NONCOMMUTATIVE RIEMANN HYPOTHESIS

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Abstract. In this note, making use of noncommutative l-adic cohomology, we extend the generalized Riemann hypothesis from the realm of algebraic geometry to the broad setting of geometric noncommutative schemes in the sense of Orlov. As a first application, we prove that the generalized Riemann hypothesis is invariant under derived equivalences and homological projective duality. As a second application, we prove the noncommutative generalized Riemann hypothesis in some new cases.

1. Introduction and statement of results

Let k be a global field and Σ_k its (infinite) set of non-archimedean places.

Let X be a smooth proper k-scheme and $0 \leq w \leq 2\dim(X)$ an integer. Following Serre's foundational work [19, 20] (consult also Manin [14]), consider the L-function $L_w(X;s) := \prod_{\nu\in\Sigma_k} L_{w,\nu}(X;s)$ of weight w. As proved in *loc. cit.*, this infinite product converges absolutely in the half-plane $\mathrm{Re}(s) > \frac{w}{2}+1$ and is non-zero in this region. Moreover, the following two conditions are expected to hold:

(C1) The L-function $L_w(X;s)$ admits a (unique) meromorphic continuation to the entire complex plane.
(C2) When $\mathrm{char}(k)=0$, the only possible pole of $L_w(X;s)$ is located at $s=\frac{w}{2}+1$ with w even.

When $\mathrm{char}(k)=0$, the conditions (C1)-(C2) have been proved in many cases: certain 0-dimensional schemes, certain elliptic curves, certain modular curves, certain abelian varieties, certain varieties of Fermat type, certain Shimura varieties, etc. When $\mathrm{char}(k)>0$, condition (C1) follows from Grothendieck's work [5].

The following conjecture, which implicitly assumes condition (C1), goes back to the work [18] of Riemann.

Generalized Riemann hypothesis $\mathrm{R}_w(X)$: *All the zeros of the L-function $L_w(X;s)$ that are contained in the critical strip $\frac{w}{2} < \mathrm{Re}(s) < \frac{w}{2}+1$ lie in the vertical line* $\mathrm{Re}(s)=\frac{w+1}{2}$.

The generalized Riemann hypothesis play a central role in mathematics. For example, when $\mathrm{char}(k)=0$ and $X=\mathrm{Spec}(k)$, the conjecture $\mathrm{R}_0(X)$ reduces to the classical *extended Riemann hypothesis* ERH_k, i.e., all the zeros of the Dedekind zeta function $\zeta_k(s) := \sum_{I\lhd\mathcal{O}_k}\frac{1}{N(I)^s}$ that are contained in the critical strip $0<\mathrm{Re}(s)<1$ lie in the vertical line $\mathrm{Re}(s)=\frac{1}{2}$; note that in the particular case where $k=\mathbb{Q}$, ERH_k is the famous Riemann hypothesis. The status of the generalized Riemann conjecture depends drastically on the characteristic of k. On the one hand, when $\mathrm{char}(k)=0$, no cases have been proved. On the other hand, when $\mathrm{char}(k)>0$, the generalized Riemann hypothesis follows from Deligne's work [3, 4].

Recall from Keller's survey [7, §4.6] that the derived category of perfect complexes $\mathrm{perf}(X)$ of a smooth proper k-scheme X admits a canonical dg enhancement $\mathrm{perf}_{\mathrm{dg}}(X)$. The following notion, introduced by Orlov in [17, Def. 4.3], plays a central role in this note:

Definition 1.1. A dg category $\mathcal{A}$ is called a *geometric noncommutative k-scheme* if there exists a smooth proper k-scheme X and an admissible triangulated subcategory $\mathfrak{A}$ of $\mathrm{perf}(X)$ such that $\mathcal{A}$ and the full dg subcategory $\mathfrak{A}_{\mathrm{dg}}$ of $\mathrm{perf}_{\mathrm{dg}}(X)$, consisting of the objects of $\mathfrak{A}$, are Morita equivalent.

Every geometric noncommutative k-scheme is a smooth proper dg category in the sense of Kontsevich [10]. Orlov asked in [17, Question 4.4] if there exist smooth proper dg categories which are *not* geometric noncommutative schemes. To the best of the author's knowledge, this question remains wide open.

Let $\mathcal{A}$ be a geometric noncommutative k-scheme; consult §3 below for several examples. In §6, making use of noncommutative l-adic cohomology, we construct the noncommutative counterparts $L_{\mathrm{even}}(\mathcal{A};s) := \prod_{\nu\in\Sigma_k} L_{\mathrm{even},\nu}(\mathcal{A};s)$ and $L_{\mathrm{odd}}(\mathcal{A};s) := \prod_{\nu\in\Sigma_k} L_{\mathrm{odd},\nu}(\mathcal{A};s)$ of the classical L-functions. Moreover, we prove the following noncommutative counterpart of Serre's convergence result:

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Theorem 1.2. *The infinite product $L_{\mathrm{even}}(\mathcal{A};s)$, resp. $L_{\mathrm{odd}}(\mathcal{A};s)$, converges absolutely in the half-plane $\mathrm{Re}(s)>1$, resp. $\mathrm{Re}(s)>\frac{3}{2}$, and is non-zero in this region.*

Similarly to the above condition (C1), it is expected that the noncommutative L-functions $L_{\mathrm{even}}(\mathcal{A};s)$ and $L_{\mathrm{odd}}(\mathcal{A};s)$ admit a (unique) meromorphic continuation to the entire complex plane. Under this assumption, the generalized Riemann hypothesis admits the following noncommutative counterpart:

Noncommutative generalized Riemann hypothesis $\mathrm{R}_{\mathrm{even}}(\mathcal{A})$ and $\mathrm{R}_{\mathrm{odd}}(\mathcal{A})$: *All the zeros of the noncommutative L-function $L_{\mathrm{even}}(\mathcal{A};s)$, resp. $L_{\mathrm{odd}}(\mathcal{A};s)$, that are contained in the critical strip $0<\mathrm{Re}(s)<1$, resp. $\frac{1}{2}<\mathrm{Re}(s)<\frac{3}{2}$, lie in the vertical line $\mathrm{Re}(s)=\frac{1}{2}$, resp. $\mathrm{Re}(s)=1$.*

The noncommutative (generalized) Riemann hypothesis was originally envisioned by Kontsevich in his seminal talks [8, 9]. The next result relates this conjecture with the generalized Riemann hypothesis:

Theorem 1.3. *Given a smooth proper k-scheme X, we have the following implications:*

$$\{\mathrm{R}_w(X)\}_{w\,\mathrm{even}} \Rightarrow \mathrm{R}_{\mathrm{even}}(\mathrm{perf}_{\mathrm{dg}}(X)) \qquad \{\mathrm{R}_w(X)\}_{w\,\mathrm{odd}} \Rightarrow \mathrm{R}_{\mathrm{odd}}(\mathrm{perf}_{\mathrm{dg}}(X))\,. \tag{1.4}$$

When $\mathrm{char}(k)>0$, the converse implications of (1.4) hold. Moreover, when $\mathrm{char}(k)=0$ and the L-functions $\{L_w(X;s)\}_{0\leq w\leq 2\dim(X)}$ satisfy condition (C2), the converse implications of (1.4) also hold.

