefore, by the strict inequality (1.7), the infimum

$$\inf_{\vec{\Phi} \in \mathcal{E}_\Sigma} \mathcal{W}(\vec{\Phi})$$

is attained. Moreover, each $\vec{\Phi} \in \mathcal{E}_\Sigma$ which is a critical point of the Willmore functional $\mathcal{W}$, is smooth (see [113]). In particular,

$$\boldsymbol{\beta}_{\mathrm{genus}(\Sigma)} = \inf_{f \in \mathcal{S}_{\mathrm{genus}(\Sigma)}} \mathcal{W}(f) = \inf_{\vec{\Phi} \in \mathcal{E}_\Sigma} \mathcal{W}(\vec{\Phi})$$

is attained by a smooth minimiser.

It remains to mention that weak convergence in the pre-Hilbert space $W_{\mathrm{loc}}^{2,2}(\Sigma \setminus \{b_1, \ldots, b_N\}, \mathbb{R}^3)$ is much stronger than the convergence of measures in the ambient approach (1.9). An advantage of such a stronger convergence is that lower semi-continuity of the Willmore functional simply follows from lower semi-continuity of the L^2-norm under weak L^2-convergence.

1.4 The Helfrich problem

The basic structural and functional unit of all known living organisms is the cell. The interior material of a cell, the cytoplasm, is enclosed by biological membranes. Most of the cell membranes of living organisms are made of lipid bilayer, which is a thin polar membrane consisting of two opposite oriented layers of lipid molecules. These molecules have a hydrophilic head and a hydrophobic tail. Exposed to water, they self-assemble into a two-layered sheet with the hydrophobic tails pointing towards the centre of the sheet.

In 1970, in order to explain the biconcave shape of red blood cells, CANHAM [21] proposed a bending energy density dependent on the squared mean curvature. He considered shape transformations under fixed area and volume. Three years later, HELFRICH [52] proposed the following curvature elastic energy per unit area of a closed lipid bilayer (see [52, Equation (12)])

$$\frac{1}{2}k_c(H - c_0)^2 + \bar{k}_c K, \tag{1.19}$$

where H is the mean curvature, K is the Gauss curvature, c_0 is the so-called *spontaneous curvature*, and $k_c, \bar{k}_c$ are the curvature elastic moduli. Based on experimental data of EVANS–FUNG [42] on red blood cells, DEULING–HELFRICH [35] found approximately $c_0 = -0.74\mu m^{-1}$.

In other words, based on experiments, the spontaneous curvature c_0 is negative and small. MUTZ–HELFRICH [102] measured k_c for dimyristoyl-phosphatidyl-ethanolamine membranes to be $k_c = 1.7 \times 10^{-12}$ at $T = 60°$C. The constant $\bar{k}_c$ is not important for the purpose of this thesis as by the Gauss–Bonnet Theorem, the integrated Gauss curvature is a topological constant.

One year later, EVANS [43] computed the variation in free energy for pure bending of a bilayer under fixed surface area. Similarly, the main ingredients in such a formula are the mean curvature and the squared mean curvature (see [43, Equation (14)]).

Lipid bilayers are very thin compared to their lateral dimensions, thus are usually modelled as surfaces. Suppose the surface and hence the membrane is represented by a weak branched conformal immersion with finite total curvature $\vec{\Phi} : \mathbb{S}^2 \to \mathbb{R}^3$. We will be concerned with the following integrated version of (1.19):

$$\mathcal{H}^{c_0}(\vec{\Phi}) := \int_{\mathbb{S}^2} \Big(\frac{H}{2} - c_0\Big)^2 \,\mathrm{d}\mu = \int_{\mathbb{S}^2} \Big(\frac{H^2}{4} - c_0 H + c_0^2\Big) \,\mathrm{d}\mu \tag{1.20}$$

where H and μ are related with $\vec{\Phi}$ as in the parametric approach (see Section 1.3). The integral in (1.20) is known as *Canham–Helfrich energy*. It is also referred to as *Canham–Evans–Helfrich* or just *Helfrich* energy. Its leading term and most important reduction for $c_0 = 0$ is the Willmore energy. According to SEIFERT [130], spontaneous curvature $c_0 \neq 0$ is mainly caused by asymmetry between the two layers of the membrane. Geometrically, the asymmetric area difference between the two layers is given by the total mean curvature, i.e. the integrated mean curvature. This is due to the fact that the infinitesimal variation of the area, i.e. the area difference between two nearby surfaces, is the total mean curvature. Indeed, considering the fact that in the following minimisation Problem 1 area will be fixed, the term to be minimised in Equation (1.20) is the Willmore energy minus (actually plus if $c_0 < 0$) the total mean curvature:

$$\mathcal{W}(\vec{\Phi}) - c_0 \int_{\mathbb{S}^2} H \,\mathrm{d}\mu.$$

In other words, the constant c_0 enters in terms of the total mean curvature. DÖBEREINER et al. [36] observed that spontaneous curvature may also arise from differences in the chemical properties of the aqueous solution on the two sides of the lipid bilayer.

Nowadays, the Canham–Helfrich energy is subject of intense research in mathematical biology and numerical analysis; for instance in modelling red blood cells [21, 91], crista junctions in mitochondria [111], folds of endoplasmatic reticulum [133], and numerical approximation [131, 84, 27].

Our goal is to minimise the Canham–Helfrich energy as well as to study the regularity of minimisers (and more generally of critical points). In the language of the calculus of variations we are concerned with the following Problem 1 stated in BERNARD–WHEELER–WHEELER [12, Introduction, Problem (P1)]. Given an embedding $\vec{\Phi} \in \mathcal{F}_{\mathbb{S}^2}$, let

$$\mathrm{area}(\vec{\Phi}) := \int_{\mathbb{S}^2} 1 \,\mathrm{d}\mu, \qquad \mathrm{vol}(\vec{\Phi}) := \frac{1}{3} \int_{\mathbb{S}^2} \vec{n} \cdot \vec{\Phi} \,\mathrm{d}\mu \tag{1.21}$$

be the area and enclosed volume, where again, $\vec{n}$ and μ are related with $\vec{\Phi}$ as in Section 1.3.

A candidate embedding $\vec{\Phi}_0$ which achieves the global minimum is called a *minimiser*. In

general it is not unique and, more dramatically, it may not exist: later in the introduction we show that for a suitable choice of parameters, the minimum is achieved by a singular immersion and it cannot be achieved by a smooth one. The constraints and the functional $\mathcal{H}^{c_0}$ are invariant under reparametrisation as well as rigid motions in $\mathbb{R}^3$. Of course, in order to have a non-empty class of competitors, the constraints have to satisfy the Euclidean isoperimetric inequality $A_0^3 \geq 36\pi V_0^2$.

Problem 1 (Helfrich problem). Let c_0, A_0, and V_0 be given constants. Minimise $\mathcal{H}^{c_0}(\vec{\Phi})$ in the class of smooth embeddings $f : \mathbb{S}^2 \to \mathbb{R}^3$ subject to the constraints

$$\text{area}(f) = A_0 \quad \text{and} \quad \text{vol}(f) = V_0. \tag{1.22}$$

That is, find an embedding $f_0 : \mathbb{S}^2 \to \mathbb{R}^3$ such that $\text{area}(f_0) = A_0$, $\text{vol}(f_0) = V_0$, and

$$\mathcal{H}^{c_0}(f_0) \leq \mathcal{H}^{c_0}(f)$$

for any other smooth embedding $f : \mathbb{S}^2 \to \mathbb{R}^3$ satisfying the constraints (1.22).

Problem 1 is the classical formulation suggested by HELFRICH [52] and DEULING–HELFRICH [35]. According to BERNARD–WHEELER–WHEELER [12], many issues for the Canham–Helfrich energy, including Problem 1, remain open and form important questions that future research should address. A similar problem in the 2-dimensional case (i.e. closed curves in the Euclidean plane) was formulated by BELLETTINI–DAL MASO–PAOLINI [7]. They proved existence of minimisers for the 2-dimensional case by a relaxation procedure.

Existence of minimisers in the special case $c_0 = 0$ of Problem 1 was proven by SCHYGULLA [129] using the *ambient approach* (see Section 1.2). The higher genus case for $c_0 = 0$ will be treated in Section 1.5. From the mathematical point of view, the spontaneous curvature c_0 causes a couple of differences between the Willmore functional and the Canham–Helfrich functional. Most obviously, the Canham–Helfrich energy cannot be bounded below by a strictly positive constant, whereas the Willmore energy is bounded below by 4π (see Section 1.1). Similarly, there is no Li–Yau type inequality for the Canham–Helfrich functional that could be used as a criterion for embeddedness. Moreover, while the Willmore functional is invariant under conformal transformations, the Canham–Helfrich functional is not even scaling invariant. We will be concerned with yet another property that seems to be difficult for the Canham–Helfrich functional due to non-zero spontaneous curvature. Namely lower semi-continuity with respect to varifold convergence: while it is well known that the Willmore functional is lower semi-continuous under varifold convergence (see [125]), the Canham–Helfrich energy in general is not. Indeed, GROSSE-BRAUCKMANN [48] constructed a sequence of non-compact infinite genus surfaces $\Sigma_1, \Sigma_2, \dots$ with constant mean curvature equal to 2 which converges as varifolds to a double plane Σ_∞ (ses [48, *Remark* (ii) on page 550]). Hence, the mean curvature H_∞ of the limit Σ_∞ is zero and

$$0 = \int_{\Sigma_k} \left(\frac{H_k}{2} - 1\right)^2 \eta \, d\mathcal{H}^2 < 2 \int_{\Sigma_\infty} \left(\frac{H_\infty}{2} - 1\right)^2 \eta \, d\mathcal{H}^2 = 2 \int_{\Sigma_\infty} \eta \, d\mathcal{H}^2 \tag{1.23}$$

for any continuous non-negative, non-zero function η on $\mathbb{R}^3$ with compact support, where $\mathcal{H}^2$ is the 2-dimensional Hausdorff measure. Hence, the general Canham–Helfrich functional is not lower

semi-continuous under varifold convergence. However, in order to solve Problem 1 by the direct method for calculus of variations, lower semi-continuity is required. According to RÖGER [1], it was an open question under which conditions and in which natural weak topology on the space of immersions one obtains lower semi-continuity of the Canham–Helfrich functional. In this thesis, we will show that the Canham–Helfrich functional is lower semi-continuous (see Theorem 2.5) on the space of bubble trees (developed by MONDINO–RIVIÈRE [95]) with uniform energy bound under the convergence of the *parametric approach* (which is a stronger notion of convergence than the varifold convergence in the ambient approach). Based on the lower semi-continuity result, we are able to prove existence of minimisers for the Helfrich problem presented shortly in Theorem 2. After our work on Helfrich spheres [96] was released, EICHMANN [41] discovered another lower semi-continuity property of the Canham–Helfrich functional. He proved lower semi-continuity on the class of minimising sequences with given fixed genus, area, and volume with respect to varifold convergence. Later BRAZDA–LUSSARDI–STEFANELLI [19] proved lower semi-continuity of a generalised Canham–Helfrich functional in the class of oriented curvature 2-varifolds under certain assumptions on the material parameters that ensure strict convexity of the integrand.

Previously, existence of minimisers for a special case of the Helfrich problem was proven by CHOKSI–VENERONI [28]. They obtained existence of minimisers in a class of axisymmetric (possibly singular) surfaces under fixed surface area and enclosed volume constraints. Five years later, DALPHIN [31] showed existence of minimisers in a class of $C^{1,1}$-regular surfaces whose principal curvatures are bounded by a given constant $1/\varepsilon$. However, in his setting, it is still unclear how to get compactness and lower semi-continuity as ε tends to zero.

We tackle Problem 1 by the direct method for calculus of variations using the mathematical setting of the parametric approach. The issue to overcome is that (unless we restrict ourselves to a very small range of spontaneous curvatures c_0) a minimising sequence $\vec{\Phi}_1, \vec{\Phi}_2, \ldots$ for the Helfrich problem does not satisfy

$$\limsup_{k \to \infty} \mathcal{W}(\vec{\Phi}_k) < 8\pi.$$

This is an important difference to the parametric approach of the Willmore problem. It means that we cannot exclude branch points for the limit of the minimising sequence. Moreover, since we need convergence of area and volume, we have to allow actual bubbling. Indeed, MONDINO–RIVIÈRE [95] proved the following:

Suppose $\vec{\Phi}_1, \vec{\Phi}_2, \ldots$ is a sequence in $\mathcal{F}_{\mathbb{S}^2}$ (see Section 1.3) such that

$$\limsup_{k \to \infty} \int_{\mathbb{S}^2} 1 + |\mathrm{d}\vec{n}_k|^2 \, \mathrm{d}\mu_k < \infty, \qquad \liminf_{k \to \infty} \operatorname{diam} \vec{\Phi}_k[\mathbb{S}^2] > 0$$

where $\vec{n}_k$ are the Gauss maps, μ_k are the corresponding Radon measures, and $\operatorname{diam} \vec{\Phi}_k[\mathbb{S}^2] := \sup_{a,b \in \mathbb{S}^2} |\vec{\Phi}(a) - \vec{\Phi}(b)|$.

Then, after passing to a subsequence, there exist a family Ψ_k of bilipschitz homeomorphisms of $\mathbb{S}^2$, a positive integer N, sequences $f_k^1, \ldots, f_k^N$ of positive conformal diffeomorphisms of $\mathbb{S}^2$,

$\vec{\xi}_\infty^1, \ldots, \vec{\xi}_\infty^N \in \mathcal{F}_{\mathbb{S}^2}$, *non-negative integers* $N_1, \ldots, N_N$, *and finitely many points on the sphere*

$$\{b^{i,j} : j = 1, \ldots, N_i, \, i = 1, \ldots, N\} \subset \mathbb{S}^2$$

such that

$$\vec{\Phi}_k \circ \Psi_k \to \vec{f}_\infty \qquad \text{as } k \to \infty \text{ strongly in } C^0(\mathbb{S}^2, \mathbb{R}^3)$$

for some $\vec{f}_\infty \in W^{1,\infty}(\mathbb{S}^2, \mathbb{R}^3)$ *and*

$$\vec{\Phi}_k \circ f_k^i \rightharpoonup \vec{\xi}_\infty^i \qquad \text{as } k \to \infty \text{ weakly in } W_{\mathrm{loc}}^{2,2}(\mathbb{S}^2 \setminus \{b^{i,1}, \ldots, b^{i,N_i}\}, \mathbb{R}^3)$$

for $i = 1, \ldots, N$. *Moreover,*

$$\sum_{i=1}^N \int_{\mathbb{S}^2} 1 \, d\mu_{\vec{\xi}_\infty^i} = \lim_{k \to \infty} \int_{\mathbb{S}^2} 1 \, d\mu_k.$$

The theorem already gives (pre-)compactness, a notion of convergence, and lower semi-continuity (actually, continuity) of the third summand in (1.20) of the Canham–Helfrich energy, that is of the area functional. Indeed, $\vec{T} := (\vec{f}_\infty, \vec{\xi}_\infty^1, \ldots, \vec{\xi}_\infty^N)$ forms what is termed a *bubble tree*, see Definition 2.4. In particular, the limit $\vec{T}$ is not in the class $\mathcal{F}_{\mathbb{S}^2}$ anymore. At an informal level, a non expert reader can think of a bubble tree $\vec{T} := (\vec{f}, \vec{\xi}^1, \ldots, \vec{\xi}^N)$ as a "pearl necklace" where each "pearl" corresponds to the image of a possibly branched weak immersion $\vec{\xi}^i[\mathbb{S}^2]$ and $\vec{f}$ is a Lipschitz map from $\mathbb{S}^2$ to $\mathbb{R}^3$ parametrising the whole pearl necklace, in particular $\vec{f}[\mathbb{S}^2] = \bigcup_{i=1}^N \vec{\xi}^i[\mathbb{S}^2]$.

To get a better understanding of why we obtain a bubble tree in the limit, we will look at an example of Problem 1. Let

$$c_0 = 1, \qquad A_0 = 2 \operatorname{area} \mathbb{S}^2, \qquad V_0 = 2 \operatorname{vol} \mathbb{S}^2.$$

Then, the infimum in Problem 1 is achieved by the bubble tree $\vec{T} = (\vec{f}, \vec{\mathrm{Id}}_{\mathbb{S}^2}, \vec{\mathrm{Id}}_{\mathbb{S}^2})$ of twice the unit sphere. Indeed, $\mathcal{H}^{c_0}(\vec{T}) = 0$ and $\mathcal{H}^{c_0}(\vec{\Phi}) \geq 0$ for any other smooth immersion $\vec{\Phi} : \mathbb{S}^2 \to \mathbb{R}^3$, so $\vec{T}$ achieves the infimum. A minimising sequence $\vec{\Phi}_k$ of smoothly embedded spheres converging to such a bubble tree can be achieved by glueing $(1 + 1/k)\mathbb{S}^2$ to $(1 - 1/k)\mathbb{S}^2$ via a small catenoidal neck of size $2/k$. Notice also that if $\vec{\Phi}$ satisfies $\mathcal{H}^{c_0}(\vec{\Phi}) = 0$, then the image $\vec{\Phi}[\mathbb{S}^2]$ is the unit sphere by a classical theorem of HOPF [54].

Getting a bubble tree in the limit is in accordance with the earlier result on existence of minimisers by CHOKSI–VENERONI [28] in the axisymmetric case: indeed the minimiser in [28, Theorem 1] is made by a finite union of axisymmetric surfaces. Moreover, the bubbling phenomenon is known as *budding transition* in biology and has been recorded with video microscopy, see SEIFERT [130] or SEIFERT–BERNDL–LIPOWSKY [131].

In Section 2.2 we sharpen the above theorem of MONDINO–RIVIÈRE [95] in a way that we get lower semi-continuity for the Canham–Helfrich functional. In Section 2.3 we compute the Euler–Lagrange equation for the Canham–Helfrich energy in divergence form. Moreover, we prove that all the weak branched conformal immersions of a minimising bubble tree (actually

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more generally for a critical bubble tree) are smooth away from their branch points. Our proof is based on the regularity theory for Willmore surfaces developed by RIVIÈRE [113]. It relies on conservation laws discovered by RIVIÈRE [113] in the context of the Willmore energy and adjusted by BERNARD [10] for the Canham–Helfrich energy. We get the following final result.

Theorem 2 (See Theorem 2.11 and MONDINO–SCHARRER [96, Theorem 1.7]). *Suppose $c_0 \in \mathbb{R}$, $A_0, V_0 > 0$, and $A_0^3 \geq 36\pi V_0^2$.*

Then, there exist a positive integer N and weak branched conformal immersions of finite total curvature $\vec{\Phi}_1, \ldots, \vec{\Phi}_N \in \mathcal{F}_{\mathbb{S}^2}$ such that $\cup_{i=1}^N \vec{\Phi}_i[\mathbb{S}^2]$ is connected,

$$\eta_{c_0}(A_0, V_0) := \inf_{\substack{\vec{\Phi} \in \mathcal{F}_{\mathbb{S}^2} \\ \text{area } \vec{\Phi} = A_0 \\ \text{vol } \vec{\Phi} = V_0}} \mathcal{H}^{c_0}(\vec{\Phi}) = \sum_{i=1}^N \mathcal{H}^{c_0}(\vec{\Phi}_i)$$

and

$$\sum_{i=1}^N \text{area } \vec{\Phi}_i = A_0, \qquad \sum_{i=1}^N \text{vol } \vec{\Phi}_i = V_0.$$

Moreover, for each $i \in \{1, \ldots, N\}$ there exist a non-negative integer N^i and finitely many points $b^{i,1}, \ldots, b^{i,N^i} \in \mathbb{S}^2$ such that $\vec{\Phi}_i$ is a C^∞ immersion of $\mathbb{S}^2 \setminus \{b^{i,1}, \ldots, b^{i,N^i}\}$ into $\mathbb{R}^3$. The total number of branch points and the number of bubbles can a priori be estimated by

$$\sum_{i=1}^N N^i \leq \eta_{c_0}(A_0, V_0) + c_0^2 A_0, \qquad N \leq \eta_{c_0}(A_0, V_0) + c_0^2 A_0.$$

Furthermore, there exists a constant $\varepsilon(A_0, V_0) > 0$ such that if $|c_0| < \varepsilon(A_0, V_0)$, then $N = 1$ and $\vec{\Phi} := \vec{\Phi}_1$ is a smooth embedding of $\mathbb{S}^2$ into $\mathbb{R}^3$.

Instead of fixing area and volume as in the classical Helfrich problem, one might consider the unconstrained minimisation of the functional $\mathcal{H}^{c_0}_{\alpha,\rho}$ where area and volume together with corresponding Lagrange multipliers are added directly to the functional $\mathcal{H}^{c_0}$:

$$\mathcal{H}^{c_0}_{\alpha,\rho}(\vec{\Phi}) := \int_{\mathbb{S}^2} \left(\frac{H_{\vec{\Phi}}}{2} - c_0\right)^2 \mathrm{d}\mu_{\vec{\Phi}} + \alpha \, \text{area } \vec{\Phi} + \rho \, \text{vol } \vec{\Phi}$$

for $\vec{\Phi} \in \mathcal{F}_{\mathbb{S}^2}$ where the parameter $\alpha > 0$ is referred to as *tensile stress*, and $\rho \geq 0$ as *osmotic pressure*. The minimisation of $\mathcal{H}^{c_0}_{\alpha,\rho}$ was formulated as an important open problem in BERNARD–WHEELER–WHEELER [12], see Problem (P2) of the introduction therein. We have the following contribution.

Theorem 3 (See Theorem 2.13 and MONDINO–SCHARRER [96, Theorem 1.9]). *Suppose $c_0 \in \mathbb{R}$, $\alpha > 0$, and $\rho \geq 0$. Then,*

$$\inf_{\vec{\Phi} \in \mathcal{F}_{\mathbb{S}^2}} \mathcal{H}^{c_0}_{\alpha,\rho}(\vec{\Phi}) \leq 4\pi. \tag{1.24}$$

Moreover, the following five statements hold.

1. *If equality holds in (1.24), then there exists a minimising sequence $\vec{\Phi}_k$ for $\inf_{\vec{\Phi} \in \mathcal{F}_{\mathbb{S}^2}} \mathcal{H}^{c_0}_{\alpha,\rho}(\vec{\Phi})$ that shrinks to a point.*

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2. *If the inequality* (1.24) *is strict, then there exist* $\vec{\Phi}_0 \in \mathcal{F}_{\mathbb{S}^2}$, *a positive integer* N, *and points* $b_1, \ldots, b_N \in \mathbb{S}^2$ *such that*

$$\inf_{\vec{\Phi} \in \mathcal{F}_{\mathbb{S}^2}} \mathcal{H}_{\alpha,\rho}^{c_0}(\vec{\Phi}) = \mathcal{H}_{\alpha,\rho}^{c_0}(\vec{\Phi}_0),$$

and $\vec{\Phi}_0$ *is a* C^∞-*immersion of* $\mathbb{S}^2 \setminus \{b_1, \ldots, b_N\}$ *into* $\mathbb{R}^3$. *Moreover, if* $|c_0| \leq \sqrt{\alpha}$, *then* $\vec{\Phi}_0$ *is a smooth embedding.*

3. *If* $c_0 < 0$ *and the infimum in* (1.24) *is attained by a member* $\vec{\Phi}_0 \in \mathcal{F}_{\mathbb{S}^2}$, *then*

$$\int_{\mathbb{S}^2} H_{\vec{\Phi}_0} \, \mathrm{d}\mu_{\vec{\Phi}_0} < 0.$$

In particular, the infimum cannot be attained by a convex surface.

4. *If* $c_0 = 0$, *then equality holds in* (1.24) *and the infimum is not attained in* $\mathcal{F}_{\mathbb{S}^2}$.

5. *If* $c_0 > 0$ *and* r_0 *is defined by*

$$r_0 := \begin{cases} \frac{c_0}{c_0^2 + \alpha} & \text{if } \rho = 0 \\ -\frac{c_0^2 + \alpha}{\rho} + \sqrt{\frac{2c_0}{\rho} + \frac{(c_0^2 + \alpha)^2}{\rho^2}} & \text{if } \rho > 0 \end{cases} \tag{1.25}$$

then $\mathcal{H}_{\alpha,\rho}^{c_0}(r_0 \mathbb{S}^2) < 4\pi$. *Moreover, if a round sphere is a critical point of* $\mathcal{H}_{\alpha,\rho}^{c_0}$, *then its radius is given by* (1.25).

Compare the strict inequality

$$\inf_{\vec{\Phi} \in \mathcal{F}_{\mathbb{S}^2}} \mathcal{H}_{\alpha,\rho}^{c_0}(\vec{\Phi}) < 4\pi \tag{1.26}$$

which by (2) guarantees existence of minimisers with the strict inequality (1.7) needed to obtain Willmore minimisers. While for the compactness of the Willmore problem the strict inequality (1.7) was needed to prevent a minimising sequence from diverging in moduli space, the strict inequality (1.26) is needed to prevent a minimising sequence from shrinking to a point. Note that in the above Theorem 3, the minimisation is amongst topological spheres. Thus, the control of the conformal class does not play a role. The additional difficulty here is due to the fact that $\mathcal{H}_{\alpha,\rho}^{c_0}$ is not scaling invariant. Recall that compactness in the parametric approach for the Willmore problem can only be achieved after composing with conformal maps (such are dilations). See also the works of MCCOY–WHEELER [86] and BLATT [16] who show that the L^2-gradient flow of the functional $\mathcal{H}_{\alpha,\rho}^{c_0}$ for $c_0 = 0$ develops singularities (shrinks to a point) in finite time. Indeed, for $c_0 = 0$, the starting energy is necessarily strictly larger than 4π.

It is an interesting open problem for what set of parameters c_0, α, ρ with $c_0 < 0$ the strict inequality (1.26) is satisfied. At this point, to the best knowledge of the author, it is not even known if the set might be empty. Another interesting open question is whether the regularity of the minimiser from (2) around the branch points can be improved. Notice also that point (3) can be put in relation with the fact that red blood cells have a biconcave shape.

1.5 The Canham problem

In the previous section we have discussed the minimisation problem formulated by HELFRICH [52] in 1973. That is, the minimisation of the energy

$$\int_\Sigma \Big(\frac{H}{2} - c_0\Big)^2 \, \mathrm{d}\mu \tag{1.27}$$

amongst spherical surfaces Σ with given fixed area and volume. The homogeneous case of the Helfrich problem where $c_0 = 0$ was first considered by CANHAM [21] in 1970. Existence of smooth minimisers of sphere type for the Canham problem was proven by SCHYGULLA [129]. In this section, we will discuss the Canham problem for higher genus surfaces. Notice that in the homogeneous case, the Canham–Helfrich functional (1.27) is given by the scaling invariant Willmore functional. Thus, the two constraints on area and volume reduce to a single constraint on the isoperimetric ratio defined by

$$\mathrm{iso}(\vec{\Phi}) := \frac{\mathrm{area}(\vec{\Phi})}{\mathrm{vol}(\vec{\Phi})^{\frac{2}{3}}} \tag{1.28}$$

whenever $\vec{\Phi} \in \mathcal{F}_\Sigma$ for some closed surface Σ. One can find different definitions of isoperimetric ratio in literature. Note that with the choice (1.28), $\mathrm{iso}(\vec{\Phi})$ is invariant under constant scaling of $\vec{\Phi}$, and it is minimised by any parametrisation of the round sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ as a consequence of the Euclidean isoperimetric inequality. Thus

$$\mathrm{im}(\mathrm{iso}) = (\sqrt[3]{36\pi}, \infty)$$

where the image is taken over the class of smoothly embedded closed surfaces with genus $g \geq 1$. We are interested in the following minimisation problem.

Problem 2 (Canham problem)**.** Let g be a non-negative integer and fix $\sigma > \sqrt[3]{36\pi}$. Minimise the Willmore functional $\mathcal{W}$ in the class of embeddings $f \in \mathcal{S}_g$ subject to the constraint $\mathrm{iso}(f) = \sigma$. That is, find $f_0 \in \{f \in \mathcal{S}_g : \mathrm{iso}(f) = \sigma\}$ such that

$$\mathcal{W}(f_0) \leq \mathcal{W}(f), \quad \text{for any } f \in \mathcal{S}_g \text{ with } \mathrm{iso}(f) = \sigma. \tag{1.29}$$

Such an immersion f_0 satisfying (1.29) is referred to as *solution* or *minimiser*. Since CANHAM [21] only considered spherical surfaces, the above Problem 2 is also referred to as *isoperimetric constrained Willmore problem*. Beyond the geometric interest, the minimisation problem is motivated by the model for closed lipid bilayer cell membranes described in the previous Section 1.4. Even if spherical membranes are most common, also higher genus membranes have been observed in nature: for toroidal shapes see [89, 101] and for higher genus see [88, 90, 132].

For all real numbers $\sigma > \sqrt[3]{36\pi}$ and non-negative integers g we define

$$\beta_g(\sigma) := \inf\{\mathcal{W}(f) : f \in \mathcal{S}_g, \, \mathrm{iso}(f) = \sigma\}.$$

First, we shall see how solutions of the classical Willmore problem already provide some solutions for the Canham problem. The following proposition which was known to experts will be of help

in that regard. For convenience of the reader, we have included a proof in Section 3.2.

Proposition 1 (See Proposition 3.9). *Suppose Σ is a closed smooth surface, and $f : \Sigma \to \mathbb{R}^3$ is a smooth embedding. Then, there exists a family of smooth maps $\{\psi_t : t > 0\}$ taking values in $\mathbb{R}^3$ whose open domains contain $\mathrm{im}\, f$ such that the isoperimetric ratio $\mathrm{iso}(\psi_t \circ f)$ varies smoothly in t,*

$$\lim_{t \to 0+} \mathrm{iso}(\psi_t \circ f) = \mathrm{iso}(\mathbb{S}^2), \qquad \lim_{t \to \infty} \mathrm{iso}(\psi_t \circ f) = \mathrm{iso}(f)$$

and

$$\mathcal{W}(\psi_t \circ f) = \mathcal{W}(f) \qquad \textit{for all } t > 0.$$

In fact, the maps ψ_t can be chosen to be so-called Möbius maps (see (1.12)) which are conformal transformations on the punctured three space. Since the isoperimetric ratio is minimised by the unit sphere $\mathbb{S}^2$, the above Proposition 1 implies that $\beta_g(\sigma)$ is a non-decreasing function in σ. Hence, the set of σ for which $\beta_g(\sigma)$ is attained is an interval which contains the non-empty interval

$$(\sqrt[3]{36\pi}, \mathrm{iso}(\Sigma_g)] \tag{1.30}$$

where Σ_g is any free Willmore minimiser, that is $\mathcal{W}(\Sigma_g) = \boldsymbol{\beta}_g = \inf_{\sigma > \sqrt[3]{36\pi}} \beta_g(\sigma)$. In particular, $\beta_g(\cdot)$ is constant on the interval $(\sqrt[3]{36\pi}, \mathrm{iso}(\Sigma_g)]$. By Marques–Neves [85] (see also Rivière [118]), the interval (1.30) for $g = 1$ reads as

$$\left(\sqrt[3]{36\pi}, \sqrt[3]{16\sqrt{2}\pi^2} \right]. \tag{1.31}$$

Thus, thanks to the proof of the Willmore conjecture, it is not only known that solutions for the toroidal Canham problem do exist for all σ in the interval (1.31), it is even known that the minimisers are given by Möbius transformations of the Clifford torus.

The first general existence result for the Canham problem was obtained by Keller–Mondino–Rivière [62]. They proved existence of smoothly embedded minimisers for all isoperimetric ratios σ satisfying

$$\beta_g(\sigma) < \min\{8\pi, \boldsymbol{\omega}_g, (\boldsymbol{\beta}_g + \beta_0(\sigma) - 4\pi)\}. \tag{1.32}$$

Compare this inequality with Simon's [136] compactness assumption (1.7). Similarly, as for the compactness proof of the classical Willmore problem, the assumption $\beta_g(\sigma) < \min\{8\pi, \boldsymbol{\omega}_g\}$ is needed to prevent the minimising sequence from diverging in moduli space. Recall again that compactness for the classical Willmore problem can only be achieved after composing with conformal maps. While the Willmore functional is conformally invariant, the isoperimetric ratio is not. Therefore, composing with conformal maps to achieve compactness is not an option for the Canham problem. Thus, what can happen is that a minimising sequence decomposes in a way that a spherical part of the surfaces carries the fixed isoperimetric ratio and the part of the surfaces that carries the genus shrinks to a point. The additional strict inequality $\beta_g(\sigma) < \boldsymbol{\beta}_g + \beta_0(\sigma) - 4\pi$ prevents this degeneration from happening.

Schygulla [129] who solved the existence part of the genus $g = 0$ case for the Canham problem also showed that $\beta_0(\sigma)$ is continuous in σ. Hence the right hand side in (1.32) is continuous in σ. By a simple variational argument using the fact that the isoperimetric ratio is

only minimised by round spheres, this implies that the interval of σ satisfying (1.32) is an open set (see also Lemma 3.10). Moreover, by rigidity of the Willmore inequality and by rigidity of the isoperimetric inequality, there holds

$$\beta_0(\sigma) > 4\pi \qquad \text{for all } \sigma > \sqrt[3]{36\pi}.$$

Consequently, the isoperimetric ratio iso(Σ_g) for any minimiser Σ_g of the free Willmore functional (with $g \geq 1$) does satisfy the inequality (1.32). Therefore, by the result of KELLER–MONDINO–RIVIÈRE [62], the interval of σ for which $\beta_g(\sigma)$ is attained is given by

$$(\sqrt[3]{36\pi}, \text{iso}(\Sigma_g) + \delta) \tag{1.33}$$

for some $0 < \delta \leq \infty$, which improves (1.30).

By the result of MARQUES–NEVES [85], the constant $\boldsymbol{\omega}_g$ on the right hand side of (1.32) is redundant since $\boldsymbol{\omega}_g > 8\pi$. This reduces the compactness assumption in (1.32) to

$$\beta_g(\sigma) < \min\{8\pi, (\boldsymbol{\beta}_g + \beta_0(\sigma) - 4\pi)\}.$$

By the following theorem, the compactness assumption can be further reduced to the strict inequality

$$\beta_g(\sigma) < 8\pi.$$

Theorem 4 (See Theorem 3.13 and MONDINO–SCHARRER [97, Theorem 1.4]). *Let $f_i : \Sigma_i \to \mathbb{R}^3$ for $i = 1, 2$ be two smoothly embedded closed surfaces neither of which parametrises a round sphere. Denote with Σ the connected sum $\Sigma_1 \# \Sigma_2$. Then there exists a smooth embedding $f : \Sigma \to \mathbb{R}^3$ such that*

$$\text{iso}(f) = \text{iso}(f_2) \tag{1.34}$$

and

$$\mathcal{W}(f) < \mathcal{W}(f_1) + \mathcal{W}(f_2) - 4\pi. \tag{1.35}$$

Recall that the inequality in (1.35) has been proven by BAUER–KUWERT [6] in order to solve the classical Willmore problem (see (1.13)). The novelty of Theorem 4 is that the same inequality remains valid under the additional constraint on the isoperimetric ratios (1.34). Indeed, in order to prove Theorem 4, we use the same connected sum construction developed by BAUER–KUWERT [6]. It will be shown in Section 3.3 that the connected sum already satisfies Equation (1.34) asymptotically (see Lemma 3.12). We then adjust the isoperimetric ratio by applying a first variation of the surface f_2 supported away from the pasting region, inspired by HUISKEN's [55] volume preserving mean curvature flow (see Lemma 3.10). Using existence of smoothly embedded Schygulla spheres (i.e. spherical solutions for the Canham problem) as well as existence of smoothly embedded Willmore minimisers, we infer the following corollary.

Corollary 1 (See Theorem 3.15 and MONDINO–SCHARRER [97, Corollary 1.6]). *Let g be a non-negative integer, and fix $\sigma > \sqrt[3]{36\pi}$. Assume that*

$$\beta_g(\sigma) = \inf\{\mathcal{W}(f) : f \in \mathcal{S}_g, \text{iso}(f) = \sigma\} < 8\pi. \tag{1.36}$$

Then $\beta_g(\sigma)$ is attained by a smoothly embedded minimiser $f_0 \in \mathcal{S}_g$, i.e. f_0 satisfies (1.29). Moreover, the function $\beta_g(\cdot)$ is non-decreasing on the whole interval iso$[\mathcal{S}_g]$ *and continuous at all σ that satisfy* (1.36).

We can now write the solution interval (1.33) as

$$(\sqrt[3]{36\pi}, \sup\{\sigma : \beta_g(\sigma) < 8\pi\}). \tag{1.37}$$

As we shall see later, the strict inequality (1.36) is now known to be always true. It is further expected that

$$\lim_{\sigma \to \infty} \beta_g(\sigma) = 8\pi. \tag{1.38}$$

Indeed, this was proven for $g = 0$ by SCHYGULLA [129] (see also KUWERT–LI [71] for a detailed blow-up analysis).

1.6 Isoperimetric constrained comparison tori and strict energy bounds

In the previous section we have seen that the Canham problem has a solution for all isoperimetric ratios σ that satisfy

$$\beta_g(\sigma) < 8\pi. \tag{1.39}$$

In many classical problems related to the Willmore functional, strict energy bounds such as (1.39) play a crucial role. Indeed, we have already seen how the strict inequalities (1.7) and (1.26) were used to prove existence of minimisers for the classical Willmore problem and for the unconstrained Helfrich problem, respectively. Another minimisation problem that we did not discuss in this thesis is the minimisation of the Willmore functional under fixed conformal class. This problem was studied for instance by KUWERT–SCHÄTZLE [79], NDIAYE–SCHÄTZLE [104], and RIVIÈRE [115, 116]. The existence result of KUWERT–SCHÄTZLE [79] holds true provided the minimal conformally constrained Willmore energy lies strictly below 8π. However, not only minimisation problems rely on these strict energy bounds. Exemplary is also the Willmore flow. KUWERT–SCHÄTZLE [76] showed that the Willmore flow of spheres exists for all times and converges to a round sphere provided the initial surface has Willmore energy less than 8π. BLATT [15] proved that this energy threshold is actually sharp. Later, DALL'ACQUA–MÜLLER–SCHÄTZLE–SPENER [30] showed that the analogous result of KUWERT–SCHÄTZLE [76] holds true for the Willmore flow of rotationally symmetric tori. Recently, RUPP [120] proved that the isoperimetric constrained Willmore flow of spheres exists for all times and converges smoothly to a critical point of the isoperimetric constrained Willmore equation provided the initial energy lies below $\min\{\frac{\sigma^3}{9}, 8\pi\}$, where σ is the isoperimetric ratio as defined in (1.28) (RUPP [120] used a different definition of isoperimetric ratio $\hat{\sigma} \in (0,1)$ such that $\frac{\sigma^3}{9} = \frac{4\pi}{\hat{\sigma}}$). See also PALMURELLA–RIVIÈRE [107] for the Willmore flow of spheres in the setting of the parametric approach. Using previous work of DE LELLIS–MÜLLER [32, 33], their result relies on an energy bound for the initial surface that guarantees existence of a suitable conformal parametrisation.

Given any $\sigma > \sqrt[3]{36\pi}$, one can prove that σ satisfies the strict inequality (1.39) by giving an example of a surface Σ such that iso$(\Sigma) = \sigma$ and $\mathcal{W}(\Sigma) < 8\pi$. The surface Σ is referred to as *competitor* or *comparison surface*. Existence of smoothly embedded spherical competitors

leading to (1.39) was proven by SCHYGULLA [129]. Inspired by the computations of CASTRO-VILLARREAL–GUVEN [22], he applied a family of sphere inversions (see (1.12)) to a complete catenoid, resulting in a family of closed surfaces with arbitrarily high isoperimetric ratios having one point of multiplicity two and Willmore energy exactly 8π. Subsequently, he applied the Willmore flow for a short time around the point of multiplicity two, to obtain a family of surfaces with Willmore energy strictly below 8π and arbitrarily high isoperimetric ratios.

For the remaining part of this section, we will investigate the strict inequality (1.39) for tori, i.e. for $g = 1$. As a starting point, one can study tori of revolution whose profile curve is a circle. Given $R > r > 0$, let $c = \frac{r}{R}$ and denote with $\mathbb{T}_c$ the torus given by the implicit equation in Cartesian coordinates

$$\left(\sqrt{x^2 + y^2} - R \right)^2 + z^2 = r^2.$$

The Willmore energy can be computed as

$$\mathcal{W}(\mathbb{T}_c) = \frac{\pi^2}{c\sqrt{1 - c^2}}$$

(see WILLMORE [139, Equation (18)]). Its minimum is attained at $c = 1/\sqrt{2}$ which results in the Clifford torus. Moreover,

$$c_1 := \inf\{c > 0 : \mathcal{W}(\mathbb{T}_c) < 8\pi\} = \sqrt{\tfrac{1}{2} - \tfrac{\sqrt{16 - \pi^2}}{8}}.$$

The isoperimetric ratio can be computed as $\mathrm{iso}(\mathbb{T}_c) = \sqrt[3]{16\pi^2/c}$. We thus obtain that the solution interval (1.37) contains the interval

$$\left(\mathrm{iso}(\mathbb{S}^2), \mathrm{iso}(\mathbb{T}_{c_1}) \right) = \left(\sqrt[3]{36\pi}, \sqrt[3]{16\pi^2 / \sqrt{\tfrac{1}{2} - \tfrac{\sqrt{16 - \pi^2}}{8}}} \right).$$

Notice that the above interval is already a strict improvement to (1.31). However, in order to get larger solution intervals, one has to find more elaborate examples.