Intuitively speaking, Theorem 1.3 shows that the generalized Riemann hypothesis belongs not only to the realm of algebraic geometry but also to the broad setting of geometric noncommutative schemes.

2. Applications to commutative geometry

Let k be a global field. In this section, making use of Theorem 1.3, we prove that the generalized Riemann hypothesis is invariant under derived equivalences and homological projective duality. Since the generalized Riemann hypothesis holds when $\mathrm{char}(k)>0$, we will restrict ourselves in this section to the case $\mathrm{char}(k)=0$.

Derived invariance. Let X and Y be two smooth proper k-schemes. In what follows, we assume that the associated L-functions $\{L_w(X;s)\}_{0\leq w\leq 2\dim(X)}$ and $\{L_w(Y;s)\}_{0\leq w\leq 2\dim(Y)}$ satisfy condition (C2).

Corollary 2.1 (Derived invariance)**.** *If the derived categories of perfect complexes $\mathrm{perf}(X)$ and $\mathrm{perf}(Y)$ are (Fourier-Mukai) equivalent, then we have the following equivalences:*

$$\{\mathrm{R}_w(X)\}_{w\,\mathrm{even}} \Leftrightarrow \{\mathrm{R}_w(Y)\}_{w\,\mathrm{even}} \qquad \{\mathrm{R}_w(X)\}_{w\,\mathrm{odd}} \Leftrightarrow \{\mathrm{R}_w(Y)\}_{w\,\mathrm{odd}}\,. \tag{2.2}$$

Proof. If the triangulated categories $\mathrm{perf}(X)$ and $\mathrm{perf}(Y)$ are (Fourier-Mukai) equivalent, then the dg categories $\mathrm{perf}_{\mathrm{dg}}(X)$ and $\mathrm{perf}_{\mathrm{dg}}(Y)$ are Morita equivalent. Consequently, we obtain the following equivalences:

$$\mathrm{R}_{\mathrm{even}}(\mathrm{perf}_{\mathrm{dg}}(X)) \Leftrightarrow \mathrm{R}_{\mathrm{even}}(\mathrm{perf}_{\mathrm{dg}}(Y)) \qquad \mathrm{R}_{\mathrm{odd}}(\mathrm{perf}_{\mathrm{dg}}(X)) \Leftrightarrow \mathrm{R}_{\mathrm{odd}}(\mathrm{perf}_{\mathrm{dg}}(Y))\,.$$

By combining them with Theorem 1.3, we hence obtain the above equivalences (2.2). □

In the literature there are numerous examples of smooth proper k-schemes X and Y for which the above Corollary 2.1 applies; consult, for example, the book [6] and the references therein.

Homological Projective Duality. Homological Projective Duality (=HPD) was introduced by Kuznetsov in [13] as a tool to study the derived categories of perfect complexes of linear sections. Let X be a smooth proper k-scheme equipped with a line bundle $\mathcal{L}_X(1)$; we write $X\to\mathbb{P}(V)$ for the associated map, where $V:=H^0(X,\mathcal{L}_X(1))^\vee$. Assume that we have a Lefschetz decomposition $\mathrm{perf}(X)=\langle\mathbb{A}_0,\mathbb{A}_1(1),\ldots,\mathbb{A}_{i-1}(i-1)\rangle$ with respect to $\mathcal{L}_X(1)$ in the sense of [13, Def. 4.1]. Following [13, Def. 6.1], let us write Y for the HP-dual of X, $\mathcal{L}_Y(1)$ for the HP-dual line bundle, and $Y\to\mathbb{P}(V^\vee)$ for the associated map. Given a generic linear subspace $L\subset V^\vee$, consider the smooth linear sections $X_L:=X\times_{\mathbb{P}(V)}\mathbb{P}(L^\perp)$ and $Y_L:=Y\times_{\mathbb{P}(V^\vee)}\mathbb{P}(L)$. In what follows, we assume that the associated L-functions $\{L_w(X_L;s)\}_{0\leq w\leq 2\dim(X_L)}$ and $\{L_w(Y_L;s)\}_{0\leq w\leq 2\dim(Y_L)}$ satisfy the above condition (C2).

Theorem 2.3 (HPD-invariance)**.** *Assume that the triangulated category $\mathbb{A}_0$ admits a full exceptional collection. Under this assumption, the following holds:*

$$\mathrm{ERH}_k \Rightarrow \big(\{\mathrm{R}_w(X_L)\}_{w\,\mathrm{even}} \Leftrightarrow \{\mathrm{R}_w(Y_L)\}_{w\,\mathrm{even}}\big) \qquad \{\mathrm{R}_w(X_L)\}_{w\,\mathrm{odd}} \Leftrightarrow \{\mathrm{R}_w(Y_L)\}_{w\,\mathrm{odd}}\,. \tag{2.4}$$

Remark 2.5. The assumption of Theorem 2.3 is quite mild since it holds in all the examples in the literature.

Roughly speaking, Theorem 2.3 "cuts in half" the difficulty of proving the generalized Riemann hypothesis, i.e., if the generalized Riemann hypothesis holds for a linear section, then it also holds for the HP-dual linear section. In the literature there are numerous examples of homological projective dualities for which Theorem 2.3 applies (e.g., Veronese-Clifford duality, Grassmannian-Pfaffian duality, Spinor duality, Determinantal duality, Symplectic duality, etc); consult, for example, the surveys [12, 24] and the references therein.

3. Applications to noncommutative geometry

Let k be a global field (of arbitrary characteristic). In this section, making use of Theorem 1.3, we prove the noncommutative generalized Riemann hypothesis in some new cases. In what follows, $\wedge$ stands for the logical symbol of conjugation.

Noncommutative gluings of schemes. Let X and Y be two smooth proper k-schemes and B a perfect dg $\mathrm{perf}_{\mathrm{dg}}(X)$-$\mathrm{perf}_{\mathrm{dg}}(Y)$ bimodule. Following Orlov [17, Def. 3.5], we can consider the gluing $X \odot_{\mathrm{B}} Y$ of the dg categories $\mathrm{perf}_{\mathrm{dg}}(X)$ and $\mathrm{perf}_{\mathrm{dg}}(Y)$ via the dg bimodule B (Orlov used a different notation). As proved by Orlov in [17, Thm. 4.11], $X \odot_{\mathrm{B}} Y$ is a geometric noncommutative k-scheme.

Theorem 3.1. *We have the following implications:*

$$\{\mathrm{R}_w(X)\}_{w\,\mathrm{even}} \wedge \{\mathrm{R}_w(Y)\}_{w\,\mathrm{even}} \Rightarrow \mathrm{R}_{\mathrm{even}}(X \odot_{\mathrm{B}} Y) \qquad \{\mathrm{R}_w(X)\}_{w\,\mathrm{odd}} \wedge \{\mathrm{R}_w(Y)\}_{w\,\mathrm{odd}} \Rightarrow \mathrm{R}_{\mathrm{odd}}(X \odot_{\mathrm{B}} Y)\,.$$

In particular, the conjectures $\mathrm{R}_{\mathrm{even}}(X \odot_{\mathrm{B}} Y)$ *and* $\mathrm{R}_{\mathrm{odd}}(X \odot_{\mathrm{B}} Y)$ *hold when* $\mathrm{char}(k) > 0$.

Calabi-Yau dg categories associated to hypersurfaces. Let $X \subset \mathbb{P}^n$ be a smooth hypersurface of degree $\deg(X) \leq n+1$. As proved by Kuznetsov in [11, Cor. 4.1], we have a semi-orthogonal decomposition $\mathrm{perf}(X) = \langle \mathcal{T}, \mathcal{O}_X, \ldots, \mathcal{O}_X(n-\deg(X))\rangle$. Moreover, the full dg subcategory $\mathcal{T}_{\mathrm{dg}}$ of $\mathrm{perf}_{\mathrm{dg}}(X)$, consisting of the objects of $\mathcal{T}$, is a Calabi-Yau dg category of fractional CY-dimension $\frac{(n+1)(\deg(X)-2)}{\deg(X)}$. Note that $\mathcal{T}_{\mathrm{dg}}$ is a geometric noncommutative k-scheme. Note also that $\mathcal{T}_{\mathrm{dg}}$ is *not* Morita equivalent to a dg category of the form $\mathrm{perf}_{\mathrm{dg}}(Y)$, with Y a smooth proper k-scheme, whenever its CY-dimension is *not* an integer.

Theorem 3.2. *We have the following implications:*

$$\{\mathrm{R}_w(X)\}_{w\,\mathrm{even}} \Rightarrow \mathrm{R}_{\mathrm{even}}(\mathcal{T}_{\mathrm{dg}}) \qquad \{\mathrm{R}_w(X)\}_{w\,\mathrm{odd}} \Rightarrow \mathrm{R}_{\mathrm{odd}}(\mathcal{T}_{\mathrm{dg}})\,. \tag{3.3}$$

In particular, the conjectures $\mathrm{R}_{\mathrm{even}}(\mathcal{T}_{\mathrm{dg}})$ *and* $\mathrm{R}_{\mathrm{odd}}(\mathcal{T}_{\mathrm{dg}})$ *hold when* $\mathrm{char}(k) > 0$.

Finite-dimensional algebras of finite global dimension. Let A be a finite-dimensional k-algebra of finite global dimension. Examples include path algebras of finite quivers without oriented cycles and their admissible quotients. As proved by Orlov in [17, Cor. 5.4], A is a geometric noncommutative k-scheme.

Example 3.4 (Dynkin quivers). Let Δ be a Dynkin quiver of type A_n, D_n, E_6, E_7 or E_8. Recall that its Coxeter number h is equal to $n+1$, $2(n-1)$, 12, 18 or 30. It is well-known that the quiver k-algebra $A := k\Delta$ has fractional CY-dimension $\frac{h-2}{h}$. Consequently, in all these cases the geometric noncommutative k-scheme A is *not* Morita equivalent to a dg category of the form $\mathrm{perf}_{\mathrm{dg}}(Y)$, with Y a smooth proper k-scheme.

Consider the largest semi-simple quotient A/J of A, where J stands for the Jacobson radical. Thanks to Artin-Wedderburn's theorem, A/J is Morita equivalent to the product $D_1 \times \cdots \times D_n$, where $V_1, \ldots, V_n$ stand for the simple (right) A/J-modules and $D_1 := \mathrm{End}_{A/J}(V_1), \ldots, D_n := \mathrm{End}_{A/J}(V_n)$ for the associated division k-algebras. Let us denote by $k_1, \ldots, k_n$ the centers of the division k-algebras $D_1, \ldots, D_n$.

Theorem 3.5. *Assume that the quotient k-algebra A/J is separable (this holds, for example, when k is perfect). Under this assumption, we have the implication* $\mathrm{R}_0(\mathrm{Spec}(k_1)) \wedge \cdots \wedge \mathrm{R}_0(\mathrm{Spec}(k_n)) \Rightarrow \mathrm{R}_{\mathrm{even}}(A)$*. In particular, the conjecture* $\mathrm{R}_{\mathrm{even}}(A)$ *holds when* $\mathrm{char}(k) > 0$*.*

Remark 3.6 (Artin L-functions). Since A/J is separable, the finite field extension k_i/k, with $1 \leq i \leq n$, is also separable. Therefore, under the classical Galois-Grothendieck correspondence, the k-scheme $\mathrm{Spec}(k_i)$ corresponds to the finite set $\mathrm{Spec}(k_i)(\overline{k})$ equipped with the continuous action of the absolute Galois group $\mathrm{Gal}(\overline{k}/k)$. Consequently, the L-function $L_0(\mathrm{Spec}(k_i); s)$ (used in conjecture $\mathrm{R}_0(\mathrm{Spec}(k_i))$) reduces to the classical Artin L-function $L(\rho_i; s)$ associated to the $\mathbb{C}$-linear representation $\rho_i\colon \mathrm{Gal}(\overline{k}/k) \to \mathrm{GL}(\mathbb{C}^{\mathrm{Spec}(k_i)(\overline{k})})$.

Finite-dimensional dg algebras. Let A be a smooth *finite-dimensional* dg k-algebra in the sense of Orlov [16]. As proved in [16, Cor. 3.4], A is a geometric noncommutative k-scheme. Following [16, Def. 2.3], consider the quotient A/J_+, where J_+ stands for the external dg Jacobson radical of A.

Theorem 3.7. *Assume that the quotient dg k-algebra A/J_+ is separable in the sense of* [16, Def. 2.11] *(this holds when k is perfect) and that* $\mathrm{char}(k) > 0$*. Under these assumptions, the conjecture* $\mathrm{R}_{\mathrm{even}}(A)$ *holds.*

Remark 3.8. In the above Theorems 3.5 and 3.7, the conjecture $\mathrm{R}_{\mathrm{odd}}(A)$ also holds; consult §10 below.

4. Preliminaries

Let k be a field. Throughout the note, we will assume some basic familiarity with the language of dg categories (consult Keller's survey [7]) and will write $\mathrm{dgcat}(k)$ for the category of (small) dg categories.

4.1. **Geometric noncommutative schemes.** Recall from Definition 1.1 the notion of a geometric noncommutative scheme in the sense of Orlov.

Lemma 4.1. *Let k'/k be a field extension and $\mathcal{A}$ a dg k-linear dg category. If $\mathcal{A}$ is a geometric noncommutative k-scheme, then the dg k'-linear category $\mathcal{A}\otimes_k k'$ is a geometric noncommutative k'-scheme.*

Proof. Let X, $\mathfrak{A}$ and $\mathfrak{A}_{\mathrm{dg}}$ be as in Definition 1.1. Since $\mathcal{A}$ and $\mathfrak{A}_{\mathrm{dg}}$ are Morita equivalent, we have an induced Morita equivalence between $\mathcal{A}\otimes_k k'$ and $\mathfrak{A}_{\mathrm{dg}}\otimes_k k'$. Consider the following Morita equivalence:

$$\mathrm{perf}_{\mathrm{dg}}(X)\otimes_k k' \longrightarrow \mathrm{perf}_{\mathrm{dg}}(X\times_k k') \qquad \mathcal{F}\mapsto \mathcal{F}\times_k k' \,. \tag{4.2}$$

Let us denote by $\mathfrak{A}'$ the smallest full triangulated subcategory of $\mathrm{perf}(X\times_k k')$ containing the objects $\mathcal{F}\times_k k'$, with $\mathcal{F}\in\mathfrak{A}$, and by $\mathfrak{A}'_{\mathrm{dg}}$ the associated full dg subcategory of $\mathrm{perf}_{\mathrm{dg}}(X\times_k k')$. By construction, the above Morita equivalence (4.2) restricts to a Morita equivalence $\mathfrak{A}_{\mathrm{dg}}\otimes_k k'\to\mathfrak{A}'_{\mathrm{dg}}$. Therefore, the proof follows now from the fact that $\mathfrak{A}'$ is an admissible triangulated subcategory of $\mathrm{perf}(X\times_k k')$. □

4.2. **Additive invariants.** Recall from [21, §2.1] that a functor $E\colon \mathrm{dgcat}(k)\to \mathrm{D}$, with values in an additive category, is called an *additive invariant* if it satisfies the following two conditions:

(i) It sends Morita equivalences to isomorphisms.