Recall that by Proposition 1, $\beta_g(\sigma)$ is non-decreasing in σ. Thus, we are looking for competitors with high isoperimetric ratio which means relatively small volume. Pancake like surfaces do have small volume and thus high isoperimetric ratio. However, the Willmore energy of the outer edge of a pancake will be too high. The idea is thus to take a surface that looks like a punctured pancake (i.e. a toroidal surface) and bend it in a way that the outer edge gets shifted to a place where it has small mean curvature. This results in a surface that looks like two concentric round spheres of nearly the same radii where the spheres are connected by two catenoidal bridges, turning the surface into a topological torus. Now, the two edges of the punctured pancake have become catenoids. Thus, they don't contribute to the Willmore energy. Moreover, each of the two spheres has Willmore energy close to 4π. Indeed, it is not too difficult to use such a construction in order to show that the non-strict inequality

$$\beta_1(\sigma) \leq 8\pi$$

is always true. In fact, this can be done for any genus by simply adding more catenoids. Similar

constructions were carried out for instance by KÜHNEL–PINKALL [66], MÜLLER–RÖGER [99], and WOJTOWYTSCH [141]. The problem is that the pasting region between the outer sphere and the catenoids contributes too much energy resulting in a total energy slightly larger than 8π.

We next illustrate a novel way to construct tori without paying any cost in terms of Willmore energy at the patching regions. Notice that both the two concentric round spheres and the catenoids are constant mean curvature surfaces of revolution. We have constructed rotationally symmetric tori out of two different kinds of constant mean curvature surfaces that are well known in literature as *Delaunay surfaces*: The inner part of the tori has constant, strictly positive mean curvature; the outer part has constant, strictly negative mean curvature. These two pieces of Delaunay surfaces have matching normal vectors along the curve of intersection, leading to $C^{1,1}$ regularity of the patched surface. For a picture of the profile curve, see Figure 1.1. The tori will be called *Delaunay tori*. Their main property is stated in the following theorem which will be proven in Section 4, using complete elliptic integrals.

Theorem 5 (See Theorem 4.1 and SCHARRER [123, Theorem 1.1]). *There exist a real number* $c_0 > 1$ *and a family of* $C^{1,1}$-*regular tori* $\mathbb{T}_{D,c}$ *corresponding to* $1 < c < c_0$ *such that*

$$\mathcal{W}(\mathbb{T}_{D,c}) < 8\pi \qquad whenever \ 1 < c < c_0$$

and

$$\lim_{c \to 1+} \mathrm{iso}(\mathbb{T}_{D,c}) = \infty.$$
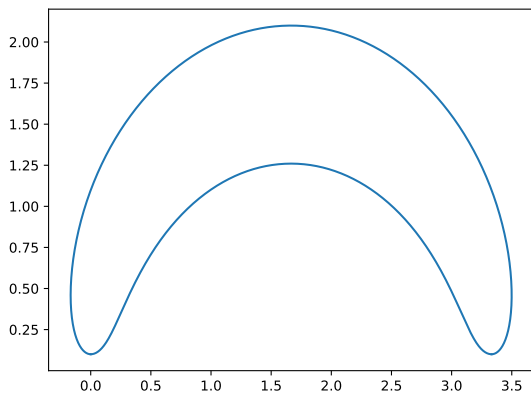
Figure 1.1: Profile curve Delaunay torus with $c = 1.1$ and the bottom line being the axis of rotation.
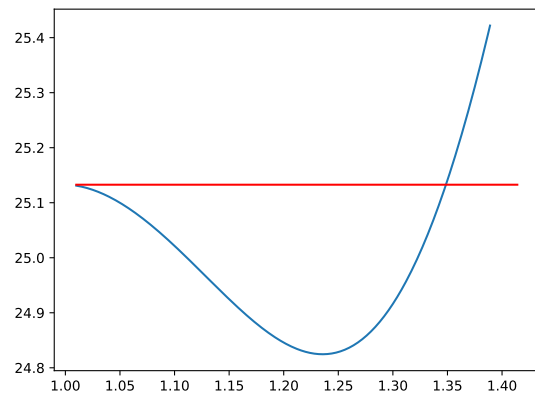
Figure 1.2: Energy curve for the family of Delaunay tori and 8π bound.

Combining the above Theorem 5 with the regularity result of KELLER–MONDINO–RIVIÈRE [62] and our Corollary 1, we infer the following consequence.

Corollary 2 (See Corollary 4.3 and SCHARRER [123, Corollary 1.3]). *Let* $\sigma > \sqrt[3]{36\pi}$. *Then,*

$$\beta_1(\sigma) = \inf\{\mathcal{W}(f) : f \in \mathcal{S}_1, \ \mathrm{iso}(f) = \sigma\}$$

is attained by a smoothly embedded minimiser $f_0 \in \mathcal{S}_1$.

This completes the solution for the existence (and regularity) part of the Canham problem in

the genus one case. In Section 4.6, we show how the Delaunay tori can be used to construct spherical competitors leading to an alternative solution of the genus $g = 0$ case by SCHYGULLA [129], see Figure 1.3.
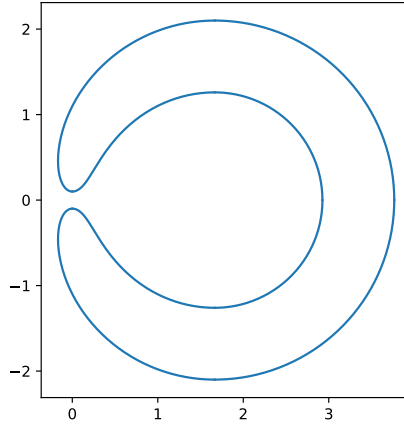
Figure 1.3: Profile of a Delaunay sphere with $c = 1.1$

1.7 Higher genus Helfrich surfaces

In the previous section we have seen how the construction of comparison tori led to existence of toroidal minimisers for the Canham problem. After my preprint on Delaunay tori [123] had appeared, KUSNER–MCGRATH [69] released a more general solution, proving the strict 8π-bound (1.39) for all $g \geq 0$. They constructed a family of genus g surfaces with arbitrarily high isoperimetric ratio and Willmore energy strictly below 8π, by gluing $g+1$ small catenoidal bridges to the bigraph of a singular solution for the linearised Willmore equation on the $(g+1)$-punctured 2-sphere. Their idea was inspired KAPOULEAS [60], who constructed minimal surfaces in $\mathbb{S}^3$ by doubling the equatorial 2-sphere. Thanks to the work of KUSNER–MCGRATH [69], the existence part of the Canham problem is now fully solved. Moreover, we could use their result to prove the following theorem on higher genus minimisers for the Helfrich problem with small spontaneous curvature.

Theorem 6 (See Theorem 4.5 and SCHARRER [123, Theorem 1.4])**.** *Suppose g is a non-negative integer, and $A_0, V_0 > 0$ satisfy the isoperimetric inequality $A_0^3 > 36\pi V_0^2$. Then, there exists $\varepsilon := \varepsilon_g(A_0, V_0) > 0$ such that the following holds.*

For each $c_0 \in (-\varepsilon, \varepsilon)$ there exists a smoothly embedded surface $f_0 \in \mathcal{S}_g$ with

$$\text{area}(f_0) = A_0, \qquad \text{vol}(f_0) = V_0$$

and

$$\mathcal{H}^{c_0}(f_0) = \inf\{\mathcal{H}^{c_0}(f) : f \in \mathcal{S}_g, \text{area}(f) = A_0, \text{vol}(f) = V_0\}.$$

Building on top of the works of SCHYGULLA [129], KELLER–MONDINO–RIVIÈRE [62], and KUSNER–MCGRATH [69], the above Theorem 6 is a combination of our Theorem 2.9, Lemma 2.10, Corollary 3.14, and Corollary 4.2, summarising some of the main results of the thesis. Notice that specialising Theorem 6 to the particular case $c_0 = 0$ provides existence of solutions for the

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Canham problem. Previously, partial results on existence of higher genus Helfrich surfaces were obtained by CHOKSI–VENERONI [28], EICHMANN [41], and BRAZDA–LUSSARDI–STEFANELLI [19].

It is very natural to expect that the blow up result of KUWERT–LI [71] (compare with (1.38)) can be generalised to the higher genus cases. To be more precise, it is expected that any sequence of genus g solutions for the Canham problem whose isoperimetric ratios diverge to infinity, converges (up to subsequences, scaling, and translating) to two concentric round spheres of nearly the same radii connected by $g+1$ catenoidal necks. It is an interesting question whether or not the catenoidal necks in the limit have to be distributed over the double sphere in a certain way. In view of the toroidal solution presented in Section 1.6, it is of course very tempting to conjecture that the catenoidal necks have to satisfy a balancing condition analogous to the one for constant mean curvature surfaces, see for instance KAPOULEAS [58, 59], or KOREVAAR–KUSNER–SOLOMON [65]. That would mean that for tori, the two catenoidal necks necessarily end up being antipodal.

1.8 Conclusion

The existence parts of the two main problems considered in this thesis, the Helfrich problem (see Problem 1) and the Canham problem (see Problem 2) are now fully solved, see Theorem 2 and Theorem 6, respectively. Moreover, Theorem 3 on the unconstrained minimisation of the Canham–Helfrich functional raises interesting questions for future research. Namely, the strict 4π-bound (1.26) as well as regularity analysis around the branch points in Part (2) for $|c_0| > \sqrt{\alpha}$. Similarly, since the existence part of the Canham problem is solved, the related blow up analysis for $\sigma \to \infty$ forms a natural continuation for future research.

2 Existence and regularity of spheres minimising the Canham–Helfrich energy

The aim of this section is to prove Theorem 2 and Theorem 3 from the introduction. This will be done in Subsection 2.4. The main ingredients are a compactness and lower semi-continuity result (see Subsection 2.2) and a regularity result (see Subsection 2.3). The content of this section corresponds to our work [96].

2.1 Preliminaries

2.1.1 Notation

We adopt the conventions of [117]. In particular, throughout Section 2, the mean curvature H denotes the arithmetic mean of the principal curvatures, see (2.6). To avoid indices and to get clearly arranged equations, we will employ the following suggestive notation. For $\mathbb{R}^3$ valued maps $\vec{e}$ and $\vec{f}$ defined on the unit disk D^2, we write

$$\nabla \vec{e} := \begin{pmatrix} \partial_{x^1}\vec{e} \\ \partial_{x^2}\vec{e} \end{pmatrix}, \qquad \nabla^\perp \vec{e} := \begin{pmatrix} -\partial_{x^2}\vec{e} \\ \partial_{x^1}\vec{e} \end{pmatrix}$$

$$\langle \vec{e}, \nabla \vec{f} \rangle := \begin{pmatrix} \vec{e} \cdot \partial_{x^1}\vec{f} \\ \vec{e} \cdot \partial_{x^2}\vec{f} \end{pmatrix}, \qquad \vec{e} \times \nabla \vec{f} := \begin{pmatrix} \vec{e} \times \partial_{x^1}\vec{f} \\ \vec{e} \times \partial_{x^2}\vec{f} \end{pmatrix}$$

as well as

$$\nabla \vec{e} \times \nabla \vec{f} := \partial_{x^1}\vec{e} \times \partial_{x^1}\vec{f} + \partial_{x^2}\vec{e} \times \partial_{x^2}\vec{f},$$

$$\vec{e} \cdot \nabla \vec{f} := \vec{e} \cdot \partial_{x^1}\vec{f} + \vec{e} \cdot \partial_{x^2}\vec{f}, \qquad \nabla \vec{e} \cdot \nabla f := \partial_{x^1}\vec{e} \cdot \partial_{x^1}\vec{f} + \partial_{x^2}\vec{e} \cdot \partial_{x^2}\vec{f},$$

where $\cdot$ denotes the Euclidean inner product and $\times$ denotes the usual vector product on $\mathbb{R}^3$. Similarly, for $\lambda : D^2 \to \mathbb{R}$ we write

$$\langle \nabla \lambda, \vec{e} \rangle := \begin{pmatrix} (\partial_{x^1}\lambda)\vec{e} \\ (\partial_{x^2}\lambda)\vec{e} \end{pmatrix}, \qquad \langle \nabla \lambda, \nabla \vec{e} \rangle := \partial_{x^1}\lambda \partial_{x^1}\vec{e} + \partial_{x^2}\lambda \partial_{x^2}\vec{e}.$$

Moreover, for a vector field

$$\vec{X} = \begin{pmatrix} \vec{X}^1 \\ \vec{X}^2 \end{pmatrix}$$

with components $\vec{X}^1, \vec{X}^2 : D^2 \to \mathbb{R}^3$, we define the divergence

$$\operatorname{div} \vec{X} := \partial_{x^1}\vec{X}^1 + \partial_{x^2}\vec{X}^2.$$

The m-dimensional Lebesgue measure is denoted by $\mathscr{L}^m$.

2.1.2 Weak possibly branched conformal immersions

We adapt the notion of weak immersions which was independently formalised by RIVIÈRE [115] and KUWERT–LI [70]. Let (Σ, c_0) be a smooth closed Riemann surface. Without loss of generality

we can assume that (Σ, c_0) is endowed with a metric g_{c_0} of constant curvature and area 4π (see for instance [57]). For the definition of the Sobolev spaces $W^{k,p}(\Sigma, \mathbb{R}^3)$ on Σ see for instance HEBEY [50]. A map $\vec{\Phi} : \Sigma \to \mathbb{R}^3$ is called a *weak branched conformal immersion with finite total curvature* if and only if there exists a positive integer N, finitely many points $b_1, \ldots, b_N \in \Sigma$ such that

$$\vec{\Phi} \in W^{1,\infty}(\Sigma, \mathbb{R}^3) \cap W^{2,2}_{\text{loc}}(\Sigma \setminus \{b_1, \cdots, b_N\}, \mathbb{R}^3), \tag{2.1}$$

there holds

$$\begin{cases} |\partial_{x^1} \vec{\Phi}| = |\partial_{x^2} \vec{\Phi}| \\ \partial_{x^1} \vec{\Phi} \cdot \partial_{x^2} \vec{\Phi} = 0 \end{cases} \tag{2.2}$$

almost everywhere for any conformal chart x of Σ,

$$\log |\mathrm{d}\vec{\Phi}| \in L^\infty_{\text{loc}}(\Sigma \setminus \{b_1, \ldots, b_N\}), \tag{2.3}$$

and its Gauss map $\vec{n}$ defined by

$$\vec{n} := \frac{\partial_{x^1} \vec{\Phi} \times \partial_{x^2} \vec{\Phi}}{|\partial_{x^1} \vec{\Phi} \times \partial_{x^2} \vec{\Phi}|}$$

in any local positive chart x of Σ satisfies

$$\vec{n} \in W^{1,2}(\Sigma, \mathbb{R}^3). \tag{2.4}$$

The space of weak branched conformal immersions with finite total curvature is denoted by $\mathcal{F}_\Sigma$ or just $\mathcal{F}$ in case $\Sigma = \mathbb{S}^2$. We define the L^∞-metric g pointwise for almost every $p \in \Sigma$ by

$$g_p(X, Y) := \mathrm{d}\vec{\Phi}_p(X) \cdot \mathrm{d}\vec{\Phi}_p(Y) \tag{2.5}$$

for elements X, Y of the tangent space $T_p\Sigma$. In the usual way, the L^∞-metric g induces a Radon measure μ_g on Σ. The conformality condition (2.2) implies that $g = e^{2\lambda} g_{c_0}$ for some $\lambda \in L^\infty_{\text{loc}}(\Sigma \setminus \{b_1, \ldots, b_N\})$ called *conformal factor*. Moreover, we define the second fundamental form $\vec{\mathbb{I}}$ pointwise for almost every $p \in \Sigma$ by

$$\vec{\mathbb{I}}_p : T_p\Sigma \times T_p\Sigma \to \mathbb{R}^3, \qquad \vec{\mathbb{I}}_p(X, Y) := -[\mathrm{d}\vec{n}_p(X) \cdot \mathrm{d}\vec{\Phi}_p(Y)]\vec{n}.$$

The mean curvature vector $\vec{H}$ and the scalar mean curvature H are given by

$$\vec{H} := \frac{1}{2} \operatorname{tr} \vec{\mathbb{I}}, \qquad H := \vec{n} \cdot \vec{H}. \tag{2.6}$$

Note that condition (2.4) ensures

$$H \in L^2(\Sigma). \tag{2.7}$$

2.1.3 Singular points and Gauss–Bonnet Theorem of weak branched immersions

First of all let us recall the following result first proved by MÜLLER-SVĚRÁK [100]. For a different proof using Hélein's moving frames technique [51], see [115, Lemma A.5]; see also [70, Theorem 3.1]) and [95, Section 2.1].

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2.1 Proposition. *Let $\vec{\Phi} : \Sigma \to \mathbb{R}^3$ be a weak branched conformal immersion with finite total curvature with singular points $b_1, \ldots, b_N \in \Sigma$. Let $\lambda \in L^\infty_{\mathrm{loc}}(\Sigma \setminus \{b_1, \ldots, b_N\})$ be the conformal factor, i.e. $g = \vec{\Phi}^* g_{\mathbb{R}^3} = e^{2\lambda} g_{c_0}$.*

Then $\vec{\Phi} \in W^{2,2}(\Sigma, \mathbb{R}^3)$ and the conformal factor λ is an element of $L^1(\Sigma)$. Moreover, for each singular point $b_j, j = 1, \ldots, N$, there exists a strictly positive integer $n_j \in \mathbb{N}$ such that the following holds:

- *For every b_j there exists a local conformal chart z centred at $\{b_j\} = \{z = 0\}$ such that*

$$\lambda(z) = (n_j - 1) \log|z| + \omega(z)$$

  for some $\omega \in C^0 \cap W^{1,2}$.

- *The multiplicity of the immersion $\vec{\Phi}$ at $\vec{\Phi}(b_j)$ is n_j. Moreover, if $n_j = 1$, then $\vec{\Phi}$ is a conformal immersion of a neighbourhood of b_j.*

- *The conformal factor λ satisfies the following singular Liouville equation in distributional sense*

$$- \Delta_{g_{c_0}} \lambda = K_{\vec{\Phi}} e^{2\lambda} - K_0 - 2\pi \sum_{j=1}^N \left[(n_j - 1) \delta_{b_j} \right], \tag{2.8}$$

  where δ_{b_j} is the Dirac delta centred at b_j, $K_{\vec{\Phi}}$ is the Gaussian curvature of $\vec{\Phi}$, and $K_0 \in \mathbb{R}$ is the (constant) curvature of (Σ, g_{c_0}).

By integrating the singular Liouville equation (2.8), we obtain the Gauss-Bonnet Theorem for weak branched immersions:

$$\int_\Sigma K_{\vec{\Phi}} \, \mathrm{d}\mu_g = 2\pi \chi(\Sigma) + 2\pi \sum_{j=1}^N (n_j - 1), \tag{2.9}$$

where $\chi(\Sigma)$ is the Euler Characteristic of Σ. Note in particular that, once the topology of Σ is fixed, the number of branch points counted with multiplicity is bounded by the Willmore energy:

$$2\pi \sum_{j=1}^N (n_j - 1) = \int_\Sigma K_{\vec{\Phi}} \, \mathrm{d}\mu_g - 2\pi \chi(\Sigma) \leq 2 \int_\Sigma H^2 \, \mathrm{d}\mu_g - 2\pi \chi(\Sigma). \tag{2.10}$$

Moreover, the Willmore energy controls the squared L^2 norm of the second fundamental form:

$$\int_\Sigma |\vec{\mathbb{I}}|^2 \, \mathrm{d}\mu_g = 4 \int_\Sigma H^2 \, \mathrm{d}\mu_g - 2 \int_\Sigma K_{\vec{\Phi}} \, \mathrm{d}\mu_g \leq 4 \int_\Sigma H^2 \, \mathrm{d}\mu_g - 4\pi \chi(\Sigma). \tag{2.11}$$

2.1.4 Simon's monotonicity formula and Li–Yau inequality for weak branched immersions

Let $\vec{\Phi} \in \mathcal{F}_\Sigma$ be any weak branched conformal immersion with finite total curvature and branch points $\{b_1, \ldots, b_N\}$. In the usual way (by splitting the vector field in its tangential and normal parts and using integration by parts) one shows

$$\int_\Sigma \mathrm{div}_{\vec{\Phi}} \vec{X} \, \mathrm{d}\mu_{\vec{\Phi}} = -2 \int_\Sigma \vec{X} \cdot \vec{H}_{\vec{\Phi}} \, \mathrm{d}\mu_{\vec{\Phi}} \tag{2.12}$$

whenever $\vec{X} \in W^{1,2}(\Sigma, \mathbb{R}^3)$ has compact support in $\Sigma \setminus \{b_1, \ldots, b_N\}$, where in a local chart x,

$$\mathrm{div}_{\vec{\Phi}}\, \vec{X} := g^{ij} \partial_{x^i} \vec{X} \cdot \partial_{x^j} \vec{\Phi}.$$

A simple cut-off argument together with (2.7) shows that the first variation formula (2.12) is true for all $\vec{X} \in W^{1,2}(\Sigma, \mathbb{R}^3)$. In the following we will gather a couple of facts that are well known for weak unbranched immersions and, due to (2.12), are also valid for weak branched conformal immersions with finite total curvature. Firstly, letting $\vec{X}(p) := \vec{\Phi}(p) - \vec{\Phi}(a_0)$ for $p \in \Sigma$ and some fixed $a_0 \in \Sigma$, one has $\mathrm{div}_{\vec{\Phi}}\, \vec{X} = 2$ and hence, see SIMON [135, Lemma 1.1] (or [124, Theorem 1.5] for varifolds on manifolds)

$$\sqrt{\mathrm{area}\, \vec{\Phi}} \leq \mathrm{diam}\, \vec{\Phi}[\Sigma] \sqrt{\int_\Sigma H_{\vec{\Phi}}^2 \, \mathrm{d}\mu_{\vec{\Phi}}}. \tag{2.13}$$

The push forward measure $\mu := \vec{\Phi}_\# \mu_{\vec{\Phi}}$ of $\mu_{\vec{\Phi}}$ defines a 2-dimensional integral varifold in $\mathbb{R}^3$ with multiplicity function $\theta^2(\mu, x) = \mathscr{H}^0(\vec{\Phi}^{-1}\{x\})$ (here $\mathscr{H}^0$ denotes the 0-dimensional Hausdorff measure, i.e. the counting measure) and approximate tangent space $T_x \mu = d\vec{\Phi}[T_p \Sigma]$ almost everywhere when $x = \vec{\Phi}(p)$. See SIMON [134, Chapter 4] for an introduction on varifolds and KUWERT–LI [70, Section 2.2] for the context of weak unbranched immersions. From (2.12), the first variation formula for the varifold μ becomes

$$\int \mathrm{div}_\mu\, \phi \, \mathrm{d}\mu = -2 \int \phi \cdot H_\mu \, \mathrm{d}\mu \qquad \text{for } \phi \in C_c^1(\mathbb{R}^3, \mathbb{R}^3) \tag{2.14}$$

where the weak mean curvature is almost everywhere given by

$$H_\mu(x) = \begin{cases} \frac{1}{\theta^2(\mu,x)} \sum_{p \in \vec{\Phi}^{-1}(x)} \vec{H}_{\vec{\Phi}}(p) & \text{if } \theta^2(\mu, x) > 0 \\ 0 & \text{else.} \end{cases}$$

The first variation formula (2.14) leads to Simon's monotonicity formula [134, 17.4] which implies (see for instance RIVIÈRE [117, Section 5.3] or KUWERT–SCHÄTZLE [76, Appendix]) the Li–Yau inequality [82, Theorem 6]

$$\theta^2(\mu, x) \leq \frac{1}{4\pi} \int_\Sigma H_{\vec{\Phi}}^2 \, \mathrm{d}\mu_{\vec{\Phi}}. \tag{2.15}$$

Consequently,

$$\inf_{\vec{\Phi} \in \mathcal{F}_\Sigma} \int_\Sigma H_{\vec{\Phi}}^2 \, \mathrm{d}\mu_{\vec{\Phi}} \geq 4\pi. \tag{2.16}$$

Moreover, if $\vec{\Phi} \in \mathcal{F}_\Sigma$ with $\int_\Sigma H_{\vec{\Phi}}^2 \, \mathrm{d}\mu_{\vec{\Phi}} < 8\pi$, then $\vec{\Phi}$ is an embedding (compare also with Proposition 2.1).

2.1.5 Canham–Helfrich energy

Given real numbers $c_0 \in \mathbb{R}$ and $\alpha, \rho \geq 0$ as well as a weak branched conformal immersion with finite total curvature $\vec{\Phi} : \Sigma \to \mathbb{R}^3$, we define the Canham–Helfrich energy $\mathcal{H}_{\alpha,\rho}^{c_0}(\vec{\Phi})$ in its most

general form by

$$\mathcal{H}_{\alpha,\rho}^{c_0}(\vec{\Phi}) := \int_\Sigma (H_{\vec{\Phi}} - c_0)^2 \, \mathrm{d}\mu_{\vec{\Phi}} + \alpha \int_\Sigma 1 \, \mathrm{d}\mu_{\vec{\Phi}} + \rho \int_\Sigma \vec{n}_{\vec{\Phi}} \cdot \vec{\Phi} \, \mathrm{d}\mu_{\vec{\Phi}}. \tag{2.17}$$

For $\alpha = \rho = 0$ we abbreviate $\mathcal{H}^{c_0} := \mathcal{H}_{0,0}^{c_0}$. Note that, in case $\vec{\Phi}: \Sigma \to \mathbb{R}^3$ is a smooth (actually Lipschitz is enough) embedding, by the Divergence Theorem the last integral equals the volume enclosed by $\vec{\Phi}(\Sigma)$. The parameter α is referred to as *tensile stress*, ρ as *osmotic pressure*. Compare this definition for instance with [10, Equation (3.6)] or [12].

2.2 Existence of minimisers

In this chapter we will prove compactness of sequences with uniformly bounded Willmore energy and area as well as lower semi-continuity of the Canham–Helfrich energy under this convergence, see Theorem 2.5. The proof of Theorem 2.5 will build on top of [95] and the next Lemma 2.2 which establishes the convergence of the constraints and the lower semi-continuity of the Willmore energy away from the branch points (Lemma 2.2 should be compared with [117, Lemma 5.2]).

2.2 Lemma (See MONDINO–SCHARRER [96, Lemma 3.1]). *Suppose $\vec{\xi}_1, \vec{\xi}_2, \ldots \in \mathcal{F}_{\mathbb{S}^2}$ is a sequence of weak branched conformal immersions with finite total curvature of the 2-sphere $\mathbb{S}^2$ into $\mathbb{R}^3$, $\mu_1, \mu_2, \ldots$ are the corresponding Radon measures on $\mathbb{S}^2$, $\vec{n}_1, \vec{n}_2, \ldots$ are the corresponding Gauss maps,*

$$\sup_{k \in \mathbb{N}} \int_{\mathbb{S}^2} |\mathrm{d}\vec{n}_k|^2 \, \mathrm{d}\mu_k < \infty, \tag{2.18}$$

there exists $\vec{\xi}_\infty \in \mathcal{F}_{\mathbb{S}^2}$, a positive integer N, and $b_1, \ldots, b_N \in \mathbb{S}^2$ such that

$$\sup_{k \in \mathbb{N}} \| \log |\mathrm{d}\vec{\xi}_k| \|_{L^\infty_{\mathrm{loc}}(\mathbb{S}^2 \setminus \{b_1, \ldots, b_N\})} < \infty, \tag{2.19}$$

$$\vec{\xi}_k \rightharpoonup \vec{\xi}_\infty \qquad as \ k \to \infty \ weakly \ in \ W^{2,2}_{\mathrm{loc}}(\mathbb{S}^2 \setminus \{b_1, \ldots, b_N\}, \mathbb{R}^3). \tag{2.20}$$

Then, there exists a sequence of positive numbers $s_1, s_2, \ldots$ converging to zero such that

$$\int_{\mathbb{S}^2} 1 \, \mathrm{d}\mu_\infty = \lim_{k \to \infty} \int_{\mathbb{S}^2 \setminus \bigcup_{i=1}^N B_{s_k}(b_i)} 1 \, \mathrm{d}\mu_k \tag{2.21}$$

$$\int_{\mathbb{S}^2} H_\infty \, \mathrm{d}\mu_\infty = \lim_{k \to \infty} \int_{\mathbb{S}^2 \setminus \bigcup_{i=1}^N B_{s_k}(b_i)} H_k \, \mathrm{d}\mu_k \tag{2.22}$$

$$\int_{\mathbb{S}^2} \vec{n}_\infty \cdot \vec{\xi}_\infty \, \mathrm{d}\mu_\infty = \lim_{k \to \infty} \int_{\mathbb{S}^2 \setminus \bigcup_{i=1}^N B_{s_k}(b_i)} \vec{n}_k \cdot \vec{\xi}_k \, \mathrm{d}\mu_k \tag{2.23}$$

where the balls are taken with respect to the geodesic distance on the standard $\mathbb{S}^2$, μ_∞ and $\vec{n}_\infty$ are the Radon measure and the Gauss map corresponding to $\vec{\xi}_\infty$, and the H_k's and H_∞ are the mean curvatures corresponding to the $\vec{\xi}_k$'s and $\vec{\xi}_\infty$. Equations (2.21)–(2.23) remain valid for s_k replaced by any sequence t_k converging to zero and satisfying $t_k \geq s_k$, for all $k \in \mathbb{N}$.

Moreover, for any sequence $s_1, s_2, \ldots$ of positive numbers converging to zero, there exists a sequence $t_k \geq s_k$ converging to zero such that

$$\int_{\mathbb{S}^2} H_\infty^2 \, \mathrm{d}\mu_\infty \leq \liminf_{k \to \infty} \int_{\mathbb{S}^2 \setminus \bigcup_{i=1}^N B_{t_k}(b_i)} H_k^2 \, \mathrm{d}\mu_k. \tag{2.24}$$

Proof. Suppose U is an open subset of $\mathbb{S}^2 \setminus \{b_1, \ldots, b_N\}$, K is a compact subset of U, and $x : U \to \mathbb{R}^2$ is a conformal chart for $\mathbb{S}^2$. Denote by

$$\lambda_k = \log |\partial_{x^1} \vec{\xi}_k|, \qquad \lambda_\infty = \log |\partial_{x^1} \vec{\xi}_\infty|$$

the conformal factors. Notice that the volume element corresponding to $\vec{\xi}_k$ is given by $e^{2\lambda_k}$. In a first step we will show that

$$e^{2\lambda_k} \to e^{2\lambda_\infty} \qquad \text{as } k \to \infty \text{ in } L^p(x[K]), \tag{2.25}$$

$$\vec{n}_k \cdot \vec{\xi}_k e^{2\lambda_k} \to \vec{n}_\infty \cdot \vec{\xi}_\infty e^{2\lambda_\infty} \qquad \text{as } k \to \infty \text{ in } L^p(x[K]) \tag{2.26}$$

for any $1 \le p < \infty$, as well as

$$\int_K H_\infty \, \mathrm{d}\mu_\infty = \lim_{k \to \infty} \int_K H_k \, \mathrm{d}\mu_k, \tag{2.27}$$

$$\int_K H_\infty^2 \, \mathrm{d}\mu_\infty \le \liminf_{k \to \infty} \int_K H_k^2 \, \mathrm{d}\mu_k. \tag{2.28}$$

A simple argument by contradiction shows that it is enough to prove the statement after passing to a subsequence of k. Since the $\vec{\xi}_k$'s and $\vec{\xi}_\infty$ are conformal and x is a conformal chart, we can write the mean curvature vector as

$$2\vec{H}_k = e^{-2\lambda_k} \Delta \vec{\xi}_k, \qquad 2\vec{H}_\infty = e^{-2\lambda_\infty} \Delta \vec{\xi}_\infty$$

where Δ is the flat Laplacian with respect to x. By Hypothesis (2.20), we have that

$$\vec{H}_k e^{2\lambda_k} = \frac{1}{2} \Delta \vec{\xi}_k \rightharpoonup \frac{1}{2} \Delta \vec{\xi}_\infty = \vec{H}_\infty e^{2\lambda_\infty}$$

as $k \to \infty$ weakly in $L^2(x[K], \mathbb{R}^3)$, which implies (2.27).

By the Rellich-Kondrachov Compactness Theorem, after passing to a subsequence, there holds

$$\partial_{x^1} \vec{\xi}_k \to \partial_{x^1} \vec{\xi}_\infty \qquad \text{as } k \to \infty \text{ in } L^p_{\text{loc}}(x[U], \mathbb{R}^3) \tag{2.29}$$

for any $1 \le p < \infty$. Therefore, using Hypothesis (2.19) and passing to a further subsequence, it follows

$$e^{-\lambda_k} = |\partial_{x^1} \vec{\xi}_k|^{-1} \to |\partial_{x^1} \vec{\xi}_\infty|^{-1} = e^{-\lambda_\infty} \qquad \text{as } k \to \infty \text{ in } L^2(x[K]).$$

It follows

$$\vec{H}_k \sqrt{e^{2\lambda_k}} = \frac{1}{2} e^{-\lambda_k} \Delta \vec{\xi}_k \rightharpoonup \frac{1}{2} e^{-\lambda_\infty} \Delta \vec{\xi}_\infty = \vec{H}_\infty \sqrt{e^{2\lambda_k}}$$

as $k \to \infty$ weakly in $L^2(x[K], \mathbb{R}^3)$, which implies (2.28) by lower semi-continuity of the L^2-norm under weak convergence.

Similarly, from Hypothesis (2.19) and the strong convergence (2.29), we infer (2.25).

Again by the strong convergence (2.29) and Hypothesis (2.19), we can extract a subsequence such that by dominated convergence,

$$\vec{n}_k = e^{-2\lambda_k}(\partial_{x^1} \vec{\xi}_k \times \partial_{x^2} \vec{\xi}_k) \to e^{-2\lambda_\infty}(\partial_{x^1} \vec{\xi}_\infty \times \partial_{x^2} \vec{\xi}_\infty) = \vec{n}_\infty$$

as $k \to \infty$ in $L^p(x[K], \mathbb{R}^3)$ for any $1 \le p < \infty$. Using this and the fact that by the Rellich–Kondrachov Compactness Theorem

$$\vec{\xi}_k \to \vec{\xi}_\infty \qquad \text{as } k \to \infty \text{ in } L^p(x[K], \mathbb{R}^3)$$

for any $1 \le p < \infty$, one verifies (2.26).

Next, let r_k be any sequence of positive numbers converging to zero and abbreviate

$$f_k = e^{2\lambda_k}, \qquad f_\infty = e^{2\lambda_\infty}, \qquad K_{r_k} = \mathbb{S}^2 \setminus \bigcup_{i=1}^N B_{r_k}(b_i).$$

First, notice that for any Borel function f on $\mathbb{S}^2$ with $\int_{\mathbb{S}^2} |f| \, d\mu_\infty < \infty$, there holds

$$\lim_{k \to \infty} \int_{K_{r_k}} f \, d\mu_\infty = \int_{\mathbb{S}^2} f \, d\mu_\infty \tag{2.30}$$

which is a consequence of the dominated convergence theorem and the fact that finite sets have μ_∞ measure zero. Let $n_0 = 1$. For each positive integer j, we use (2.25) to inductively choose $n_j > n_{j-1}$ such that

$$\int_{x[K_{r_j}]} |f_k - f_\infty| \, d\mathscr{L}^2 \le \frac{1}{j} \qquad \text{for all } k \ge n_j.$$

Moreover, define $l_k = j$ for all integers k with $n_{j-1} < k \le n_j$ and define $s_k = r_{l_k}$. Then, we have that $s_k \to 0$ as $k \to \infty$ as well as

$$\int_{x[K_{s_k}]} |f_k - f_\infty| \, d\mathscr{L}^2 \to 0 \qquad \text{as } k \to \infty \tag{2.31}$$

which in particular remains valid for s_k replaced by any $t_k \ge s_k$. Hence, by (2.30) we can deduce (2.21). Using the convergence on compact sets (2.26)–(2.28), Equations (2.22)–(2.24) follow similarly. It only remains to show that Equation (2.22) is still valid after replacing s_k by any sequence $t_k \ge s_k$ converging to zero. Hence, we only have to show that

$$\int_{K_{s_k} \setminus K_{t_k}} H_k \, d\mu_k \to 0 \qquad \text{as } k \to \infty.$$

This follows as by Hölder's inequality

$$\Big| \int_{K_{s_k} \setminus K_{t_k}} H_k \, d\mu_k \Big| \le \Big(\int_{\mathbb{S}^2} H_k^2 \, d\mu_k \Big)^{1/2} \Big(\int_{K_{s_k} \setminus K_{t_k}} 1 \, d\mu_k \Big)^{1/2}.$$

The first factor on the right hand side is bounded by (2.18). To see that the second factor goes to zero as k tends to infinity, we apply (2.31) and the fact that $\mu_\infty(\bigcup_{i=1}^N B_{t_k}(b_i)) \to 0$ as $k \to \infty$. $\qquad\qquad \square$

Our main compactness and lower semi-continuity result (see Theorem 2.5) builds on top of the following theorem of MONDINO–RIVIÈRE [95].

2.3 Theorem (See [95, Theorem 1.5]). *Suppose $\vec{\Phi}_1, \vec{\Phi}_2, \dots$ is a sequence in $\mathcal{F}_{\mathbb{S}^2}$ of conformal*

weak (possibly branched) immersions such that

$$\limsup_{k \to \infty} \int_{\mathbb{S}^2} 1 + |\mathrm{d}\vec{n}_k|^2 \, \mathrm{d}\mu_k < \infty, \qquad \liminf_{k \to \infty} \operatorname{diam} \vec{\Phi}_k[\mathbb{S}^2] > 0$$

where $\vec{n}_k$ are the Gauss maps, μ_k are the corresponding Radon measures, and $\operatorname{diam} \vec{\Phi}_k[\mathbb{S}^2] := \sup_{a,b \in \mathbb{S}^2} |\vec{\Phi}(a) - \vec{\Phi}(b)|$.

Then, after passing to a subsequence, there exist a family Ψ_k of bilipschitz homeomorphisms of $\mathbb{S}^2$, a positive integer N, sequences $f_k^1, \ldots, f_k^N$ of positive conformal diffeomorphisms of $\mathbb{S}^2$, $\vec{\xi}_\infty^1, \ldots, \vec{\xi}_\infty^N \in \mathcal{F}_{\mathbb{S}^2}$, non-negative integers $N_1, \ldots, N_N$, and finitely many points on the sphere

$$\{b^{i,j} : j = 1, \ldots, N_i, \, i = 1, \ldots, N\} \subset \mathbb{S}^2$$

such that

$$\vec{\Phi}_k \circ \Psi_k \to \vec{f}_\infty \qquad \text{as } k \to \infty \text{ strongly in } C^0(\mathbb{S}^2, \mathbb{R}^3) \tag{2.32}$$

for some $\vec{f}_\infty \in W^{1,\infty}(\mathbb{S}^2, \mathbb{R}^3)$ and

$$\vec{\Phi}_k \circ f_k^i \rightharpoonup \vec{\xi}_\infty^i \qquad \text{as } k \to \infty \text{ weakly in } W_{\mathrm{loc}}^{2,2}(\mathbb{S}^2 \setminus \{b^{i,1}, \ldots, b^{i,N_i}\}, \mathbb{R}^3)$$

for $i = 1, \ldots, N$. Moreover,

$$\sum_{i=1}^{N} \int_{\mathbb{S}^2} 1 \, \mathrm{d}\mu_{\vec{\xi}_\infty^i} = \lim_{k \to \infty} \int_{\mathbb{S}^2} 1 \, \mathrm{d}\mu_k.$$

The limit in the previous theorem is not in the class $\mathcal{F}_{\mathbb{S}^2}$ anymore. It is what we will define as a bubble tree. The idea is that the different bubbles can be parametrised by decomposing a single 2-sphere. The bubbles can then be attached to each other by a Lipschitz map, see (2.33) and (2.34).

2.4 Definition (Bubble tree of weak immersions, see [95, Definition 7.1]). An $N + 1$ tuple $\vec{T} = (\vec{f}, \vec{\Phi}^1, \ldots, \vec{\Phi}^N)$ is called a *bubble tree of weak immersions* if and only if N is a positive integer, $\vec{f} \in W^{1,\infty}(\mathbb{S}^2, \mathbb{R}^3)$, and $\vec{\Phi}^1, \ldots, \vec{\Phi}^N \in \mathcal{F}_{\mathbb{S}^2}$ are weak branched conformal immersions with finite total curvature such that the following holds.

There exist open geodesic balls $B^1, \ldots, B^N \subset \mathbb{S}^2$ such that

- $\overline{B^1} = \mathbb{S}^2$ and for all $i \neq i'$ either $\overline{B^i} \subset B^{i'}$ or $\overline{B^{i'}} \subset B^i$.

For all $i \in \{1, \ldots, N\}$ there exists a positive integer N^i and disjoint open geodesic balls $B^{i,1}, \ldots, B^{i,N^i} \subset \mathbb{S}^2$ whose closures are included in B^i such that

- for all $i' \neq i$ either $\overline{B^i} \subset B^{i'}$ or $\overline{B^{i'}} \subset B^{i,j}$ for some $j \in \{1, \ldots, N^i\}$.

For all $i \in \{1, \ldots, N\}$ there exist distinct points $b^{i,1}, \ldots, b^{i,N^i} \in \mathbb{S}^2$ and a Lipschitz diffeomorphism

$$\Xi^i : B^i \setminus \bigcup_{j=1}^{N^i-1} \overline{B^{i,j}} \to \mathbb{S}^2 \setminus \{b^{i,1}, \ldots, b^{i,N^i}\}$$

which extends to a Lipschitz map

$$\overline{\Xi}_i : \overline{B^i} \setminus \bigcup_{j=1}^{N^i-1} B^{i,j} \to \mathbb{S}^2$$

such that

$$\overline{\Xi}_i[\partial B^{i,j}] = b^{i,j} \text{ whenever } j \in \{1,\dots,N^i-1\}, \qquad \overline{\Xi}_i[\partial B^i] = b^{i,N^i}.$$

Moreover, for all $i \in \{1,\dots,N\}$,

$$\vec{f}(x) = (\vec{\Phi}^i \circ \Xi^i)(x) \qquad \text{whenever } x \in B^i \setminus \bigcup_{j=1}^{N^i-1} \overline{B^{i,j}} \tag{2.33}$$

and for all $j \in \{1,\dots,N^i\}$ there exists $p^{i,j} \in \mathbb{R}^3$ such that

$$\vec{f}(x) = p^{i,j} \qquad \text{whenever } x \in B^{i,j} \setminus \bigcup_{i' \in J^{i,j}} \overline{B^{i'}} \tag{2.34}$$

where $J^{i,j} = \{i' : \overline{B^{i'}} \subset B^{i,j}\}$.