(ii) Let $\mathcal{B},\mathcal{C}\subseteq\mathcal{A}$ be dg categories inducing a semi-orthogonal decompositions[1] $H^0(\mathcal{A})=\langle H^0(\mathcal{B}),H^0(\mathcal{C})\rangle$ in the sense of Bondal-Orlov [2, Def. 2.4]. Under these notations, the inclusions $\mathcal{B}\subseteq\mathcal{A}$ and $\mathcal{C}\subseteq\mathcal{A}$ induce an isomorphism $E(\mathcal{B})\oplus E(\mathcal{C})\to E(\mathcal{A})$.

Lemma 4.3. *Let k'/k be a field extension. Given an additive invariant $E\colon \mathrm{dgcat}(k')\to\mathrm{D}$, the composed functor $E(-\otimes_k k')\colon \mathrm{dgcat}(k)\to\mathrm{D}$ is also an additive invariant.*

Proof. Condition (i) follows from the fact that the functor $-\otimes_k k'$ preserves Morita equivalences; consult [15, Prop. 7.1]. Concerning condition (ii), let $\mathcal{B},\mathcal{C}\subseteq\mathcal{A}$ be dg categories inducing a semi-orthogonal decomposition $H^0(\mathcal{A})=\langle H^0(\mathcal{B}),H^0(\mathcal{C})\rangle$. The associated dg categories $\mathrm{pre}(\mathcal{B}\otimes_k k'),\mathrm{pre}(\mathcal{C}\otimes_k k')\subseteq\mathrm{pre}(\mathcal{A}\otimes_k k')$, where $\mathrm{pre}(-)$ stands for Bondal-Kapranov's pretriangulated envelope [1], also induce a semi-orthogonal decomposition $H^0(\mathrm{pre}(\mathcal{A}\otimes_k k'))=\langle\mathrm{pre}(\mathcal{B}\otimes_k k'),\mathrm{pre}(\mathcal{C}\otimes_k k')\rangle$. Therefore, since the canonical dg functors $\mathcal{A}\otimes_k k'\to\mathrm{pre}(\mathcal{A}\otimes_k k')$, $\mathcal{B}\otimes_k k'\to\mathrm{pre}(\mathcal{B}\otimes_k k')$, and $\mathcal{C}\otimes_k k'\to\mathrm{pre}(\mathcal{C}\otimes_k k')$, are Morita equivalences, the proof of condition (ii) follows now from the fact that the functor E satisfies condition (ii). □

We now recall from [21, §2.3] the construction of the universal additive invariant $U\colon\mathrm{dgcat}(k)\to\mathrm{Hmo}_0(k)$. Given two dg categories $\mathcal{A}$ and $\mathcal{B}$, let us write $\mathcal{D}(\mathcal{A}^{\mathrm{op}}\otimes_k\mathcal{B})$ for the derived category of dg $\mathcal{A}$-$\mathcal{B}$-bimodules and $\mathrm{rep}(\mathcal{A},\mathcal{B})$ for the full triangulated subcategory of $\mathcal{D}(\mathcal{A}^{\mathrm{op}}\otimes_k\mathcal{B})$ consisting of those dg $\mathcal{A}$-$\mathcal{B}$-bimodules B such that for every object $x\in\mathcal{A}$ the associated right dg $\mathcal{B}$-module $\mathrm{B}(x,-)$ belongs to the full triangulated subcategory of compact objects $\mathcal{D}_c(\mathcal{B})$. Given a dg functor $F\colon\mathcal{A}\to\mathcal{B}$, note that the associated dg $\mathcal{A}$-$\mathcal{B}$-bimodule ${}_F\mathcal{B}$, defined as $\mathcal{B}(-,F(-))$, belongs to the subcategory $\mathrm{rep}(\mathcal{A},\mathcal{B})$. The objects of the category $\mathrm{Hmo}_0(k)$ are the (small) dg categories, the abelian group of morphisms $\mathrm{Hom}_{\mathrm{Hmo}_0(k)}(U(\mathcal{A}),U(\mathcal{B}))$ is given by the Grothendieck group $K_0\mathrm{rep}(\mathcal{A},\mathcal{B})$, and the composition law is induced by the (derived) tensor product of dg bimodules. Moreover, the functor U is the identity on objects and sends a dg functor $F\colon\mathcal{A}\to\mathcal{B}$ to the Grothendieck class of the associated dg $\mathcal{A}$-$\mathcal{B}$-bimodule ${}_F\mathcal{B}$. As explained in *loc. cit.*, the functor U is the

[1] Recall that $H^0(\mathcal{A})$ stands for the category obtained by taking the 0^{th} cohomology of the complexes of morphisms of $\mathcal{A}$.

universal additive invariant in the sense that given any additive invariant E, there exists a unique $\mathbb{Z}$-linear functor $\overline{E}$ making the following diagram commute:

$$\begin{array}{ccc} \mathrm{dgcat}(k) & \xrightarrow{E} & \mathrm{D} \\ {\scriptstyle U}\downarrow & \nearrow_{\overline{E}} & \\ \mathrm{Hmo}_0(k) & & \end{array} \quad . \tag{4.4}$$

5. Noncommutative l-adic cohomology

Let k be a field. Given a prime number $l \neq \mathrm{char}(k)$, recall from [22, §2.5] the construction of the l-adic étale K-theory functor with values in the (homotopy) category of spectra:

$$K^{\mathrm{et}}(-)_{\widehat{l}}\colon \mathrm{dgcat}(k) \longrightarrow \mathrm{Spt} \qquad \mathcal{A} \mapsto \mathrm{holim}_{n\geq 0} K^{\mathrm{et}}(\mathcal{A};\mathbb{Z}/l^n)\,. \tag{5.1}$$

By construction, the homotopy groups $\pi_*(K^{\mathrm{et}}(\mathcal{A})_{\widehat{l}})$ are modules over the ring of l-adic integers $\mathbb{Z}_l$.