Finally, we define

$$\mathcal{W}(\vec{T}) := \sum_{i=1}^{N} \int_{\mathbb{S}^2} H_{\vec{\Phi}^i}^2 \, \mathrm{d}\mu_{\vec{\Phi}^i}, \qquad \text{area}(\vec{T}) := \sum_{i=1}^{N} \int_{\mathbb{S}^2} 1 \, \mathrm{d}\mu_{\vec{\Phi}^i},$$

$$\text{vol}(\vec{T}) := \sum_{i=1}^{N} \int_{\mathbb{S}^2} \vec{n}_{\vec{\Phi}^i} \cdot \vec{\Phi}^i \, \mathrm{d}\mu_{\vec{\Phi}^i}.$$

The next theorem establishes the weak closure of bubble trees, as well as the convergence of the constraints in the Helfrich problem and the lower semi-continuity of the Willmore energy.

2.5 Theorem (See MONDINO–SCHARRER [96, Theorem 3.3]). *Suppose $\vec{T}_k = (\vec{f}_k, \vec{\Phi}_k^1, \dots, \vec{\Phi}_k^{N_k})$ is a sequence of bubble trees of weak immersions and*

$$\limsup_{k \to \infty} \sum_{i=1}^{N_k} \int_{\mathbb{S}^2} 1 + |\mathrm{d}\vec{n}_{\vec{\Phi}_k^i}|^2 \, \mathrm{d}\mu_{\vec{\Phi}_k^i} < \infty, \qquad \liminf_{k \to \infty} \sum_{i=1}^{N_k} \text{diam}\, \vec{\Phi}_k^i[\mathbb{S}^2] > 0. \tag{2.35}$$

Then, there exists a subsequence of $\vec{T}_k$ which we again denote by $\vec{T}_k$ such that $N_k = N$ for some positive integer N and there exists a sequence of diffeomorphisms Ψ_k of $\mathbb{S}^2$ such that

$$\vec{f}_k \circ \Psi_k \to \vec{u}_\infty \qquad \text{as } k \to \infty \text{ uniformly in } C^0(\mathbb{S}^2, \mathbb{R}^3),$$

$$\text{area}\, \vec{f}_k[\mathbb{S}^2] \to \text{area}\, \vec{u}_\infty[\mathbb{S}^2] \qquad \text{as } k \to \infty$$

for some $\vec{u}_\infty \in W^{1,\infty}(\mathbb{S}^2, \mathbb{R}^3)$. Moreover, for all $i \in \{1,\dots,N\}$ there exists a positive integer Q^i and sequences $f_k^{i,1},\dots,f_k^{i,Q^i}$ of positive conformal diffeomorphisms of $\mathbb{S}^2$ such that for each $j \in \{1,\dots,Q^i\}$ there exist finitely many points $b^{i,j,1},\dots b^{i,j,Q^{i,j}} \in \mathbb{S}^2$ with

$$\vec{\Phi}_k^i \circ f_k^{i,j} \rightharpoonup \vec{\xi}_\infty^{i,j} \qquad \text{as } k \to \infty \text{ weakly in } W_{\text{loc}}^{2,2}(\mathbb{S}^2 \setminus \{b^{i,j,1},\dots b^{i,j,Q^{i,j}}\}, \mathbb{R}^3) \tag{2.36}$$

for some branched Lipschitz conformal immersion $\vec{\xi}_\infty^{i,j} \in \mathcal{F}_{\mathbb{S}^2}$. Furthermore,

$$\vec{T}_\infty := \big(\vec{u}_\infty, (\vec{\xi}_\infty^{1,j})_{j=1,\ldots,Q^1}, \ldots, (\vec{\xi}_\infty^{N,j})_{j=1,\ldots,Q^N}\big)$$

is a bubble tree of weak immersions and

$$\mathcal{W}(\vec{T}_\infty) \leq \liminf_{k\to\infty} \mathcal{W}(\vec{T}_k), \quad \mathrm{area}(\vec{T}_\infty) = \lim_{k\to\infty} \mathrm{area}(\vec{T}_k), \quad \mathrm{vol}(\vec{T}_\infty) = \lim_{k\to\infty} \mathrm{vol}(\vec{T}_k)$$

as well as

$$\sum_{i=1}^N \sum_{j=1}^{Q^i} \int_{\mathbb{S}^2} H_{\vec{\xi}_\infty^{i,j}} \, \mathrm{d}\mu_{\vec{\xi}_\infty^{i,j}} = \lim_{k\to\infty} \sum_{i=1}^N \int_{\mathbb{S}^2} H_{\vec{\Phi}_k^i} \, \mathrm{d}\mu_{\vec{\Phi}_k^i}.$$

Proof. We first consider the special case where $N_k = 1$ for all positive integers k. By [95, Theorem 1.5], it then only remains to show the convergence properties of the Willmore energy $\mathcal{W}$, the volume, and the integral of the mean curvature. In view of Lemma 2.2, we can add Equations (2.22)–(2.24) for $\vec{\xi}_k$ replaced by $\vec{\Phi}_k \circ f_k^i$ to the conclusion of the Domain Decomposition Lemma [95, Theorem 6.1]. Therefore, adapting the proof of [95, Theorem 1.5], we get the following statement.

After passing to a subsequence and denoting $\vec{\Phi}_k := \vec{\Phi}_k^1$, there exists a positive integer N, sequences $f_k^1, \ldots, f_k^N$ of positive conformal diffeomorphisms of $\mathbb{S}^2$, and for each $i \in \{1, \ldots, N\}$ there exist points $b^{i,1}, \ldots, b^{i,N^i} \in \mathbb{S}^2$ such that (2.36) and (2.32) hold. Moreover, there exists a sequence of positive numbers s_k converging to zero such that for $i = 1, \ldots, N$ Equations (2.21)–(2.24) are satisfied for $\vec{\xi}_k$ replaced by $\vec{\Phi}_k \circ f_k^i$. Furthermore, defining

$$S_k^i := \mathbb{S}^2 \setminus \bigcup_{j=1}^{N^i} B_{s_k}(b^{i,j}),$$

and for $j = 1, \ldots, N^i$ the sets of indices

$$J^{i,j} := \{i' : \forall k \in \mathbb{N} : ((f_k^i)^{-1} \circ f_k^{i'})[S_k^{i'}] \subset B_{s_k}(b^{i,j})\},$$

and

$$\hat{J}^{i,j} := \{i' \in J^{i,j} : \forall k \in \mathbb{N} : \not\exists i'' : ((f_k^i)^{-1} \circ f_k^{i'})[S_k^{i'}] \subset \mathrm{Conv}((f_k^i)^{-1} \circ f_k^{i''})[S_k^{i''}]\},$$

and the necks

$$S_k^{i,j} := B_{s_k}(b^{i,j}) \setminus \bigcup_{i' \in \hat{J}^{i,j}} ((f_k^i)^{-1} \circ f_k^{i'})[\mathbb{S}^2 \setminus B_{s_k}(b^{i',N^{i'}})],$$

there holds

$$\lim_{k\to\infty} \int_{S_k^{i,j}} 1 \, \mathrm{d}\mu_{\vec{\Phi}_k \circ f_k^i} = 0, \qquad \lim_{k\to\infty} \mathrm{diam}(\vec{\Phi}_k \circ f_k^i)[S_k^{i,j}] = 0. \tag{2.37}$$

Finally, for any $\mu_{\vec{\Phi}_k}$ integrable Borel function φ on $\mathbb{S}^2$, we get

$$\int_{\mathbb{S}^2} \varphi \, \mathrm{d}\mu_{\vec{\Phi}_k} = \sum_{i=1}^N \int_{\mathbb{S}^2 \setminus \bigcup_{j=1}^{N_i} B_{s_k}(b^{i,j})} \varphi \circ f_k^i \, \mathrm{d}\mu_{\vec{\Phi}_k \circ f_k^i} + \sum_{i=1}^N \sum_{j=1}^{N^i-1} \int_{S_k^{i,j}} \varphi \circ f_k^i \, \mathrm{d}\mu_{\vec{\Phi}_k \circ f_k^i}. \tag{2.38}$$

We notice that by the strong convergence (2.32),

$$\sup_{k\in\mathbb{N}}\sup_{S_k^{i,j}}|\vec{\Phi}_k\circ f_k^i|\leq C<\infty$$

for some finite number $C>0$. Hence, by Hölder's inequality

$$\Big|\int_{S_k^{i,j}}H_{\vec{\Phi}_k\circ f_k^i}\,\mathrm{d}\mu_{\vec{\Phi}_k\circ f_k^i}\Big|\leq\Big(\int_{S_k^{i,j}}1\,\mathrm{d}\mu_{\vec{\Phi}_k\circ f_k^i}\Big)^{1/2}\Big(\int_{\mathbb{S}^2}H^2_{\vec{\Phi}_k\circ f_k^i}\,\mathrm{d}\mu_{\vec{\Phi}_k\circ f_k^i}\Big)^{1/2},$$

$$\Big|\int_{S_k^{i,j}}\vec{n}_{\vec{\Phi}_k\circ f_k^i}\cdot(\vec{\Phi}_k\circ f_k^i)\,\mathrm{d}\mu_{\vec{\Phi}_k\circ f_k^i}\Big|\leq\Big(\int_{S_k^{i,j}}1\,\mathrm{d}\mu_{\vec{\Phi}_k\circ f_k^i}\Big)^{1/2}\Big(\int_{S_k^{i,j}}C^2\,\mathrm{d}\mu_{\vec{\Phi}_k\circ f_k^i}\Big)^{1/2}.$$

By (2.35) and (2.37), the right hand side of each line goes to zero as k tends to infinity. That means the last term of Equation (2.38) goes to zero as k tends to infinity when φ is replaced by $H_{\vec{\Phi}_k}$ as well as when φ is replaced by $\vec{n}_{\vec{\Phi}_k}\cdot\vec{\Phi}_k$. Therefore, using (2.22) and (2.23), we can conclude the convergence of the integrated mean curvature and the convergence of the volume from (2.38). Similarly, we can conclude the lower semi-continuity of the Willmore energy $\mathcal{W}$ from (2.38) by replacing φ with $H^2_{\vec{\Phi}_k}$, using super linearity of the limit inferior and by ignoring the non-negative second term in (2.38).

Now, the general case follows analogously to the proof of [95, Theorem 7.2]. □

2.3 Regularity of minimisers

Throughout this section, Σ denotes a smooth, oriented, and closed 2-dimensional manifold. Moreover, c_0, α, and ρ are the parameters of the Canham–Helfrich energy, i.e. $c_0\in\mathbb{R}$ and $\alpha,\rho\geq 0$, see (2.17). A (possibly branched) weak immersion $\vec{\Phi}\in\mathcal{F}_\Sigma$ is called *weak Canham–Helfrich immersion* if

$$\frac{d}{\mathrm{d}t}\Big|_{t=0}\mathcal{H}^{c_0}_{\alpha,\rho}(\vec{\Phi}+t\vec{\omega})=0 \tag{2.39}$$

for all $\vec{\omega}\in C^\infty(\Sigma,\mathbb{R}^3)$.

In the following, we will first compute the Canham–Helfrich equation in divergence form, see Lemma 2.6. Then, we will prove that a weak immersion satisfying the Canham–Helfrich equation is smooth away from its branch points, see Theorem 2.9. The proof is based on the regularity theory for weak Willmore immersions developed by RIVIÈRE [113, 117]. An important step in Riviere's regularity theory is the discovery of hidden conservation laws for weak Willmore immersions. In the framework of Canham–Helfrich immersions, the corresponding hidden conservation laws were discovered by BERNARD [10].

2.6 Lemma (See MONDINO–SCHARRER [96, Lemma 4.1])**.** *Suppose $\vec{\Phi}\in\mathcal{F}_\Sigma$ is a weak Canham–Helfrich immersion with branch points $b_1,\ldots,b_N$. Then, away from its branch points, i.e. in conformal parametrisations from the open unit disk D^2 into a subset of $\Sigma\setminus\bigcup_{i=1}^N B_\varepsilon(b_i)$ for any $\varepsilon>0$, there holds*

$$\vec{\mathcal{W}}=-\operatorname{div}\Big[c_0\nabla\vec{n}+(2c_0H-c_0^2-\alpha)\nabla\vec{\Phi}-\frac{\rho}{2}\vec{\Phi}\times\nabla^\perp\vec{\Phi}\Big] \tag{2.40}$$

in $\mathcal{D}'(D^2, \mathbb{R}^3)$, where

$$\vec{\mathcal{W}} := \operatorname{div} \frac{1}{2} \left[2\nabla \vec{H} - 3H\nabla \vec{n} + \vec{H} \times \nabla^\perp \vec{n} \right] \tag{2.41}$$

corresponds to the first variation of the Willmore energy.

Proof. After composing with a conformal chart away from the branch points, we may assume that $\vec{\Phi}$ is a map $D^2 \to \mathbb{R}^3$. Let $\vec{\omega} \in C_c^\infty(D, \mathbb{R}^3)$ and define $\vec{\Phi}_t := \vec{\Phi} + t\vec{\omega}$ for $t \in \mathbb{R}$. The conformal factor λ is given by $2e^{2\lambda} = |\nabla \vec{\Phi}|^2$ and the metric coefficients $(g_t)_{ij}$ by $(g_t)_{ij} = \partial_i \vec{\Phi}_t \cdot \partial_j \vec{\Phi}_t$. Standard computations (see for instance [117, (7.8)–(7.10)]) give

$$\frac{d}{dt}\Big|_{t=0}(g_t)^{ij} = -e^{-4\lambda}(\partial_i \vec{\omega} \cdot \partial_j \vec{\Phi} + \partial_i \vec{\Phi} \cdot \partial_j \vec{\omega})$$

$$g^{ij}\Big(\partial_i \frac{d}{dt}\Big|_{t=0} \vec{n}_t \cdot \partial_j \vec{\Phi}\Big) = -e^{-2\lambda}(\partial_1(\partial_1 \vec{\omega} \cdot \vec{n}) + \partial_2(\partial_2 \vec{\omega} \cdot \vec{n}))$$

$$\frac{d}{dt}\Big|_{t=0}\sqrt{\det(g_t)_{ij}} = \partial_1 \vec{\Phi} \cdot \partial_1 \vec{\omega} + \partial_2 \vec{\Phi} \cdot \partial_2 \vec{\omega}.$$

Therefore, using

$$\frac{d}{dt}\Big|_{t=0} H_t = -\frac{d}{dt}\Big|_{t=0} \frac{1}{2}(g_t)^{ij}\Big(\partial_i \vec{n}_t \cdot \partial_j \vec{\Phi}_t\Big)$$

$$= -\frac{1}{2}\Big(\frac{d}{dt}\Big|_{t=0}(g_t)^{ij}\Big)\partial_i \vec{n} \cdot \partial_j \vec{\Phi} - g^{ij}\frac{1}{2}\Big(\partial_i \frac{d}{dt}\Big|_{t=0} \vec{n}_t \cdot \partial_j \vec{\Phi} + \partial_i \vec{n} \cdot \partial_j \vec{\omega}\Big),$$

we obtain

$$\frac{d}{dt}\Big|_{t=0}\int_{D^2} H_t \, d\mu_t = \int_{D^2}\Big(\frac{d}{dt}\Big|_{t=0} H_t\Big)\sqrt{\det g_{ij}} + H\frac{d}{dt}\Big|_{t=0}\sqrt{\det(g_t)_{ij}} \, d\mathscr{L}^2$$

$$= \int_{D^2} \frac{1}{2}e^{-2\lambda}\sum_{i,j=1}^{2}(\partial_i \vec{\omega} \cdot \partial_j \vec{\Phi} + \partial_i \vec{\Phi} \cdot \partial_j \vec{\omega})(\partial_i \vec{n} \cdot \partial_j \vec{\Phi})$$

$$+ \frac{1}{2}(\partial_1(\partial_1 \vec{\omega} \cdot \vec{n}) + \partial_2(\partial_2 \vec{\omega} \cdot \vec{n})) - \frac{1}{2}g^{ij}\partial_i \vec{n} \cdot \partial_j \vec{\omega}e^{2\lambda}$$

$$+ H(\partial_1 \vec{\Phi} \cdot \partial_1 \vec{\omega} + \partial_2 \vec{\Phi} \cdot \partial_2 \vec{\omega}) \, d\mathscr{L}^2.$$

Using that $\vec{\omega}$ has compact support in D^2,

$$g^{ij}\partial_i \vec{n} \cdot \partial_j \vec{\omega}e^{2\lambda} = e^{-2\lambda}\sum_{i,j=1}^{2}(\partial_i \vec{n} \cdot \partial_j \vec{\Phi})(\partial_j \vec{\Phi} \cdot \partial_i \vec{\omega}),$$

and using the symmetry of the second fundamental form, i.e. $\partial_i \vec{n} \cdot \partial_j \vec{\Phi} = \partial_j \vec{n} \cdot \partial_i \vec{\Phi}$, we compute further

$$\frac{d}{dt}\Big|_{t=0}\int_{D^2} H_t \, d\mu_t = \int_{D^2} -\frac{1}{2}\vec{\omega} \cdot \partial_1\Big[e^{-2\lambda}((\partial_1 \vec{n} \cdot \partial_1 \vec{\Phi})\partial_1 \vec{\Phi} + (\partial_1 \vec{n} \cdot \partial_2 \vec{\Phi})\partial_2 \vec{\Phi})\Big]$$

$$- \frac{1}{2}\vec{\omega} \cdot \partial_2\Big[e^{-2\lambda}((\partial_2 \vec{n} \cdot \partial_1 \vec{\Phi})\partial_1 \vec{\Phi} + (\partial_2 \vec{n} \cdot \partial_2 \vec{\Phi})\partial_2 \vec{\Phi})\Big]$$

$$- \vec{\omega} \cdot (\partial_1(H\partial_1 \vec{\Phi}) + \partial_2(H\partial_2 \vec{\Phi})) \, d\mathscr{L}^2$$

$$= -\int_{D^2} \vec{\omega} \cdot \partial_1\Big(\frac{1}{2}\pi_T(\partial_1 \vec{n}) + H\partial_1 \vec{\Phi}\Big) + \vec{\omega} \cdot \partial_2\Big(\frac{1}{2}\pi_T(\partial_2 \vec{n}) + H\partial_2 \vec{\Phi}\Big) \, d\mathscr{L}^2$$

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which gives

$$\frac{d}{dt}\Big|_{t=0} \int_{D^2} H_t \, d\mu_t = -\int_{D^2} \vec{\omega} \cdot \operatorname{div}\left[\frac{1}{2}\nabla\vec{n} + H\nabla\vec{\Phi}\right] d\mathscr{L}^2. \tag{2.42}$$

From [117, Corollary 7.3] we know

$$\frac{d}{dt}\Big|_{t=0} \int_{D^2} H_t^2 \, d\mu_t = \int_{D^2} \vec{\omega} \cdot \operatorname{div}\frac{1}{2}\left[2\nabla\vec{H} - 3H\nabla\vec{n} + \vec{H} \times \nabla^{\perp}\vec{n}\right] d\mathscr{L}^2 \tag{2.43}$$

see also [113]. Moreover (see for instance [10, Chapter 3.3])

$$\frac{d}{dt}\Big|_{t=0} \int_{D^2} 1 \, d\mu_t = -\int_{D^2} \vec{\omega} \cdot \operatorname{div}\nabla\vec{\Phi} \, d\mathscr{L}^2 \tag{2.44}$$

and

$$\frac{d}{dt}\Big|_{t=0} \int_{D^2} \vec{n}_t \cdot \vec{\Phi}_t \, d\mu_t = -\int_{D^2} \vec{\omega} \cdot \operatorname{div}\left[\frac{1}{2}\vec{\Phi} \times \nabla^{\perp}\vec{\Phi}\right] d\mathscr{L}^2. \tag{2.45}$$

Putting (2.42) – (2.45) into (2.39) yields (2.40). $\qquad\square$

2.7 Remark. The reason why a weak Canham–Helfrich immersion $\vec{\Phi} \in \mathcal{F}_{\Sigma}$ does not necessarily satisfy the Euler–Lagrange Equation (2.40) around a branch point is the potential degeneracy of the conformal factor $\lambda = \frac{1}{2}\log(\frac{1}{2}|d\vec{\Phi}|^2)$. By our definition of the space $\mathcal{F}_{\Sigma}$, the conformal factor only satisfies $\lambda \in L^{\infty}_{\text{loc}}(\Sigma \setminus \{b_1, \ldots, b_N\})$, see (2.3). If $\lambda \in L^{\infty}(\Sigma)$, then the induced metric $g_{\vec{\Phi}}$ (see (2.5)) has a bounded inverse and satisfies Equation (1.10) in [115]. Indeed, in this case, $\vec{\Phi}$ is what is termed a *Lipschitz immersion* and around each point in the domain Σ, there exist local isothermal coordinates (see [115, Theorem 1.4]) which is exactly what was needed in the previous proof. In other words, a weak Canham–Helfrich immersion which is also a Lipschitz immersion does satisfy the Euler–Lagrange Equation (2.40) around each point. By Proposition 2.1, we have that any $\vec{\Phi} \in \mathcal{F}_{\Sigma}$ which is injective satisfies $\lambda \in L^{\infty}(\Sigma)$. We conclude by observing that by the Li–Yau inequality (2.15), any weak Canham–Helfrich immersion $\vec{\Phi} \in \mathcal{F}_{\Sigma}$ which satisfies

$$\int_{\Sigma} H_{\vec{\Phi}}^2 \, d\mu_{\vec{\Phi}} < 8\pi$$

is injective, thus satisfies the Euler–Lagrange Equation (2.40) around each point.

Nevertheless, exploiting the variational nature of the problem, it is natural to expect that the branch points showing up in the minimising branched immersions do satisfy improved regularity properties. See the works of Kuwert–Schätzle [77] for the ambient approach and Bernard–Rivière [11], Bernard [9] for the parametric approach. Bernard–Rivière [11] show improved regularity at branch points depending on the values of two residues, where the first residue is given by the boundary integral corresponding to the divergence theorem applied to the expression in the Euler–Lagrange Equation (2.41), and the second residue is given by the boundary integral of a meromorphic approximation for the mean curvature. If both residues vanish, then the weak immersion is smooth.

2.8 Remark. Usually in the literature (see for instance [10, Chapter 3.3]) one finds the expression of the first variation for $\int H d\mu_g$ written as

$$\frac{d}{dt}\Big|_{t=0} \int_{D^2} H_t \, d\mu_t = \int_{D^2} (\vec{\omega} \cdot \vec{n})\left(\frac{1}{2}\mathbb{I}_j^i \mathbb{I}_i^j - 2H^2\right) d\mu_g. \tag{2.46}$$

It is not hard to check the equivalence of (2.46) with (2.42) proved above. The advantage of the expression (2.42) is two fold: first it invokes less regularity of the immersion map $\vec{\Phi}$, second it is already in divergence form. Both advantages will be useful in establishing the regularity of weak Canham–Helfrich immersions: indeed, (2.46) would correspond to an L^1 term in the Euler–Lagrange equation (which is usually a problematic right hand side for elliptic regularity theory) while (2.42) corresponds to the divergence of an L^2 term (which is a much better right hand side in elliptic regularity).

2.9 Theorem (See MONDINO–SCHARRER [96, Theorem 4.3]). *Suppose $\vec{\Phi} \in \mathcal{F}_\Sigma$ is a weak Canham–Helfrich immersion. Then $\vec{\Phi}$ is a C^∞ immersion away from the branch points.*

Proof. After composing with a conformal chart of Σ away from the branch points onto the unit disk D^2, we may assume that $\vec{\Phi}$ is a map $D^2 \to \mathbb{R}^3$ without branch points and $\vec{\Phi}$ satisfies the Canham–Helfrich equation (2.40). It is enough to show that $\vec{\Phi} \in C^\infty(B_{1/2}(0))$. The proof splits into three parts.

Step 1: Conservation laws. In view of the Canham–Helfrich equation (2.40), we define [1] $\vec{T} \in L^2(D^2, (\mathbb{R}^3)^2)$ by letting

$$\vec{T} := c_0 \nabla \vec{n} + (2c_0 H - c_0^2 - \alpha)\nabla\vec{\Phi} - \frac{\rho}{2}\vec{\Phi} \times \nabla^\perp\vec{\Phi}.$$

Then, $\operatorname{div}\vec{T} = -\vec{\mathcal{W}}$ where $\vec{\mathcal{W}}$ is as in (2.41). Hence $\vec{\mathcal{W}} \in H^{-1}(D^2, \mathbb{R}^3)$ and there exists a solution $\vec{V}$ of

$$\begin{cases} \Delta\vec{V} = -\vec{\mathcal{W}} \\ \vec{V} \in H_0^1(D^2, \mathbb{R}^3). \end{cases}$$

Therefore, we can find

$$\vec{X} \in W^{2,2}(D^2, \mathbb{R}^3), \qquad Y \in W^{2,2}(D^2, \mathbb{R}) \tag{2.47}$$

such that

$$\begin{cases} \Delta\vec{X} = \nabla\vec{V} \times \nabla\vec{\Phi} & \text{in } D^2 \\ \vec{X} = 0 & \text{on } \partial D^2 \end{cases}$$

and

$$\begin{cases} \Delta Y = \nabla\vec{V} \cdot \nabla\vec{\Phi} & \text{in } D^2 \\ Y = 0 & \text{on } \partial D^2. \end{cases}$$

After the breakthrough of RIVIÈRE [113], BERNARD [10, Chapter 2.2] showed that by invariance of the Willmore functional under conformal transformation and the weak Poincaré Lemma, one can find potentials $\vec{L}, \vec{R} \in W^{1,2}(D^2, \mathbb{R}^3)$, and $S \in W^{1,2}(D^2, \mathbb{R})$ such that

$$\begin{cases} \nabla^\perp\vec{L} = \vec{T} - \nabla\vec{V} \\ \nabla^\perp\vec{R} = \vec{L} \times \nabla^\perp\vec{\Phi} - \vec{H} \times \nabla\vec{\Phi} - \nabla\vec{X} \\ \nabla^\perp S = \langle\vec{L}, \nabla^\perp\vec{\Phi}\rangle - \nabla Y. \end{cases}$$

Indeed, from [10, Chapter 3.3] we find that $\vec{R}, S, \vec{X}, Y,$ and $\vec{\Phi}$ satisfy the following system of

---

[1]Note that here, $\vec{T}$ is not a bubble tree.

conservation laws

$$
\begin{cases}
\Delta \vec{R} = \langle \nabla^\perp \vec{n}, \nabla S \rangle + \nabla^\perp \vec{n} \times \nabla \vec{R} + \mathrm{div}\Big[\langle \vec{n}, \nabla Y \rangle + \frac{\rho}{4} |\vec{\Phi}|^2 \nabla \vec{\Phi} \Big] & (2.48) \\[2mm]
\Delta S = \nabla^\perp \vec{n} \cdot \nabla \vec{R} & (2.49) \\[2mm]
\Delta Y = |\nabla \vec{\Phi}|^2 \big(-(c_0^2 + \alpha) + c_0 H + \frac{\rho}{2} \vec{\Phi} \cdot \vec{n} \big) & (2.50) \\[2mm]
\Delta \vec{\Phi} = -\langle \nabla^\perp S, \nabla \vec{\Phi} \rangle - \nabla^\perp \vec{R} \times \nabla \vec{\Phi} + \langle \nabla \vec{\Phi}, \nabla Y \rangle + \frac{\rho}{4} |\vec{\Phi}|^2 |\nabla \vec{\Phi}|^2 \vec{n}. & (2.51)
\end{cases}
$$

Step 2: Morrey decrease. We will show that for some number $\alpha > 0$, there holds

$$
\sup_{r < 1/4,\, a \in B_{1/2}(0)} r^{-\alpha} \int_{B_r(a)} |\nabla \vec{R}|^2 + |\nabla S|^2 \, \mathrm{d}\mathscr{L}^2 < \infty. \tag{2.52}
$$

We let $\varepsilon_0 > 0$ and fix its value later. Choose $0 < r_0 < 1/4$ such that

$$
\sup_{a \in B_{1/2}(0)} \int_{B_{r_0}(a)} |\nabla \vec{n}|^2 \, \mathrm{d}\mathscr{L}^2 < \varepsilon_0. \tag{2.53}
$$

Let a be any point in $B_{1/2}(0)$. Denote by $\vec{R}_0$ the solution of

$$
\begin{cases}
\Delta \vec{R}_0 = \mathrm{div}\Big[\langle \vec{n}, \nabla Y \rangle + \frac{\rho}{4} |\vec{\Phi}|^2 \nabla \vec{\Phi} \Big] & \text{in } D^2 \\[2mm]
\vec{R}_0 = 0 & \text{on } \partial D^2.
\end{cases}
$$

Then, from (2.47) we obtain $\vec{R}_0 \in W^{2,2}(B_1(0), \mathbb{R}^3)$ and hence $\nabla \vec{R}_0 \in L^p(B_1(0), (\mathbb{R}^3)^2)$ for any $1 \leq p < \infty$. Therefore, by Hölder's inequality

$$
\int_{B_r(a)} |\nabla \vec{R}_0|^2 \, \mathrm{d}\mathscr{L}^2 \leq r \boldsymbol{\alpha}(2)^{1/2} \Big(\int_{B_1(0)} |\nabla \vec{R}_0|^4 \, \mathrm{d}\mathscr{L}^2 \Big)^{1/2} =: r C_1 \tag{2.54}
$$

whenever $0 < r \leq r_0$ where $\boldsymbol{\alpha}(2)$ is the area of the unit disk. Let $0 < r \leq r_0$ and let $\vec{\Psi}_{\vec{R}}$ and Ψ_S be the solutions of

$$
\begin{cases}
\Delta \vec{\Psi}_{\vec{R}} = \langle \nabla^\perp \vec{n}, \nabla S \rangle + \nabla^\perp \vec{n} \times \nabla \vec{R} & \text{in } B_r(a) \\[2mm]
\vec{\Psi}_{\vec{R}} = 0 & \text{on } \partial B_r(a)
\end{cases}
$$

and

$$
\begin{cases}
\Delta \Psi_S = \nabla^\perp \vec{n} \cdot \nabla \vec{R} & \text{in } B_r(a) \\[2mm]
\Psi_S = 0 & \text{on } \partial B_r(a).
\end{cases}
$$

Then, the maps

$$
\vec{\nu}_{\vec{R}} := \vec{R} - \vec{R}_0 - \vec{\Psi}_{\vec{R}}, \qquad \nu_S := S - \Psi_S
$$

are harmonic and satisfy

$$
\vec{\nu}_{\vec{R}} = \vec{R} - \vec{R}_0, \quad \vec{\nu}_S = S \qquad \text{on } \partial B_r(a).
$$

Therefore, by monotonicity (see [117, Lemma 7.10]), the Dirichlet principle, and (2.54):

$$\int_{B_{r/3}(a)} |\nabla \vec{\nu}_{\vec{R}}|^2 + |\nabla \nu_S|^2 \, d\mathscr{L}^2 \leq \frac{1}{9} \int_{B_r(a)} |\nabla(\vec{R} - \vec{R}_0)|^2 + |\nabla S|^2 \, d\mathscr{L}^2$$

$$\leq \frac{2}{9} \int_{B_r(a)} |\nabla \vec{R}|^2 + |\nabla S|^2 \, d\mathscr{L}^2 + \frac{2}{9} r C_1. \tag{2.55}$$

By Wente's theorem (see for instance [117, Theorem 3.7]) and the definition of r_0 (2.53) we find

$$\int_{B_r(a)} |\nabla \vec{\Psi}_{\vec{R}}|^2 + |\nabla \Psi_S|^2 \, d\mathscr{L}^2 \leq C_2 \int_{B_{r_0}(a)} |\nabla \vec{n}|^2 \, d\mathscr{L}^2 \int_{B_r(a)} |\nabla \vec{R}|^2 + |\nabla S|^2 \, d\mathscr{L}^2$$

$$\leq C_2 \varepsilon_0 \int_{B_r(a)} |\nabla \vec{R}|^2 + |\nabla S|^2 \, d\mathscr{L}^2 \tag{2.56}$$

for some constant $0 < C_2 < \infty$ independent of $r, r_0, \vec{\Psi}_{\vec{R}}, \Psi_S$. Using the inequalities (2.54)–(2.56) we compute

$$\int_{B_{r/3}(a)} |\nabla \vec{R}|^2 + |\nabla S|^2 \, d\mathscr{L}^2$$

$$\leq 3 \int_{B_{r/3}(a)} |\nabla \vec{\Psi}_{\vec{R}}|^2 + |\nabla \Psi_S|^2 \, d\mathscr{L}^2$$

$$+ 3 \int_{B_{r/3}(a)} |\nabla \vec{\nu}_{\vec{R}}|^2 + |\nabla \nu_S|^2 \, d\mathscr{L}^2 + 3 \int_{B_{r/3}(a)} |\nabla \vec{R}_0|^2 \, d\mathscr{L}^2$$

$$\leq (3 C_2 \varepsilon_0 + 6/9) \int_{B_r(a)} |\nabla \vec{R}|^2 + |\nabla S|^2 \, d\mathscr{L}^2 + \frac{6}{9} r C_1 + r C_1.$$

Therefore, taking $\varepsilon_0 = (3 C_2 9)^{-1}$ yields

$$\int_{B_{r/3}(a)} |\nabla \vec{R}|^2 + |\nabla S|^2 \, d\mathscr{L}^2 \leq \frac{7}{9} \int_{B_r(a)} |\nabla \vec{R}|^2 + |\nabla S|^2 \, d\mathscr{L}^2 + r 2 C_1 \tag{2.57}$$

for all $0 < r \leq r_0$. We next show by induction that

$$\int_{B_{3^{-n}r_0}(a)} |\nabla \vec{R}|^2 + |\nabla S|^2 \, d\mathscr{L}^2$$

$$\leq \left(\frac{7}{9}\right)^n \int_{B_{r_0}(a)} |\nabla \vec{R}|^2 + |\nabla S|^2 \, d\mathscr{L}^2 + r_0 2 C_1 \sum_{i=1}^{n} 3^{-i+1} \left(\frac{7}{9}\right)^{n-i} \tag{2.58}$$

for all $n \in \mathbb{N}$. Indeed, letting

$$A(s) := \int_{B_s(a)} |\nabla \vec{R}|^2 + |\nabla S|^2 \, d\mathscr{L}^2 \qquad \text{for } 0 < s \leq r_0$$

we have from (2.57) that $A(r_0/3^1) \leq \left(\frac{7}{9}\right)^1 A(r_0) + r_0 2 C_1 \sum_{i=1}^{1} 3^{-i+1} \left(\frac{7}{9}\right)^{1-i}$. Assuming (2.58) to

be true for some integer n, we get from (2.57) that

$$
\begin{aligned}
A(r_0/3^{n+1}) &\le \frac{7}{9} A(r_0/3^n) + r_0 3^{-n} 2C_1 \\
&\le \frac{7}{9} \Big[\Big(\frac{7}{9}\Big)^n A(r_0) + r_0 2C_1 \sum_{i=1}^{n} 3^{-i+1} \Big(\frac{7}{9}\Big)^{n-i} \Big] + r_0 3^{-n} 2C_1 \\
&\le \Big(\frac{7}{9}\Big)^{n+1} A(r_0) + r_0 2C_1 \sum_{i=1}^{n+1} 3^{-i+1} \Big(\frac{7}{9}\Big)^{n+1-i}.
\end{aligned}
$$

Thus, by induction, (2.58) holds true for all $n \in \mathbb{N}$. Since

$$
2C_1 \sum_{i=1}^{n} 3^{-i+1} \Big(\frac{7}{9}\Big)^{n-i} \le \Big(\frac{7}{9}\Big)^n 2C_1 3 \sum_{i=1}^{n} \frac{9^i}{3^i 7^i} \le \Big(\frac{7}{9}\Big)^n 12 C_1,
$$

it follows that

$$
\fint_{B_{3^{-n} r_0}(a)} |\nabla \vec{R}|^2 + |\nabla S|^2 \, d\mathscr{L}^2 \le \Big(\frac{r_0}{3^n}\Big)^{\alpha} C_0
$$

for $\alpha = \log_3(9/7)$ and

$$
C_0 = r_0^{-\alpha} \Big(\fint_{B_1(0)} |\nabla \vec{R}|^2 + |\nabla S|^2 \, d\mathscr{L}^2 + 12 C_1 \Big)
$$

which implies (2.52) as C_0 and α are independent of a. From (2.48), (2.49), the definition of $\vec{R}_0$, and Hölder's inequality it follows

$$
\sup_{r < 1/4, \, a \in B_{1/2}(0)} r^{-\alpha/2} \int_{B_r(a)} |\Delta(\vec{R} - \vec{R}_0)| + |\Delta S| \, d\mathscr{L}^2 < \infty
$$

and hence, by a classical estimate on Riesz potentials [2],

$$
\nabla(\vec{R} - \vec{R}_0) \in L^p_{\mathrm{loc}}(B_{1/2}(0), (\mathbb{R}^3)^2), \qquad \nabla S \in L^p_{\mathrm{loc}}(B_{1/2}(0), \mathbb{R}^2)
$$

for some $p > 2$. Since $\nabla \vec{R}_0 \in L^q(B_{1/2}(0), (\mathbb{R}^3)^2)$ for all $1 \le q < \infty$, we obtain

$$
\nabla \vec{R} \in L^p_{\mathrm{loc}}(B_{1/2}(0), (\mathbb{R}^3)^2), \qquad \nabla S \in L^p_{\mathrm{loc}}(B_{1/2}(0), \mathbb{R}^2). \tag{2.59}
$$

Step 3: Bootstrapping. Putting (2.47) and (2.59) into (2.51), we infer

$$
\nabla \vec{n} \in L^p_{\mathrm{loc}}(B_{1/2}(0), (\mathbb{R}^3)^2),
$$

for some $p > 2$ given in the previous step. By Hölder's inequality and (2.48), (2.49) we first get

$$
|\Delta(\vec{R} - \vec{R}_0)| \in L^q_{\mathrm{loc}}(B_{1/2}(0)), \qquad |\Delta S| \in L^q_{\mathrm{loc}}(B_{1/2}(0))
$$

for $q := p/2 > 1$ and then, by Sobolev embedding,

$$
\nabla(\vec{R} - \vec{R}_0) \in L^{q^*}_{\mathrm{loc}}(B_{1/2}(0), (\mathbb{R}^3)^2), \qquad \nabla S \in L^{q^*}_{\mathrm{loc}}(B_{1/2}(0), \mathbb{R}^2)
$$

where $q^* := 2q/(2 - q) = 2p/(4 - p)$ satisfies $q^* > 2q = p$ as $p > 2$.

Since $\nabla \vec{R}_0 \in L^q_{\text{loc}}(B_{1/2}(0)$ for all $1 \leq q < \infty$, we infer

$$\nabla \vec{R} \in L^{q^*}_{\text{loc}}(B_{1/2}(0), (\mathbb{R}^3)^2), \qquad \nabla S \in L^{q^*}_{\text{loc}}(B_{1/2}(0), \mathbb{R}^2).$$

Notice that q^* as above induces a recursively defined sequence of real numbers. Given a starting point $q_0 > 1$, this sequence is unbounded as $q^* > 2q$. Hence, we can repeat this procedure to obtain

$$\nabla \vec{R} \in L^q_{\text{loc}}(B_{1/2}(0), (\mathbb{R}^3)^2), \quad \nabla S \in L^q_{\text{loc}}(B_{1/2}(0), \mathbb{R}^2) \qquad \text{for all } 1 \leq q < \infty.$$

Therefore, from the system of conservation laws (2.48)–(2.51) we get step by step for all $1 \leq q < \infty$

$$\vec{\Phi} \in W^{2,q}_{\text{loc}}(B_{1/2}(0), \mathbb{R}^3), \qquad \nabla \vec{n} \in L^q_{\text{loc}}(B_{1/2}(0), (\mathbb{R}^3)^2), \qquad Y \in W^{2,q}_{\text{loc}}(B_{1/2}(0), \mathbb{R}),$$
$$\vec{R} \in W^{2,q}_{\text{loc}}(B_{1/2}(0), \mathbb{R}^3), \qquad S \in W^{2,q}_{\text{loc}}(B_{1/2}(0), \mathbb{R}).$$

Iteration gives

$$\vec{\Phi} \in W^{k,p}_{\text{loc}}(B_{1/2}(0), \mathbb{R}^3) \qquad \text{for all } k \in \mathbb{N}, 1 \leq p < \infty$$

and hence,

$$\vec{\Phi} \in C^\infty(B_{1/2}(0))$$

which finishes the proof. $\qquad\qquad\square$

2.4 Main theorems

Let g be a non-negative integer, and Σ be a smooth, oriented, and closed 2-dimensional manifold with genus g. Let $A_0, V_0 > 0$ such that $A_0^3 \geq 36\pi V_0^2$ if $g = 0$ and $A_0^3 > 36\pi V_0^2$ if $g \geq 1$. Define

$$\mathcal{F}_\Sigma(A_0, V_0) := \mathcal{F}_\Sigma \cap \{\vec{\Phi} : \text{area } \vec{\Phi} = A_0, \text{vol } \vec{\Phi} = V_0\}$$

where vol $\vec{\Phi}$ denotes the enclosed volume if $\vec{\Phi}$ is an embedding (i.e. injective) and vol $\vec{\Phi} := \infty$ if $\vec{\Phi}$ is not an embedding (see also Section 2.1.5). Anticipating Section 3, we define

$$\boldsymbol{\beta}_g := \inf_{\vec{\Phi} \in \mathcal{F}_\Sigma} \int_\Sigma H^2_{\vec{\Phi}} \, \mathrm{d}\mu_{\vec{\Phi}}$$

as well as

$$\beta_g(\sigma) := \inf \left\{ \int_\Sigma H^2_{\vec{\Phi}} \, \mathrm{d}\mu_{\vec{\Phi}} : \vec{\Phi} \in \mathcal{F}_\Sigma, \, \frac{\text{area}(\vec{\Phi})}{\text{vol}(\vec{\Phi})^{2/3}} = \sigma \right\}$$

for all $\sigma > \sqrt[3]{36\pi}$. Finally, we define the constant

$$\varepsilon_g(A_0, V_0) := \frac{\sqrt{\min\{8\pi, \omega_g(A_0/V_0^{2/3})\}} - \sqrt{\beta_g(A_0/V_0^{2/3})}}{2\sqrt{A_0}}, \tag{2.60}$$

where

$$\omega_g(\sigma) := \begin{cases} \infty & \text{if } g = 0 \\ \beta_g + \beta_0(\sigma) - 4\pi & \text{if } g \geq 1 \end{cases}$$

for all $\sigma > \sqrt[3]{36\pi}$.