Definition 5.2 (Noncommutative l-adic cohomology). Given a dg category $\mathcal{A}$, its noncommutative l-adic cohomology is defined as follows ($\overline{k}$ stands for a fixed separable closure of k):

$$\mathrm{H}_{\mathrm{even},l}(\mathcal{A}) := \pi_0(K^{\mathrm{et}}(\mathcal{A}\otimes_k \overline{k})_{\widehat{l}})\otimes_{\mathbb{Z}}\mathbb{Z}[1/l] \qquad \mathrm{H}_{\mathrm{odd},l}(\mathcal{A}) := \pi_1(K^{\mathrm{et}}(\mathcal{A}\otimes_k \overline{k})_{\widehat{l}})\otimes_{\mathbb{Z}}\mathbb{Z}[1/l]\,. \tag{5.3}$$

Note that, by construction, the noncommutative l-adic cohomology groups (5.3) are $\mathbb{Q}_l$-vector spaces. Moreover, they are equipped with a continuous action of the absolute Galois group $\mathrm{Gal}(\overline{k}/k)$. Consequently, we obtain the following well-defined functors with values in the category of $\mathbb{Q}_l$-linear $\mathrm{Gal}(\overline{k}/k)$-modules:

$$\mathrm{H}_{\mathrm{even},l}(-), \mathrm{H}_{\mathrm{odd},l}(-)\colon \mathrm{dgcat}(k) \longrightarrow \mathrm{Gal}(\overline{k}/k)\text{-}\mathrm{Mod}\,. \tag{5.4}$$

Proposition 5.5. *The functors* (5.4) *are additive invariants.*

Proof. The l-adic étale K-theory functor (5.1) is an additive invariant; consult [22, §2.5]. Therefore, the proof follows from the above general Lemma 4.3. □

Proposition 5.6. *Given a smooth proper k-scheme X, we have isomorphisms of* $\mathrm{Gal}(\overline{k}/k)$*-modules*

$$\mathrm{H}_{\mathrm{even},l}(\mathrm{perf}_{\mathrm{dg}}(X)) \simeq \bigoplus_{w\,\mathrm{even}} H^w_{\mathrm{et}}(X\times_k \overline{k};\mathbb{Q}_l(\frac{w}{2})) \qquad \mathrm{H}_{\mathrm{odd},l}(\mathrm{perf}_{\mathrm{dg}}(X)) \simeq \bigoplus_{w\,\mathrm{odd}} H^w_{\mathrm{et}}(X\times_k \overline{k};\mathbb{Q}_l(\frac{w-1}{2}))\,,$$

where $H_{\mathrm{et}}(-)$ stands for étale cohomology.

Proof. Since the k-scheme X is smooth and proper, the associated $\overline{k}$-scheme $X\times_k \overline{k}$ is, in particular, regular and separated. These conditions imply that Thomason's étale descent spectral sequence [25, Thm. 4.1] is well-defined and degenerates rationally; consult Soulé [26, §3.3.2]. Consequently, we obtain an isomorphism of $\mathrm{Gal}(\overline{k}/k)$-modules between $\pi_0(K^{\mathrm{et}}(\mathrm{perf}_{\mathrm{dg}}(X\times_k\overline{k}))_{\widehat{l}})\otimes_{\mathbb{Z}}\mathbb{Z}[1/l]$ and the direct sum $\bigoplus_{w\,\mathrm{even}} H^w_{\mathrm{et}}(X\times_k\overline{k};\mathbb{Q}_l(\frac{w}{2}))$ and between $\pi_1(K^{\mathrm{et}}(\mathrm{perf}_{\mathrm{dg}}(X\times_k\overline{k}))_{\widehat{l}})\otimes_{\mathbb{Z}}\mathbb{Z}[1/l]$ and the direct sum $\bigoplus_{w\,\mathrm{odd}} H^w_{\mathrm{et}}(X\times_k\overline{k};\mathbb{Q}_l(\frac{w-1}{2}))$. The proof follows now from the Morita equivalence $\mathrm{perf}_{\mathrm{dg}}(X)\otimes_k\overline{k}\to\mathrm{perf}_{\mathrm{dg}}(X\times_k\overline{k}), \mathcal{F}\mapsto\mathcal{F}\times_k\overline{k}$. □

Lemma 5.7. *Given a geometric noncommutative k-scheme $\mathcal{A}$, the associated $\mathbb{Q}_l$-linear* $\mathrm{Gal}(\overline{k}/k)$*-modules* (5.3) *are finite-dimensional.*

Proof. Let X, $\mathfrak{A}$ and $\mathfrak{A}_{\mathrm{dg}}$ be as in Definition 1.1. In what follows, we denote by ${}^\perp\mathfrak{A}$ the left orthogonal complement of $\mathfrak{A}$ in $\mathrm{perf}(X)$ and by ${}^\perp\mathfrak{A}_{\mathrm{dg}}$ the associated full dg subcategory of $\mathrm{perf}_{\mathrm{dg}}(X)$. By construction, we have a semi-orthogonal decomposition $H^0(\mathrm{perf}_{\mathrm{dg}}(X)) = \langle H^0(\mathfrak{A}_{\mathrm{dg}}), H^0({}^\perp\mathfrak{A}_{\mathrm{dg}})\rangle$, i.e., $\mathrm{perf}(X) = \langle \mathfrak{A}, {}^\perp\mathfrak{A}\rangle$. Therefore, making use of Proposition 5.5, we obtain the following computations:

$$\mathrm{H}_{\mathrm{even},l}(\mathrm{perf}_{\mathrm{dg}}(X)) \simeq \mathrm{H}_{\mathrm{even},l}(\mathcal{A})\oplus\mathrm{H}_{\mathrm{even},l}({}^\perp\mathfrak{A}_{\mathrm{dg}}) \qquad \mathrm{H}_{\mathrm{odd},l}(\mathrm{perf}_{\mathrm{dg}}(X)) \simeq \mathrm{H}_{\mathrm{odd},l}(\mathcal{A})\oplus\mathrm{H}_{\mathrm{odd},l}({}^\perp\mathfrak{A}_{\mathrm{dg}})\,.$$

The proof follows now from Proposition 5.6 and from the fact that the étale cohomology $\mathbb{Q}_l$-vector spaces $\{H^w_{\mathrm{et}}(X\times_k\overline{k};\mathbb{Q}_l(\frac{w}{2}))\}_{w\,\mathrm{even}}$ and $\{H^w_{\mathrm{et}}(X\times_k\overline{k};\mathbb{Q}_l(\frac{w-1}{2}))\}_{w\,\mathrm{odd}}$ are finite-dimensional. □

6. Noncommutative L-functions

Let k be a global field. We start by fixing some important notations:

Notation 6.1. Given a non-archimedean place $\nu \in \Sigma_k$, let us write k_ν for the completion of k at ν, $\mathcal{O}_\nu$ for the valuation ring of k_ν, κ_ν for the residue field of $\mathcal{O}_\nu$, p_ν for the characteristic of κ_ν, N_ν for the cardinality of the finite field κ_ν, I_ν for the *inertia subgroup*, i.e., the kernel of the canonical surjective map $\mathrm{Gal}(\overline{k_\nu}/k_\nu) \twoheadrightarrow \mathrm{Gal}(\overline{\kappa_\nu}/\kappa_\nu)$, and $\pi_\nu \in \mathrm{Gal}(\overline{\kappa_\nu}/\kappa_\nu) \simeq \widehat{\mathbb{Z}}$ for the *geometric Frobenius*, i.e., the inverse of the arithmetic Frobenius $\lambda \mapsto \lambda^{N_\nu}$.

Notation 6.2. Let us choose a prime number $l_\nu \neq p_\nu$ and a field embedding $\iota_\nu \colon \mathbb{Q}_{l_\nu} \hookrightarrow \mathbb{C}$ for every $\nu \in \Sigma_k$.