2.10 Lemma (See MONDINO–SCHARRER [96, Lemma 4.4])**.** *Suppose g is a non-negative integer, Σ is a smooth genus g surface, and $A_0, V_0 > 0$ satisfy the isoperimetric inequality: $A_0^3 > 36\pi V_0^2$.*

Then, for all $c_0 \in \mathbb{R}$, there holds

$$\left| \inf_{\vec{\Phi} \in \mathcal{F}_\Sigma(A_0, V_0)} \sqrt{\int_\Sigma H_{\vec{\Phi}}^2 \, \mathrm{d}\mu_{\vec{\Phi}}} - \inf_{\vec{\Phi} \in \mathcal{F}_\Sigma(A_0, V_0)} \sqrt{\int_\Sigma (H_{\vec{\Phi}} - c_0)^2 \, \mathrm{d}\mu_{\vec{\Phi}}} \right| \leq |c_0| \sqrt{A_0}. \tag{2.61}$$

Moreover, if $\varepsilon_g(A_0, V_0)$ defined as in (2.60) satisfies

$$\varepsilon := \varepsilon_g(A_0, V_0) > 0 \tag{2.62}$$

then, for all $c_0 \in (-\varepsilon, \varepsilon)$ and any minimising sequence $\vec{\Phi}_k$ of $\inf_{\vec{\Phi} \in \mathcal{F}_\Sigma(A_0, V_0)} \int_\Sigma (H - c_0)^2 \, \mathrm{d}\mu_{\vec{\Phi}}$ there holds

$$\limsup_{k \to +\infty} \int_\Sigma H_{\vec{\Phi}_k}^2 \, \mathrm{d}\mu_{\vec{\Phi}_k} < \min\{8\pi, \omega_g(A_0/V_0^{2/3})\}. \tag{2.63}$$

Proof. From the Cauchy-Schwartz inequality, we have $|\int_\Sigma H \, \mathrm{d}\mu_{\vec{\Phi}}| \leq \sqrt{\int_\Sigma H^2 \, \mathrm{d}\mu_{\vec{\Phi}}} \sqrt{\text{area}(\vec{\Phi})}$. Thus:

$$\left(\sqrt{\int_\Sigma H^2 \, \mathrm{d}\mu_{\vec{\Phi}}} - |c_0| \sqrt{\text{area}(\vec{\Phi})} \right)^2 \leq \int_\Sigma (H - c_0)^2 \, \mathrm{d}\mu_{\vec{\Phi}} \leq \left(\sqrt{\int_\Sigma H^2 \, \mathrm{d}\mu_{\vec{\Phi}}} + |c_0| \sqrt{\text{area}(\vec{\Phi})} \right)^2$$

which yields

$$\left| \sqrt{\int_\Sigma H^2 \, \mathrm{d}\mu_{\vec{\Phi}}} - \sqrt{\int_\Sigma (H - c_0)^2 \, \mathrm{d}\mu_{\vec{\Phi}}} \right| \leq |c_0| \sqrt{\text{area}(\vec{\Phi})}. \tag{2.64}$$

In particular, we deduce that

$$\left| \inf_{\vec{\Phi} \in \mathcal{F}_\Sigma(A_0, V_0)} \sqrt{\int_\Sigma H^2 \, \mathrm{d}\mu_{\vec{\Phi}}} - \inf_{\vec{\Phi} \in \mathcal{F}_\Sigma(A_0, V_0)} \sqrt{\int_\Sigma (H - c_0)^2 \, \mathrm{d}\mu_{\vec{\Phi}}} \right| \leq |c_0| \sqrt{A_0} \tag{2.65}$$

which proves (2.61). Let $c_0 \in (-\varepsilon, \varepsilon)$ and let $\vec{\Phi}_k$ be a minimising sequence of the minimal Helfrich energy $\inf_{\vec{\Phi} \in \mathcal{F}_\Sigma(A_0, V_0)} \int_\Sigma (H - c_0)^2 \, \mathrm{d}\mu_{\vec{\Phi}}$. For k large enough it holds

$$\sqrt{\int_\Sigma (H_k - c_0)^2 \, \mathrm{d}\mu_{\vec{\Phi}_k}} \leq \sqrt{\inf_{\vec{\Phi} \in \mathcal{F}_\Sigma(A_0, V_0)} \int_\Sigma (H - c_0)^2 \, \mathrm{d}\mu_{\vec{\Phi}}} + (\varepsilon - |c_0|) \sqrt{A_0}. \tag{2.66}$$

Combining (2.64), (2.66), and (2.65), we get

$$\sqrt{\int_\Sigma H_k^2 \, \mathrm{d}\mu_{\vec{\Phi}_k}} \le \sqrt{\int_\Sigma (H_k - c_0)^2 \, \mathrm{d}\mu_{\vec{\Phi}_k}} + |c_0|\sqrt{A_0}$$

$$\le \sqrt{\inf_{\vec{\Phi} \in \mathcal{F}_\Sigma(A_0, V_0)} \int_\Sigma (H - c_0)^2 \, \mathrm{d}\mu_{\vec{\Phi}}} + (\varepsilon - |c_0|)\sqrt{A_0} + |c_0|\sqrt{A_0}$$

$$< \sqrt{\inf_{\vec{\Phi} \in \mathcal{F}_\Sigma(A_0, V_0)} \int_{\vec{\Phi}_k} H^2 \, \mathrm{d}\mu_{\vec{\Phi}}} + 2\varepsilon\sqrt{A_0} = \sqrt{\min\{8\pi, \omega_g(A_0/V_0^{2/3})\}},$$

where in the last identity we plugged in the definition of $\varepsilon = \varepsilon_g(A_0, V_0)$ as in (2.60). $\qquad\square$

We recall that for $\vec{\Phi} \in \mathcal{F}_{\mathbb{S}^2}$ and $c_0 \in \mathbb{R}$.

$$\mathcal{H}^{c_0}(\vec{\Phi}) = \int_{\mathbb{S}^2} (H_{\vec{\Phi}} - c_0)^2 \, \mathrm{d}\mu_{\vec{\Phi}}.$$

2.11 Theorem (See MONDINO–SCHARRER [96, Theorem 1.7]). *Suppose $c_0 \in \mathbb{R}$, $A_0, V_0 > 0$, and $A_0^3 \ge 36\pi V_0^2$.*

Then, there exist a positive integer N and weak branched conformal immersions of finite total curvature $\vec{\Phi}_1, \dots, \vec{\Phi}_N \in \mathcal{F}_{\mathbb{S}^2}$ such that $\cup_{i=1}^N \vec{\Phi}_i[\mathbb{S}^2]$ is connected,

$$\eta_{c_0}(A_0, V_0) := \inf_{\substack{\vec{\Phi} \in \mathcal{F}_{\mathbb{S}^2} \\ \mathrm{area}\, \vec{\Phi} = A_0 \\ \mathrm{vol}\, \vec{\Phi} = V_0}} \mathcal{H}^{c_0}(\vec{\Phi}) = \sum_{i=1}^N \mathcal{H}^{c_0}(\vec{\Phi}_i) \tag{2.67}$$

and

$$\sum_{i=1}^N \mathrm{area}\, \vec{\Phi}_i = A_0, \qquad \sum_{i=1}^N \mathrm{vol}\, \vec{\Phi}_i = V_0.$$

Moreover, for each $i \in \{1, \dots, N\}$ there exist a non-negative integer N^i and finitely many points $b^{i,1}, \dots, b^{i,N^i} \in \mathbb{S}^2$ such that $\vec{\Phi}_i$ is a C^∞ immersion of $\mathbb{S}^2 \setminus \{b^{i,1}, \dots, b^{i,N^i}\}$ into $\mathbb{R}^3$ and $b^{i,1}, \dots, b^{i,N^i}$ are branch points for $\vec{\Phi}_i$. The total number of branch points and the number of bubbles can a priori be bounded in terms of

$$\sum_{i=1}^N N^i \le \eta_{c_0}(A_0, V_0) + c_0^2 A_0, \qquad N \le \eta_{c_0}(A_0, V_0) + c_0^2 A_0. \tag{2.68}$$

Furthermore, there exists a constant $\varepsilon(A_0, V_0) > 0$ such that if $|c_0| < \varepsilon(A_0, V_0)$, then $N = 1$ and $\vec{\Phi} := \vec{\Phi}_1$ is a smooth embedding of $\mathbb{S}^2$ into $\mathbb{R}^3$.

Proof. Let $\vec{\Phi}_1, \vec{\Phi}_2, \dots$ be a minimising sequence of (2.67). There holds

$$\int_{\mathbb{S}^2} H_{\vec{\Phi}_k}^2 \, \mathrm{d}\mu_{\vec{\Phi}_k} = \int_{\mathbb{S}^2} 2(H_{\vec{\Phi}_k} - c_0)^2 - (H_{\vec{\Phi}_k} - 2c_0)^2 + 2c_0^2 \, \mathrm{d}\mu_{\vec{\Phi}_k} \le 2\mathcal{H}^{c_0}(\vec{\Phi}_k) + 2c_0^2 A_0. \tag{2.69}$$

By the Gauss–Bonnet theorem (see (2.9) for the precise statement in case of weak branched immersions and (2.11) for the estimate below),

$$\int_{\mathbb{S}^2} |\mathrm{d}\vec{n}_{\vec{\Phi}_k}|^2 \, \mathrm{d}\mu_{\vec{\Phi}_k} \le 4 \int_{\mathbb{S}^2} H_{\vec{\Phi}_k}^2 \, \mathrm{d}\mu_{\vec{\Phi}_k}$$

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and thus

$$\int_{\mathbb{S}^2} 1 + |\mathrm{d}\vec{n}_{\vec{\Phi}_k}|^2 \,\mathrm{d}\mu_{\vec{\Phi}_k} \le 8\mathcal{H}^{c_0}(\vec{\Phi}_k) + (1 + 8c_0^2)A_0$$

which means the first inequality of (2.35) is satisfied. Moreover, (2.13) implies the second inequality of (2.35). Hence, we can apply Theorem 2.5 and Theorem 2.9 to conclude the proofs of the first and second part of the theorem.

To prove the last part, we let $\varepsilon(A_0, V_0) := \varepsilon_0(A_0, V_0)$, where $\varepsilon_0(A_0, V_0)$ is defined as in (2.60). Notice that by Theorem 4.4 in combination with Theorem 4.1 (see also [129, Lemma 2.1]), we have $\varepsilon_0(A_0, V_0) > 0$. Thus, we can employ Lemma 2.10, (2.16) and (2.15) in combination with Remark 2.7 and Theorem 2.9 to conclude the proof of the last part.

The last statement in particular proves (2.68) for the case $c_0 = 0$. Hence, in order to proof (2.68), we my assume $c_0 \ne 0$. Let $\vec{\Phi} \in \mathcal{F}_{\mathbb{S}^2}$ with $c_0^2 \operatorname{area}(\vec{\Phi}) \le 2$. Then, by Hölder's inequality and Willmore's inequality we infer

$$\mathcal{H}^{c_0}(\vec{\Phi}) \ge \int_{\mathbb{S}^2} H_{\vec{\Phi}}^2 \,\mathrm{d}\mu_{\vec{\Phi}} - 2|c_0|\sqrt{\operatorname{area}(\vec{\Phi})}\sqrt{\int_{\mathbb{S}^2} H_{\vec{\Phi}}^2 \,\mathrm{d}\mu_{\vec{\Phi}}} \ge \sqrt{4\pi}\left(\sqrt{4\pi} - 2|c_0|\sqrt{\operatorname{area}(\vec{\Phi})}\right) \ge 1.$$

Hence, letting

$$N_1 := \#\left\{i : \operatorname{area}(\vec{\Phi}_i) \le \frac{2}{c_0^2}\right\}, \qquad N_2 := \#\left\{i : \operatorname{area}(\vec{\Phi}_i) > \frac{2}{c_0^2}\right\}$$

it follows

$$N = N_1 + N_2 \le \eta_{c_0}(A_0, V_0) + c_0^2 A_0.$$

On the other hand, combining (2.10) with (2.69), we have for all $i \in \{1, \dots, N\}$

$$N^i \le \frac{1}{\pi}\int_{\mathbb{S}^2} H_{\vec{\Phi}_i}^2 \,\mathrm{d}\mu_{\vec{\Phi}_i} \le \frac{2}{\pi}\left(\mathcal{H}^{c_0}(\vec{\Phi}_i) + c_0^2 \operatorname{area}(\vec{\Phi}_i)\right).$$

Summing over $i = 1, \dots, N$ yields the conclusion $\qquad\square$

2.12 Remark. The arguments in the proof of Theorem 2.11 yield also that the minimum of $\mathcal{H}^{c_0}$ is achieved in the class of bubble trees of possibly branched weak immersions, by a bubble tree of possibly branched immersions which are smooth out of the branch points.

A more general form of the Canham–Helfrich energy is given by

$$\mathcal{H}_{\alpha,\rho}^{c_0}(\vec{\Phi}) := \int_{\mathbb{S}^2} (H_{\vec{\Phi}} - c_0)^2 \,\mathrm{d}\mu_{\vec{\Phi}} + \alpha \operatorname{area}\vec{\Phi} + \rho \operatorname{vol}\vec{\Phi},$$

for $\vec{\Phi} \in \mathcal{F}_{\mathbb{S}^2}$ where the parameter $\alpha \ge 0$ is referred to as *tensile stress*, and $\rho \ge 0$ as *osmotic pressure*. We get the following solution of Problem (P2) from the introduction in [12].

2.13 Theorem (See MONDINO–SCHARRER [96, Theorem 1.9])**.** *Suppose $c_0 \in \mathbb{R}$, $\alpha > 0$, and $\rho \ge 0$. Then,*

$$\inf_{\vec{\Phi} \in \mathcal{F}_{\mathbb{S}^2}} \mathcal{H}_{\alpha,\rho}^{c_0}(\vec{\Phi}) \le 4\pi. \tag{2.70}$$

Moreover, the following five statements hold.

1. *If equality holds in (2.70), then there exists a minimising sequence $\vec{\Phi}_k$ of $\inf_{\vec{\Phi} \in \mathcal{F}_{\mathbb{S}^2}} \mathcal{H}^{c_0}_{\alpha,\rho}(\vec{\Phi})$ that shrinks to a point.*

2. *If the inequality (2.70) is strict, then there exist $\vec{\Phi}_0 \in \mathcal{F}_{\mathbb{S}^2}$, a positive integer N_0, and points $b_1, \ldots, b_{N_0} \in \mathbb{S}^2$ such that*
$$\inf_{\vec{\Phi} \in \mathcal{F}_{\mathbb{S}^2}} \mathcal{H}^{c_0}_{\alpha,\rho}(\vec{\Phi}) = \mathcal{H}^{c_0}_{\alpha,\rho}(\vec{\Phi}_0),$$
and $\vec{\Phi}_0$ is a C^∞-immersion of $\mathbb{S}^2 \setminus \{b_1, \ldots, b_{N_0}\}$ into $\mathbb{R}^3$. Moreover, if $|c_0| \leq \sqrt{\alpha}$, then $\vec{\Phi}_0$ is a smooth embedding.

3. *If $c_0 < 0$ and the infimum in (1.24) is attained by a member $\vec{\Phi}_0 \in \mathcal{F}_{\mathbb{S}^2}$, then*
$$\int_{\mathbb{S}^2} H_{\vec{\Phi}_0} \, \mathrm{d}\mu_{\vec{\Phi}_0} < 0.$$
In particular, the infimum cannot be attained by a convex surface.

4. *If $c_0 = 0$, then equality holds in (2.70) and the infimum is neither attained in $\mathcal{F}_{\mathbb{S}^2}$ nor in the class of bubble trees of weak immersions.*

5. *If $c_0 > 0$ and r_0 is defined by*
$$r_0 := \begin{cases} \frac{c_0}{c_0^2 + \alpha} & \text{if } \rho = 0 \\ -\frac{c_0^2 + \alpha}{\rho} + \sqrt{\frac{2c_0}{\rho} + \frac{(c_0^2 + \alpha)^2}{\rho^2}} & \text{if } \rho > 0 \end{cases} \tag{2.71}$$
then $\mathcal{H}^{c_0}_{\alpha,\rho}(r_0 \mathbb{S}^2) < 4\pi$. Moreover, if a round sphere is a critical point of $\mathcal{H}^{c_0}_{\alpha,\rho}$, then its radius is given by (2.71).

Proof. Taking $\vec{\Phi}_k = \frac{1}{k}\mathbb{S}^2$ for each integer k leads to
$$\inf_{\vec{\Phi} \in \mathcal{F}_{\mathbb{S}^2}} \mathcal{H}^{c_0}_{\alpha,\rho}(\vec{\Phi}) \leq \liminf_{k \to \infty} \mathcal{H}^{c_0}_{\alpha,\rho}(\vec{\Phi}_k) = \int_{\mathbb{S}^2} H^2_{\mathbb{S}^2} \, \mathrm{d}\mu_{\mathbb{S}^2} = 4\pi$$

which proves the inequality in (2.70) as well as statement (1). Moreover, by (2.16), we have that for $c_0 = 0$, there holds $4\pi < \mathcal{H}^{c_0}_{\alpha,\rho}$ which by (2.70) implies statement (4). Now assume $\vec{\Phi}_1, \vec{\Phi}_2, \ldots$ is a sequence in $\mathcal{F}_{\mathbb{S}^2}$ such that
$$\lim_{k \to \infty} \mathcal{H}^{c_0}_{\alpha,\rho}(\vec{\Phi}_k) = \inf_{\vec{\Phi} \in \mathcal{F}_{\mathbb{S}^2}} \mathcal{H}^{c_0}_{\alpha,\rho}(\vec{\Phi}) < 4\pi.$$

As $\alpha > 0$, we have $\sup_k \text{area}\, \vec{\Phi}_k < \infty$ and thus, using (2.69), also $\sup_k \int_{\mathbb{S}^2} H^2_{\vec{\Phi}_k} \, \mathrm{d}\mu_{\vec{\Phi}_k} < \infty$. Hence, (2.13) leads to
$$\liminf_{k \to \infty} \text{diam}\, \vec{\Phi}_k[\mathbb{S}^2] > 0,$$
as otherwise we had
$$\lim_{k \to \infty} \text{area}\, \vec{\Phi}_k = 0, \qquad \lim_{k \to \infty} \mathcal{H}^{c_0}_{\alpha,\rho}(\vec{\Phi}_k) = \lim_{k \to \infty} \int_{\mathbb{S}^2} H^2_{\vec{\Phi}_k} \, \mathrm{d}\mu_{\vec{\Phi}_k} \geq 4\pi.$$

Therefore, analogously to the proof of Theorem 2.11, we can apply Theorem 2.5 to obtain an

integer N and $\vec{\Phi}_1, \ldots, \vec{\Phi}_N \in \mathcal{F}_{\mathbb{S}^2}$ such that

$$\inf_{\vec{\Phi} \in \mathcal{F}_{\mathbb{S}^2}} \mathcal{H}_{\alpha,\rho}^{c_0}(\vec{\Phi}) = \sum_{i=1}^{N} \mathcal{H}_{\alpha,\rho}^{c_0}(\vec{\Phi}_i).$$

Obviously,

$$\mathcal{H}_{\alpha,\rho}^{c_0}(\vec{\Phi}_1) \leq \sum_{i=1}^{N} \mathcal{H}_{\alpha,\rho}^{c_0}(\vec{\Phi}_i)$$

and since there are no constraints, we simply get $N = 1$. Letting $\vec{\Phi}_0 := \vec{\Phi}_1$, we infer from (2.69) that in case $|c_0| \leq \sqrt{\alpha}$

$$\int_{\mathbb{S}^2} H_{\vec{\Phi}_0}^2 \, \mathrm{d}\mu_{\vec{\Phi}_0} \leq 2\mathcal{H}^{c_0}(\vec{\Phi}_0) + 2c_0^2 \, \text{area} \, \vec{\Phi}_0 \leq 2\mathcal{H}_{\alpha,\rho}^{c_0}(\vec{\Phi}_0) < 8\pi.$$

Now, statement (2) follows from Remark 2.7, and Theorem 2.9.

Statement (3) follows from (2.70) by simply expanding the square in $\mathcal{H}^{c_0}$.

To prove statement (5), we define

$$E : (0, \infty) \to (0, \infty), \qquad E(r) = \mathcal{H}_{\alpha,\rho}^{c_0}(r\mathbb{S}^2).$$

A simple computation shows that E is a polynomial:

$$E(r) = 4\pi - 8\pi c_0 r + 4\pi(c_0^2 + \alpha)r^2 + \frac{4\pi}{3}\rho r^3.$$

Moreover,

$$E'(r) = -8\pi c_0 + 8\pi(c_0^2 + \alpha)r + 4\pi\rho r^2 \tag{2.72}$$

and

$$E(r) < 4\pi \quad \iff \quad (c_0^2 + \alpha)r + \frac{\rho}{3}r^2 < 2c_0. \tag{2.73}$$

Case $\rho = 0$. If $\rho = 0$, then (2.72) implies

$$E'(r_0) = 0 \quad \iff \quad r_0 = \frac{c_0}{c_0^2 + \alpha}$$

and

$$(c_0^2 + \alpha)r_0 = c_0 < 2c_0$$

which by (2.73) concludes the case $\rho = 0$.

Case $\rho > 0$. If $\rho > 0$, then (2.72) implies

$$E'(r_0) = 0 \quad \iff \quad r_0 = -\frac{c_0^2 + \alpha}{\rho} + \sqrt{\frac{2c_0}{\rho} + \frac{(c_0^2 + \alpha)^2}{\rho^2}}.$$

Moreover,

$$(c_0^2 + \alpha)r_0 + \frac{\rho}{3}r_0^2 = \frac{c_0^2 + \alpha}{3}\sqrt{\frac{2c_0}{\rho} + \frac{(c_0^2 + \alpha)^2}{\rho^2}} - \frac{(c_0^2 + \alpha)^2}{3\rho} + \frac{2c_0}{3}$$

and thus

$$(c_0^2 + \alpha)r_0 + \frac{\rho}{3}r_0^2 < 2c_0 \quad \Longleftrightarrow \quad \sqrt{\frac{2c_0}{\rho} + \frac{(c_0^2 + \alpha)^2}{\rho^2}} < \frac{4c_0}{c_0^2 + \alpha} + \frac{c_0^2 + \alpha}{\rho}$$

which is indeed always true. Now, (2.73) concludes the proof of the first part of statement (5). The Euler–Lagrange Equation (2.40) for $\mathcal{H}_{\alpha,\rho}^{c_0}$ is designed to prove regularity of weak solutions. A more practical way to write it is

$$\vec{\mathcal{W}} = 2c_0\Big(\frac{1}{2}\mathbb{I}_j^i\mathbb{I}_i^j - 2H^2\Big) + 2(c_0^2 + \alpha)H + \rho \tag{2.74}$$

where H is the arithmetic mean of the principle curvatures, see for instance Equation (3.8) and Equation (3.9) in [10]. Round spheres solve the Euler–Lagrange equation of the classical Willmore functional, which means $\vec{\mathcal{W}} = 0$. Hence, for round spheres, (2.74) becomes

$$0 = -\frac{2c_0}{r^2} + \frac{2(c_0^2 + \alpha)}{r} + \rho$$

where r is the radius of the sphere. Therefore, in view of (2.72), a round sphere is a critical point of $\mathcal{H}_{\alpha,\rho}^{c_0}$ if and only if its radius is a critical point for E which concludes the proof. $\qquad\square$

3 A strict inequality for the minimisation of the Willmore functional under isoperimetric constraint

The aim of this section is to prove Theorem 4 and Corollary 1 from the introduction. This will be done in Subsection 3.4. To prove the main ingredients, we will stick close to the connected sum construction of BAUER–KUWERT [6], see Subsection 3.3. Moreover, we will need a variational lemma to adjust the isoperimetric ratio, see Subsection 3.2. The content of this section corresponds to our work [97].

3.1 Notation

Throughout Section 3, the mean curvature H of an immersed closed surface $f : \Sigma \to \mathbb{R}^3$ is defined as the trace of the second fundamental form. The Willmore functional $\mathcal{W}$ thus takes the form

$$\mathcal{W}(f) = \frac{1}{4} \int_\Sigma H^2 \, \mathrm{d}\mu \,,$$

where μ is the Radon measure corresponding to the pull back metric of the Euclidean metric along f. The isoperimetric ratio is defined by

$$\mathrm{iso}(f) = \frac{\mathrm{area}(f)}{\mathrm{vol}(f)^{\frac{2}{3}}} \,,$$

where

$$\mathrm{area}(f) = \int_\Sigma 1 \, \mathrm{d}\mu, \qquad \mathrm{vol}(f) = \frac{1}{3} \int_\Sigma n \cdot f \, \mathrm{d}\mu \tag{3.1}$$

are the area and volume, and $n : \Sigma \to \mathbb{S}^2$ is the Gauss map. Denote with $\mathcal{S}_g$ the set of smooth immersions $f : \Sigma \to \mathbb{R}^3$ where Σ is a closed surface (i.e. compact without boundary) with $\mathrm{genus}(\Sigma) = g$. Define the constants

$$\beta_g := \inf\{\mathcal{W}(f) : f \in \mathcal{S}_g\}$$

and, for all $\sigma > \sqrt[3]{36\pi}$,

$$\beta_g(\sigma) := \inf\{\mathcal{W}(f) : f \in \mathcal{S}_g, \, \mathrm{iso}(f) = \sigma\}.$$

We will use Bachmann–Landau notation.

3.2 Isoperimetric ratio and conformal transformation

In this subsection, we will gather some facts about isoperimetric ratio and conformal maps. These facts will be used to prove Proposition 3.9 which is needed to prove monotonicity of $\beta_g(\cdot)$. Moreover, inspired by the volume preserving mean curvature flow by HUISKEN [55], we will prove a variational lemma (see Lemma 3.10) which allows to adjust the isoperimetric ratio of a surface by changing the surface only in a small neighbourhood.

3.1 Definition. A smooth immersion $\psi : M \to N$ between Riemannian manifolds (M, g) and (N, h) is called *conformal*, if and only of there exists a smooth function $\mu : M \to \mathbb{R}$ such that the

pull back metric $\psi^* h$ satisfies

$$\psi^* h = e^{2\mu} g.$$

The following theorem is due to CHEN [25]. To be more precise, it is a combination of the Gauss–Bonnet theorem with [25, Theorem 1].

3.2 Theorem (CHEN [25])**.** *Suppose Σ is a closed smooth surface, $f : \Sigma \to \mathbb{R}^n$ is a smooth immersion, $U \subset \mathbb{R}^n$ is an open set containing Σ, and $\psi : U \to \mathbb{R}^n$ is conformal.*

Then, there holds

$$\mathcal{W}(f) = \mathcal{W}(\psi \circ f).$$

3.3 Theorem (WILLMORE [139, Theorem 1])**.** *Suppose $f : \mathbb{S}^2 \to \mathbb{R}^3$ is a smooth embedding.*

Then, there holds

$$\mathcal{W}(f) \geq 4\pi$$

with equality if and only if f parametrises a round sphere.

3.4 Proposition (LIOUVILLE [83])**.** *Suppose S is a round sphere in $\mathbb{R}^3$, that is $S = r\mathbb{S}^2 + a$ for some $r > 0$ and $a \in \mathbb{R}^3$.*

Then, the image of S under conformal transformation is again a round sphere.

Proof. This is a direct consequence of Theorem 3.2 and Theorem 3.3. $\qquad\square$

3.5 Definition. For all $a \in \mathbb{R}^3$ we define the *inversion I_a* at the unit sphere centred at a by

$$I_a : \mathbb{R}^3 \setminus \{a\} \to \mathbb{R}^3, \qquad I_a(x) = \frac{x - a}{|x - a|^2}$$

and abbreviate $I := I_0$.

The following theorem is originally due to LIOUVILLE 1850. Its proof can be found for instance in [13].

3.6 Theorem. *Inversions are conformal maps. Moreover, any one-to-one conformal transformation on an open subset of $\mathbb{R}^3$ can be written as composition of dilations, similarities and inversions.*

3.7 Lemma. *Suppose Σ is a closed smooth surface and $f : \Sigma \to \mathbb{R}^3$ is a smooth embedding.*

Then, there holds

$$\lim_{|a| \to \infty} \mathrm{iso}(I_a \circ f) = \mathrm{iso}(f).$$

Proof. Denoting with Id the identity map $\mathbb{R}^3 \to \mathbb{R}^3$, we have

$$\mathrm{D}I(x) = \frac{1}{|x|^2} \left(\mathrm{Id} - 2\frac{x \otimes x}{|x|^2} \right). \tag{3.2}$$

We abbreviate $f^a := I_a \circ f$ as well as $f_i := \partial_i f$, $f_i^a := \partial_i f^a$ for $i = 1, 2$ and compute

$$g_{ij}^a = \frac{1}{|f - a|^4} \left(f_i - 2(f - a)\frac{(f - a) \cdot f_i}{|f - a|^2} \right) \cdot \left(f_j - 2(f - a)\frac{(f - a) \cdot f_j}{|f - a|^2} \right) = \frac{1}{|f - a|^4} g_{ij}.$$

Hence,
$$\mathrm{d}\mu^a = \frac{1}{|f-a|^4}\mathrm{d}\mu, \qquad |f_1^a \times f_2^a| = |f_1 \times f_2|\frac{1}{|f-a|^4}.$$

We compute further

$$
\begin{aligned}
f_1^a \times f_2^a &= \frac{1}{|f-a|^4}\left(f_1 - 2(f-a)\frac{(f-a)\cdot f_1}{|f-a|^2}\right) \times \left(f_2 - 2(f-a)\frac{(f-a)\cdot f_2}{|f-a|^2}\right) \\
&= \frac{1}{|f-a|^4}\left(f_1 \times f_2 - f_1 \times 2(f-a)\frac{(f-a)\cdot f_2}{|f-a|^2} - 2(f-a) \times f_2\frac{(f-a)\cdot f_1}{|f-a|^2}\right)
\end{aligned}
$$

and thus

$$(f_1^a \times f_2^a)\cdot(f-a) = \frac{1}{|f-a|^4}(f_1 \times f_2)\cdot(f-a)$$

and

$$n^a \cdot f^a\,\mathrm{d}\mu^a = \frac{f_1^a \times f_2^a}{|f_1^a \times f_2^a|}\cdot\frac{f-a}{|f-a|^2}\frac{1}{|f-a|^4}\,\mathrm{d}\mu = \frac{f_1 \times f_2}{|f_1 \times f_2|}\cdot\frac{f-a}{|f-a|^6}\,\mathrm{d}\mu = \frac{n\cdot(f-a)}{|f-a|^6}\,\mathrm{d}\mu.$$

Using (3.2), one readily verifies $\det \mathrm{D}I(x) = -1/|x|^4$. This means inversion changes the orientation. Summarising, we have

$$\mathrm{area}(I_a \circ f) = \int_\Sigma \frac{1}{|f-a|^4}\,\mathrm{d}\mu, \qquad \mathrm{vol}(I_a \circ f) = -\frac{1}{3}\int_\Sigma \frac{n\cdot(f-a)}{|f-a|^6}\,\mathrm{d}\mu. \tag{3.3}$$

Next, we first notice that for $|a|$ large enough,

$$\frac{|a|}{2} \le |f-a| \le 2|a|.$$

Moreover, by the fundamental theorem of calculus and by boundedness of f, we have for $p \ge 1$

$$||a|^p - |f-a|^p| \le C(p,f)|a|^{p-1}$$

and hence,

$$\left|\frac{1}{|f-a|^p} - \frac{1}{|a|^p}\right| = \left|\frac{|a|^p - |f-a|^p}{|a|^p|f-a|^p}\right| = O\left(\frac{1}{|a|^{p+1}}\right) \qquad \text{as } |a| \to \infty. \tag{3.4}$$

One computes

$$\partial_i \frac{x_i - a_i}{|x-a|^6} = \frac{1}{|x-a|^6} + (x_i - a_i)\partial_i\frac{1}{|x-a|^6} = \frac{1}{|x-a|^6} - 6\frac{(x_i - a_i)^2}{|x-a|^8}$$

and thus

$$\mathrm{div}\,\frac{x-a}{|x-a|^6} = \frac{-3}{|x-a|^6}.$$

Let $\Omega \subset \mathbb{R}^3$ be the open and bounded set such that $\partial\Omega = \Sigma$. In other words, $\mathrm{vol}(f) = \mathcal{L}^3(\Omega)$ where $\mathcal{L}^3$ denotes the 3-dimensional Lebesgue measure. By the Gauss–Green theorem (see for instance [45]) and (3.3), we compute

$$\mathrm{vol}(f^a) = -\frac{1}{3}\int_\Omega \mathrm{div}\,\frac{x-a}{|x-a|^6}\,\mathrm{d}\mathcal{L}^3 x = \int_\Omega \frac{1}{|x-a|^6}\,\mathrm{d}\mathcal{L}^3 x. \tag{3.5}$$

Thus, combining (3.3), (3.5), and (3.4), we infer

$$\text{area}(I_a \circ f) = \frac{\text{area}(f)}{|a|^4} + O\left(\frac{1}{|a|^5}\right) \qquad \text{as } |a| \to \infty,$$

$$\text{vol}(I_a \circ f) = \frac{\text{vol}(f)}{|a|^6} + O\left(\frac{1}{|a|^7}\right) \qquad \text{as } |a| \to \infty.$$

We can now deduce

$$\text{iso}(I_a \circ f) = \frac{\text{area}(f) + O\left(\frac{1}{|a|}\right)}{\left(\text{vol}(f) + O\left(\frac{1}{|a|}\right)\right)^{2/3}} \frac{\frac{1}{|a|^4}}{\left(\frac{1}{|a|^6}\right)^{2/3}} \xrightarrow{|a| \uparrow \infty} \text{iso}(f)$$

which concludes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The following lemma can be found in [143, Theorem 3.1]. For completeness, we will include its proof here.

3.8 Lemma. *Suppose Σ is a closed smooth surface, $f : \Sigma \to \mathbb{R}^3$ is a smooth embedding, $p \in \Sigma$, $n : \Sigma \to \mathbb{S}^2$ is the Gauss map, and*

$$\gamma : [0, \infty) \to \mathbb{R}^3, \qquad \gamma(t) = p + tn(p).$$

Then, there holds

$$\lim_{t \to 0+} \text{iso}(I_{\gamma(t)} \circ f) = \text{iso}(\mathbb{S}^2).$$

Proof. Recall that by the isoperimetric inequality, $\text{iso}(I_{\gamma(t) \circ f}) \geq \text{iso}(\mathbb{S}^2)$ for all $t > 0$. Hence, it will be enough to estimate the area of $I_{\gamma(t)} \circ f$ from above and the volume from below. After a rigid motion, we may assume that $p = 0$ and the surface is given by a local graph representation

$$f : D_R := \{z \in \mathbb{R}^2 : |z| < R\} \to \mathbb{R}^3, \qquad f(z) = (z, u(z))$$

for some $R > 0$ and some smooth function u with

$$u(0) = 0, \qquad \mathrm{D}u(0) = 0. \tag{3.6}$$

In particular, we have that $n(0) = (0, 0, 1)$ and, by (3.3),

$$\text{area}(I_{\gamma(t)} \circ f|_{D_R}) = \int_{D_R} \frac{\sqrt{1 + |\mathrm{D}u(z)|^2}}{(|z|^2 + (u(z) - t)^2)^2} \, \mathrm{d}z. \tag{3.7}$$

Let $\frac{1}{2} < \alpha < 1$ and define $t^* := t^\alpha$. From (3.6) it follows $|u(z)| \leq C|z|^2$ for some $C > 0$ and thus, on the disk D_{t^*} for small t:

$$|u(z)^2 - 2u(z)t| \leq C|z|^2(|z|^2 + 2t) \leq C|z|^2(t^* + 2t) \leq Ct^*|z|^2,$$
$$(u(z) - t)^2 \geq t^2 - Ct^*|z|^2.$$

Therefore,

$$\frac{|z|^2 + t^2}{|z|^2 + (u(z) - t)^2} \leq 1 + \frac{Ct^*|z|^2}{|z|^2 + t^2 - Ct^*|z|^2} \leq 1 + \frac{Ct^*}{1 - Ct^*} = 1 + o(\sqrt{t}) \qquad (3.8)$$

as $t \to 0$. Moreover, by (3.6),

$$\sqrt{1 + |\mathrm{D}u(z)|^2} = 1 + o(1) \qquad \text{as } t \to 0. \qquad (3.9)$$

Now, by (3.8) and (3.9) it follows

$$\begin{aligned}
\int_{D_{t^*}} \frac{\sqrt{1 + |\mathrm{D}u(z)|^2}}{(|z|^2 + (u(z) - t)^2)^2} \, \mathrm{d}z &\leq (1 + o(1)) \int_{D_{t^*}} \frac{1}{(|z|^2 + t^2)^2} \, \mathrm{d}z \\
&= (1 + o(1))2\pi \int_0^{t^*} \frac{x}{(x^2 + t^2)^2} \, \mathrm{d}x = (1 + o(1))2\pi \frac{1}{2} \left[\frac{1}{t^2} - \frac{1}{t^2 + t^{*2}} \right] \\
&= (1 + o(1)) \frac{\pi}{t^2}
\end{aligned} \qquad (3.10)$$

as $t \to 0$. On the other hand, since $|\mathrm{D}u|$ is bounded,

$$\int_{D_R \setminus D_{t^*}} \frac{\sqrt{1 + |\mathrm{D}u(z)|^2}}{(|z|^2 + (u(z) - t)^2)^2} \, \mathrm{d}z \leq C \int_{D_R \setminus D_{t^*}} \frac{1}{|z|^4} \, \mathrm{d}z = C \int_{t^*}^R \frac{\mathrm{d}x}{x^3} = C + O\left(\frac{1}{t^{*2}}\right) \qquad (3.11)$$

as $t \to 0$. Putting (3.10) and (3.11) into (3.7), we infer

$$\mathrm{area}(I_{\gamma(t)} \circ f) \leq C + O\left(\frac{1}{t^{*2}}\right) + (1 + o(1)) \frac{\pi}{t^2} \qquad \text{as } t \to 0. \qquad (3.12)$$

Next, we will estimate the volume from below. Let S be a round sphere of radius $r_0 > 0$, that is $S = r_0 \mathbb{S}^2 + q$, where r_0 and q are chosen such that S lies inside of $f[\Sigma]$ and S is tangential to $f[\Sigma]$ at p. By monotonicity of the volume and Proposition 3.4, there holds

$$\mathrm{vol}(I_{\gamma(t)} \circ f) \geq \mathrm{vol}(I_{\gamma(t)}[S]) = \mathrm{vol}(r(t)\mathbb{S}^2) = \frac{4\pi}{3} r(t)^3,$$

where the radius $r(t)$ of the inverted sphere $I_{\gamma(t)}[S]$ can be computed as

$$r(t) = \frac{1}{2} \left| \frac{1}{t} - \frac{1}{t + 2r_0} \right| = \frac{1}{2t}(1 + O(t)) \qquad \text{as } t \to 0.$$

Hence,

$$\mathrm{vol}(I_{\gamma(t)} \circ f) \geq \frac{\pi}{6} \frac{1}{t^3}(1 + O(t)) \qquad \text{as } t \to 0. \qquad (3.13)$$

Now, by (3.12) and (3.13), it follows

$$\sqrt[3]{36\pi} \leq \mathrm{iso}(I_{\gamma(t)} \circ f) \leq \frac{(1 + o(1))\frac{\pi}{t^2}}{\left(\frac{\pi}{6} \frac{1}{t^3}(1 + o(1))\right)^{2/3}} + o(1) = \sqrt[3]{36\pi}(1 + o(1)) \qquad \text{as } t \to 0$$

which concludes the proof. $\qquad \square$

3.9 Proposition. *Suppose Σ is a closed smooth surface, and $f : \Sigma \to \mathbb{R}^3$ is a smooth embedding.*

Then, there exists a smooth curve $\gamma : (0, \infty) \to \mathbb{R}^3$ such that $(\mathrm{im}\,\gamma) \cap (\mathrm{im}\,f) = \varnothing$, the isoperimetric ratio $\mathrm{iso}(I_{\gamma(t)} \circ f)$ varies smoothly in t,

$$\lim_{t \to 0+} \mathrm{dist}(\gamma(t), \mathrm{im}\,f) = 0, \qquad \lim_{t \to \infty} |\gamma(t)| = \infty,$$

and

$$\lim_{t \to 0+} \mathrm{iso}(I_{\gamma(t)} \circ f) = \mathrm{iso}(\mathbb{S}^2), \qquad \lim_{t \to \infty} \mathrm{iso}(I_{\gamma(t)} \circ f) = \mathrm{iso}(f).$$

Moreover, $\mathrm{iso}[\mathcal{S}_0] = [\mathrm{iso}(\mathbb{S}^2), \infty)$ and $\mathrm{iso}[\mathcal{S}_g] = (\mathrm{iso}(\mathbb{S}^2), \infty)$ for $g \geq 1$.