The above Lemmas 4.1 and 5.7 enable the following definition:

Definition 6.3 (Noncommutative L-functions). Given a geometric noncommutative k-scheme $\mathcal{A}$, its noncommutative L-functions are defined as follows (we are implicitly using the field embedding ι_ν):

$$L_{\mathrm{even}}(\mathcal{A};s) := \prod_{\nu\in\Sigma_k} L_{\mathrm{even},\nu}(\mathcal{A};s) \qquad L_{\mathrm{even},\nu}(\mathcal{A};s) := \frac{1}{\det\big(\mathrm{id} - N_\nu^{-s}(\pi_\nu \otimes_{\mathbb{Q}_{l_\nu}} \mathbb{C}) \,|\, \mathrm{H}_{\mathrm{even},l_\nu}(\mathcal{A}\otimes_k k_\nu)^{I_\nu} \otimes_{\mathbb{Q}_{l_\nu}} \mathbb{C}\big)}$$

$$L_{\mathrm{odd}}(\mathcal{A};s) := \prod_{\nu\in\Sigma_k} L_{\mathrm{odd},\nu}(\mathcal{A};s) \qquad L_{\mathrm{odd},\nu}(\mathcal{A};s) := \frac{1}{\det\big(\mathrm{id} - N_\nu^{-s}(\pi_\nu \otimes_{\mathbb{Q}_{l_\nu}} \mathbb{C}) \,|\, \mathrm{H}_{\mathrm{odd},l_\nu}(\mathcal{A}\otimes_k k_\nu)^{I_\nu} \otimes_{\mathbb{Q}_{l_\nu}} \mathbb{C}\big)}\,.$$

Proposition 6.4. *Let $\mathcal{B},\mathcal{C} \subseteq \mathcal{A}$ be geometric noncommutative k-schemes inducing a semi-orthogonal decomposition $H^0(\mathcal{A}) = \langle H^0(\mathcal{B}), H^0(\mathcal{C})\rangle$. Under these notations, we have the following equalities:*

$$L_{\mathrm{even}}(\mathcal{A};s) = L_{\mathrm{even}}(\mathcal{B};s)\cdot L_{\mathrm{even}}(\mathcal{C};s) \qquad L_{\mathrm{odd}}(\mathcal{A};s) = L_{\mathrm{odd}}(\mathcal{B};s)\cdot L_{\mathrm{odd}}(\mathcal{C};s)\,. \tag{6.5}$$

Proof. Let $\nu \in \Sigma_k$ be a non-archimedean place. Thanks to Proposition 5.5 and Lemma 4.3, we have an isomorphism of $\mathrm{Gal}(\overline{k_\nu}/k_\nu)$-modules between $\mathrm{H}_{\mathrm{even},l_\nu}(\mathcal{A}\otimes_k k_\nu)$ and $\mathrm{H}_{\mathrm{even},l_\nu}(\mathcal{B}\otimes_k k_\nu) \oplus \mathrm{H}_{\mathrm{even},l_\nu}(\mathcal{C}\otimes_k k_\nu)$. This implies that $L_{\mathrm{even},\nu}(\mathcal{A};s) = L_{\mathrm{even},\nu}(\mathcal{B};s)\cdot L_{\mathrm{even},\nu}(\mathcal{C};s)$. Consequently, the left-hand side of (6.5) follows from Definition 6.3. The proof of the odd case is similar: simply replace the word "even" by the word "odd". ☐

Proposition 6.6. *Given a smooth proper k-scheme X, we have the following equalities:*

$$L_{\mathrm{even}}(\mathrm{perf}_{\mathrm{dg}}(X);s) = \prod_{w\ \mathrm{even}} L_w(X; s + \frac{w}{2}) \qquad L_{\mathrm{odd}}(\mathrm{perf}_{\mathrm{dg}}(X);s) = \prod_{w\ \mathrm{odd}} L_w(X; s + \frac{w-1}{2})\,. \tag{6.7}$$

Roughly speaking, Proposition 6.6 shows that the noncommutative even/odd L-function of $\mathrm{perf}_{\mathrm{dg}}(X)$ may be understood as the "weight normalization" of the product of the L-functions of X of even/odd weight.

Proof. Recall first that the L-function of X of weight w is defined as follows (consult Notations 6.1-6.2):

$$L_w(X;s) := \prod_{\nu\in\Sigma_k} L_{w,\nu}(X;s) \quad L_{w,\nu}(X;s) := \frac{1}{\det(\mathrm{id} - N_\nu^{-s}(\pi_\nu \otimes_{\mathbb{Q}_{l_\nu}} \mathbb{C}) \,|\, H^w_{\mathrm{et}}((X\times_k k_\nu)\times_{k_\nu} \overline{k_\nu}; \mathbb{Q}_{l_\nu})^{I_\nu} \otimes_{\mathbb{Q}_{l_\nu}} \mathbb{C})}$$

Note that we have the following isomorphisms of $\mathrm{Gal}(\overline{\kappa_\nu}/\kappa_\nu)$-modules

$$\mathrm{H}_{\mathrm{even},l_\nu}(\mathrm{perf}_{\mathrm{dg}}(X)\otimes_k k_\nu)^{I_\nu} \simeq \mathrm{H}_{\mathrm{even},l_\nu}(\mathrm{perf}_{\mathrm{dg}}(X\times_k k_\nu))^{I_\nu} \tag{6.8}$$

$$\simeq \Big(\bigoplus_{w\ \mathrm{even}} H^w_{\mathrm{et}}((X\times_k k_\nu)\times_{k_\nu}\overline{k_\nu}; \mathbb{Q}_{l_\nu}(\frac{w}{2}))\Big)^{I_\nu} \tag{6.9}$$

$$\simeq \bigoplus_{w\ \mathrm{even}} H^w_{\mathrm{et}}((X\times_k k_\nu)\times_{k_\nu}\overline{k_\nu}; \mathbb{Q}_{l_\nu}(\frac{w}{2}))^{I_\nu}$$

$$\simeq \bigoplus_{w\ \mathrm{even}} \big(H^w_{\mathrm{et}}((X\times_k k_\nu)\times_{k_\nu}\overline{k_\nu}; \mathbb{Q}_{l_\nu})^{I_\nu} \otimes_{\mathbb{Q}_{l_\nu}} \mathbb{Q}_{l_\nu}(\frac{w}{2})\big)\,, \tag{6.10}$$

where (6.8) follows from the Morita equivalence $\mathrm{perf}_{\mathrm{dg}}(X)\otimes_k k_\nu \to \mathrm{perf}_{\mathrm{dg}}(X\times_k k_\nu), \mathcal{F} \mapsto \mathcal{F}\times_k k_\nu$, (6.9) from Proposition 5.6, and (6.10) from the fact that the $\mathrm{Gal}(\overline{k_\nu}/k_\nu)$-module $\mathbb{Q}_{l_\nu}(\frac{w}{2})$ is unramified (i.e., I_ν acts trivially). Moreover, since the action of the geometric Frobenius on $\mathbb{Q}_{l_\nu}(\frac{w}{2})$ is given by multiplication by $N_\nu^{-\frac{w}{2}}$, the $\mathrm{Gal}(\overline{\kappa_\nu}/\kappa_\nu)$-module (6.10) may be identified with $\bigoplus_{w\,\text{even}} H^w_{\mathrm{et}}((X\times_k k_\nu)\times_{k_\nu}\overline{k_\nu};\mathbb{Q}_{l_\nu})^{I_\nu}$ where the geometric Frobenius acts diagonally as $\bigoplus_{w\,\text{even}} \pi_\nu\cdot N_\nu^{-\frac{w}{2}}$. This implies the following equalities:

$$\begin{aligned} L_{\mathrm{even},\nu}(\mathrm{perf}_{\mathrm{dg}}(X);s) &= \prod_{w\,\text{even}} \frac{1}{\det(\mathrm{id}-N_\nu^{-(s+\frac{w}{2})}(\pi_\nu\otimes_{\mathbb{Q}_{l_\nu}}\mathbb{C})\,|\,H^w_{\mathrm{et}}((X\times_k k_\nu)\times_{k_\nu}\overline{k_\nu};\mathbb{Q}_{l_\nu})^{I_\nu}\otimes_{\mathbb{Q}_{l_\nu}}\mathbb{C})} \\ &= \prod_{w\,\text{even}} L_{w,\nu}(X;s+\frac{w}{2})\,. \end{aligned}$$