Proof. This is a consequence of Lemma 3.7 and Lemma 3.8. $\qquad\qquad\square$

3.10 Lemma (See MONDINO–SCHARRER [97, Lemma 2.2]). *Suppose $f : \Sigma \to \mathbb{R}^3$ is a smoothly immersed closed surface which is not a round sphere and $q \in \Sigma$ is any given point. Then, there exists $q \neq p \in \Sigma$ with the following property.*

For each neighbourhood U of p there exists a smooth normal vector field $\xi : \Sigma \to \mathbb{R}^3$ compactly supported in U such that for $f_t := f + t\xi$ with $t \in \mathbb{R}$, the function $t \mapsto \mathrm{iso}(f_t)$ is differentiable at $t = 0$, and

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{iso}(f_t) \neq 0.$$

Moreover,

$$\mathcal{W}(f_t) = \mathcal{W}(f) + O(t) \qquad \text{as } t \to 0.$$

Proof. First of all, recall that for any smooth vector field $\xi : \Sigma \to \mathbb{R}^3$, the family $f_t = f + t\xi$ defines a variation of the immersion f. In particular, for small $t \in \mathbb{R}$, the map $f_t : \Sigma \to \mathbb{R}^3$ is again a smooth immersion. Thus area, volume, and Willmore energy are defined for f_t with t small.

By a classical theorem of ALEXANDROV [4], since $f : \Sigma \to \mathbb{R}^3$ is not a round sphere, the mean curvature H cannot be constant. Therefore, we can choose a point p in the non-empty boundary of the set

$$\{x \in \Sigma : H(x) = \max \mathrm{im}\,H\},$$

where $\mathrm{im}\,H$ is the image of the mean curvature H. In fact, given any $c \in \mathrm{im}\,H$, we might as well have chosen p in the non-empty boundary of the level set $\{H = c\}$. In particular, we can make sure that $p \neq q$. Now, given any neighbourhood U of p, we pick a smooth function $\varphi : \Sigma \to \mathbb{R}$ compactly supported in U such that $\varphi \geq 0$ and $\varphi(p) = 1$. Let $n : \Sigma \to \mathbb{S}^2$ be the Gauss map and define the constant h and the vector field ξ by

$$h = \int_\Sigma \varphi H \, d\mu \Big/ \int_\Sigma \varphi \, d\mu \quad \text{and} \quad \xi = \varphi(H - h)n.$$

Then, $\xi : \Sigma \to \mathbb{R}^3$ is a smooth vector field compactly supported in U. Using the first variation formula of the volume, we compute for $f_t = f + t\xi$ that

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{vol}(f_t) = -\int_\Sigma n \cdot \xi \, d\mu = -\int_\Sigma \varphi(H - h) \, d\mu = 0. \tag{3.14}$$

Moreover, using the first variation formula of the area, it follows

$$\frac{d}{dt}\bigg|_{t=0} \text{area}(f_t) = -\int_\Sigma Hn \cdot \xi \, d\mu$$
$$= -\int_\Sigma \varphi H(H-h) \, d\mu = -\int_\Sigma \varphi(H-h)^2 \, d\mu < 0. \tag{3.15}$$

The last expression is non-zero due to our choice of the point p and the function φ. Using (3.14) and (3.15), we infer that

$$\frac{d}{dt}\bigg|_{t=0} \text{iso}(f_t) = -\int_\Sigma \varphi(H-h)^2 \, d\mu \bigg/ \text{vol}(f)^{\frac{2}{3}} < 0.$$

Finally, using the first variation formula for the Willmore energy, we see that the function $t \mapsto \mathcal{W}(f_t)$ is differentiable at $t=0$ which implies the conclusion. $\qquad\square$

3.3 Isoperimetric balance of the connected sum

In this subsection we recall the connected sum construction developed by BAUER–KUWERT [6] and estimate its change of isoperimetric ratio (see Lemma 3.12).

Let $f_i : \Sigma_i \to \mathbb{R}^3$ for $i=1,2$ be two smoothly immersed closed surfaces neither of which is a round sphere such that

$$f_i^{-1}\{0\} = \{p_i\}, \quad \text{for some } p_i \in \Sigma_i, \qquad \text{im } Df_i(p_i) = \mathbb{R}^2 \times \{0\}. \tag{3.16}$$

For some $\rho > 0$, one can then pick smooth local graph representations

$$f_1(z) = (z, u(z)), \quad f_2(z) = (z, v(z)) \qquad \text{for } z \in D_\rho,$$

where D_ρ is the open disk $\{z \in \mathbb{R}^2 : |z| < \rho\}$. Letting $P, Q : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be the second fundamental forms at the origin of f_1 and f_2, respectively, we define the error terms ϕ and ψ such that

$$u(z) = p(z) + \varphi(z), \quad \text{where } p(z) = \frac{1}{2}P(z,z) \qquad \text{for } z \in D_\rho,$$
$$v(z) = q(z) + \psi(z), \quad \text{where } q(z) = \frac{1}{2}Q(z,z) \qquad \text{for } z \in D_\rho.$$

We denote the trace-free parts of the second fundamental forms with

$$P^\circ(w,z) = P(w,z) - \frac{(\text{tr } P)}{2}w \cdot z, \qquad Q^\circ(w,z) = Q(w,z) - \frac{(\text{tr } Q)}{2}w \cdot z.$$

In view of [6, Lemma 4.5], we may assume that in addition to (3.16), there also holds

$$\langle P^\circ, Q^\circ \rangle > 0. \tag{3.17}$$

By [6, Lemma 2.3], the inverted and translated surface

$$f_1^\circ : \Sigma_1 \setminus \{p_1\} \to \mathbb{R}^3, \qquad f_1^\circ(p) = \frac{f_1(p)}{|f_1(p)|^2} - \frac{(\text{tr } P)}{4}e_3,$$

where $e_3 = (0, 0, 1)$ is the third unit vector in $\mathbb{R}^3$, has a graph representation at infinity. That is, outside of a large ball around zero, f_1° is given by the graph of a smooth function u° on $\mathbb{R}^2 \setminus D_R$ for some $R > 0$ with

$$u^\circ(z) = p^\circ(z) + \varphi^\circ(z), \quad \text{where } p^\circ(z) = \frac{1}{2} P^\circ \left(\frac{z}{|z|}, \frac{z}{|z|} \right) \tag{3.18}$$

such that the error term satisfies

$$|z||\varphi^\circ(z)| + |z|^2 |D\varphi^\circ(z)| + |z|^3 |D^2\varphi^\circ(z)| \leq C \qquad \text{for } z \in \mathbb{R}^2 \setminus D_R. \tag{3.19}$$

Given any function $w : \Omega \to \mathbb{R}$ for $\Omega \subset \mathbb{R}^2$ and given any scalar $\lambda > 0$, we define the scaled function w_λ by

$$w_\lambda : \Omega_\lambda = \{ z \in \mathbb{R}^2 : \lambda^{-1} z \in \Omega \} \to \mathbb{R}, \qquad w_\lambda(z) = \lambda w(\lambda^{-1} z).$$

Hence, for small $\alpha, \beta > 0$, the graph representations of the scaled surfaces αf_1° and $(1/\beta) f_2$ are given by

$$u_\alpha^\circ(z) = p_\alpha^\circ(z) + \varphi_\alpha^\circ(z) \qquad\qquad \text{for } z \in \mathbb{R}^2 \setminus D_{\alpha R},$$
$$v_{1/\beta}(z) = q_{1/\beta}(z) + \psi_{1/\beta}(z) \qquad\qquad \text{for } z \in D_{\rho/\beta}.$$

Next, pick a smooth function $\eta : \mathbb{R} \to \mathbb{R}$ such that

$$\eta(t) = \begin{cases} 0 & t \leq (1/4)\sqrt{\alpha} \\ 1 & t \geq (3/4)\sqrt{\alpha} \end{cases}$$

and such that $|\eta| + \sqrt{\alpha}|\eta'| + \alpha|\eta''| \leq C$ for some $0 < C < \infty$ independent of α. Then, for a third parameter γ with $0 < \alpha, \beta \ll \gamma \ll 1$, define for $r = |z|$,

$$w(z) = \begin{cases} p_\alpha^\circ(z) + \eta(\gamma - r)\varphi_\alpha^\circ(z) & \alpha R < r \leq \gamma \\ q_{1/\beta}(z) + \eta(r - 1)\psi_{1/\beta}(z) & 1 \leq r < \rho/\beta \end{cases}$$

and notice that $w = u_\alpha^\circ$ for $r \leq \gamma - (3/4)\sqrt{\alpha}$ as well as $w = v_{1/\beta}$ for $r \geq 1 + (3/4)\sqrt{\alpha}$. Moreover, on $D_1 \setminus D_\gamma$, let w be the unique solution of the bi-harmonic Dirichlet–Neumann problem (see Lemma 3.1 and Lemma 3.2 in [6])

$$\begin{cases} \Delta^2 w = 0 & \text{in } D_1 \setminus D_\gamma, \\ w = p_\alpha^\circ, \quad \partial_r w = \partial_r p_\alpha^\circ & \text{on } |z| = \gamma, \\ w = q_{1/\beta}, \quad \partial_r w = \partial_r q_{1/\beta} & \text{on } |z| = 1. \end{cases} \tag{3.20}$$

To define the pasted surface, let U be the complement in Σ_1 of the preimage of the set $\{z \in \mathbb{R}^2 : \gamma - \sqrt{\alpha} < |z| < \infty\}$ under the map $\alpha \cdot \pi \circ f_1^\circ$, where $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ denotes the orthogonal projection. Analogously, let V be the complement in Σ_2 of the preimage of the set $\{z \in \mathbb{R}^2 : |z| < 1 + \sqrt{\alpha}\}$ under the map $(1/\beta) \cdot \pi \circ f_2$. Moreover, let $W = \{z \in \mathbb{R}^2 : \gamma - \sqrt{\alpha} \leq |z| \leq 1 + \sqrt{\alpha}\}$. Then, we

can write the connected sum $\Sigma = \Sigma_1 \# \Sigma_2$ as $\Sigma = (U \cup V \cup W)/\sim$, where the identification $\sim$ is given by

$$
\begin{aligned}
p \sim z &= \alpha\pi(f_1^\circ(p)) & & \text{for } p \in U,\, z \in W, \\
q \sim z &= (1/\beta)\pi(f_2(q)) & & \text{for } q \in V,\, z \in W.
\end{aligned}
$$

Now, the immersion of the patched surface can be defined by

$$
f : \Sigma \to \mathbb{R}^3, \qquad f(x) = \begin{cases} \alpha f_1^\circ(p) & x = p \in U \subset \Sigma_1, \\ (1/\beta)f_2(q) & x = q \in V \subset \Sigma_2, \\ (z, w(z)) & x = z \in W. \end{cases} \tag{3.21}
$$

The connected sum satisfies the following energy saving proven by BAUER–KUWERT [6].

3.11 Lemma (See BAUER–KUWERT [6, Lemma 4.4]). *Taking $\beta = t\alpha$ for any $t > 0$, and letting α tend to zero, there holds*

$$
\begin{aligned}
&\mathcal{W}(f) - (\mathcal{W}(f_1) + \mathcal{W}(f_2) - 4\pi) \\
&\quad = \pi\alpha^2\Big(|P^\circ|^2 - t\langle P^\circ, Q^\circ\rangle + O_t(\gamma^2\log(\gamma)^2) + O_{t,\gamma}(\alpha^{1/2})\Big),
\end{aligned} \tag{3.22}
$$

where the constants in O_t and $O_{t,\gamma}$ depend on t, respectively t and γ.

We will show that the isoperimetric ratio of the connected sum behaves as follows.

3.12 Lemma (See MONDINO–SCHARRER [97, Lemma 3.2]). *Taking $\beta = t\alpha$ for any $t > 0$, and letting α tend to zero, there holds*

$$
\mathrm{iso}(f) = \mathrm{iso}(f_2) + O_{t,\gamma}(\alpha^{2+\frac{1}{2}}), \tag{3.23}
$$

where the constant in $O_{t,\gamma}$ depends on t and γ.

Proof. First, we will compute the area of the surface $f : \Sigma \to \mathbb{R}^3$. By definition of the connected sum in Equation (3.21), we can split the area into

$$
\mathrm{area}(f) = \mathrm{area}(f|_U) + \mathrm{area}(f|_W) + \mathrm{area}(f|_V). \tag{3.24}
$$

Let

$$
U_1 = \Sigma_1 \setminus (\pi \circ f_1^\circ)^{-1}\{z \in \mathbb{R}^2 : R < |z| < \infty\}
$$

where again, π denotes the orthogonal projection of $\mathbb{R}^3$ onto $\mathbb{R}^2$. Then, we can write

$$
\mathrm{area}(f|_U) = \alpha^2 \, \mathrm{area}(f_1^\circ|_{U_1}) + \int_{D_{\gamma-\sqrt{\alpha}} \setminus D_{\alpha R}} \sqrt{1 + |\mathrm{D}u_\alpha^\circ|^2}\, \mathrm{d}\mathcal{L}^2, \tag{3.25}
$$

where $\mathcal{L}^2$ denotes the 2-dimensional Lebesgue measure. For p° defined as in Equation (3.18), we have

$$
\mathrm{D}p^\circ(z) = P^\circ\left(\frac{z}{|z|}, \mathrm{D}\left(\frac{z}{|z|}\right)\right), \qquad \mathrm{D}\left(\frac{z}{|z|}\right) = \frac{\mathrm{Id}}{|z|} - \frac{\langle z, \cdot\rangle}{|z|^3}z.
$$

Hence, $|p^\circ(z)| + |z||\mathrm{D}p^\circ(z)| \le C$ for $z \in \mathbb{R}^2 \setminus D_R$ and after scaling,

$$|p_\alpha^\circ(z)| + |z||\mathrm{D}p_\alpha^\circ(z)| \le C\alpha \qquad \text{for } z \in \mathbb{R}^2 \setminus D_{\alpha R}.$$

Moreover, from the error estimation in Equation (3.19),

$$|z||\varphi_\alpha^\circ(z)| + |z|^2|\mathrm{D}\varphi_\alpha^\circ(z)| \le C\alpha^2 \qquad \text{for } z \in \mathbb{R}^2 \setminus D_{\alpha R}.$$

Using $u_\alpha^\circ = p_\alpha^\circ + \varphi_\alpha^\circ$ we thus infer

$$|u_\alpha^\circ(z)| + |\mathrm{D}u_\alpha^\circ(z)| \le C \qquad\qquad\qquad \text{for } z \in D_{\sqrt{\alpha}R} \setminus D_{\alpha R}, \qquad (3.26)$$

$$|u_\alpha^\circ(z)| + |\mathrm{D}u_\alpha^\circ(z)| \le C\sqrt{\alpha} \qquad\qquad \text{for } z \in \mathbb{R}^2 \setminus D_{\sqrt{\alpha}R}. \qquad (3.27)$$

Therefore, the area in Equation (3.25) can be estimated by

$$\text{area}(f|_U) \le C\alpha^2 + C\mathcal{L}^2(D_{R\sqrt{\alpha}} \setminus D_{R\alpha}) + (1 + C\sqrt{\alpha})\mathcal{L}^2(D_\gamma) = \mathcal{L}^2(D_\gamma) + O(\sqrt{\alpha})$$

as $\alpha \to 0$. On the other hand,

$$\text{area}(f|_U) \ge \mathcal{L}^2(D_{\gamma - \sqrt{\alpha}}) = \mathcal{L}^2(D_\gamma) - O(\sqrt{\alpha}) \qquad \text{as } \alpha \to 0$$

and thus

$$\text{area}(f|_U) = \mathcal{L}^2(D_\gamma) + O(\sqrt{\alpha}) \qquad \text{as } \alpha \to 0. \qquad (3.28)$$

From [6, Equation (4.13)] it follows that

$$|v_{1/\beta}(z)| + |\mathrm{D}v_{1/\beta}(z)| \le C(t)\alpha \qquad \text{for } z \in D_{1+\sqrt{\alpha}} \qquad (3.29)$$

and hence,

$$\text{area}((1/\beta)f_2) - \text{area}(f|_V) = \int_{D_{1+\sqrt{\alpha}}} \sqrt{1 + |\mathrm{D}v_{1/\beta}|^2}\,\mathrm{d}\mathcal{L}^2 = \mathcal{L}^2(D_1) + O_t(\sqrt{\alpha}) \qquad (3.30)$$

as $\alpha \to 0$. Because of the homogeneity of p° and q (notice that $p_\alpha^\circ = \alpha p^\circ$, $q_{1/\beta} = \beta q$), the parameters α and β enter linearly into the boundary values of w on $D_1 \setminus D_\gamma$ and thus linearly into the solution (3.20) (see (3.29), (3.30), and (3.35) in [6]). Therefore, using $\beta = t\alpha$ as well as (4.21), (4.22), and (4.25) in [6], we infer

$$|w(z)| + |\mathrm{D}w(z)| \le C(t,\gamma)\alpha \qquad \text{for } z \in D_{1+\sqrt{\alpha}} \setminus D_{\gamma - \sqrt{\alpha}}. \qquad (3.31)$$

It follows

$$\text{area}(f|_W) = \int_{D_{1+\sqrt{\alpha}} \setminus D_{\gamma - \sqrt{\alpha}}} \sqrt{1 + |\mathrm{D}w|^2}\,\mathrm{d}\mathcal{L}^2 = \mathcal{L}^2(D_1 \setminus D_\gamma) + O_{t,\gamma}(\sqrt{\alpha}) \qquad (3.32)$$

as $\alpha \to 0$. Putting (3.28), (3.30), and (3.32) into (3.24), leads to

$$\text{area}(f) = \text{area}((1/\beta)f_2) + O_{t,\gamma}(\sqrt{\alpha}) \qquad \text{as } \alpha \to 0$$

and thus

$$\mathrm{area}(f) = (t\alpha)^{-2}\big(\mathrm{area}(f_2) + O_{t,\gamma}(\alpha^{2+\frac{1}{2}})\big) \qquad \text{as } \alpha \to 0. \tag{3.33}$$

Next, we will estimate the volume of the patched surface $f : \Sigma \to \mathbb{R}^3$. Using the definition of the volume (3.1), as well as the formula for the Gauss map of graphical surfaces, we estimate

$$|\mathrm{vol}(f) - \mathrm{vol}((1/\beta)f_2)| \le \alpha^3 \mathrm{vol}(f_1^\circ|_{U_1}) + \int_{D_{\gamma-\sqrt{\alpha}} \setminus D_{\alpha R}} |z| |\mathrm{D}u_\alpha^\circ| + |u_\alpha^\circ| \, \mathrm{d}\mathcal{L}^2(z)$$
$$+ \int_{D_{1+\sqrt{\alpha}} \setminus D_{\gamma-\sqrt{\alpha}}} |z| |\mathrm{D}w| + |w| \, \mathrm{d}\mathcal{L}^2(z) + \int_{D_{1+\sqrt{\alpha}}} |z| |\mathrm{D}v_{1/\beta}| + |v_{1/\beta}| \, \mathrm{d}\mathcal{L}^2(z).$$

In view of (3.26), (3.27), (3.29), and (3.31), we can see that the right hand side is uniformly bounded in α for $0 < \alpha \ll \gamma \ll 1$ and $\beta = t\alpha$. That means,

$$\mathrm{vol}(f) = \mathrm{vol}((1/\beta)f_2) + O_{t,\gamma}(1) \qquad \text{as } \alpha \to 0$$

and therefore,

$$\mathrm{vol}(f) = (t\alpha)^{-3}\big(\mathrm{vol}(f_2) + O_{t,\gamma}(\alpha^3)\big) \qquad \text{as } \alpha \to 0. \tag{3.34}$$

Notice that by differentiability of the function $s \mapsto (\mathrm{vol}(f_2) + s)^{-2/3}$ at $s = 0$, there holds

$$\frac{1}{\mathrm{vol}(f_2)^{\frac{2}{3}}} = \frac{1}{(\mathrm{vol}(f_2) + s)^{\frac{2}{3}}} + O(s) \qquad \text{as } s \to 0.$$

Thus, using $\beta = t\alpha$, (3.33), and (3.34), we infer

$$\mathrm{iso}(f) = \frac{\mathrm{area}(f_2)}{(\mathrm{vol}(f_2) + O_{t,\gamma}(\alpha^3))^{\frac{2}{3}}} + \frac{O_{t,\gamma}(\alpha^{2+\frac{1}{2}})}{(\mathrm{vol}(f_2) + O_{t,\gamma}(\alpha^3))^{\frac{2}{3}}} = \mathrm{iso}(f_2) + O_{t,\gamma}(\alpha^{2+\frac{1}{2}})$$

as $\alpha \to 0$, which finishes the proof. $\qquad\square$

3.4 Isoperimetric constrained minimisation and the strict inequality

In this subsection we prove the main theorem and its corollaries.

3.13 Theorem (See MONDINO–SCHARRER [97, Theorem 1.4]). *Suppose Σ_1, Σ_2 are two closed surfaces, $f_1 : \Sigma_1 \to \mathbb{R}^3$ is a smooth embedding, $f_2 : \Sigma_2 \to \mathbb{R}^3$ is a smooth immersion, and neither f_1 nor f_2 parametrise a round sphere. Denote with Σ the connected sum $\Sigma_1 \# \Sigma_2$. Then there exists a smooth immersion $f : \Sigma \to \mathbb{R}^3$ such that*

$$\mathrm{iso}(f) = \mathrm{iso}(f_2) \tag{3.35}$$

and

$$\mathcal{W}(f) < \mathcal{W}(f_1) + \mathcal{W}(f_2) - 4\pi. \tag{3.36}$$

Moreover, if also f_2 is an embedding, then f is an embedding as well.

Proof. Let Σ_1, Σ_2 be two closed surfaces, $f_1 : \Sigma_1 \to \mathbb{R}^3$ be a smooth embedding, and $f_2 : \Sigma_2 \to \mathbb{R}^3$ be a smooth immersion such that neither f_1 nor f_2 parametrise a round sphere. Notice that the

multiplicity of f_2 does not affect the construction of the connected sum. First, pick $p_i \in \Sigma_i$ for $i = 1, 2$ according to (3.16) and (3.17), with $p_2 \in f_2^{-1}\{0\}$ instead of $\{p_2\} = f_2^{-1}\{0\}$.

Apply Lemma 3.10 to the surface $f_2 : \Sigma_2 \to \mathbb{R}^3$: denote $f_{2,s} = f_2 + s\xi$ with ξ compactly supported away from the point p_2 and

$$\mathcal{W}(f_{2,s}) = \mathcal{W}(f_2) + O(s) \qquad\qquad \text{as } s \to 0, \qquad\qquad (3.37)$$

$$\mathrm{iso}(f_{2,s}) = \mathrm{iso}(f_2) - c_2 s + o(s) \qquad\qquad \text{as } s \to 0, \qquad\qquad (3.38)$$

for some $c_2 > 0$. Now, apply the connected sum construction described in this section to the surfaces $f_1 : \Sigma_1 \to \mathbb{R}^3$ and $f_{2,s} : \Sigma_2 \to \mathbb{R}^3$; in this way we obtain the glued surface $f_{s,\alpha} : \Sigma \to \mathbb{R}^3$, where Σ is the connected sum of Σ_1 and Σ_2. Notice that the right hand side in Equation (3.22) does not depend on s as the vector field ξ is compactly supported away from the patching area. Therefore, we can first choose $t > 0$ large enough such that $|P^\circ|^2 - t\langle P^\circ, Q^\circ \rangle < 0$ and then choose $0 < \gamma < 1$ small enough such that still $|P^\circ|^2 - t\langle P^\circ, Q^\circ \rangle + O_t(\gamma^2 \log(\gamma)^2) < 0$ to obtain from Lemma 3.11 that

$$\mathcal{W}(f_{s,\alpha}) - (\mathcal{W}(f_1) + \mathcal{W}(f_{2,s}) - 4\pi) = -c\alpha^2 + O(\alpha^{2+\frac{1}{2}}) \qquad\qquad (3.39)$$

as $\alpha \to 0$ for some $c > 0$. Putting (3.37) into (3.39) and (3.38) into (3.23), we infer

$$\mathcal{W}(f_{s,\alpha}) - (\mathcal{W}(f_1) + \mathcal{W}(f_2) - 4\pi) = -c\alpha^2 + O(\alpha^{2+\frac{1}{2}}) + O(s) \qquad\qquad (3.40)$$

$$\mathrm{iso}(f_{s,\alpha}) - \mathrm{iso}(f_2) = O(\alpha^{2+\frac{1}{2}}) - c_2 s + o(s) \qquad\qquad (3.41)$$

as $s, \alpha \to 0$. Picking any $2 < m < 2 + \frac{1}{2}$, we see that for small $\alpha > 0$ and for $|s| \leq \alpha^m$, the right hand side in (3.40) is strictly negative, while for $s = \alpha^m$ the right hand side in (3.41) is strictly negative and for $s = -\alpha^m$ the right hand side in (3.41) is strictly positive. Notice that once α is fixed, $\mathrm{iso}(f_{s,\alpha})$ depends continuously on s. Therefore, there exists $\alpha > 0$ small and $-\alpha^m < s < \alpha^m$ such that the right hand side in (3.40) is strictly negative, while the right hand side in (3.41) is zero. In other words, $f_{s,\alpha}$ satisfies (3.35) and (3.36). Notice that the immersion $f_{s,\alpha}$ is smooth everywhere except on the boundary of $D_1 \setminus D_\gamma$, where the bi-harmonic function meets the second fundamental forms with Dirichlet and Neumann conditions, see (3.20). In general, $f_{s,\alpha}$ is only $C^{1,1}$-regular. It remains to show that one can approximate $f_{s,\alpha}$ by a smooth immersion without loosing the conditions (3.35) and (3.36). In view of its construction, we can choose a local graph representation of $f_{s,\alpha}$ given by a function u defined on an open subset of $\mathbb{R}^2$ that contains the boundary of $D_1 \setminus D_\gamma$. By multiplying with a cut-off function, one can write $u = u_\mathrm{s} + u_\mathrm{r}$ such that u_s is smooth and u_r is $C^{1,1}$-regular as well as compactly supported. The standard mollification u_r^ε of u_r is smooth, compactly supported, and converges to u_r as $\varepsilon \to 0$ in the Sobolev space $W^{2,p}$ for all $1 \leq p < \infty$. The immersions f^ε corresponding to $u^\varepsilon := u_\mathrm{s} + u_\mathrm{r}^\varepsilon$ are smooth and differ from $f_{s,\alpha}$ only on a small neighbourhood of the boundary of $D_1 \setminus D_\gamma$. Moreover, there holds

$$|\mathcal{W}(f^\varepsilon) - \mathcal{W}(f_{s,\alpha})| + |\mathrm{iso}(f^\varepsilon) - \mathrm{iso}(f_{s,\alpha})| \to 0 \qquad \text{as } \varepsilon \to 0.$$

Hence there exists $\eta > 0$ such that, for $\varepsilon > 0$ small enough, f^ε satisfies the following quantified

version of (3.36):

$$\mathcal{W}(f^\varepsilon) < \mathcal{W}(f_1) + \mathcal{W}(f_2) - 4\pi - \eta.$$

Finally, we once again apply Lemma 3.10 away from the support of u_r^ε to re-establish (3.35) still keeping the validity of (3.36). $\qquad\square$

3.14 Corollary (See MONDINO–SCHARRER [97, Corollary 1.5])**.** *Given any integer $g \geq 1$, there holds*

$$\beta_g(\sigma) < \boldsymbol{\beta}_g + \beta_0(\sigma) - 4\pi \qquad \text{for all } \sigma > \sqrt[3]{36\pi}.$$

Proof. By [136, 67, 6], $\boldsymbol{\beta}_g$ is attained by a smoothly embedded surface. Moreover, by [129], also $\beta_0(\sigma)$ is attained by a smoothly embedded surface. Thus, the corollary is an application of Theorem 3.13. $\qquad\square$

3.15 Theorem (See MONDINO–SCHARRER [97, Corollary 1.6])**.** *Let g be a non-negative integer, and fix $\sigma > \sqrt[3]{36\pi}$. Assume that*

$$\beta_g(\sigma) = \inf\{\mathcal{W}(f) : f \in \mathcal{S}_g, \text{iso}(f) = \sigma\} < 8\pi. \tag{3.42}$$

Then $\beta_g(\sigma)$ is attained by a smoothly embedded minimiser $f_0 \in \mathcal{S}_g$, i.e. f_0 satisfies (1.29). Moreover, the function $\beta_g(\cdot)$ is non-decreasing on the whole interval $\text{iso}[\mathcal{S}_g]$ and continuous at all σ that satisfy (3.42).

Proof. The genus zero case was treated by SCHYGULLA [129]. Notice that by the proof of the Willmore conjecture by MARQUES–NEVES [85], there holds

$$\boldsymbol{\beta}_p \geq 2\pi^2 \qquad \text{for all } p \geq 1.$$

Moreover, $2(2\pi^2 - 4\pi) = 2\pi(2\pi - 4) \geq 4\pi$. Hence, the constant

$$\boldsymbol{\omega}_g = \min\left\{ 4\pi + \sum_{i=1}^{N}(\boldsymbol{\beta}_{g_i} - 4\pi) : g = g_1 + \ldots + g_N, 1 \leq g_i < g \right\}.$$

satisfies

$$\boldsymbol{\omega}_g \geq 8\pi. \tag{3.43}$$

Existence of smooth minimisers follows from Theorem 1.2 and Corollary 1.3 of KELLER–MONDINO–RIVIÈRE [62] in combination with Corollary 3.14, assumption (3.42), and (3.43).

Monotonicity is a combination of Proposition 3.9 with Theorem 3.6 and Theorem 3.2. Knowing that $\beta_g(\cdot)$ is non-decreasing, we can deduce its left continuity at σ by compactness of the isoperimetric constrained Willmore functional proven in [62, Section 4] together with lower semi-continuity of the Willmore functional. Right continuity at σ follows from Lemma 3.10. $\qquad\square$

4 Delaunay tori and their Willmore energy

The aim of this section is to prove the following theorems from the introduction: Theorem 5 (see Subsection 4.5) and Theorem 6 (see Subsection 4.7). The Dalunay tori that are subject in Theorem 5 are constructed out of constant mean curvature surfaces of revolution. These are introduced in the subsections 4.2–4.4. The computations are done using elliptic integrals introduced in Subsection 4.1. The content of this section corresponds to my work [123].

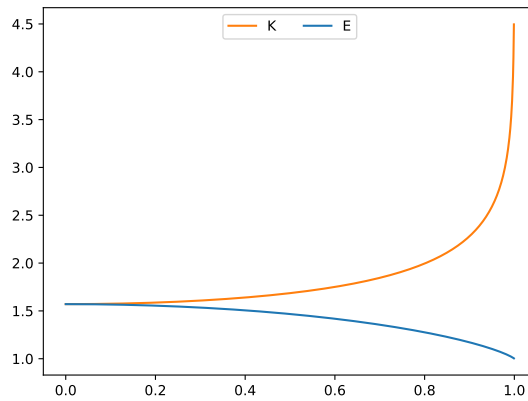## 4.1 Elliptic integrals

Figure 4.1: Complete elliptic integrals of the first and the second kind.

Elliptic integrals are functions defined as the value of common types of integrals that cannot be expressed in terms of simple functions. They arise when computing geometric quantities such as the arc length of an ellipse or a hyperbola. In particular, they naturally occur in the context of constant mean curvature (Delaunay) surfaces of revolution. This is because the rotating curves of Delaunay surfaces are given by the roulette generated by ellipses and hyperbolas. In fact, all the quantities that are needed to construct the family of embedded Delaunay tori (see Subsection 4.5) as well as their Willmore energy can be expressed in terms of *complete elliptic integrals*. Given a so-called *elliptic modulus* k, that is a real number $0 < k < 1$, the *complete elliptic integral of the first kind* K and the *complete elliptic integral of the second kind* E are defined by

$$K(k) = \int_0^{\pi/2} \frac{\mathrm{d}\theta}{\sqrt{1 - k^2 \sin^2(\theta)}}, \qquad E(k) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2(\theta)}\, \mathrm{d}\theta.$$

All the formulas for elliptic integrals used in this thesis can be found in the book of BYRD–FRIEDMAN [20]. The derivatives are given by

$$\frac{\mathrm{d}K(k)}{\mathrm{d}k} = \frac{E(k)}{k(1 - k^2)} - \frac{K(k)}{k}, \qquad \frac{\mathrm{d}E(k)}{\mathrm{d}k} = \frac{E(k) - K(k)}{k}.$$

The *Gauss transformation* works as follows. Define the *complementary modulus* k' and the transformed modulus k_1 by

$$k' = \sqrt{1 - k^2}, \qquad k_1 = \frac{1 - k'}{1 + k'}.$$

Then, there holds (see [20, (164.02)])

$$K(k) = (1 + k_1)K(k_1), \qquad E(k) = (1 + k')E(k_1) - k'(1 + k_1)K(k_1). \tag{4.1}$$

Moreover, K grows like $\log(1/k')$, namely

$$\lim_{k \to 1-} \left(K(k) - \log(4/\sqrt{1 - k^2}) \right) = 0 \tag{4.2}$$

and E is bounded:

$$1 \le E \le \pi/2. \tag{4.3}$$

4.2 Surfaces of revolution

A surface of revolution in $\mathbb{R}^3$ is given by a parametrisation X of the type

$$X(t, \theta) = (f(t)\cos(\theta), f(t)\sin(\theta), g(t))$$

with parameters t lying in an open interval and $0 \le \theta \le 2\pi$, where f, g are real valued functions. The rotating curve $c := (f, g)$ is referred to as *meridian* or *profile curve*. The underlying geometry is described by the coefficients of the first fundamental form

$$E = X_t \cdot X_t = \dot{f}^2 + \dot{g}^2 = |\dot{c}|^2, \qquad F = X_t \cdot X_\theta = 0, \qquad G = X_\theta \cdot X_\theta = f^2$$

and the second fundamental form

$$L = X_{tt} \cdot n = \frac{\dot{f}\ddot{g} - \ddot{f}\dot{g}}{|\dot{c}|}, \qquad M = X_{t\theta} \cdot n = 0, \qquad N = X_{\theta\theta} \cdot n = \frac{f\dot{g}}{|\dot{c}|}$$

where the Gauss map n is given by

$$n = \frac{X_t \times X_\theta}{|X_t \times X_\theta|}.$$

The mean curvature H is defined as the arithmetic mean of the principal curvatures κ_1, κ_2, that is

$$2H = \kappa_1 + \kappa_2 = \frac{L}{E} + \frac{N}{G} = \frac{\dot{f}\ddot{g} - \ddot{f}\dot{g}}{|\dot{c}|^3} + \frac{\dot{g}}{f|\dot{c}|}.$$

We will focus on surfaces of revolution with constant mean curvature

$$H = \frac{1}{2a} \tag{4.4}$$

for some given $0 \ne a \in \mathbb{R}$. These surfaces arise as critical points of the volume constrained area functional. Outside of a discrete set, one has $\dot{g} \ne 0$ and thus $c(\varphi(t)) = (\rho(t), t)$ for some parameter transform φ and some real valued function ρ. Hence, outside of a discrete set, Equation (4.4) can be turned into a second order ODE. Its solutions were first described by DELAUNAY [34] and are now named after him. More precisely, solutions for $a > 0$ are called *unduloids* and will be discussed in Section 4.3; solutions for $a < 0$ are called *nodoids* and will be discussed in Section 4.4.
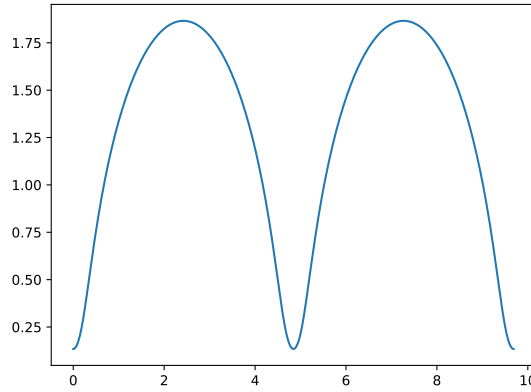
4.3 Unduloids

Figure 4.2: Profile curve of an unduloid with 2 periods, $a = 1$, $b = 0.5$ and the bottom line being the axis of rotation.

Unduloids are surfaces of revolution with constant, strictly positive mean curvature. Their rotating curve (f, g) is given by the roulette of an ellipse with generating point being one of the foci. To be more precise, let $a > b > 0$ and define $c = \sqrt{a^2 - b^2}$. Then, the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

describes a standard ellipse centred at the origin with width $2a$, height $2b$, and foci $\pm c$ on the x-axis. Rolling the ellipse without slipping along a line, each of the two focus points will describe a periodic curve, *the roulette*. Let (f, g) be the parametrisation of one of the roulettes such that the period is 2π. BENDITO–BOWICK–MEDINA [8] found the following representation:

$$f(t) = b\frac{a - c\cos(t)}{\sqrt{a^2 - c^2\cos^2(t)}} \tag{4.5}$$

$$g(t) = \int_0^t \sqrt{a^2 - c^2\cos^2(x)}\,\mathrm{d}x - c\sin(t)\frac{a - c\cos(t)}{\sqrt{a^2 - c^2\cos^2(t)}} \tag{4.6}$$

with coefficients of the first fundamental form

$$E = \frac{a^2 b^2}{(a + c\cos(t))^2}, \qquad G = b^2\frac{a - c\cos(t)}{a + c\cos(t)},$$

and mean curvature

$$H = \frac{1}{2a}. \tag{4.7}$$

The extrema are given by

$$\min\operatorname{im} f = f(0) = a - c, \qquad \max\operatorname{im} f = f(\pi) = a + c. \tag{4.8}$$

Next, a formula for the area will be determined. Using the Weierstraß substitution, the area

$\mathcal{A}$ of the rotational symmetric surface corresponding to one period can be computed by

$$\int_0^{2\pi}\int_0^{2\pi}\sqrt{EG}\,\mathrm{d}\theta\,\mathrm{d}t = 2\pi ab^2\int_0^{2\pi}\sqrt{\frac{a-c\cos(t)}{(a+c\cos(t))^3}}\,\mathrm{d}t = 4\pi ab^2\int_0^{\pi}\sqrt{\frac{a+c\cos(t)}{(a-c\cos(t))^3}}\,\mathrm{d}t$$

$$= 4\pi ab^2\int_0^{\infty}\sqrt{\frac{a+c\frac{1-x^2}{1+x^2}}{(a-c\frac{1-x^2}{1+x^2})^3}}\frac{2\mathrm{d}x}{1+x^2} = 8\pi ab^2\int_0^{\infty}\sqrt{\frac{a(1+x^2)+c(1-x^2)}{(a(1+x^2)-c(1-x^2))^3}}\,\mathrm{d}x$$

$$= 8\pi ab^2\int_0^{\infty}\sqrt{\frac{(a+c)+(a-c)x^2}{((a-c)+(a+c)x^2)^3}}\,\mathrm{d}x = 8\pi ab^2\sqrt{\frac{a-c}{(a+c)^3}}\int_0^{\infty}\sqrt{\frac{\tilde{a}^2+t^2}{(\tilde{b}^2+t^2)^3}}\,\mathrm{d}t$$

for $\tilde{a}^2 = (a+c)/(a-c)$ and $\tilde{b}^2 = (a-c)/(a+c)$. The last integral can be transformed into a complete elliptic integral of the second kind using [20, (221.01)] with $k = 1 - \tilde{b}^2/\tilde{a}^2$ and $g = 1/\tilde{a}$:

$$\int_0^{\infty}\sqrt{\frac{\tilde{a}^2+t^2}{(\tilde{b}^2+t^2)^3}}\,\mathrm{d}t = \frac{g}{k'^2}E(k).$$

One can show that

$$k^2 = \frac{4ac}{(a+c)^2}, \qquad k' = \frac{a-c}{a+c}, \qquad \frac{g}{k'^2} = \frac{(a+c)^{3/2}}{(a-c)^{3/2}}$$

and thus

$$\mathcal{A} = 8\pi a(a+c)E(k); \qquad k = \frac{2\sqrt{ac}}{a+c}. \tag{4.9}$$

Notice that this coincides with the area formula for unduloids computed for a different parametrisation in [49] and [92].

Finally, we compute the extrinsic length L of one period. It is given by

$$L = |g(2\pi) - g(0)| = a\int_0^{2\pi}\sqrt{1 - \frac{c^2}{a^2}\cos^2(x)}\,\mathrm{d}x = 4a\int_0^{\pi/2}\sqrt{\left(1 - \frac{c^2}{a^2}\right) + \frac{c^2}{a^2}\sin^2(x)}\,\mathrm{d}x$$

$$= 4a\sqrt{1 - \frac{c^2}{a^2}}\int_0^{\pi/2}\sqrt{1 + n^2\sin^2(x)}\,\mathrm{d}x$$

for $n = c/b$. Letting $k^2 = n^2/(1+n^2)$, (282.03) and (315.02) in [20] imply

$$\int_0^{\pi/2}\sqrt{1 + n^2\sin^2(x)}\,\mathrm{d}x = \frac{1}{k'}E(k).$$

Thus, since

$$k = \frac{c}{a}, \qquad k' = \sqrt{1 - \frac{c^2}{a^2}},$$

it follows that

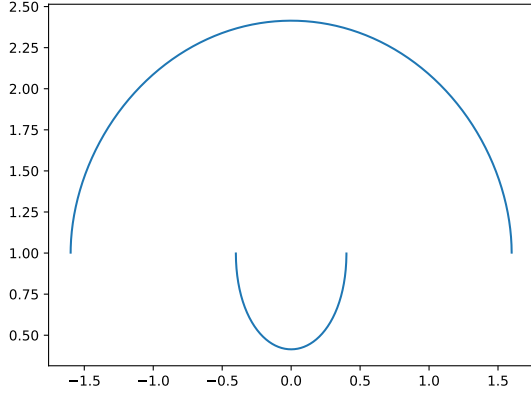$$L = 4aE(k); \qquad k = \frac{c}{a}. \tag{4.10}$$

4.4 Nodoids

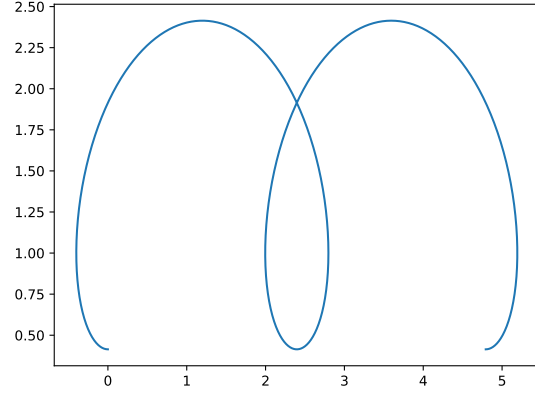Figure 4.3: Separate roulettes $(f_\pm, g_\pm)$ (bottom/top) for $a = b = 1$.

Figure 4.4: Both roulettes patched together for 2 periods and $a = b = 1$.

Nodoids are surfaces of revolution with constant, strictly negative mean curvature. Their rotating curve (f, g) is given by the roulette of a hyperbola with generating points given by the foci. To be more precise, let $a, b > 0$ and define $c = \sqrt{a^2 + b^2}$. Then, the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

describes a hyperbola in canonical form with distance a to the centre and foci $\pm c$ on the x-axis. Rolling the right branch of the hyperbola without slipping along a line, each of the two focus points will describe a curve, *the roulette*. BENDITO–BOWICK–MEDINA [8] found parametrisations $(f_\pm, g_\pm)$ of the roulettes, where (f_+, g_+) corresponds to the focus $(c, 0)$ and (f_-, g_-) is the reflected roulette corresponding to the focus $(-c, 0)$:

$$f_\pm(t) = b \frac{c \cosh(t) \mp a}{\sqrt{c^2 \cosh^2(t) - a^2}} \tag{4.11}$$

$$g_\pm(t) = \int_0^t \sqrt{c^2 \cosh^2(x) - a^2} \, \mathrm{d}x - c \sinh(t) \frac{c \cosh(t) \mp a}{\sqrt{c^2 \cosh^2(t) - a^2}} \tag{4.12}$$

with parameter t running through all of $\mathbb{R}$, coefficients of the first fundamental form

$$E_\pm = \frac{a^2 b^2}{(c \cosh(t) \pm a)^2}, \qquad G_\pm = b^2 \frac{c \cosh(t) \mp a}{c \cosh(t) \pm a},$$

and mean curvature

$$H_\pm = -\frac{1}{2a}. \tag{4.13}$$

One has

$$\max \operatorname{im} f_+ = \lim_{t \to \pm\infty} f_+(t) = b, \qquad \min \operatorname{im} f_+ = f_+(0) = c - a$$

$$\min \operatorname{im} f_- = \lim_{t \to \pm\infty} f_-(t) = b \qquad \max \operatorname{im} f_- = f_-(0) = c + a \tag{4.14}$$

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and thus, after translation along the axis of rotation, the two roulettes corresponding to the foci $(\pm c, 0)$ can be glued together into one periodic curve (see Figure 4.4).

Next, the area $\mathcal{A}$ of the rotational symmetric surface corresponding to one period will be computed. Using the parameter transformations $t = \operatorname{artanh}(x)$ and $x = \sin(t)$, one infers

$$\int_{-\infty}^{\infty} \sqrt{E_+ G_+}\, dt = ab^2 \int_{-\infty}^{\infty} \sqrt{\frac{c\cosh(t) - a}{(c\cosh(t) + a)^3}}\, dt = 2ab^2 \int_0^1 \sqrt{\frac{c\frac{1}{\sqrt{1-x^2}} - a}{(c\frac{1}{\sqrt{1-x^2}} + a)^3}}\, \frac{dx}{1 - x^2}$$

$$= 2ab^2 \int_0^1 \sqrt{\frac{c - a\sqrt{1-x^2}}{(c + a\sqrt{1-x^2})^3}}\, \frac{dx}{\sqrt{1 - x^2}} = 2ab^2 \int_0^{\pi/2} \sqrt{\frac{c - a\cos(t)}{(c + a\cos(t))^3}}\, dt$$

$$= 2ab^2 \int_{\pi/2}^{\pi} \sqrt{\frac{c + a\cos(t)}{(c - a\cos(t))^3}}\, dt$$

and similarly,

$$\int_{-\infty}^{\infty} \sqrt{E_- G_-}\, dt = 2ab^2 \int_0^{\pi/2} \sqrt{\frac{c + a\cos(t)}{(c - a\cos(t))^3}}\, dt.$$

Consequently,

$$\mathcal{A} = \int_0^{2\pi} \int_{-\infty}^{\infty} \sqrt{E_+ G_+} + \sqrt{E_- G_-}\, dt\, d\theta = 4\pi ab^2 \int_0^{\pi} \sqrt{\frac{c + a\cos(t)}{(c - a\cos(t))^3}}\, dt.$$

In Subsection 4.3 it was shown that

$$\int_0^{\pi} \sqrt{\frac{c + a\cos(t)}{(c - a\cos(t))^3}}\, dt = \frac{2E(k)}{c - a}; \qquad k = \frac{2\sqrt{ac}}{a + c}.$$

Thus, it follows that

$$\mathcal{A} = 8\pi a(a + c)E(k); \qquad k = \frac{2\sqrt{ac}}{a + c}. \tag{4.15}$$

Next, in order to determine the extrinsic length L of one period, a parameter transformation will be carried out. First, for $\alpha = a^2/c^2$, $u = \tan t$, and $s = \operatorname{arsinh}(u)$, one computes

$$f_{\pm}(s) = b\frac{c\sqrt{1 + u^2} \mp a}{\sqrt{c^2(1 + u^2) - a^2}} = b\frac{c \mp a\cos(t)}{\sqrt{c^2 - a^2\cos^2(t)}}, \tag{4.16}$$

for $-\pi/2 < t < \pi/2$, and

$$\int_0^s \sqrt{\cosh^2(x) - \alpha}\, dx = \int_0^u \frac{\sqrt{1 + x^2 - \alpha}}{\sqrt{1 + x^2}}\, dx = \int_0^u \sqrt{1 - \frac{\alpha}{1 + x^2}}\, dx$$

$$= \int_0^t \frac{\sqrt{1 - \alpha\cos^2(x)}}{\cos^2(x)}\, dx = -\int_0^t \frac{\alpha\cos(x)\sin(x)}{\sqrt{1 - \alpha\cos^2(x)}}\tan(x)\, dx + \sqrt{1 - \alpha\cos^2(t)}\tan(t)$$

$$= -\sqrt{\alpha} \int_0^t \frac{\sin^2(x)\, dx}{\sqrt{\sin^2(x) + b^2/a^2}} + \tan(t)\sqrt{1 - \alpha\cos^2(t)},$$

as well as

$$\tan(t)\left[\sqrt{1 - \frac{a^2}{c^2}\cos^2(t)} - \frac{c \mp a\cos(t)}{\sqrt{c^2 - a^2\cos^2(t)}}\right] = \frac{a}{c}\sin(t)\frac{\pm c - a\cos(t)}{\sqrt{c^2 - a^2\cos^2(t)}}.$$

It follows

$$g_\pm(s) = -a\int_0^t \frac{\sin^2(x)\,dx}{\sqrt{\sin^2(x) + b^2/a^2}} \pm a\sin(t)\sqrt{\frac{c \mp a\cos(t)}{c \pm a\cos(t)}} \tag{4.17}$$

for $-\pi/2 < t < \pi/2$. Notice that, up to translation along the axis of rotation, both curves $\mathbb{R} \to \mathbb{R}^2$ with $s \mapsto (f_+(s), g_+(s))$ and $s \mapsto (f_-(s), g_-(s))$ as given by (4.16), (4.17) parametrise the whole periodic curve resulting from the patched roulettes in Figure 4.4.

Finally, we compute the extrinsic length L of one period. It is given by

$$L = |g_+(s(\pi/2)) - g_+(s(-\pi/2))| + |g_-(s(\pi/2)) - g_-(s(-\pi/2))|$$
$$= 4an\int_0^{\pi/2} \frac{\sin^2(x)\,dx}{\sqrt{1 + n^2\sin^2(x)}}$$

for $n^2 = a^2/b^2$. Letting $k^2 = n^2/(1 + n^2)$, it follows

$$k = \frac{a}{c}, \qquad k' = \frac{b}{c}, \qquad ank'^3\frac{1}{k^2k'^2} = c.$$

Therefore, by (282.04) and (318.02) in [20],

$$L = 4c\left[E(k) - k'^2K(k)\right]; \qquad k = \frac{a}{c}. \tag{4.18}$$
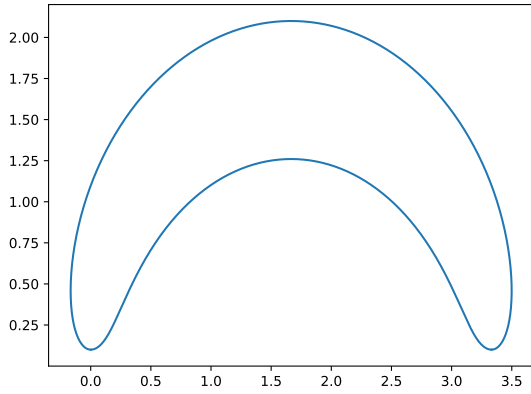
4.5 Embedded Delaunay tori

Figure 4.5: Profile curve Delaunay torus with $c = 1.1$ and the bottom line being the axis of rotation.
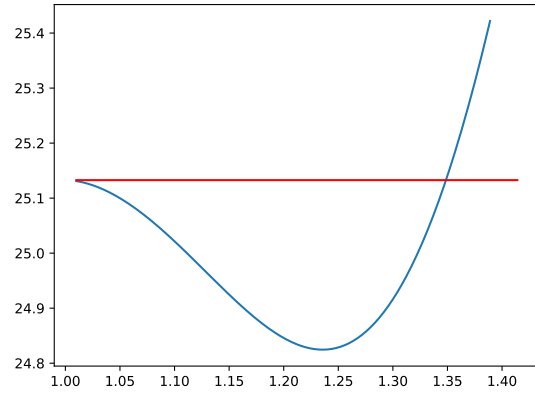
Figure 4.6: Energy curve for the family of Delaunay tori and 8π bound.

In this subsection, we will construct the family of embedded Delaunay tori $\mathbb{T}_{D,c}$ with $1 < c < c_0$ for some constant c_0.

4.1 Theorem (See SCHARRER [123, Theorem 1.1]). *There exist a real number $c_0 > 1$ and a family of $C^{1,1}$-regular tori $\mathbb{T}_{D,c}$ corresponding to $1 < c < c_0$ such that*

$$\mathcal{W}(\mathbb{T}_{D,c}) < 8\pi \qquad whenever\ 1 < c < c_0$$

and

$$\lim_{c \to 1+} \mathrm{iso}(\mathbb{T}_{D,c}) = \infty.$$

Proof. Each Delaunay torus $\mathbb{T}_{D,c}$ is a rotationally symmetric surface whose profile curve (see Figure 4.5) consists of one period of an unduloid roulette (see Figure 4.2) and one period of a nodoid roulette (see Figure 4.4). The construction works as follows. Start with a one parameter family of patched nodoids (see Section 4.4) running for one period and starting at the minimum according to (4.14), where $a = 1$ and $c > 1$ is the free parameter, thus b is given by $b = \sqrt{c^2 - 1}$ and the minimum according to (4.14) is $c-1$. Next, depending on the parameter c, find $a > y > 0$ such that the unduloid corresponding to the ellipse with foci $\pm y$, width $2a$, and height $2b$ for $b = \sqrt{a^2 - y^2}$ (see Section 4.3) running for one period and starting at its minimum $a-y$ according (4.8), fits right into the given nodoid. That means the two end points where the patched nodoids reach their minimum need to match the two end points where the unduloid reaches its minimum. Notice that, in this way, the profile curve is $C^{1,1}$ regular. The coordinates of the two patching points can be determined using the equations (4.8), (4.10) for the unduloid and (4.14), (4.18) for the nodoid. Thus, a, y are given as the solution of the system of equations

$$\begin{cases} 4aE\left(\frac{y}{a}\right) = 4c\left[E\left(\frac{1}{c}\right) - \left(1 - \frac{1}{c^2}\right)K\left(\frac{1}{c}\right)\right] & (4.19) \\[2mm] a - y = c - 1. & (4.20) \end{cases}$$

Abbreviating

$$\varepsilon := c - 1, \qquad L := c\left[E\left(\frac{1}{c}\right) - \varepsilon\frac{c+1}{c^2}K\left(\frac{1}{c}\right)\right],$$

the system of equation reads as

$$\begin{cases} (y + \varepsilon)E\left(\frac{y}{y+\varepsilon}\right) = L \\[2mm] a = y + \varepsilon. \end{cases}$$

Define the function

$$F : (1, \infty) \times (0, \infty) \to \mathbb{R}, \qquad F(c,y) = (y + \varepsilon(c))E\left(\frac{y}{y+\varepsilon(c)}\right) - L(c).$$

There holds

$$\varepsilon \log \frac{4}{\sqrt{1 - 1/c^2}} = \varepsilon \log \frac{4c}{\sqrt{\varepsilon}\sqrt{c+1}} \le \frac{4c}{\sqrt{c+1}}\sqrt{\varepsilon} \xrightarrow{c \downarrow 1} 0.$$

Hence, by (4.2),

$$\lim_{c \to 1+} L(c) = 1. \tag{4.21}$$

Moreover,

$$\partial_c L = [E(\tfrac{1}{c}) - (1 - \tfrac{1}{c^2})K(\tfrac{1}{c})] + c[E(\tfrac{1}{c}) - K(\tfrac{1}{c})]c\tfrac{-1}{c^2}$$
$$+ c\left[\tfrac{-2}{c^3}K(\tfrac{1}{c}) - (1 - \tfrac{1}{c^2})[E(\tfrac{1}{c})/(1 - \tfrac{1}{c^2}) - K(\tfrac{1}{c})]c(\tfrac{-1}{c^2})\right] = E(\tfrac{1}{c}) - K(\tfrac{1}{c})$$

and, for $k = y/(y + \varepsilon)$

$$\partial_c(F + L) = E(k) + (y + \varepsilon)[E(k) - K(k)]\frac{y + \varepsilon}{y}\frac{-y}{(y + \varepsilon)^2} = K(k)$$

which implies

$$\partial_c F = K\Big(\frac{y}{y + \varepsilon}\Big) + K\Big(\frac{1}{c}\Big) - E\Big(\frac{1}{c}\Big).$$

Writing $k = y/(y + \varepsilon)$, there holds

$$\partial_y F = E(k) + (y + \varepsilon)[E(k) - K(k)]\frac{y + \varepsilon}{y}\frac{\varepsilon}{(y + \varepsilon)^2} = (1 + \frac{\varepsilon}{y})E(k) - \frac{\varepsilon}{y}K(k).$$

Moreover, for fixed y,

$$\varepsilon \log \frac{4}{\sqrt{1 - y^2/(y + \varepsilon)^2}} = \varepsilon \log \frac{4(y + \varepsilon)}{\sqrt{2\varepsilon y + \varepsilon^2}} \le \frac{4(y + \varepsilon)}{\sqrt{2y + \varepsilon}}\sqrt{\varepsilon} \xrightarrow{c\downarrow 1} 0 \qquad (4.22)$$

and thus, by (4.2),

$$\lim_{c \to 1+} \partial_y F(c, y) = 1.$$

Hence, using (4.21) and (4.3), it follows that there exists $c_0 > 1$ such that for all $1 < c < c_0$ there exists a unique $y(c) > 0$ with $F(c, y(c)) = 0$ and derivative

$$y' = \frac{E(\tfrac{1}{c}) - K(\tfrac{1}{c}) - K(\tfrac{y}{y + \varepsilon})}{(y + \varepsilon)E(\tfrac{y}{y + \varepsilon}) - \varepsilon K(\tfrac{y}{y + \varepsilon})} y. \qquad (4.23)$$

Using (4.7), (4.9) for the unduloid and (4.13), (4.15) for the nodoid, one obtains the Willmore energy of the Delaunay tori $\mathbb{T}_{D,c}$:

$$\mathcal{W}(\mathbb{T}_{D,c}) = \mathcal{W}_{\mathrm{nod}} + \mathcal{W}_{\mathrm{und}} \qquad (4.24)$$

where

$$\mathcal{W}_{\mathrm{nod}} = 2\pi(1 + c)E\Big(\frac{2\sqrt{c}}{1 + c}\Big), \qquad \mathcal{W}_{\mathrm{und}} = 2\pi\Big(1 + \frac{y}{y + \varepsilon}\Big)E\Big(\frac{2\sqrt{y(y + \varepsilon)}}{2y + \varepsilon}\Big).$$

Using (4.3) and (4.21), one can see that $C \le y \le 1/C$ for some $C > 0$ which implies by (4.22) and (4.2) that

$$\lim_{c \to 1+} \frac{y}{y + \varepsilon} = 1, \qquad \lim_{c \to 1+} y = 1, \qquad \lim_{c \to 1+} \varepsilon K\Big(\frac{y}{y + \varepsilon}\Big) = 0, \qquad \lim_{c \to 1+} \mathcal{W}(\mathbb{T}_{D,c}) = 8\pi.$$

Next, we show that $\partial_c \mathcal{W}(\mathbb{T}_{D,c}) < 0$ for c close to 1 which then implies $\mathcal{W}(\mathbb{T}_{D,c}) < 8\pi$ for c

close to 1. First, we compute $\partial_c \mathcal{W}_{\mathrm{nod}}$. For this purpose, let $k = 2\sqrt{c}/(1+c)$. Then,

$$\partial_c k = \frac{1}{(1+c)\sqrt{c}} - \frac{2\sqrt{c}}{(1+c)^2} = \frac{1-c}{(1+c)^2\sqrt{c}}$$

and

$$\partial_c \mathcal{W}_{\mathrm{nod}} = 2\pi E(k) + 2\pi(1+c)[E(k) - K(k)]\frac{1+c}{2\sqrt{c}}\frac{1-c}{(1+c)^2\sqrt{c}}$$
$$= \pi\Big(\Big(1 + \frac{1}{c}\Big)E(k) + \Big(1 - \frac{1}{c}\Big)K(k)\Big).$$

By the Gauss transformation (4.1) there holds

$$k' = \sqrt{1 - \frac{4c}{(1+c)^2}} = \frac{c-1}{c+1}, \qquad k_1 = \frac{1-k'}{1+k'} = \frac{1}{c},$$

and

$$\frac{\partial_c \mathcal{W}_{\mathrm{nod}}}{\pi} = (1 + \tfrac{1}{c})\Big[(1 + \tfrac{c-1}{c+1})E(\tfrac{1}{c}) - \tfrac{c-1}{c+1}(1 + \tfrac{1}{c})K(\tfrac{1}{c})\Big] + (1 - \tfrac{1}{c})(1 + \tfrac{1}{c})K(\tfrac{1}{c}) = 2E(\tfrac{1}{c}).$$

Hence,

$$\partial_c \mathcal{W}_{\mathrm{nod}} = 2\pi E\Big(\frac{1}{c}\Big). \tag{4.25}$$

In order to compute $\partial_y \mathcal{W}_{\mathrm{und}}$, let $k = 2\sqrt{y(y+\varepsilon)}/(2y+\varepsilon)$. Then, there holds

$$\partial_y k = \frac{2y+\varepsilon}{(2y+\varepsilon)\sqrt{y(y+\varepsilon)}} - \frac{4\sqrt{y(y+\varepsilon)}}{(2y+\varepsilon)^2} = \frac{\varepsilon^2}{(2y+\varepsilon)^2\sqrt{y(y+\varepsilon)}}, \qquad \partial_y \frac{y}{y+\varepsilon} = \frac{\varepsilon}{(y+\varepsilon)^2}$$

and

$$\partial_y \mathcal{W}_{\mathrm{und}} = 2\pi\frac{\varepsilon}{(y+\varepsilon)^2}E(k) + 2\pi\frac{2y+\varepsilon}{y+\varepsilon}[E(k) - K(k)]\frac{2y+\varepsilon}{2\sqrt{y(y+\varepsilon)}}\frac{\varepsilon^2}{\sqrt{y(y+\varepsilon)}(2y+\varepsilon)^2}$$
$$= \frac{\pi\varepsilon}{(y+\varepsilon)^2}\Big(2E(k) + \frac{\varepsilon}{y}[E(k) - K(k)]\Big) = \frac{\pi\varepsilon}{y(y+\varepsilon)^2}\big((2y+\varepsilon)E(k) - \varepsilon K(k)\big).$$

By the Gauss transformation there holds

$$k' = \sqrt{1 - \frac{4y(y+\varepsilon)}{(2y+\varepsilon)^2}} = \frac{\varepsilon}{2y+\varepsilon}, \qquad k_1 = \frac{1-k'}{1+k'} = \frac{y}{y+\varepsilon}$$

and

$$(2y+\varepsilon)E(k) - \varepsilon K(k)$$
$$= (2y+\varepsilon)[(1 + \tfrac{\varepsilon}{2y+\varepsilon})E(\tfrac{y}{y+\varepsilon}) - \tfrac{\varepsilon}{2y+\varepsilon}(1 + \tfrac{y}{y+\varepsilon})K(\tfrac{y}{y+\varepsilon})] - \varepsilon(1 + \tfrac{y}{y+\varepsilon})K(\tfrac{y}{y+\varepsilon}) \tag{4.26}$$
$$= 2\big((y+\varepsilon)E(\tfrac{y}{y+\varepsilon}) - \varepsilon(1 + \tfrac{y}{y+\varepsilon})K(\tfrac{y}{y+\varepsilon})\big).$$

Therefore,

$$\partial_y \mathcal{W}_{\mathrm{und}} = \frac{2\pi\varepsilon}{y(y+\varepsilon)^2}\Big[(y+\varepsilon)E\Big(\frac{y}{y+\varepsilon}\Big) - \varepsilon\Big(1 + \frac{y}{y+\varepsilon}\Big)K\Big(\frac{y}{y+\varepsilon}\Big)\Big]. \tag{4.27}$$

Abbreviate $z = y'/y$, and $k = y/(y+\varepsilon)$. Then, (4.27) and (4.23) imply

$$
\begin{aligned}
\partial_y \mathcal{W}_{\text{und}} \cdot y' &= \tfrac{2\pi\varepsilon}{y(y+\varepsilon)^2} \left[(y+\varepsilon)E(k) - \varepsilon(1 + \tfrac{y}{y+\varepsilon})K(k)\right] \cdot \tfrac{E(\frac{1}{c}) - K(\frac{1}{c}) - K(k)}{(y+\varepsilon)E(k) - \varepsilon K(k)} y \\
&= \tfrac{2\pi\varepsilon}{(y+\varepsilon)^2} \left[E(\tfrac{1}{c}) - K(\tfrac{1}{c}) - K(\tfrac{y}{y+\varepsilon})\right] - \tfrac{2\pi\varepsilon}{(y+\varepsilon)^2} \tfrac{\varepsilon y}{y+\varepsilon} z K(\tfrac{y}{y+\varepsilon}).
\end{aligned}
\tag{4.28}
$$

Finally we compute $\partial_\varepsilon \mathcal{W}_{\text{und}}$. For this purpose let $k = 2\sqrt{y(y+\varepsilon)}/(2y+\varepsilon)$. Then, there holds

$$
\partial_\varepsilon k = \frac{y}{(2y+\varepsilon)\sqrt{y(y+\varepsilon)}} - \frac{2\sqrt{y(y+\varepsilon)}}{(2y+\varepsilon)^2} = \frac{-y\varepsilon}{(2y+\varepsilon)^2\sqrt{y(y+\varepsilon)}}
$$

and

$$
\begin{aligned}
\partial_\varepsilon \mathcal{W}_{\text{und}} &= 2\pi \tfrac{-y}{(y+\varepsilon)^2} E(k) + 2\pi \tfrac{2y+\varepsilon}{y+\varepsilon} [E(k) - K(k)] \tfrac{2y+\varepsilon}{2\sqrt{y(y+\varepsilon)}} \tfrac{-y\varepsilon}{(2y+\varepsilon)^2\sqrt{y(y+\varepsilon)}} \\
&= -\tfrac{\pi}{(y+\varepsilon)^2} \big((2y+\varepsilon)E(\tfrac{y}{y+\varepsilon}) - \varepsilon K(\tfrac{y}{y+\varepsilon})\big).
\end{aligned}
$$

Thus, by (4.26),

$$
\partial_\varepsilon \mathcal{W}_{\text{und}} = -\frac{2\pi}{(y+\varepsilon)^2}\left[(y+\varepsilon)E\Big(\frac{y}{y+\varepsilon}\Big) - \varepsilon\Big(1 + \frac{y}{y+\varepsilon}\Big)K\Big(\frac{y}{y+\varepsilon}\Big)\right].
$$

Recall that, by the choice of y, there holds

$$
(y+\varepsilon)E\Big(\frac{y}{y+\varepsilon}\Big) = c\left[E\Big(\frac{1}{c}\Big) - \Big(1 - \frac{1}{c^2}\Big)K\Big(\frac{1}{c}\Big)\right].
$$

Therefore,

$$
\partial_\varepsilon \mathcal{W}_{\text{und}} = -\frac{2\pi}{(y+\varepsilon)^2}\left[cE\Big(\frac{1}{c}\Big) - \varepsilon\frac{c+1}{c}K\Big(\frac{1}{c}\Big) - \varepsilon\Big(1 + \frac{y}{y+\varepsilon}\Big)K\Big(\frac{y}{y+\varepsilon}\Big)\right].
\tag{4.29}
$$

Putting (4.25), (4.28), and (4.29) into (4.24), it follows

$$
\begin{aligned}
\partial_c \mathcal{W}(\mathbb{T}_{\text{D},c}) &= \partial_c \mathcal{W}_{\text{nod}} + \partial_\varepsilon \mathcal{W}_{\text{und}} + \partial_y \mathcal{W}_{\text{und}} \cdot y' \\
&= 2\pi E(\tfrac{1}{c}) - \tfrac{2\pi}{(y+\varepsilon)^2}\left[cE\Big(\tfrac{1}{c}\Big) - \varepsilon\tfrac{c+1}{c}K\Big(\tfrac{1}{c}\Big) - \varepsilon\Big(1 + \tfrac{y}{y+\varepsilon}\Big)K\Big(\tfrac{y}{y+\varepsilon}\Big)\right] \\
&\quad + \tfrac{2\pi\varepsilon}{(y+\varepsilon)^2}\left[E(\tfrac{1}{c}) - K(\tfrac{1}{c}) - K(\tfrac{y}{y+\varepsilon})\right] - \tfrac{2\pi\varepsilon^2 y}{(y+\varepsilon)^3} z K(\tfrac{y}{y+\varepsilon}) \\
&= 2\pi E(\tfrac{1}{c})\left[1 - \tfrac{c}{(y+\varepsilon)^2} + \tfrac{\varepsilon}{(y+\varepsilon)^2}\right] + \tfrac{2\pi\varepsilon}{(y+\varepsilon)^2} K(\tfrac{1}{c})\left[\tfrac{c+1}{c} - 1\right] + \tfrac{2\pi\varepsilon y}{(y+\varepsilon)^3} K(\tfrac{y}{y+\varepsilon})[1 - \varepsilon z] \\
&= 2\pi(1 - \tfrac{1}{(y+\varepsilon)^2})E(\tfrac{1}{c}) + \tfrac{2\pi}{c(y+\varepsilon)^2}\varepsilon K(\tfrac{1}{c}) + \tfrac{2\pi y}{(y+\varepsilon)^3}(1 - \varepsilon z)\varepsilon K(\tfrac{y}{y+\varepsilon}).
\end{aligned}
$$

Therefore,

$$
\frac{\partial_c \mathcal{W}(\mathbb{T}_{\text{D},c})}{2\pi} = a_1\Big(1 - \frac{1}{(y+\varepsilon)^2}\Big) + a_2\varepsilon K\Big(\frac{1}{c}\Big) + a_3\varepsilon K\Big(\frac{y}{y+\varepsilon}\Big)
\tag{4.30}
$$

where $a_1, a_2, a_3 \to 1$ as $c \to 1$. In particular, $\partial_c \mathcal{W}(\mathbb{T}_{\text{D},c}) \to 0$ as $c \to 1$. We claim that

$$
\lim_{c \to 1+} \frac{\varepsilon K(\tfrac{1}{c}) + \varepsilon K(\tfrac{y}{y+\varepsilon})}{1 - \tfrac{1}{(y+\varepsilon)^2}} = -\frac{1}{2}.
\tag{4.31}
$$

First, recall that

$$\varepsilon K(\tfrac{1}{c}) \xrightarrow{c\downarrow 1} 0, \qquad \varepsilon K(\tfrac{y}{y+\varepsilon}) \xrightarrow{c\downarrow 1} 0, \qquad \varepsilon y' \xrightarrow{c\downarrow 1} 0, \qquad y' \xrightarrow{c\downarrow 1} -\infty, \qquad y \xrightarrow{c\downarrow 1} 1.$$

It follows

$$\tfrac{\varepsilon}{1+y'}\partial_c K(\tfrac{1}{c}) = \tfrac{\varepsilon}{1+y'}\left[\tfrac{E(\tfrac{1}{c})}{1-\tfrac{1}{c^2}} - K(\tfrac{1}{c})\right]c\tfrac{-1}{c^2} \xrightarrow{c\downarrow 1} 0,$$

$$\partial_c \tfrac{y}{y+\varepsilon} = y'\partial_y \tfrac{y}{y+\varepsilon} + \partial_\varepsilon \tfrac{y}{y+\varepsilon} = \tfrac{\varepsilon y' - y}{(y+\varepsilon)^2} \xrightarrow{c\downarrow 1} -1,$$

$$\tfrac{\varepsilon}{1+y'}\partial_c K(\tfrac{y}{y+\varepsilon}) = \tfrac{\varepsilon}{1+y'}\left[\tfrac{E(\tfrac{y}{y+\varepsilon})}{1-\tfrac{y^2}{(y+\varepsilon)^2}} - K(\tfrac{y}{y+\varepsilon})\right]\tfrac{y+\varepsilon}{y}\partial_c \tfrac{y}{y+\varepsilon} \xrightarrow{c\downarrow 1} 0.$$

Thus, by L'Hôspital's rule,

$$\lim_{c\to 1+} \frac{\varepsilon K(\tfrac{1}{c}) + \varepsilon K(\tfrac{y}{y+\varepsilon})}{1 - \tfrac{1}{(y+\varepsilon)^2}} = \lim_{c\to 1+} \frac{K(\tfrac{1}{c}) + K(\tfrac{y}{y+\varepsilon})}{\tfrac{2}{(y+\varepsilon)^3}[1+y']} = -\frac{1}{2}$$

which proves (4.31). By (4.31) and (4.30), one infers

$$\partial_c \mathcal{W}(\mathbb{T}_{\mathrm{D},c}) < 0 \qquad \text{for } c \text{ close to 1.}$$

Therefore, for some $\delta > 0$, there holds

$$\mathcal{W}(\mathbb{T}_{\mathrm{D},c}) < 8\pi \qquad \text{whenever } 1 < c < 1 + \delta. \tag{4.32}$$

The Delaunay tori $\mathbb{T}_{\mathrm{D},c}$ converge to a round sphere of multiplicity 2 and radius 2 as $c \to 1$ in varifold convergence. In particular,

$$\lim_{c\to 1+} \mathrm{iso}(\mathbb{T}_{\mathrm{D},c}) = \infty.$$

Together with (4.32), this proves Theorem 4.1. It remains to mention that, since the two periodic profile curves of nodoids and unduloids are patched together at their minimum, the resulting Delaunay torus is a $C^{1,1}$ regular closed genus-1 surface. $\square$

4.2 Corollary (See SCHARRER [123, Corollary 1.2])**.** *Let* $\sigma > \sqrt[3]{36\pi}$*. Then, there holds*

$$\beta_1(\sigma) := \inf\{\mathcal{W}(f) : f \in \mathcal{S}_1, \mathrm{iso}(f) = \sigma\} < 8\pi.$$

Proof. First of all, notice that by Theorem 4.1, the minimal isoperimetric constrained Willmore energy amongst $C^{1,1}$-regular tori lies strictly below 8π. We are now going to show that by a result of KELLER–MONDINO–RIVIÈRE [62], the same holds true for the minimal isoperimetric constrained Willmore energy smooth tori. Let $\mathcal{E}_{\mathbb{S}^2}$ be the space of Lipschitz immersions of $\mathbb{S}^2$ into $\mathbb{R}^3$ as defined in Section 2.2 of [62]. Similarly, let $\mathbb{T}^2$ be an abstract 2-dimensional torus and denote with $\mathcal{E}_{\mathbb{T}^2}$ the space of Lipschitz immersions of $\mathbb{T}^2$ into $\mathbb{R}^3$ (see [62, Section 2.2]). By SCHYGULLA [129] and [62, Theorem 1.1] the following holds true. For each $\sigma > \sqrt[3]{36\pi}$, there

exists a smoothly embedded spherical surface $\mathbb{S}_{\mathrm{S},\sigma}$ with

$$\mathcal{W}(\mathbb{S}_{\mathrm{S},\sigma}) = \beta_0(\sigma) := \inf\{\mathcal{W}(\vec{\Phi}) : \vec{\Phi} \in \mathcal{E}_{\mathbb{S}^2}, \mathrm{iso}(\vec{\Phi}) = \sigma\}.$$

Moreover, by Theorem 1.6 in [62], for each σ_0 in the set

$$I_1 := \left\{\sigma \in \mathbb{R} : \inf_{\substack{\vec{\Phi} \in \mathcal{E}_{\mathbb{T}^2} \\ \mathrm{iso}(\vec{\Phi})=\sigma}} \mathcal{W}(\vec{\Phi}) < \min\{8\pi, 2\pi^2 + \beta_0(\sigma) - 4\pi\}\right\} \subset (\sqrt[3]{36\pi}, \infty)$$

there exists a smoothly embedded torus Σ_0 in $\mathbb{R}^3$ with

$$\mathcal{W}(\Sigma_0) = \beta_1(\sigma_0) := \inf_{\substack{\vec{\Phi} \in \mathcal{E}_{\mathbb{T}^2} \\ \mathrm{iso}(\vec{\Phi})=\sigma_0}} \mathcal{W}(\vec{\Phi}). \tag{4.33}$$

From Corollary 3.14 it follows

$$I_1 = \{\sigma \in \mathbb{R} : \beta_1(\sigma) < 8\pi\}.$$

Recall that by Proposition 3.9 and the conformal invariance of the Willmore functional (see Theorem 3.2) the function $\beta_1(\cdot)$ is non-decreasing on the set I_1 (see also Theorem 3.15). Moreover, each $C^{1,1}$-regular torus embedded in $\mathbb{R}^3$ is a member of $\mathcal{E}_{\mathbb{T}^2}$. Thus, by Theorem 4.1,

$$I_1 = (\sqrt[3]{36\pi}, \infty).$$

Now, (4.33) concludes the proof. $\qquad\square$

4.3 Corollary (See SCHARRER [123, Corollary 1.3])**.** *Let* $\sigma > \sqrt[3]{36\pi}$*. Then,*

$$\beta_1(\sigma) := \inf\{\mathcal{W}(f) : f \in \mathcal{S}_1, \mathrm{iso}(f) = \sigma\}$$

is attained by a smoothly embedded minimiser $f_0 \in \mathcal{S}_1$*.*

Proof. This is a consequence of Theorem 3.15 in combination with Corollary 4.2. $\qquad\square$

4.6 Delaunay spheres of high isoperimetric ratio

4.4 Theorem. *There exist a real number* $c_0 > 1$ *and a family of* $C^{1,1}$*-regular spheres* $\mathbb{S}_{\mathrm{D},c}$ *corresponding to* $1 < c < c_0$ *such that*

$$\mathcal{W}(\mathbb{S}_{\mathrm{D},c}) = 4\pi + \frac{\mathcal{W}(\mathbb{T}_{\mathrm{D},c})}{2} \qquad \text{for all } 1 < c < c_0$$

as well as

$$\lim_{c \to 1+} \mathrm{iso}(\mathbb{S}_{\mathrm{D},c}) = \infty.$$

Proof. The Delaunay tori $\mathbb{T}_{\mathrm{D},c}$ corresponding to $1 < c < c_0$ can be used to construct spheres with analogous properties. The first part of the construction works just like the construction of
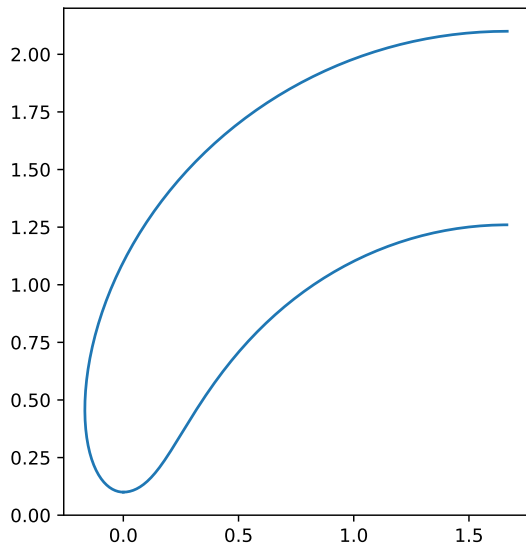
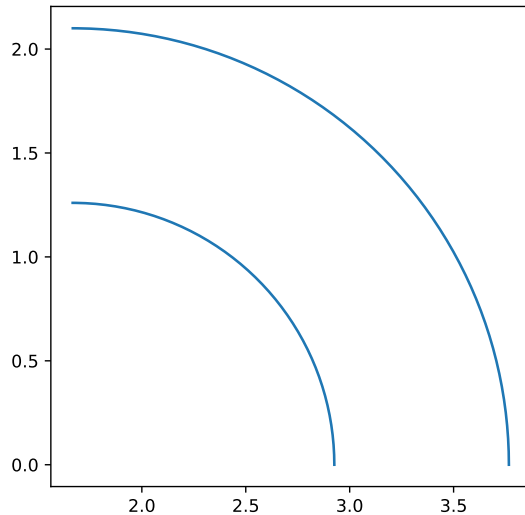Figure 4.7: Profile curve of half a Delaunay torus with $c = 1.1$

Figure 4.8: Concentric quarter circles fitting into half a Delaunay torus.

the Delaunay tori only that now, both the nodoid and the unduloid run only for half a period instead of one full period. To be more precise, both the nodoid and the unduloid now only run from their minimum according to (4.8), (4.14) until they reach their maximum (according to (4.8), (4.14)) but not until they reach their minimum again. This results in half a Delaunay torus, see Figure 4.7. Notice that unduloids and nodoids are symmetric around their maxima ($t = \pi$ in (4.5), (4.6) for unduloids; $t = 0$ in (4.11), (4.12) for nodoids). Thus, the Willmore energy of this particular half of a Delaunay torus is indeed half the Willmore energy of a whole Delaunay torus. Let c, y, a be the balancing parameters according to (4.19) and (4.20). Then the maxima of the nodoid and the unduloid are given by $c + 1$ and $a + y$, respectively. Next, take two concentric circular sectors with radii $c + 1$ and $a + y$ both of which being one quarter of a full circle, see Figure 4.8. Choose the centre of the two circular sectors at $L/2$ on the axis of rotation of the half Delaunay torus, where $L = 4aE(y/a)$ (see (4.10)). Then, the two circular sectors fit right into the half Delaunay torus, resulting in a $C^{1,1}$ curve. Since the two circular sectors meet the axis of rotation perpendicular, the resulting surface of revolution is $C^{1,1}$ regular too. It is of sphere type. The full profile can be seen in Figure 4.9. The resulting family of surfaces is called *Delaunay spheres*. Since $a + y, c + 1 \xrightarrow{c \downarrow 1} 2$, the Delaunay spheres converge as varifolds to a sphere of multiplicity 2 as $c \to 1$. Their Willmore energy is given by

$$\mathcal{W}(\mathbb{S}_{\mathrm{D},c}) = 4\pi + \frac{\mathcal{W}(\mathbb{T}_{\mathrm{D},c})}{2}$$

which concludes the proof. □

4.7 Higher genus Helfrich surfaces with small spontaneous curvature

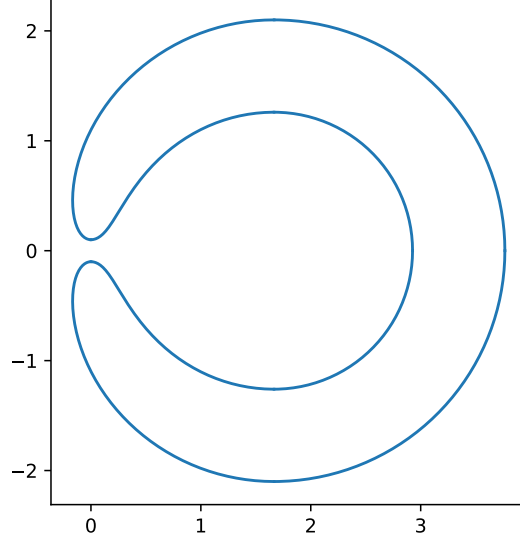Recall the definition of the Canham–Helfrich functional (2.17).