Consequently, the left-hand side of (6.7) follows now from the following equalities:

$$L_{\mathrm{even}}(\mathrm{perf}_{\mathrm{dg}}(X);s) := \prod_{\nu\in\Sigma_k} L_{\mathrm{even},\nu}(\mathrm{perf}_{\mathrm{dg}}(X);s) = \prod_{\nu\in\Sigma_k}\prod_{w\,\text{even}} L_{w,\nu}(X;s+\frac{w}{2}) = \prod_{w\,\text{even}} L_w(X;s+\frac{w}{2})\,.$$

The proof of the odd case is similar: simply replace the word "even" by the word "odd" and $\frac{w}{2}$ by $\frac{w-1}{2}$. $\square$

7. Proof of Theorem 1.2

Recall that k is a finite field extension of $\mathbb{Q}$ (when $\mathrm{char}(k)=0$) or a finite field extension of $\mathbb{F}_q(t)$ (when $\mathrm{char}(k)>0$), where $\mathbb{F}_q$ is the finite field with q elements. In what follows, we will write $\mathcal{O}_k\subset k$ for the integral closure of $\mathbb{Z}$ in k (when $\mathrm{char}(k)=0$) or for the integral closure of $\mathbb{F}_q[t]$ in k (when $\mathrm{char}(k)>0$).

Recall also that $\mathcal{A}$ is a geometric noncommutative k-scheme and let X, $\mathfrak{A}$ and $\mathfrak{A}_{\mathrm{dg}}$ be as in Definition 1.1.

Notation 7.1. Given an integer $0\leq w\leq 2\dim(X)$ and a prime number $l\neq\mathrm{char}(k)$, let us write β_w for the dimension of the $\mathbb{Q}_l$-vector space $H^w_{\mathrm{et}}(X\times_k\overline{k};\mathbb{Q}_l)$; it is well-known that this dimension is independent of l.

Lemma 7.2. *For every non-archimedean place $\nu\in\Sigma_k$, we have the equalities (consult Notation 6.2):*

$$(7.3)\qquad \dim_{\mathbb{Q}_{l_\nu}}\mathrm{H}_{\mathrm{even},l_\nu}(\mathrm{perf}_{\mathrm{dg}}(X)\otimes_k k_\nu) = \sum_{w\,\text{even}}\beta_w \qquad \dim_{\mathbb{Q}_{l_\nu}}\mathrm{H}_{\mathrm{odd},l_\nu}(\mathrm{perf}_{\mathrm{dg}}(X)\otimes_k k_\nu) = \sum_{w\,\text{odd}}\beta_w\,.$$

Proof. We have the following isomorphisms of $\mathbb{Q}_{l_\nu}$-vector spaces

$$\begin{aligned} (7.4)\qquad \mathrm{H}_{\mathrm{even},l_\nu}(\mathrm{perf}_{\mathrm{dg}}(X)\otimes_k k_\nu) &\simeq \mathrm{H}_{\mathrm{even},l_\nu}(\mathrm{perf}_{\mathrm{dg}}(X\times_k k_\nu)) \\ (7.5)\qquad &\simeq \bigoplus_{w\,\text{even}} H^w_{\mathrm{et}}((X\times_k k_\nu)\times_{k_\nu}\overline{k_\nu};\mathbb{Q}_{l_\nu}(\frac{w}{2})) \\ &\simeq \bigoplus_{w\,\text{even}} H^w_{\mathrm{et}}((X\times_k k_\nu)\times_{k_\nu}\overline{k_\nu};\mathbb{Q}_{l_\nu})\,, \end{aligned}$$

where (7.4) follows from the Morita equivalence $\mathrm{perf}_{\mathrm{dg}}(X)\otimes_k k_\nu\to\mathrm{perf}_{\mathrm{dg}}(X\times_k k_\nu), \mathcal{F}\mapsto\mathcal{F}\times_k k_\nu$, and (7.5) from Proposition 5.6. Consequently, the left-hand side of (7.3) follows from the (well-known) fact that the canonical homomorphism $H^w_{\mathrm{et}}(X\times_k\overline{k};\mathbb{Q}_{l_\nu})\to H^w_{\mathrm{et}}((X\times_k k_\nu)\times_{k_\nu}\overline{k_\nu};\mathbb{Q}_{l_\nu})$ is invertible. The proof of the odd case is similar: simply replace the word "even" by the word "odd" and $\frac{w}{2}$ by $\frac{w-1}{2}$. $\square$

Notation 7.6. Let S_X be the (finite) set of prime ideals of $\mathcal{O}_k$ where X has bad reduction, $\mathcal{O}_k[S_X^{-1}]$ the localized ring, and Σ_X the subset of Σ_k corresponding to the prime ideals of $\mathcal{O}_k[S_X^{-1}]$. Given a prime ideal $\mathcal{P}\lhd\mathcal{O}_k[S_X^{-1}]$, we will denote by $\nu_{\mathcal{P}}\in\Sigma_X$ the corresponding non-archimedean place. Similarly, given a non-archimedean place $\nu\in\Sigma_X$, we will denote by $\mathcal{P}_\nu\lhd\mathcal{O}_k[S_X^{-1}]$ the corresponding prime ideal.

Lemma 7.7. *Let $\nu\in\Sigma_X$ be a non-archimedean place and λ an eigenvalue of the automorphism $\pi_\nu\otimes_{\mathbb{Q}_{l_\nu}}\mathbb{C}$ of the $\mathbb{C}$-vector space $\mathrm{H}_{\mathrm{even},l_\nu}(\mathrm{perf}_{\mathrm{dg}}(X)\otimes_k k_\nu)^{I_\nu}\otimes_{\mathbb{Q}_{l_\nu}}\mathbb{C}$, resp. $\mathrm{H}_{\mathrm{odd},l_\nu}(\mathrm{perf}_{\mathrm{dg}}(X)\otimes_k k_\nu)^{I_\nu}\otimes_{\mathbb{Q}_{l_\nu}}\mathbb{C}$. Under these notations, we have $|\lambda|=1$, resp. $|\lambda|=N_\nu^{\frac{1}{2}}$.*