Figure 4.9: Profile of a Delaunay sphere with $c = 1.1$

4.5 Theorem (See SCHARRER [123, Theorem 1.4])**.** *Suppose g is an integer, and $A_0, V_0 > 0$ satisfy the isoperimetric inequality $A_0^3 > 36\pi V_0^2$. Then, there exists $\varepsilon := \varepsilon_g(A_0, V_0) > 0$ such that the following holds.*

For each $c_0 \in (-\varepsilon, \varepsilon)$ there exists a smoothly embedded surface $f_0 \in \mathcal{S}_g$ with

$$\text{area}(f_0) = A_0, \qquad \text{vol}(f_0) = V_0$$

and

$$\mathcal{H}^{c_0}(f_0) = \inf\{\mathcal{H}^{c_0}(f) : f \in \mathcal{S}_g, \text{area}(f) = A_0, \text{vol}(f) = V_0\}.$$

Proof. Define the set

$$\mathcal{S}_g(A_0, V_0) = \{f \in \mathcal{S}_g : \text{area}(f) = A_0, \text{vol}(f) = V_0\}.$$

Notice that if $c_0 = 0$, then the Canham–Helfrich functional reduces to the Willmore functional: $\mathcal{H}^0 = \mathcal{W}$. Moreover, by (2.61) of Lemma 2.10,

$$\left| \inf_{f \in \mathcal{S}_1(A_0, V_0)} \sqrt{\mathcal{W}(f)} - \inf_{f \in \mathcal{S}_1(A_0, V_0)} \sqrt{\mathcal{H}^{c_0}(f)} \right| \leq |c_0| \sqrt{A_0}.$$

In particular, the minimal Canham–Helfrich energy is continuous with respect to c_0 at $c_0 = 0$. For the case $c_0 = 0$, existence of smoothly embedded minimisers with given fixed area and volume corresponds to KELLER–MONDINO–RIVIÈRE [62, Corollary 1.3]. Theorem 4.5 states that minimisers remain embedded for c_0 close to zero. Moreover, by Lemma 2.10, a minimising sequence for $c_0 = 0$ has the same uniform bounds on the Willmore energy as a minimising sequence for c_0 close to zero. Indeed, we will see that the compactness proof in [62] still works for c_0 close to zero. However, for general c_0, minimisers are given by bubble trees, thus are no longer embedded, see Theorem 2.11. The following proof is a combination of five independent results: the strict inequalities Theorem 4.1, KUSNER–MCGRATH [69, Theorem 1.2], and Corollary 3.14

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are needed to deduce that $\varepsilon_g(A_0, V_0)$ as defined in (2.60) is strictly positive; then, one can apply the compactness proof of [62]; finally, one can conclude the regularity from Theorem 2.9 (after RIVIÈRE [113]).

In order to prove Theorem 4.5, let Σ_g be an abstract 2-dimensional genus g surface and let $\mathcal{E}_{\Sigma_g}$ be the space of Lipschitz immersions of Σ_g into $\mathbb{R}^3$ as defined in [62, Section 2.2] (that is, $\vec{\Phi} \in \mathcal{E}_{\Sigma_g}$ if and only $\vec{\Phi}$ satisfies the conditions (2.1) and (2.4) of Subsection 2.1.2 and, additionally, $\log|\mathrm{d}\vec{\Phi}| \in L^\infty(\Sigma_g)$). Let $\vec{\Phi}_k$ be a minimising sequence of

$$\inf\{\mathcal{H}^{c_0}(\vec{\Phi}) : \vec{\Phi} \in \mathcal{E}_{\Sigma_g}, \ \mathrm{area}(\vec{\Phi}) = A_0, \ \mathrm{vol}(\vec{\Phi}) = V_0\}.$$

Recall the definition of $\varepsilon_g(A_0, V_0)$ in Equation (2.60):

$$\varepsilon_g(A_0, V_0) := \frac{\sqrt{\min\{8\pi, \boldsymbol{\beta}_g + \beta_0(A_0/V_0^{2/3}) - 4\pi\}} - \sqrt{\beta_g(A_0/V_0^{2/3})}}{2\sqrt{A_0}}.$$

By Theorem 4.1, [69, Theorem 1.2], and Corollary 3.14, there holds $\varepsilon_g(A_0, V_0) > 0$. By the inequality (2.63) of Lemma 2.10 we have for $|c_0| < \varepsilon_g(A_0, V_0)$ that

$$\limsup_{k\to\infty} \mathcal{W}(\vec{\Phi}_k) < 8\pi \tag{4.34}$$

and

$$\limsup_{k\to\infty} \mathcal{W}(\vec{\Phi}_k) < \boldsymbol{\beta}_g + \beta_0(A_0/V_0^{2/3}) - 4\pi. \tag{4.35}$$

In Section 4.3 of [62] it is shown that due to the strict inequality in (4.35), the conformal factors of $\vec{\Phi}_k$ are bounded away from finitely many concentration points $a_1, \ldots, a_N$ in Σ_g. Hence, by the uniform energy bound in (4.34), there exists $\vec{\Phi}_\infty \in \mathcal{E}_{\Sigma_g}$ such that (after passing to a subsequence and after re-parametrising) for all $\delta > 0$

$$\vec{\Phi}_k \to \vec{\Phi}_\infty \qquad \text{as } k \to \infty \text{ weakly in } W^{2,2}(\Sigma_g \setminus \bigcup_{i=1}^N B_\delta(a_i), \mathbb{R}^3). \tag{4.36}$$

Moreover, in Section 4.2 of [62] it is shown that due to (4.34), there holds that

$$\lim_{k\to\infty} \mathrm{vol}(\vec{\Phi}_k) = \mathrm{vol}(\vec{\Phi}_\infty), \qquad \lim_{k\to\infty} \mathrm{area}(\vec{\Phi}_k) = \mathrm{area}(\vec{\Phi}_\infty).$$

After the mentioned re-parametrisations, the $\vec{\Phi}_k$'s are weakly conformal which implies $\Delta_k \vec{\Phi}_k = 2H_{\vec{\Phi}_k}$ for the intrinsic Laplacian Δ_k. Therefore, by the weak convergence (4.36) it follows that for all $\delta > 0$,

$$\lim_{k\to\infty} \int_{\Sigma_g \setminus \bigcup_{i=1}^N B_\delta(a_i)} H_{\vec{\Phi}_k} \, \mathrm{d}\mu_{\vec{\Phi}_k} = \int_{\Sigma_g \setminus \bigcup_{i=1}^N B_\delta(a_i)} H_{\vec{\Phi}_\infty} \, \mathrm{d}\mu_{\vec{\Phi}_\infty}.$$

Moreover, by [62, Equation (4.7)],

$$\liminf_{\delta\to 0} \liminf_{k\to\infty} \int_{B_\delta(a_i)} 1 \, \mathrm{d}\mu_{\vec{\Phi}_k} = 0$$

for all $i \in \{1, \ldots, N\}$. Using the Cauchy–Schwarz inequality and the uniform bound on the

Willmore energy (4.34), it follows that after passing to a subsequence

$$\lim_{k \to \infty} \int_{\Sigma_g} H_{\vec{\Phi}_k} \, \mathrm{d}\mu_{\vec{\Phi}_k} = \int_{\Sigma_g} H_{\vec{\Phi}_\infty} \, \mathrm{d}\mu_{\vec{\Phi}_\infty}.$$

Thus, by lower semi continuity of the Willmore functional under the convergence of (4.36),

$$\mathcal{H}^{c_0}(\vec{\Phi}_\infty) \leq \liminf_{k \to \infty} \mathcal{H}^{c_0}(\vec{\Phi}_k), \qquad \mathcal{W}(\vec{\Phi}_\infty) < 8\pi.$$

Therefore, $\vec{\Phi}_\infty$ is a minimiser and, by the Li–Yau inequality, $\vec{\Phi}_\infty \in W^{2,2}(\Sigma_g, \mathbb{R}^3)$ is an embedding without branch points. Finally, by the regularity result Theorem 2.9 (after [113]), $\vec{\Phi}_\infty \in C^\infty(\Sigma_g, \mathbb{R}^3)$ which completes the proof. $\square$

5 Li–Yau inequalities for varifolds on Riemannian manifolds

Many inequalities that relate the mean curvature of submanifolds with other geometric quantities such as the diameter can be obtained in some way from monotonicity identities, which are formulas that can be used to deduce monotonicity of weighted density ratios. In the Euclidean case, these identities are typically proven by testing the first variation formula with certain vector fields. One of the main ingredients in the construction of these vector fields is the inclusion map of the submanifold into the ambient Euclidean space. A key observation in the computations is that its relative divergence equals the dimension of the submanifold. In the Riemannian case, the inclusion map of a submanifold is not a vector field; however one can perform analogous arguments by using the vector field $r\nabla r$, where r is the distance function to a given point (see for instance ANDERSON [5]). Indeed, its relative divergence is not constant but can be bounded below on small geodesic balls by Rauch's comparison theorem (see Lemma 5.6). Such an idea revealed to be very fruitful; for instance, it enabled HOFFMAN–SPRUCK [53] to derive a Sobolev inequality for Riemannian manifolds. The idea of testing the first variation formula with the vector field $r\nabla r$ in combination with Hessian comparison theorems for the distance function that give a lower bound of the relative divergence was used again in the works of KARCHER–WOOD [61] and XIN [142]. Their resulting monotonicity inequalities imply Liouville type vanishing theorems for harmonic vector bundle valued p-forms. Later, the same idea was used by several authors to prove vanishing theorems in various settings, see for instance DONG–WEI [37]. The technique was recently applied by MONDINO–SPADARO [98] to derive an inequality that relates the radius of balls with the volume and area of the boundary. See also NARDULLI–OSORIO ACEVEDO [103] who used the technique to prove monotonicity inequalities for varifolds on Riemannian manifolds. A weighted monotonicity inequality was obtained by NGUYEN [106].

In the present section, we apply the described technique to prove a general Li–Yau inequality (see Theorem 5.13). In particular, Theorem 1 from the introduction will be proven. We start with a brief introduction to intrinsic varifolds on Riemannian manifolds in Section 5.1. All our monotonicity inequalities (see Section 5.2) as well as our main theorem (see Section 5.3) are valid for general varifolds. The content of this section corresponds to my work [124].

5.1 Varifolds on Riemannian manifolds

5.1.1 Introduction of varifolds on Riemannian manifolds

Let m, n be positive integers satisfying $m \leq n$. Given any n-dimensional vector space V, we define the *Grassmann manifold* $\mathbb{G}(V, m)$ to be the set of all m-dimensional linear subspaces of V. For $V = \mathbb{R}^n$, we write $\mathbb{G}(n, m) := \mathbb{G}(\mathbb{R}^n, m)$. One can show that $\mathbb{G}(n, m)$ is a smooth Euclidean submanifold, see for instance [46, 3.2.29(4)].

Let (N, g) be an n-dimensional Riemannian manifold. We denote with $\mathbb{G}_m(TN)$ the *Grassmann m-plane bundle* of the tangent bundle TN of N. That is, there exists a map $\pi : \mathbb{G}_m(TN) \to N$ such that for each $p \in N$, the fibre $\pi^{-1}(p)$ is given by the Grassmannian manifold $\mathbb{G}(T_pN, m)$. Given any open set U in N and a chart $x : U \to \mathbb{R}^n$ of N, we note that

$\pi^{-1}[U]$ is homeomorphically mapped onto an open subset of $\mathbb{R}^n \times \mathbb{G}(n, m)$ via

$$\mathbb{G}_m(TN) \supset \pi^{-1}[U] \xrightarrow{(x \circ \pi, \mathrm{d}x_\pi)} \mathbb{R}^n \times \mathbb{G}(n, m).$$

This turns $\mathbb{G}_m(TN)$ into a differentiable manifold. We define

$$\mathbb{G}_m(N) := \{(p, T) \in N \times \mathbb{G}_m(TN) : p = \pi(T)\}$$

and note that $\mathbb{G}_m(N)$ and $\mathbb{G}_m(TN)$ are homeomorphic. In particular, $\mathbb{G}_m(N)$ is a locally compact and separable metric space.

With an m-dimensional *varifold* in N we mean a Radon measure V over $\mathbb{G}_m(N)$. The space of all m-dimensional varifolds on N is denoted with $\mathbb{V}_m(N)$. The *weight* measure $\|V\|$ of a varifold V is defined by

$$\|V\|(A) = V\{(p, T) \in \mathbb{G}_m(N) : p \in A\} \qquad \text{whenever } A \subset N.$$

It is the push forward measure of the varifold under the projection $\mathbb{G}_m(N) \to N$. In particular, $\|V\|$ is a Radon measure on N (see [87, Lemma 2.6]).

The space of compactly supported vector fields on N is denoted with $\mathscr{X}(N)$. Given any $X \in \mathscr{X}(N)$, $p \in N$, and $T \in \mathbb{G}(T_p N, m)$ with orthonormal basis $\{e_1, \ldots, e_m\}$, we let

$$\operatorname{div}_T X(p) = \sum_{i=1}^{m} g_p(\nabla_{e_i} X(p), e_i),$$

where ∇ denotes the Levi-Civita connection. Moreover, we denote with $\operatorname{spt} X$ the support of X. The *first variation* of a varifold V is defined as the linear functional

$$\delta V : \mathscr{X}(N) \to \mathbb{R}, \qquad \delta V(X) = \int \operatorname{div}_T X(p) \, \mathrm{d}V(p, T).$$

The total variation $\|\delta V\|$ of δV is defined by

$$\|\delta V\|(U) = \sup\{\delta V(X) : X \in \mathscr{X}(N), \operatorname{spt} X \subset U, g(X, X) \leq 1\}$$

whenever U is an open subset of N, and

$$\|\delta V\|(A) = \inf\{\|\delta V\|(U) : U \text{ is open in } N, A \subset U\}$$

whenever A is any subset of N.

Finally, we say that H is the *generalised mean curvature* of V in (N, g), if and only if $H : N \to TN$ is $\|V\|$ measurable, $\|\delta V\|$ is a Radon measure over N, there exists a $\|\delta V\|$ measurable map η taking values in TN such that $\|\delta V\|$ almost everywhere, $g(\eta, \eta) \leq 1$, and

$$\delta V(X) = -\int g(X, H) \, \mathrm{d}\|V\| + \int g(X, \eta) \, \mathrm{d}\|\delta V\|_{\text{sing}}, \tag{5.1}$$

where $\|\delta V\|_{\text{sing}} = \|\delta V\| - \|\delta V\|_{\|V\|}$, and $\|\delta V\|_{\|V\|}$ is the absolutely continuous part of $\|\delta V\|$ with respect to $\|V\|$, see [46, 2.9.1].

It remains to mention that each isometrically immersed Riemannian manifold $M \to N$ can be considered as a varifold in N. For more details, see Example 5.3.

5.1.2 Notation and definitions

Suppose (X, d) is a metric space, μ is a Radon measure on X, and m is a positive integer. Denote with $\boldsymbol{\alpha}(m)$ the volume of the unit ball in $\mathbb{R}^m$. Given any $p \in X$ and $r > 0$, we define the balls

$$B_r(p) := \{x \in X : d(p, x) < r\}, \qquad \bar{B}_r(p) := \{x \in X : d(p, x) \le r\}.$$

The m-dimensional *lower density* $\Theta_*^m(\mu, p)$ and *upper density* $\Theta^{*m}(\mu, p)$ of μ at $p \in X$ are defined by

$$\Theta_*^m(\mu, p) = \liminf_{r \to 0+} \frac{\mu(\bar{B}_r(p))}{\boldsymbol{\alpha}(m) r^m}, \qquad \Theta^{*m}(\mu, p) = \limsup_{r \to 0+} \frac{\mu(\bar{B}_r(p))}{\boldsymbol{\alpha}(m) r^m}.$$

Moreover, if $\Theta_*^m(\mu, p) = \Theta^{*m}(\mu, p)$, we let $\Theta^m(\mu, p) := \Theta_*^m(\mu, p)$. The *support* $\operatorname{spt} \mu$ of the measure μ is defined by

$$\operatorname{spt} \mu := X \setminus \bigcup \{U : U \text{ is open in } X, \mu(U) = 0\}.$$

5.1 Definition. Suppose N is a Riemannian manifold and $p \in N$.

We say that U is an open *geodesically star-shaped neighbourhood* of p if and only if there exists an open star-shaped neighbourhood D of 0 in $T_p N$ such that the exponential map $\exp_p : D \to U$ is a diffeomorphism with $\exp_p[D] = U$, and all geodesics emanating from p are length-minimising in U.

Similarly, we say that the open ball $B_r(p)$ with radius $r > 0$ is a *geodesic ball* if it is a geodesically star-shaped neighbourhood of p.

Typically, we denote with $|\cdot|_g$ the norm induced by a Riemannian metric g.

5.1.3 Basic examples and Hessian comparison theorems

In this section, we prove that any smoothly immersed manifold is a varifold, see Lemma 5.2 and Example 5.3. Moreover, we state the Hessian comparison theorems for the distance function (see Lemma 5.4 and Lemma 5.6) that are crucial to derive the monotonicity inequalities in Section 5.2.

The following lemma is the Riemannian counterpart of [87, Lemma 2.8].

5.2 Lemma (See SCHARRER [124, Lemma 2.3])**.** *Suppose m, n are positive integers, $m \le n$, M is a compact m-dimensional connected differentiable manifold with boundary, (N, g) is an n-dimensional Riemannian manifold, and $f : M \to N$ is a smooth proper immersion. Denote with $\mathcal{H}_g^m$ the m-dimensional Hausdorff measure on N with respect to the distance induced by the metric g, and denote with $\mu_{f^* g}$ the Riemannian measure on M corresponding to the pull back metric $f^* g$ of g along f.*

Then, there holds

$$\int_M k \, \mathrm{d}\mu_{f^* g} = \int_N \sum_{p \in f^{-1}(x)} k(p) \, \mathrm{d}\mathcal{H}_g^m x \tag{5.2}$$

for all compactly supported continuous functions $k : M \to \mathbb{R}$. In particular, the push forward measure $f_\# \mu_{f^* g}$ of $\mu_{f^* g}$ under f is a Radon measure on N and satisfies

$$f_\# \mu_{f^* g}(B) = \int_B \mathcal{H}^0(f^{-1}\{x\}) \, \mathrm{d}\mathcal{H}^m_g x \qquad \textit{for all Borel sets } B \subset N, \tag{5.3}$$

where $\mathcal{H}^0$ denotes the counting measure. Moreover, for all $x \in f[M \setminus \partial M]$, there holds

$$\Theta^m(f_\# \mu_{f^* g}, x) = \mathcal{H}^0(f^{-1}\{x\}) \tag{5.4}$$

and for $f_\# \mu_{f^* g}$ almost all x,

$$\mathrm{d}f_p[T_p M] = \mathrm{d}f_q[T_q M] \qquad \textit{whenever } p, q \in f^{-1}\{x\}. \tag{5.5}$$

Proof. First, suppose that $\partial M = \varnothing$, M is a submanifold of N and $f = i$ is the inclusion map. Denote with d_M and d_N the distance functions on $(M, i^* g)$ and (N, g), respectively. Clearly, $d_M(a, b) \geq d_N(a, b)$ for all $a, b \in M$. By [46, 3.2.46], $\mu_{i^* g}$ coincides with the m-dimensional Hausdorff measure $\mathcal{H}^m_{i^* g}$ on M corresponding to the distance d_M. Thus, we have

$$i_\# \mu_{i^* g}(S) = \mathcal{H}^m_{i^* g}(M \cap S) \geq \mathcal{H}^m_g(M \cap S) \qquad \text{for all } S \subset N.$$

To prove the reverse inequality, let $\lambda > 1$ and $p \in M$. Choose an open neighbourhood U of p in N together with a submanifold chart $x : U \to \mathbb{R}^n$, i.e. $x[M \cap U] = x[U] \cap (\mathbb{R}^m \times \{0\})$. Composing x with a linear map $\mathbb{R}^m \times \mathbb{R}^{n-m} \to \mathbb{R}^m \times \mathbb{R}^{n-m}$, we may assume that $\mathrm{d}x_p$ maps an orthonormal basis of $T_p N$ onto an orthonormal basis of $\mathbb{R}^n$. In particular, $\|\mathrm{d}x_p\| = \|\mathrm{d}x^{-1}_{x(p)}\| = 1$. Hence, there exists $\rho > 0$ such that $B_\rho(p) := \{q \in N : d_N(p, q) < \rho\} \subset U$, as well as $\|\mathrm{d}x|_{B_\rho(p)}\| \leq \sqrt{\lambda}$ and $\|\mathrm{d}(x|_{B_\rho(p)})^{-1}\| \leq \sqrt{\lambda}$. Thus, $x|_{B_\rho(p)}$ is Lipschitz continuous with Lipschitz constant bounded above by $\sqrt{\lambda}$. Next, choose $\rho_0 > 0$ such that $\rho_0 < \rho$ and the convex hull of $x[M \cap B_{\rho_0}(p)]$ in $\mathbb{R}^m \times \{0\}$ is contained in $x[M \cap B_\rho(p)]$. Given any $a, b \in M \cap B_{\rho_0}(p)$, let

$$\gamma : [0, 1] \to \mathbb{R}^m \times \{0\}, \qquad \gamma(t) = (1 - t)x(a) + tx(b).$$

Then, $c := x^{-1} \circ \gamma$ is a smooth curve in $M \cap B_\rho(p)$, connecting a with b. Therefore,

$$d_M(a, b) \leq \int_0^1 \sqrt{(i^* g)_c(\dot{c}, \dot{c})} \, \mathrm{d}t \leq \sqrt{\lambda} \int_0^1 |\dot{\gamma}| \, \mathrm{d}t = \sqrt{\lambda}|x(a) - x(b)| \leq \lambda d_N(a, b).$$

This implies $\mathcal{H}^m_{i^* g}(M \cap S) \leq \lambda^m \mathcal{H}^m_g(M \cap S)$ for all $S \subset B_{\rho_0}(p)$. Thus,

$$\lim_{r \to 0+} \frac{\mu_{i^* g}(M \cap \{q : d_M(p, q) \leq r\})}{r^m} = \lim_{r \to 0+} \frac{\mathcal{H}^m_g(M \cap \{q : d_N(p, q) \leq r\})}{r^m} \tag{5.6}$$

in the sense that the left hand side exists if and only if the right hand side exists in which case both sides coincide. Moreover, it follows $i_\# \mu_{i^* g}(B) = \mathcal{H}^m_{i^* g}(M \cap B) = \mathcal{H}^m_g(M \cap B)$ for all Borel sets $B \subset N$, which proves (5.3) for the special case.

Next, suppose that f is an embedding and $\partial M = \varnothing$. Then, $f[M]$ is a submanifold of N. Denote with $i : f[M] \to N$ the inclusion map. Then, $f : (M, f^*(i^* g)) \to (f[M], i^* g)$ is an

isometry. This means $f_{\#}\mu_{f^*(i^*g)} = \mu_{i^*g}$ and, by the first case,

$$f_{\#}\mu_{f^*g}(B) = i_{\#}\mu_{i^*g}(B) = \mathcal{H}_g^m(f[M] \cap B) \qquad \text{for all Borel sets } B \subset N.$$

Hence, (5.3) is valid if $\partial M = \varnothing$ and f is an embedding. Moreover, in this case, (5.4) follows from [121, Chapter II, Corollary 5.5] in combination with Equation (5.6).

Now, suppose that f is an immersion and $\partial M = \varnothing$. Let $k : M \to \mathbb{R}$ be a continuous function with compact support $\mathrm{spt}\,k$ and choose finitely many open sets $U_1, \ldots, U_\Lambda$ whose union contains $\mathrm{spt}\,k$ such that $f|_{U_\lambda}$ is an embedding for $\lambda = 1, \ldots, \Lambda$. Pick a subordinate partition of unity $\{\varphi_\lambda\}_{\lambda=1}^\Lambda$, i.e. $\sum_{\lambda=1}^\Lambda \varphi_\lambda(p) = 1$ for all $p \in \mathrm{spt}\,k$, and $\mathrm{spt}\,\varphi_\lambda \subset U_\lambda$ for $\lambda = 1, \ldots, \Lambda$. Given any $x \in f[M]$, then, by injectivity of $f|_{U_\lambda}$ for $\lambda = 1, \ldots, \Lambda$, the union

$$\bigcup_{p \in f^{-1}\{x\}} \{\lambda : p \in U_\lambda\}$$

is disjoint. In particular, denoting with χ_A the characteristic function of any given set A,

$$\sum_{\lambda=1}^\Lambda \varphi_\lambda((f|_{U_\lambda})^{-1}(x))k((f|_{U_\lambda})^{-1}(x))\chi_{f[U_\lambda]}(x) = \sum_{p \in f^{-1}\{x\}} \sum_{\lambda=1}^\Lambda \varphi_\lambda(p)k(p) = \sum_{p \in f^{-1}\{x\}} k(p).$$

Hence, using (5.3) for the special case,

$$\int_M k \, \mathrm{d}\mu_{f^*g} = \sum_{\lambda=1}^\Lambda \int_{U_\lambda} \varphi_\lambda \cdot k \, \mathrm{d}\mu_{f^*g} = \sum_{\lambda=1}^\Lambda \int_{f[U_\lambda]} [\varphi_\lambda \circ (f|_{U_\lambda})^{-1}] \cdot [k \circ (f|_{U_\lambda})^{-1}] \, \mathrm{d}f_{\#}\mu_{f^*g}$$
$$= \int_N \sum_{p \in f^{-1}\{x\}} k(p) \, \mathrm{d}\mathcal{H}_g^m x.$$

This proves (5.2) which readily implies (5.3). It remains to mention that if $\partial M \neq \varnothing$, then we have that $\partial(\partial M) = \varnothing$ and $f|_{\partial M}$ is an immersion. By the first cases it follows that $\mathcal{H}_g^{m-1}(f[\partial M] \cap K) < \infty$ for all compact sets $K \subset N$. Thus, $\mathcal{H}_g^m(f[\partial M]) = 0$.

To prove (5.4), we first assume that f is an embedding. Then, the statement follows from [121, Chapter II, Corollary 5.5] in combination with Equation (5.6). If f is an immersion let $x \in f[M \setminus \partial M]$ and let $p_1, \ldots, p_k$ be distinct points such that $f^{-1}\{x\} = \{p_1, \ldots, p_k\}$. Choose pairwise disjoint open sets $U_1, \ldots, U_k$ such that for $i = 1, \ldots, k$ there holds $p_i \in U_i$. Then, for small $r > 0$, there holds

$$f_{\#}\mu_{f^*g}(B_r(x)) = \sum_{i=1}^k (f|_{U_i})_{\#}\mu_{(f|_{U_i})^*g}(B_r(x)).$$

Hence, (5.4) follows from the special case by linearity of the limit operator.

To prove (5.5), suppose $x \in f[M \setminus \partial M]$, $p_1, p_2 \in f^{-1}\{x\}$, and U_1, U_2 are disjoint open neighbourhoods of p_1, p_2 in M, respectively, such that $f|_{U_1}, f|_{U_2}$ are embeddings. Assume that $\mathrm{d}f_{p_1}[T_{p_1}M] \neq \mathrm{d}f_{p_2}[T_{p_2}M]$. Then, we can pick a unit vector $v_1 \in T_xN$ such that $v_1 \in \mathrm{d}f_{p_1}[T_{p_1}M] \setminus \mathrm{d}f_{p_2}[T_{p_2}M]$ and there exists $0 < \varepsilon < 1$ such that the cone

$$C := \{w \in T_xN : |rw - v_1|_g \leq \varepsilon \text{ for some } r \in \mathbb{R}\}$$

satisfies $C \cap \mathrm{d}f_{p_2}[T_{p_2}M] = \varnothing$. Next, we pick $\varepsilon_1 > 0$ such that $\exp_{p_1} : B_{\varepsilon_1} \subset T_{p_1}M \to M$ is a diffeomorphism on $B_{\varepsilon_1} := \{\xi \in T_{p_1}M : (f^*g)_{p_1}(\xi, \xi) < \varepsilon_1^2\}$ and introduce polar coordinates

$$\Xi : (0, \varepsilon_1) \times S^{m-1} \to M, \qquad \Xi(t, u) = \exp_{p_1}(tu),$$

where $S^{m-1} := \{\xi \in T_{p_1}M : (f^*g)_{p_1}(\xi, \xi) = 1\}$, as well as the density function

$$\theta : (0, \varepsilon_1) \times S^{m-1} \to \mathbb{R}, \qquad \theta(t, u) = t^{m-1}\sqrt{\det(f^*g)_{ij}(\Xi(t, u))}.$$

By [121, Chapter II, Lemma 5.4], there holds $\mu_{\Xi^*(f^*g)} = \theta\mu_{g_0}$, where g_0 is the canonical product metric on $(0, \varepsilon_1) \times S^{m-1}$. Hence, for $E := (\Xi \circ (\mathrm{d}f_{p_1})^{-1})[C]$ and $u_1 := (\mathrm{d}f_{p_1})^{-1}(v_1)$, we have by Fubini's theorem

$$\mu_{f^*g}(E \cap B_\rho(p_1)) = \int_{S^{m-1} \cap B_\varepsilon(u_1)} \int_0^\rho \theta(t, u) \, \mathrm{d}t \, \mathrm{d}\mu_{S^{m-1}}u$$

for all $0 < \rho < \varepsilon_1$. Noting that $\theta(t, u) = t^{m-1} + O(t^{m+1})$ as $t \to 0+$, it follows

$$\Theta^m(\mu_{f^*g}\llcorner E, p_1) = \frac{\mu_{S^{m-1}}(B_\varepsilon(u_1))}{\boldsymbol{\alpha}(m)m} > 0,$$

where $\mu_{f^*g}\llcorner E$ denotes the Radon measure on M given by $(\mu_{f^*g}\llcorner E)(B) = \mu_{f^*g}(E \cap B)$ for all Borel sets $B \subset M$. Hence, by (5.3) applied to $f|_{U_1}$, there holds $\Theta_*^m(\mathcal{H}_g^m\llcorner f[E], x) > 0$. Notice that if γ is a curve in $E \cup \{p_1\}$ with $\gamma(0) = p_1$, then $(f \circ \gamma)^{\cdot}(0) \in C \cup \{0\}$. Moreover, we make the following observation. Choose $\rho > 0$ such that $B_\rho(x)$ is a geodesic ball around x in N. Given any unit vector $v \in T_x N$ and $\delta > 0$, we denote with $C(v, \delta)$ the image of the set

$$B_\rho(0) \cap \{w \in T_x N : |rw - v|_g < \delta \text{ for some } r \in \mathbb{R}\}$$

under the exponential map $\exp_x : B_\rho(0) \cap T_x N \to N$. Then, given any smooth curve γ in N with

$$\gamma(0) = x \qquad \text{and} \qquad \frac{\dot\gamma(0)}{|\dot\gamma(0)|_g} = v,$$

one can use normal coordinates and differentiability of γ to show that for some $t_0 > 0$, there holds

$$\gamma(t) \in C(v, \delta) \cup \{x\} \qquad \text{for all } -t_0 < t < t_0.$$

This observation together with compactness of $\bar{B}_\rho(x)$ can be used to show that for small $\rho > 0$, $f[E] \cap f[U_2] \cap B_\rho(x) = \varnothing$. Thus, $\Theta_*^m(\mathcal{H}_g^m\llcorner f[M], x) > 1$. Now, the conclusion follows from [46, 2.10.19(5)]. $\qquad\square$

The following example is the Riemannian counterpart to the Euclidean case [87, Definition 2.14]. Compare also with [70, Section 2.2], where smoothness of f is replaced with $W^{2,2}$-regularity.

5.3 Example (See Scharrer [124, Example 2.4]). Let f, M, N be as in Lemma 5.2. Define $V \in \mathbb{V}_m(N)$ by letting

$$V(k) = \int_N \sum_{p \in f^{-1}\{x\}} k(x, \mathrm{d}f_p[T_pM]) \, \mathrm{d}\mathcal{H}_g^m x$$

for all continuous functions $k : \mathbb{G}_m(N) \to \mathbb{R}$ with compact support. In view of Lemma 5.2, we have

$$\|V\| = f_\# \mu_{f^*g}, \qquad \mathrm{spt}\,\|V\| = \mathrm{closure}\, f[M].$$

In particular, $\mathrm{spt}\,\|V\| = f[M]$ if M is closed. Moreover, for all $x \in f[M \setminus \partial M]$,

$$\Theta^m(\|V\|, x) = \mathcal{H}^0(f^{-1}\{x\})$$

and

$$\|V\|(N) = \int_M 1 \, \mathrm{d}\mu_{f^*g} = |M|.$$

Identify $\mathrm{d}f_p[T_p M]$ with $T_p M$. Let NM be the normal bundle of the immersion f. That is, there exists $\pi : NM \to M$ such that for each $p \in M$, the fibre $\pi^{-1}(p)$ is given by the orthogonal complement of $T_p M$ in $T_{f(p)} N$. Denote with $H_f : M \to NM$ the mean curvature vector field of f, i.e. the trace of the second fundamental form (see [53, Definition 3.1]). Define the $\|V\|$ measurable map $H : N \to TN$ by

$$H(x) = \begin{cases} \frac{1}{\Theta^m(\|V\|, x)} \sum_{p \in f^{-1}\{x\}} H_f(p) & \text{if } \Theta^m(\|V\|, x) > 0 \\ 0 & \text{if } \Theta^m(\|V\|, x) = 0. \end{cases}$$

Let $X \in \mathscr{X}(N)$. By a simple computation (see [53, Lemma 3.2(i)]),

$$\mathrm{div}_{T_p M}\, X(x) = -g_x(X(x), H_f(p)) + \mathrm{div}(X \circ f)^t(p)$$

whenever $p \in M$ and $f(p) = x$, where $(X \circ f)^t$ denotes the orthogonal projection of $(X \circ f)$ onto the tangent bundle TM. Integrating this equation and using Lemma 5.2 as well as the usual Divergence Theorem on M (see [121, Chapter II, Theorem 5.11]), we infer

$$\begin{aligned}
\delta V(X) &= \int_N \sum_{p \in f^{-1}\{x\}} \mathrm{div}_{T_p M}\, X(x) \, \mathrm{d}\mathcal{H}_g^m x \\
&= -\int_N \sum_{p \in f^{-1}\{x\}} g_x(X(x), H_f(p)) \, \mathrm{d}\mathcal{H}_g^m x + \int_N \sum_{p \in f^{-1}\{x\}} \mathrm{div}(X \circ f)^t(p) \, \mathrm{d}\mathcal{H}_g^m x \\
&= -\int_N g(X, H) \Theta^m(\|V\|, \cdot) \, \mathrm{d}\mathcal{H}_g^m + \int_M \mathrm{div}(X \circ f)^t \, \mathrm{d}\mu_{f^*g} \\
&= -\int_N g(X, H) \, \mathrm{d}\|V\| + \int_{\partial M} (f|_{\partial M}{}^*g)\big((X \circ f)^t, \nu\big) \, \mathrm{d}\mu_{f|_{\partial M}{}^*g},
\end{aligned}$$

where ν is the outward unit normal vector field on ∂M. In particular, V has generalised mean curvature H,

$$\|\delta V\|(B) \leq \int_B |H|_g \, \mathrm{d}\|V\| + \int_B \mathcal{H}^0((f|_{\partial M})^{-1}\{x\}) \, \mathrm{d}\mathcal{H}_g^{m-1}$$

for all Borel sets $B \subset N$, and by (5.5),

$$H(x) \perp T \qquad \text{for } V \text{ almost all } (x, T) \in \mathbb{G}_m(N).$$

By definition of H, we have trivially

$$\int_N |H|_g \, \mathrm{d}\|V\| \leq \int_N \sum_{p \in f^{-1}\{x\}} |H_f(p)|_g \, \mathrm{d}\mathcal{H}^m_g x = \int_M |H_f|_g \, \mathrm{d}\mu_{f^*g}$$

and

$$\int_N |H|^2_g \, \mathrm{d}\|V\| \leq \int_N \frac{1}{\mathcal{H}^0(f^{-1}\{x\})^2} \left(\sum_{p \in f^{-1}\{x\}} |H_f(p)|_g \right)^2 \mathrm{d}\|V\|x$$

$$\leq \int_N \frac{1}{\mathcal{H}^0(f^{-1}\{x\})} \sum_{p \in f^{-1}\{x\}} |H_f(p)|^2_g \, \mathrm{d}\|V\|x$$

$$= \int_N \sum_{p \in f^{-1}\{x\}} |H_f(p)|^2_g \, \mathrm{d}\mathcal{H}^m_g x = \int_M |H_f|^2_g \, \mathrm{d}\mu_{f^*g}.$$

5.4 Lemma (See [108, Theorem 6.4.3])**.** *Suppose (N,g) is a Riemannian manifold, $p \in N$, U is a geodesically star-shaped open neighbourhood of p, the metric is represented in geodesic polar coordinates $g = \mathrm{d}r \otimes \mathrm{d}r + g_r$ on U, $b \in \mathbb{R}$, the sectional curvature satisfies $K \leq b$ on U, and either $b \leq 0$ or $U \subset B_{\frac{\pi}{\sqrt{b}}}(p)$.*

Then, the Hessian $\nabla^2 r$ of r can be bounded below on U by

$$\nabla^2 r \geq \begin{cases} \sqrt{b} \cot(\sqrt{b}r) g_r & \text{if } b > 0 \\ \sqrt{-b} \coth(\sqrt{-b}r) g_r & \text{if } b \leq 0. \end{cases}$$

5.5 Remark. Define the continuous function

$$a : [0, \pi) \to \mathbb{R}, \qquad a(x) = x \cot(x)$$

where $a(0) = 1$. Using the series expansion

$$\cot(x) = \frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - \dots \qquad \text{for } 0 < x < \pi \tag{5.7}$$

where all higher order terms are negative, we see that a is strictly decreasing. In particular, we have

$$x \cot x \leq 1 \qquad \text{for } 0 \leq x < \pi. \tag{5.8}$$

5.6 Lemma. *Suppose m, n are positive integers, $m \leq n$, (N, g) is an n-dimensional Riemannian manifold, $p \in N$, U is a geodesically star-shaped open neighbourhood of p, $b > 0$, the sectional curvature satisfies $K \leq b$ on U, and $U \subset B_{\frac{\pi}{\sqrt{b}}}(p)$.*

Then, writing $r = d(p, \cdot)$, there holds

$$\mathrm{div}_T(r\nabla r) \geq m\sqrt{b}r \cot(\sqrt{b}r)$$

for all $T \in \mathbb{G}_m(TU)$.

Proof. Writing the metric in polar coordinates $g = \mathrm{d}r \otimes \mathrm{d}r + g_r$ and using Lemma 5.4 in

combination with (5.8), we compute for $b > 0$

$$\nabla(r\nabla r) = \mathrm{d}r \otimes \mathrm{d}r + r\nabla^2 r \geq \mathrm{d}r \otimes \mathrm{d}r + \sqrt{b}r\cot(\sqrt{b}r)g_r \geq \sqrt{b}r\cot(\sqrt{b}r)g.$$

Given any $T \in \mathbb{G}_m(TU)$ with orthonormal basis $\{e_1, \ldots, e_m\}$, it follows

$$\mathrm{div}_T(r\nabla r) = \sum_{i=1}^{m} \nabla(r\nabla r)(e_i, e_i) \geq \sqrt{b}r\cot(\sqrt{b}r) \sum_{i=1}^{m} g(e_i, e_i) = m\sqrt{b}r\cot(\sqrt{b}r)$$

which concludes the proof. $\qquad\square$

5.2 Monotonicity inequalities

In this section, we prove several monotonicity inequalities. Moreover, we prove existence and upper semi-continuity of the density (see Theorem 5.10).

The proof of the following Lemma is based on the ideas of the monotonicity formula in SIMON [136] in combination with a technique of ANDERSON [5]. See also [114, Lemma A.3] for a proof in the presence of boundary, and [93] for higher dimensional varifolds.

5.7 Lemma (See SCHARRER [124, Lemma 3.1]). *Suppose n is an integer, $n \geq 2$, (N, g) is an n-dimensional Riemannian manifold, $p \in N$, $V \in \mathbb{V}_2(N)$ has generalised mean curvature H, H is square integrable with respect to $\|V\|$, $H(x) \perp T$ for V almost all $(x, T) \in \mathbb{G}_2(N)$, $b > 0$, $0 < \rho < \frac{\pi}{\sqrt{b}}$, the sectional curvature satisfies $K \leq b$ on $\mathrm{spt}\|V\| \cap B_\rho(p)$, U is a geodesically star-shaped open neighbourhood of p, and $\mathrm{spt}\|V\| \cap \bar{B}_\rho(p) \subset U$.*

Then, writing $r = d(p, \cdot)$, there holds

$$\frac{\|V\|\bar{B}_\sigma(p)}{\sigma^2} \leq \frac{\|V\|\bar{B}_\rho(p)}{\rho^2} + \frac{1}{16}\int_{\bar{B}_\rho(p)\setminus\bar{B}_\sigma(p)} |H|_g^2 \,\mathrm{d}\|V\| + \int_{\bar{B}_\rho(p)} \frac{1-a_b(r)}{r^2} \,\mathrm{d}\|V\|$$

$$+ \int_{\bar{B}_\sigma(p)} \frac{|H|_g}{2\sigma} \,\mathrm{d}\|V\| + \int_{\bar{B}_\rho} \frac{|H|_g}{2\rho} \,\mathrm{d}\|V\| - \int_{\pi^{-}[\bar{B}_\rho(p)\setminus\bar{B}_\sigma(p)]} \Big|\frac{1}{4}H + \frac{\nabla^\perp r}{r}\Big|_g^2 \,\mathrm{d}V$$

$$+ \int_{\bar{B}_\sigma(p)} \frac{r}{2\sigma^2} \,\|\delta V\|_{\mathrm{sing}} + \int_{\bar{B}_\rho(p)\setminus\bar{B}_\sigma(p)} \frac{1}{2r} \,\mathrm{d}\|\delta V\|_{\mathrm{sing}} + \int_{\bar{B}_\rho(p)} \frac{r}{2\rho^2} \,\mathrm{d}\|\delta V\|_{\mathrm{sing}}$$

for all $0 < \sigma < \rho$, where $a_b(r) = \sqrt{b}r\cot(\sqrt{b}r)$.

Proof. Given any $\sigma < t < \rho$ and any non-negative smooth function $\varphi : \mathbb{R} \to \mathbb{R}$ whose support is contained in an open neighbourhood of the interval $(-\infty, 1]$, we let $X = \varphi(\frac{r}{t})r\nabla r$ and compute

$$\mathrm{div}_T X = \varphi\Big(\frac{r}{t}\Big)\mathrm{div}_T(r\nabla r) + \dot{\varphi}\Big(\frac{r}{t}\Big)\frac{r}{t}|\nabla^T r|_g^2$$

for all $T \in \mathbb{G}_2(TN)$, where $\nabla^T r$ denotes the orthogonal projection of ∇r onto T. We write

$$\nabla^\perp r : \mathbb{G}_2(N) \to TN, \qquad (\nabla^\perp r)(x, T) = (\nabla r)(x) - (\nabla^T r)(x),$$

and notice that

$$1 = |\nabla r|_g^2 = |\nabla^T r|_g^2 + |\nabla^\perp r|_g^2.$$

Therefore, testing the first variation equation (see (5.1)) with X, we infer by Lemma 5.6

$$2 \int_N \varphi\left(\frac{r}{t}\right) a_b(r) \, \mathrm{d}\|V\| + \int_{\mathbb{G}_2(N)} \dot\varphi\left(\frac{r}{t}\right)\frac{r}{t}\left[1 - |\nabla^\perp r|_g^2\right] \mathrm{d}V$$

$$\leq -\int_N \varphi\left(\frac{r}{t}\right) g(r\nabla r, H) \, \mathrm{d}\|V\| + \int_N \varphi\left(\frac{r}{t}\right) g(r\nabla r, \eta) \, \mathrm{d}\|\delta V\|_{\mathrm{sing}}.$$

There holds

$$-\frac{d}{dt}\left[\frac{1}{t^2}\varphi\left(\frac{r}{t}\right)\right] = \frac{1}{t^3}\left[2\varphi\left(\frac{r}{t}\right) + \dot\varphi\left(\frac{r}{t}\right)\frac{r}{t}\right].$$

Hence, adding $\int_N 2\varphi(\frac{r}{t})(1 - a_b(r)) \, \mathrm{d}\|V\|$ on both sides of the inequality and multiplying with $\frac{1}{t^3}$, it follows

$$-\frac{d}{dt}\int \frac{1}{t^2}\varphi\left(\frac{r}{t}\right) \mathrm{d}\|V\| - \int_{\mathbb{G}_2(N)} \dot\varphi\left(\frac{r}{t}\right)\frac{r}{t^4}|\nabla^\perp r|_g^2 \, \mathrm{d}V$$

$$\leq 2\int_N \varphi\left(\frac{r}{t}\right)\frac{1 - a_b(r)}{t^3} \, \mathrm{d}\|V\| - \int_N \varphi\left(\frac{r}{t}\right)\frac{g(r\nabla r, H)}{t^3} \, \mathrm{d}\|V\| \qquad (5.9)$$

$$+ \int_N \varphi\left(\frac{r}{t}\right)\frac{g(r\nabla r, \eta)}{t^3} \, \mathrm{d}\|\delta V\|_{\mathrm{sing}}.$$

Given any $\lambda > 1$, choose φ such that $\dot\varphi \leq 0$, $\varphi(s) = 1$ for all $s \leq 1$, and $\varphi(s) = 0$ for all $s \geq \lambda$. In other words, φ approaches the characteristic function of the interval $(-\infty, 1]$ from above as $\lambda \downarrow 1$. In particular, if $\dot\varphi(\frac{r}{t}) \neq 0$, then $\frac{r}{t} \geq 1$. Hence,

$$\int_{\mathbb{G}_2(N)} \dot\varphi\left(\frac{r}{t}\right)\frac{r}{t^4}|\nabla^\perp r|_g^2 \, \mathrm{d}V \leq \int_{\mathbb{G}_2(N)} \dot\varphi\left(\frac{r}{t}\right)\frac{r}{t^2}\frac{1}{r^2}|\nabla^\perp r|_g^2 \, \mathrm{d}V - \frac{d}{dt}\int_{\mathbb{G}_2(N)} \varphi\left(\frac{r}{t}\right)\frac{1}{r^2}|\nabla^\perp r|_g^2 \, \mathrm{d}V. \quad (5.10)$$

Moreover, given any $\|V\|$ integrable real valued function f, one computes using Fubini's theorem, writing $r_\sigma := \max\{\sigma, r\}$, and denoting with χ_A the characteristic function of any set A,

$$\int_\sigma^\rho \int_{\bar B_t(p)} \frac{f(x)}{t^3} \, \mathrm{d}\|V\|x \, \mathrm{d}t = \int_{\bar B_\rho(p)} \int_\sigma^\rho \frac{f(x)}{t^3}\chi_{\{r \leq t\}}(x) \, \mathrm{d}t \, \mathrm{d}\|V\|x$$

$$= \int_{\bar B_\rho(p)} f(x) \int_{r_\sigma(x)}^\rho \frac{1}{t^3} \, \mathrm{d}t \mathrm{d}\|V\|x = \frac{1}{2}\int_{\bar B_\rho(p)}\left(\frac{1}{r_\sigma^2} - \frac{1}{\rho^2}\right) f \, \mathrm{d}\|V\| \qquad (5.11)$$

Therefore, putting (5.10) into (5.9), integrating with respect to t from σ to ρ, letting $\lambda \downarrow 1$, and using (5.11), we infer

$$\frac{\|V\|\bar B_\sigma(p)}{\sigma^2} \leq \frac{\|V\|\bar B_\rho(p)}{\rho^2} - \int_{\pi^{-1}[\bar B_\rho(p)\setminus\bar B_\sigma(p)]} \frac{|\nabla^\perp r|_g^2}{r^2} \, \mathrm{d}V$$

$$+ \int_{\bar B_\rho(p)}\left(\frac{1}{r_\sigma^2} - \frac{1}{\rho^2}\right)(1 - a_b(r)) \, \mathrm{d}\|V\| - \frac{1}{2}\int_{\bar B_\rho(p)}\left(\frac{1}{r_\sigma^2} - \frac{1}{\rho^2}\right) g(r\nabla r, H) \, \mathrm{d}\|V\|$$

$$+ \frac{1}{2}\int_{\bar B_\rho(p)}\left(\frac{1}{r_\sigma^2} - \frac{1}{\rho^2}\right) g(r\nabla r, \eta) \, \mathrm{d}\|\delta V\|_{\mathrm{sing}},$$

where $\pi : \mathbb{G}_2(N) \to N$ denotes the canonical projection. Observe that

$$\left|\frac{1}{4}H + \frac{\nabla^\perp r}{r}\right|_g^2 = \frac{1}{2r}g(\nabla r, H) + \frac{|\nabla^\perp r|_g^2}{r^2} + \frac{1}{16}|H|_g^2.$$

Thus, it follows

$$\frac{\|V\|\bar{B}_\sigma(p)}{\sigma^2} \le \frac{\|V\|\bar{B}_\rho(p)}{\rho^2} - \int_{\pi^-[\bar{B}_\rho(p)\setminus\bar{B}_\sigma(p)]} \Big|\frac{1}{4}H + \frac{\nabla^\perp r}{r}\Big|_g^2 \mathrm{d}V + \frac{1}{16}\int_{\bar{B}_\rho(p)\setminus\bar{B}_\sigma(p)} |H|_g^2 \,\mathrm{d}\|V\|$$

$$+ \int_{\bar{B}_\sigma(p)} \frac{1-a_b(r)}{\sigma^2} \,\mathrm{d}\|V\| + \int_{\bar{B}_\rho(p)\setminus\bar{B}_\sigma(p)} \frac{1-a_b(r)}{r^2} \,\mathrm{d}\|V\| - \int_{\bar{B}_\rho(p)} \frac{1-a_b(r)}{\rho^2} \,\mathrm{d}\|V\|$$

$$- \int_{\bar{B}_\sigma(p)} \frac{g(r\nabla r, H)}{2\sigma^2} \,\mathrm{d}\|V\| + \int_{\bar{B}_\rho(p)} \frac{g(r\nabla r, H)}{2\rho^2} \,\mathrm{d}\|V\|$$

$$+ \int_{\bar{B}_\sigma(p)} \frac{r}{2\sigma^2} \,\|\delta V\|_{\mathrm{sing}} + \int_{\bar{B}_\rho(p)\setminus\bar{B}_\sigma(p)} \frac{1}{2r} \,\mathrm{d}\|\delta V\|_{\mathrm{sing}} + \int_{\bar{B}_\rho(p)} \frac{r}{2\rho^2} \,\mathrm{d}\|\delta V\|_{\mathrm{sing}}$$

which, in view of (5.8), implies the conclusion. $\qquad\square$

5.8 Remark. Define the function

$$c : [0, \pi) \to \mathbb{R}, \qquad c(x) = \frac{1 - x\cot x}{x^2}.$$

Then, using the series expansion of $\cot(x)$ (see (5.7)), we obtain the series expansion for c:

$$c(x) = \frac{1}{3} + \frac{x^2}{45} + \dots$$

with all higher order terms being positive. In particular, $c(0) = \frac{1}{3}$ and c is strictly increasing. Since $c(\frac{\pi}{2}) = \frac{4}{\pi^2}$, the curvature depending term in Lemma 5.7 can be estimated by

$$\int_{\bar{B}_\rho(p)} \frac{1-a_b(r)}{r^2} \,\mathrm{d}\|V\| \le \frac{4}{\pi^2} b\|V\|(\bar{B}_\rho(p)) \le b\|V\|(\bar{B}_\rho(p)) \tag{5.12}$$

whenever $0 < \rho < \frac{\pi}{2\sqrt{b}}$.

The following lemma is a consequence of Lemma 5.7. It can also be derived directly from the first variation formula, see [122, Theorem 5.5].

5.9 Lemma. *Suppose n is an integer, $n \ge 2$, N is an n-dimensional Riemannian manifold, $V \in \mathbb{V}_2(N)$ has generalised mean curvature H, $H(x)\perp T$ for V almost all $(x, T) \in \mathbb{G}_2(N)$, and H is locally square integrable with respect to $\|V\|$.*

Then, there holds $\|V\|\{p\} = 0$ for all $p \in N$.

Proof. Let $p \in N$. For small $\rho > 0$ there exists $b > 0$ such that we can apply Lemma 5.7. Multiplying the inequality in Lemma 5.7 with σ where $0 < \sigma < \rho$, and using (5.12), we infer

$$\frac{\|V\|\bar{B}_\sigma(p)}{\sigma} \le \frac{\|V\|\bar{B}_\rho(p)}{\rho} + \int_{\bar{B}_\rho(p)} |H|_g^2 \,\mathrm{d}\|V\| + b\|V\|(\bar{B}_\rho(p))$$

$$+ \int_{\bar{B}_\rho(p)} |H|_g \,\mathrm{d}\|V\| + \|\delta V\|_{\mathrm{sing}}(\bar{B}_\rho(p)).$$

The right hand side is finite and does not depend on σ. Thus the conclusion follows. $\qquad\square$

The proof of the following theorem is based on the Euclidean version in the appendix of [76]. See also [117, Corollary 5.8] for the existence of the density.

5.10 Theorem (See SCHARRER [124, Theorem 3.6])**.** *Suppose n is an integer, $n \geq 2$, N is an n-dimensional Riemannian manifold, $V \in \mathbb{V}_2(N)$ has generalised mean curvature H, $H(x) \perp T$ for V almost all $(x, T) \in \mathbb{G}_2(N)$, and H is locally square integrable with respect to $\|V\|$.*

Then, for all $p \in N \setminus \operatorname{spt} \|V\|_{\mathrm{sing}}$, there holds:

1. *The density $\Theta^2(\|V\|, p)$ exists.*

2. *The function $\Theta^2(\|V\|, \cdot)$ is upper semi-continuous at p.*

Proof. By (5.12) in combination with Lemma 5.9, we have

$$\int_{\bar{B}_\rho(p)} \frac{1 - a_b(r)}{r^2} \, \mathrm{d}\|V\| = o(1) \qquad \text{as } \rho \to 0, \tag{5.13}$$

where a_b is defined as in Lemma 5.7. We abbreviate

$$W(t) := \int_{\bar{B}_t(p)} |H|_g^2 \, \mathrm{d}\|V\| \quad \text{and} \quad A(t) := \frac{\|V\|\bar{B}_t(p)}{t^2}$$

for $t > 0$. Using Hölder's inequality, we deduce

$$\int_{\bar{B}_t(p)} \frac{|H|_g}{2t} \, \mathrm{d}\|V\| \leq \sqrt{A(t)}\sqrt{W(t)} \leq (1 + A(t))\sqrt{W(t)}. \tag{5.14}$$

Moreover, since H is locally square integrable,

$$W(t) = o(1) \qquad \text{as } t \to 0. \tag{5.15}$$

Choose $\rho_0 > 0$ small enough such that $B_{\rho_0}(p) \cap \operatorname{spt} \|\delta V\|_{\mathrm{sing}} = \varnothing$ and such that Lemma 5.7 can be applied for some $b > 0$. Then, there holds

$$\int_{\bar{B}_\rho(p)} \frac{1}{2r} \, \mathrm{d}\|\delta V\|_{\mathrm{sing}} = \int_{\bar{B}_\rho(p)} \frac{r}{2\rho^2} \, \mathrm{d}\|\delta V\|_{\mathrm{sing}} = 0$$

for all $0 < \rho < \rho_0$. Hence, putting (5.13), (5.14), and (5.15) into the inequality of Lemma 5.7, we infer

$$(1 - o_\sigma(1))A(\sigma) - o_\sigma(1) \leq (1 + o_\rho(1))A(\rho) + o_\rho(1)$$

for all $0 < \sigma < \rho < \rho_0$. Applying $\limsup_{\sigma \to 0+}$ on the left and $\liminf_{\rho \to 0+}$ on the right, it follows

$$\Theta^{*2}(\|V\|, p) \leq \Theta_*^2(\|V\|, p)$$

which proves (1). Hence, letting $\sigma \to 0$ in the inequality of Lemma 5.7, and using $2\sqrt{A(t)}\sqrt{W(t)} \leq A(t) + W(t)$, we have

$$\pi\Theta^2(\|V\|, p) \leq \frac{\|V\|\bar{B}_\rho(p)}{\rho^2} + W(\rho) + b\|V\|(\bar{B}_\rho(p)) + A(\rho) + W(\rho)$$

for small $0 < \rho < \rho_0$. It follows

$$\limsup_{q \to p} \frac{\|V\|\bar{B}_\rho(q)}{\rho^2} \geq \limsup_{q \to p} \pi\Theta^2(\|V\|, q) - W(2\rho) - b\|V\|(\bar{B}_{2\rho}(p)) - A(2\rho) - W(2\rho)$$
$$= \limsup_{q \to p} \pi\Theta^2(\|V\|, q) - o_\rho(1). \tag{5.16}$$

On the other hand,

$$\limsup_{q \to p} \frac{\|V\|\bar{B}_\rho(q)}{\rho^2} \leq \lim_{\varepsilon \to 0} \frac{\|V\|\bar{B}_{\rho+\varepsilon}(p)}{\rho^2} = \frac{\|V\|\bar{B}_\rho(p)}{\rho^2} \tag{5.17}$$

where we used the limit formula for the measure of decreasing sets (see [46, 2.1.3(5)]). Putting (5.16) and (5.17) together and taking the limit $\rho \to 0$ implies statement (2). $\qquad\square$

The following lemma is a generalisation of [23, Theorem 7].

5.11 Lemma. *Suppose n is an integer, $n \geq 2$, (N, g) is an n-dimensional Riemannian manifold, $p \in N$, $V \in \mathbb{V}_2(N)$ has generalised mean curvature H, H is square integrable with respect to $\|V\|$, $H(x) \perp T$ for V almost all $(x, T) \in \mathbb{G}_2(N)$, $b > 0$, $0 < \rho_0 < \frac{\pi}{\sqrt{b}}$, the sectional curvature satisfies $K \leq b$ on $\mathrm{spt}\,\|V\| \cap \bar{B}_{\rho_0}(p)$, U is a geodesically star-shaped open neighbourhood of p, and $\mathrm{spt}\,\|V\| \cap \bar{B}_{\rho_0}(p) \subset U$. Define the functions*

$$s_b : (0, \infty) \to \mathbb{R}, \qquad s_b(t) = \frac{\sin(\sqrt{b}t)}{\sqrt{b}},$$

$c_b := s_b'$, and $\phi := b/(1 - c_b)$.

Then, writing $r = d(p, \cdot)$, there holds

$$2\phi(\sigma) \int_{\bar{B}_\sigma(p)} c_b(r)\,\mathrm{d}\|V\|$$
$$\leq 2\phi(\rho) \int_{\bar{B}_\rho(p)} c_b(r)\,\mathrm{d}\|V\| + b\|V\|(\bar{B}_\rho(p) \setminus \bar{B}_\sigma(p)) + \frac{1}{4} \int_{\bar{B}_\rho(p) \setminus \bar{B}_\sigma(p)} |H|_g^2\,\mathrm{d}\|V\|$$
$$- \phi(\sigma) \int_{\bar{B}_\sigma(p)} s_b(r)g(\nabla r, H)\,\mathrm{d}\|V\| + \phi(\rho) \int_{\bar{B}_\sigma(p)} s_b(r)g(\nabla r, H)\,\mathrm{d}\|V\|$$
$$+ \phi(\sigma) \int_{\bar{B}_\sigma(p)} s_b(r)\,\mathrm{d}\|\delta V\|_{\mathrm{sing}} + \int_{\bar{B}_\rho(p)} 2\sqrt{b}\cot\left(\frac{\sqrt{b}r}{2}\right)\,\mathrm{d}\|\delta V\|_{\mathrm{sing}}$$

for almost all $0 < \sigma < \rho < \rho_0$.

Proof. Given any non-negative smooth function $\varphi : \mathbb{R} \to \mathbb{R}$ whose support is contained in the open interval $(-\infty, \rho_0)$, we define the vector field $X := \varphi(r)s_b(r)\nabla r$. Write the metric in polar coordinates $g = \mathrm{d}r \otimes \mathrm{d}r + g_r$ on U and use Lemma 5.4 to estimate

$$\nabla(s_b(r)\nabla r) = c_b(r)\mathrm{d}r \otimes \mathrm{d}r + s_b(r)\nabla^2 r \geq c_b(r)\mathrm{d}r \otimes \mathrm{d}r + s_b(r)\frac{c_b(r)}{s_b(r)}g_r = c_b(r)g.$$

Hence,

$$\nabla X \geq \varphi'(r)s_b(r)\mathrm{d}r \otimes \mathrm{d}r + \varphi(r)c_b(r)g$$

which implies

$$\text{div}_T X \geq \varphi'(r) s_b(r) |\nabla^T r|_g^2 + 2\varphi(r) c_b(r)$$

for all $T \in \mathbb{G}_2(TU)$, where $\nabla^T r$ denotes the orthogonal projection of ∇r onto T. Writing

$$\nabla^\perp r : \mathbb{G}_2(N) \to TN, \qquad (\nabla^\perp r)(x, T) = (\nabla r)(x) - (\nabla^T r)(x),$$

and testing the first variation equation (see (5.1)) with X, we infer

$$
\begin{aligned}
2 \int_N \varphi(r) c_b(r) \, \mathrm{d}\|V\| &+ \int_{\mathbb{G}_2(N)} \varphi'(r) s_b(r)(1 - |\nabla^\perp r|_g^2) \, \mathrm{d}V \\
&\leq -\int_N \varphi(r) g(Y, H) \, \mathrm{d}\|V\| + \int_N \varphi(r) s_b(r) g(\nabla r, \eta) \, \mathrm{d}\|\delta V\|_{\text{sing}},
\end{aligned}
\tag{5.18}
$$

where we abbreviated $Y := s_b \nabla^\perp r$. Notice that the function $t \mapsto \|V\| \bar{B}_t(p)$ is continuous at $t_0 > 0$ if and only if $\|V\|(\{r = t_0\}) = 0$. Since the function $t \mapsto \|V\| \bar{B}_t(p)$ is non-decreasing, it can only have countably many discontinuity points. Choose $0 < \sigma < \rho < \rho_0$ to be continuity points. Define the non-increasing Lipschitz function

$$\phi_\sigma : (0, \infty) \to \mathbb{R}, \qquad \phi_\sigma(t) = \phi(\max\{t, \sigma\})$$

and let φ approach $(\phi_\sigma(\cdot) - \phi(\rho))_+$, where $(\cdot)_+ := \max\{\cdot, 0\}$, such that on $[0, \rho_0) \setminus \{\sigma, \rho\}$, the function φ' approaches $(\phi_\sigma(\cdot) - \phi(\rho))'_+$. Then, by the dominated convergence theorem, the inequality (5.18) becomes

$$
\begin{aligned}
2\phi(\sigma) \int_{\bar{B}_\sigma(p)} c_b(r) \, \mathrm{d}\|V\| &+ 2 \int_{\bar{B}_\rho(p) \setminus \bar{B}_\sigma(p)} \phi(r) c_b(r) \, \mathrm{d}\|V\| \\
&\leq 2\phi(\rho) \int_{\bar{B}_\rho(p)} c_b(r) \, \mathrm{d}\|V\| - \int_{\pi^{-1}[\bar{B}_\rho(p) \setminus \bar{B}_\sigma(p)]} \phi'(r) s_b(r) [1 - |\nabla^\perp r|_g^2] \, \mathrm{d}V \\
&\quad - \int_{\bar{B}_\rho(p)} (\phi_\sigma(r) - \phi(\rho))_+ g(Y, H) \, \mathrm{d}\|V\| + \phi(\sigma) \int_{\bar{B}_\sigma(p)} s_b(r) g(\nabla r, \eta) \, \mathrm{d}\|\delta V\|_{\text{sing}} \\
&\quad + \int_{\bar{B}_\rho(p) \setminus \bar{B}_\sigma(p)} \phi(r) s_b(r) g(\nabla r, \eta) \, \mathrm{d}\|\delta V\|_{\text{sing}} - \phi(\rho) \int_{\bar{B}_\rho(p)} s_b(r) g(\nabla r, \eta) \, \mathrm{d}\|\delta V\|_{\text{sing}}
\end{aligned}
\tag{5.19}
$$

where $\pi : \mathbb{G}_2(N) \to N$ is the canonical projection. Next, we compute

$$2\phi c_b + \phi' s_b = \frac{b}{(1 - c_b)^2} \left[-2c_b^2 + 2c_b + c_b' s_b \right] = -b$$

as well as $\phi'(r) s_b(r) = -(\phi(r) s_b(r))^2$. Hence, the inequality 5.19 becomes

$$
\begin{aligned}
2\phi(\sigma) \int_{\bar{B}_\sigma(p)} c_b(r) \, \mathrm{d}\|V\| &- b\|V\|(\bar{B}_\rho(p) \setminus \bar{B}_\sigma(p)) \\
&\leq 2\phi(\rho) \int_{\bar{B}_\rho(p)} c_b(r) \, \mathrm{d}\|V\| - \int_{\pi^{-1}[\bar{B}_\rho(p) \setminus \bar{B}_\sigma(p)]} \phi(r)^2 |Y|_g^2 \, \mathrm{d}\|V\| \\
&\quad - \int_{\bar{B}_\rho(p)} (\phi_\sigma(r) - \phi(\rho))_+ g(Y, H) \, \mathrm{d}\|V\| \\
&\quad + \phi(\sigma) \int_{\bar{B}_\sigma(p)} s_b(r) \, \mathrm{d}\|\delta V\|_{\text{sing}} + 2 \int_{\bar{B}_\rho(p)} \phi(r) s_b(r) \, \mathrm{d}\|\delta V\|_{\text{sing}}
\end{aligned}
\tag{5.20}
$$

We claim that on $\bar{B}_\rho(p) \setminus \bar{B}_\sigma(p)$,

$$-\phi(r)^2|Y|_g^2 - (\phi_\sigma(r) - \phi(\rho))_+ g(Y,H) \leq \frac{1}{4}|H|_g^2. \tag{5.21}$$

Indeed, this is clear if $g(Y,H) \geq 0$. If on the other hand $g(Y,H) \leq 0$, then $\phi(\rho)g(Y,H) \leq 0$ and thus

$$-\phi(r)^2|Y|_g^2 - (\phi_\sigma(r) - \phi(\rho))_+ g(Y,H) = -\phi(r)^2|Y|_g^2 - \phi(r)g(Y,H) + \phi(\rho)g(Y,H)$$

$$\leq -\left|\phi(r)Y + \frac{1}{2}H\right|_g^2 + \frac{1}{4}|H|_g^2$$

which implies that (5.21) is always true. Putting (5.21) into (5.20) and noting that $\phi(r)s_b(r) = \sqrt{b}\cot(\frac{\sqrt{b}r}{2})$ concludes the proof. $\qquad\square$

The following lemma is a generalisation of [23, Equation (10)].

5.12 Lemma (See SCHARRER [124, Lemma 3.7]). *Suppose n is an integer, $n \geq 2$, (N,g) is an n-dimensional Riemannian manifold, $p \in N$, $V \in \mathbb{V}_2(N)$ has generalised mean curvature H, H is square integrable with respect to $\|V\|$, $H(x) \perp T$ for V almost all $(x,T) \in \mathbb{G}_2(N)$, $b < 0$, $\rho_0 > 0$, the sectional curvature satisfies $K \leq b$ on $\operatorname{spt}\|V\| \cap \bar{B}_{\rho_0}(p)$, U is a geodesically star-shaped open neighbourhood of p, and $\operatorname{spt}\|V\| \cap \bar{B}_{\rho_0}(p) \subset U$. Define the functions*

$$s_b : (0,\infty) \to \mathbb{R}, \qquad s_b(t) = \frac{\sinh(\sqrt{|b|}t)}{\sqrt{|b|}},$$

$c_b := s_b'$, *and* $\phi := |b|/(c_b - 1)$.

Then, writing $r = d(p,\cdot)$, there holds

$$2\phi(\sigma)\int_{\bar{B}_\sigma(p)} c_b(r)\,\mathrm{d}\|V\| + |b|\|V\|(\bar{B}_\rho(p) \setminus \bar{B}_\sigma(p))$$

$$\leq 2\phi(\rho)\int_{\bar{B}_\rho(p)} c_b(r)\,\mathrm{d}\|V\| + \frac{1}{4}\int_{\bar{B}_\rho(p)\setminus\bar{B}_\sigma(p)} |H|_g^2\,\mathrm{d}\|V\|$$

$$- \phi(\sigma)\int_{\bar{B}_\sigma(p)} s_b(r)g(\nabla r,H)\,\mathrm{d}\|V\| + \phi(\rho)\int_{\bar{B}_\rho(p)} s_b(r)g(\nabla r,H)\,\mathrm{d}\|V\|$$

$$+ \phi(\sigma)\int_{\bar{B}_\sigma(p)} s_b(r)g(\nabla r,\eta)\,\mathrm{d}\|\delta V\|_{\mathrm{sing}} - \phi(\rho)\int_{\bar{B}_\rho(p)} s_b(r)g(\nabla r,\eta)\,\mathrm{d}\|\delta V\|_{\mathrm{sing}}$$

$$+ \int_{\bar{B}_\rho(p)\setminus\bar{B}_\sigma(p)} \phi(r)s_b(r)g(\nabla r,\eta)\,\mathrm{d}\|\delta V\|_{\mathrm{sing}}$$

for almost all $0 < \sigma < \rho < \rho_0$.

Proof. Given any non-negative smooth function $\varphi : \mathbb{R} \to \mathbb{R}$ whose support is contained in the open interval $(-\infty, \rho_0)$, we define the vector field $X := \varphi(r)s_b(r)\nabla r$. Write the metric in polar coordinates $g = \mathrm{d}r \otimes \mathrm{d}r + g_r$ on U and use Lemma 5.4 to estimate

$$\nabla(s_b(r)\nabla r) = c_b(r)\mathrm{d}r \otimes \mathrm{d}r + s_b(r)\nabla^2 r \geq c_b(r)\mathrm{d}r \otimes \mathrm{d}r + s_b(r)\frac{c_b(r)}{s_b(r)}g_r \geq c_b(r)g.$$

Hence,

$$\nabla X \geq \varphi'(r)s_b(r)\mathrm{d}r \otimes \mathrm{d}r + \varphi(r)c_b(r)g$$

which implies

$$\mathrm{div}_T X \geq \varphi'(r)s_b(r)|\nabla^T r|_g^2 + 2\varphi(r)c_b(r)$$

for all $T \in \mathbb{G}_2(TU)$, where $\nabla^T r$ denotes the orthogonal projection of ∇r onto T. Writing

$$\nabla^\perp r : \mathbb{G}_2(N) \to TN, \qquad (\nabla^\perp r)(x, T) = (\nabla r)(x) - (\nabla^T r)(x),$$

and testing the first variation equation (see (5.1)) with X, we infer

$$
\begin{aligned}
2\int_N \varphi(r)c_b(r)\,\mathrm{d}\|V\| &+ \int_{\mathbb{G}_2(N)} \varphi'(r)s_b(r)(1 - |\nabla^\perp r|_g^2)\,\mathrm{d}V \\
&\leq -\int_N \varphi(r)s_b(r)g(\nabla r, H)\,\mathrm{d}\|V\| + \int_N \varphi(r)s_b(r)g(\nabla r, \eta)\,\mathrm{d}\|\delta V\|_{\mathrm{sing}}.
\end{aligned}
\tag{5.22}
$$

Notice that the function $t \mapsto \|V\|\bar{B}_t(p)$ is continuous at t_0 if and only if $\|V\|(\{r = t_0\}) = 0$. Since the function $t \mapsto \|V\|\bar{B}_t(p)$ is non-decreasing, it can only have countably many discontinuity points. Choose $0 < \sigma < \rho < \rho_0$ to be continuity points. Define the non-increasing Lipschitz function

$$\phi_\sigma : (0, \infty) \to \mathbb{R}, \qquad \phi_\sigma(t) = \phi(\max\{t, \sigma\})$$

and let φ approach $(\phi_\sigma(\cdot) - \phi(\rho))_+$, where $(\cdot)_+ := \max\{\cdot, 0\}$. Then, by the dominated convergence theorem, (5.22) becomes

$$
\begin{aligned}
2\phi(\sigma)\int_{\bar{B}_\sigma(p)} c_b(r)\,\mathrm{d}\|V\| &+ 2\int_{\bar{B}_\rho(p)\setminus\bar{B}_\sigma(p)} \phi(r)c_b(r)\,\mathrm{d}\|V\| \\
\leq 2\phi(\rho)\int_{\bar{B}_\rho(p)} c_b(r)\,\mathrm{d}\|V\| &- \int_{\pi^{-1}[\bar{B}_\rho(p)\setminus\bar{B}_\sigma(p)]} \phi'(r)s_b(r)[1 - |\nabla^\perp r|_g^2]\,\mathrm{d}\|V\| \\
- \phi(\sigma)\int_{\bar{B}_\sigma(p)} s_b(r)g(\nabla r, H)\,\mathrm{d}\|V\| &- \int_{\bar{B}_\rho(p)\setminus\bar{B}_\sigma(p)} \phi(r)s_b(r)g(\nabla r, H)\,\mathrm{d}\|V\| \\
+ \phi(\rho)\int_{\bar{B}_\rho(p)} s_b(r)g(\nabla r, H)\,\mathrm{d}\|V\| &+ \phi(\sigma)\int_{\bar{B}_\sigma(p)} s_b(r)g(\nabla r, \eta)\,\mathrm{d}\|\delta V\|_{\mathrm{sing}} \\
+ \int_{\bar{B}_\rho(p)\setminus\bar{B}_\sigma(p)} \phi(r)s_b(r)g(\nabla r, \eta)\,\mathrm{d}\|\delta V\|_{\mathrm{sing}} &- \phi(\rho)\int_{\bar{B}_\rho(p)} s_b(r)g(\nabla r, \eta)\,\mathrm{d}\|\delta V\|_{\mathrm{sing}}
\end{aligned}
\tag{5.23}
$$

where $\pi : \mathbb{G}_2(N) \to N$ is the canonical projection. We compute

$$2\phi c_b + \phi' s_b = \frac{|b|}{(c_b - 1)^2}\left[2c_b^2 - 2c_b - c_b' s_b\right] = |b| \tag{5.24}$$

as well as $\phi'(r)s_b(r) = -(\phi(r)s_b(r))^2$, and

$$\phi'(r)s_b(r)|\nabla^\perp r|_g^2 - \phi(r)s_b(r)g(\nabla r, H) = -\left|\phi(r)s_b(r)\nabla^\perp r + \frac{1}{2}H\right|_g^2 + \frac{1}{4}|H|_g^2. \tag{5.25}$$

Putting (5.24) and (5.25) into (5.23) and neglecting negative terms on the right hand side implies the conclusion. $\qquad\square$

5.3 Li–Yau inequalities

In this section, we prove the general Li–Yau inequality (see Theorem 5.13). For its smooth version, see Corollary 5.15.

5.13 Theorem (See SCHARRER [124, Theorem 1.7])**.** *Suppose n is an integer, $n \geq 2$, (N, g) is an n-dimensional Riemannian manifold, $p \in N$, U is a geodesically star-shaped open neighbourhood of p, $V \in \mathbb{V}_2(N)$ has generalised mean curvature H, H is square integrable with respect to $\|V\|$, $H(x) \perp T$ for V almost all $(x, T) \in \mathbb{G}_2(N)$, $p \notin \operatorname{spt} \|\delta V\|_{\mathrm{sing}}$, $\operatorname{spt} \|V\|$ is compact, $\operatorname{spt} \|V\| \subset U$, $b \in \mathbb{R}$, and the sectional curvature of N satisfies $\sup_{\operatorname{spt} \|V\|} K \leq b$.*

Then, writing $r = d(p, \cdot)$, the following two statements hold:

1. *If $b \geq 0$ and $\sup_{x \in \operatorname{spt} \|V\|} r(x) < \frac{\pi}{\sqrt{b}}$, then*

$$\frac{1}{4} \int_N |H|_g^2 \, \mathrm{d}\|V\| + b \int_N \cos(\sqrt{b}r) \, \mathrm{d}\|V\| + \int_N t_b(r) \, \mathrm{d}\|\delta V\|_{\mathrm{sing}} + b\|V\|(N) \geq 4\pi\Theta^2(\|V\|, p),$$

   where $t_b(r) = 2\sqrt{b}\cot(\frac{\sqrt{b}r}{2})$ if $b > 0$ and $t_b(r) = \frac{4}{r}$ if $b = 0$.

2. *If $b < 0$, then*

$$\frac{1}{4} \int_N |H|_g^2 \, \mathrm{d}\|V\| + \int_N t_b(r) \, \mathrm{d}\|\delta V\|_{\mathrm{sing}} + b\|V\|(N) \geq 4\pi\Theta^2(\|V\|, p),$$

   where $t_b(r) = \sqrt{|b|}\coth(\frac{\sqrt{|b|}r}{2})$.

Proof. We first notice that if the statement (1) holds for all $b > 0$, then the case $b = 0$ follows by letting $b \to 0+$. Hence, to prove (1), we may assume $b > 0$. We are going to determine the limits in Lemma 5.11 as $\sigma \to 0+$ and $\rho \to \frac{\pi}{\sqrt{b}}-$. Using L'Hôspital's rule twice, one readily verifies

$$\frac{\sigma^2}{1 - \cos(\sqrt{b}\sigma)} \to \frac{2}{b} \qquad \text{as } \sigma \to 0+.$$

Therefore, by Theorem 5.10,

$$2\phi(\sigma) \int_{\bar{B}_\sigma(p)} c_b(r) \, \mathrm{d}\|V\| = 2\pi b \frac{\sigma^2}{1 - \cos(\sqrt{b}\sigma)} \frac{1}{\pi\sigma^2} \int_{\bar{B}_\sigma(p)} \cos(\sqrt{b}r) \, \mathrm{d}\|V\| \to 4\pi\Theta^2(\|V\|, p).$$

Similarly, by L'Hôspital's rule,

$$\frac{\sqrt{b}\pi \sin(\sqrt{b}\sigma)\sigma}{1 - \cos(\sqrt{b}\sigma)} \to 2\sqrt{\pi} \qquad \text{as } \sigma \to 0+.$$

Hence, by Hölder's inequality and square integrability of the generalised mean curvature,

$$\phi(\sigma) \int_{\bar{B}_\sigma(p)} s_b(r)g(\nabla r, H) \, \mathrm{d}\|V\| \leq \frac{\sqrt{b}\pi \sin(\sqrt{b}\sigma)\sigma}{1 - \cos(\sqrt{b}\sigma)} \left(\frac{\|V\|\bar{B}_\sigma(p)}{\pi\sigma^2} \right)^{1/2} \left(\int_{\bar{B}_\sigma(p)} |H|_g^2 \, \mathrm{d}\|V\| \right)^{1/2}$$

$$\to \left(4\pi\Theta^2(\|V\|, p) \right)^{1/2} \limsup_{\sigma \to 0+} \left(\int_{\bar{B}_\sigma(p)} |H|_g^2 \, \mathrm{d}\|V\| \right)^{1/2} = 0.$$

The other limits can be easily determined using that $p \notin \operatorname{spt} \|\delta V\|_{\mathrm{sing}}$.

To prove (2), assume that $b < 0$. We are going to determine the limits in Lemma 5.12 as $\sigma \to 0+$ and $\rho \to \infty$. Using L'Hôspital's rule twice, one readily verifies

$$\frac{\sigma^2}{\cosh(\sqrt{|b|}\sigma) - 1} \to \frac{2}{|b|} \qquad \text{as } \sigma \to 0+.$$

Therefore, by Theorem 5.10,

$$2\phi(\sigma) \int_{\bar{B}_\sigma(p)} c_b(r) \, \mathrm{d}\|V\| = 2\pi|b| \frac{\sigma^2}{\cosh(\sqrt{|b|}\sigma) - 1} \frac{1}{\pi\sigma^2} \int_{\bar{B}_\sigma(p)} \cosh(\sqrt{|b|}r) \, \mathrm{d}\|V\| \to 4\pi\Theta^2(\|V\|, p).$$

Similarly, by L'Hôspital's rule,

$$\frac{\sqrt{|b|\pi} \sinh(\sqrt{|b|}\sigma)\sigma}{\cosh(\sqrt{|b|}\sigma) - 1} \to 2\sqrt{\pi} \qquad \text{as } \sigma \to 0+.$$

Hence, by Hölder's inequality and square integrability of the generalised mean curvature,

$$\phi(\sigma) \int_{\bar{B}_\sigma(p)} s_b(r) g(\nabla r, H) \, \mathrm{d}\|V\| \leq \frac{\sqrt{|b|\pi} \sinh(\sqrt{|b|}\sigma)\sigma}{\cosh(\sqrt{|b|}\sigma) - 1} \left(\frac{\|V\|\bar{B}_\sigma(p)}{\pi\sigma^2}\right)^{1/2} \left(\int_{\bar{B}_\sigma(p)} |H|_g^2 \, \mathrm{d}\|V\|\right)^{1/2}$$
$$\to \left(4\pi\Theta^2(\|V\|, p)\right)^{1/2} \limsup_{\sigma \to 0+} \left(\int_{\bar{B}_\sigma(p)} |H|_g^2 \, \mathrm{d}\|V\|\right)^{1/2} = 0.$$

All the other limits can be easily determined using that $\operatorname{spt} \|V\|$ is compact and using that $p \notin \operatorname{spt} \|\delta V\|_{\mathrm{sing}}$. $\qquad\square$

5.14 Remark. Notice that the existence of the density $\Theta^2(\|V\|, p)$ is part of the statement. Indeed, existence of the density as well as its upper semi-continuity are local statements that do not require any global upper bounds on the curvature nor do they require positive injectivity radius, see Theorem 5.10.

If $N = \mathbb{R}^n$, then the condition on the generalised mean curvature to bo normal:

$$H(x) \perp T \qquad \text{for } V \text{ almost all } (x, t) \in \mathbb{G}_2(N)$$

is satisfied for all integral varifolds, see [17, Section 5.8].

5.15 Corollary (See SCHARRER [124, Corollary 1.9]). *Suppose $n \geq 3$ is an integer, (N, g) is an n-dimensional Riemannian manifold, Σ is a smooth closed surface, $f : \Sigma \to N$ is a smooth immersion, $p \in N$, $f^{-1}\{p\} = \{x_1, \ldots x_k\}$ where the x_i's are distinct points in Σ, $b \in \mathbb{R}$, and the sectional curvature K of N satisfies $K \leq b$ on the image of f. Let H be the trace of the second fundamental form of the immersion f, μ be the Radon measure on Σ induced by the pull-back metric of g along f, and $|\Sigma| := \int_\Sigma 1 \, \mathrm{d}\mu$ be the area of Σ in N. Then, the following two statements hold.*

1. *If $b > 0$, and the image of f is contained in a geodesic ball around p of radius strictly less than $\frac{\pi}{\sqrt{b}}$, then*

$$\frac{1}{4} \int_\Sigma |H|_g^2 \, \mathrm{d}\mu + b \int_\Sigma \cos(\sqrt{b}r) \, \mathrm{d}\mu + b|\Sigma| \geq 4\pi k,$$

where $r = d(p, \cdot)$ is the distance to p in N.

2. *If $b \leq 0$ and the image of f is contained in a geodesically star-shaped open neighbourhood of p, then*

$$\frac{1}{4} \int_\Sigma |H|_g^2 \, \mathrm{d}\mu + b|\Sigma| \geq 4\pi k.$$

In particular, if the left hand side is strictly smaller than 8π, then f is an embedding.

Proof. This is a combination of Theorem 5.13 and Example 5.3. $\square$

5.16 Remark. If N is a Cartan–Hadamard manifold, then N itself is a geodesically star-shaped open neighbourhood of any point. In particular, there is no condition on f in (2).

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