Proof. As explained in the proof of Proposition 6.6, we have an isomorphism of $\mathrm{Gal}(\overline{\kappa_\nu}/\kappa_\nu)$-modules

$$(7.8)\qquad \mathrm{H}_{\mathrm{even},l_\nu}(\mathrm{perf}_{\mathrm{dg}}(X)\otimes_k k_\nu)^{I_\nu}\otimes_{\mathbb{Q}_{l_\nu}}\mathbb{C} \simeq \bigoplus_{w\,\text{even}} H^w_{\mathrm{et}}((X\times_k k_\nu)\times_{k_\nu}\overline{k_\nu})^{I_\nu}\otimes_{\mathbb{Q}_{l_\nu}}\mathbb{C}\,,$$

where the automorphism $\pi_\nu\otimes_{\mathbb{Q}_{l_\nu}}\mathbb{C}$ on the left-hand side of (7.8) corresponds to the diagonal automorphism $\bigoplus_{w\,\text{even}}(\pi_\nu\otimes_{\mathbb{Q}_{l_\nu}}\mathbb{C})\cdot N_\nu^{-\frac{w}{2}}$ on the right-hand side. Let λ' be an eigenvalue of the automorphism $\pi_\nu\otimes_{\mathbb{Q}_{l_\nu}}\mathbb{C}$ of the $\mathbb{C}$-vector space $H^w_{\mathrm{et}}((X\times_k k_\nu)\times_{k_\nu}\overline{k_\nu})^{I_\nu}\otimes_{\mathbb{Q}_{l_\nu}}\mathbb{C}$, with $0\leq w\leq 2\dim(X)$. Since X has good reduction at the prime ideal $\mathcal{P}_\nu$, it follows from Deligne's proof of the Weil conjecture (consult [4]) and from the smooth proper base-change property of étale cohomology that $|\lambda'| = N_\nu^{\frac{w}{2}}$. Consequently, we conclude that $|\lambda| = N_\nu^{\frac{w}{2}} N_\nu^{-\frac{w}{2}} = 1$. The proof of the odd case is similar: simply replace the word "even" by the word "odd" and the diagonal automorphism $\bigoplus_{w\,\mathrm{even}}(\pi_\nu \otimes_{\mathbb{Q}_{l_\nu}} \mathbb{C}) \cdot N_\nu^{-\frac{w}{2}}$ by $\bigoplus_{w\,\mathrm{odd}}(\pi_\nu \otimes_{\mathbb{Q}_{l_\nu}} \mathbb{C}) \cdot N_\nu^{-\frac{w-1}{2}}$. □

Now, note that since the set $\Sigma_k \backslash \Sigma_X$ is finite, in order to prove Theorem 1.2, we can (and will) replace the infinite products $\prod_{\nu \in \Sigma_k} L_{\mathrm{even},\nu}(\mathcal{A}; s)$ and $\prod_{\nu \in \Sigma_k} L_{\mathrm{odd},\nu}(\mathcal{A}; s)$ by the infinite products $\prod_{\nu \in \Sigma_X} L_{\mathrm{even},\nu}(\mathcal{A}; s)$ and $\prod_{\nu \in \Sigma_X} L_{\mathrm{odd},\nu}(\mathcal{A}; s)$, respectively.

Notation 7.9. Given a non-archimedean place $\nu \in \Sigma_k$ and an integer $n \geq 1$, consider the complex numbers:

$$\begin{aligned} \#_{(+,\nu,n)} &:= \mathrm{trace}\big((\pi_\nu \otimes_{\mathbb{Q}_{l_\nu}} \mathbb{C})^{\circ n} \,|\, \mathrm{H}_{\mathrm{even},l_\nu}(\mathcal{A} \otimes_k k_\nu)^{I_\nu} \otimes_{\mathbb{Q}_{l_\nu}} \mathbb{C}\big) \\ \#_{(-,\nu,n)} &:= \mathrm{trace}\big((\pi_\nu \otimes_{\mathbb{Q}_{l_\nu}} \mathbb{C})^{\circ n} \,|\, \mathrm{H}_{\mathrm{odd},l_\nu}(\mathcal{A} \otimes_k k_\nu)^{I_\nu} \otimes_{\mathbb{Q}_{l_\nu}} \mathbb{C}\big)\,. \end{aligned}$$

Proposition 7.10. *Given a non-archimedean place $\nu \in \Sigma_X$ and an integer $n \geq 1$, we have the inequalities:*

$$|\#_{(+,\nu,n)}| \leq \sum_{w\,\mathrm{even}} \beta_w \qquad |\#_{(-,\nu,n)}| \leq (\sum_{w\,\mathrm{odd}} \beta_w) \cdot N_\nu^{\frac{n}{2}}\,.$$

Proof. Let us write $\chi_{(+,\nu)}$, resp. $\chi_{(-,\nu)}$, for the dimension of the $\mathbb{C}$-vector space

$$\mathrm{H}_{\mathrm{even},l_\nu}(\mathcal{A} \otimes_k k_\nu)^{I_\nu} \otimes_{\mathbb{Q}_{l_\nu}} \mathbb{C}\,, \qquad \text{resp.} \quad \mathrm{H}_{\mathrm{odd},l_\nu}(\mathcal{A} \otimes_k k_\nu)^{I_\nu} \otimes_{\mathbb{Q}_{l_\nu}} \mathbb{C}\,, \tag{7.11}$$

and $\{\lambda_{(+,\nu,1)}, \ldots, \lambda_{(+,\nu,\chi_{(+,\nu)})}\}$, resp. $\{\lambda_{(-,\nu,1)}, \ldots, \lambda_{(-,\nu,\chi_{(-,\nu)})}\}$, for the set of eigenvalues (with multiplicities) of the automorphism $\pi_\nu \otimes_{\mathbb{Q}_{l_\nu}} \mathbb{C}$ of (7.11). Moreover, let us write ${}^\perp\mathfrak{A}$ for the left orthogonal complement of $\mathfrak{A}$ in $\mathrm{perf}(X)$ and ${}^\perp\mathfrak{A}_{\mathrm{dg}}$ for the associated full dg subcategory of $\mathrm{perf}_{\mathrm{dg}}(X)$. By construction, we have a semi-orthogonal decomposition $H^0(\mathrm{perf}_{\mathrm{dg}}(X)) = \langle H^0(\mathfrak{A}_{\mathrm{dg}}), H^0({}^\perp\mathfrak{A}_{\mathrm{dg}})\rangle$, i.e., $\mathrm{perf}(X) = \langle \mathfrak{A}, {}^\perp\mathfrak{A}\rangle$. Therefore, by combining Proposition 5.5 with Lemma 4.3, we obtain an isomorphism of $\mathrm{Gal}(\overline{k_\nu}/k_\nu)$-modules between $\mathrm{H}_{\mathrm{even},l_\nu}(\mathrm{perf}_{\mathrm{dg}}(X) \otimes_k k_\nu)$ and the direct sum $\mathrm{H}_{\mathrm{even},l_\nu}(\mathcal{A} \otimes_k k_\nu) \oplus \mathrm{H}_{\mathrm{even},l_\nu}({}^\perp\mathfrak{A}_{\mathrm{dg}} \otimes_k k_\nu)$ and between $\mathrm{H}_{\mathrm{odd},l_\nu}(\mathrm{perf}_{\mathrm{dg}}(X) \otimes_k k_\nu)$ and the direct sum $\mathrm{H}_{\mathrm{odd},l_\nu}(\mathcal{A} \otimes_k k_\nu) \oplus \mathrm{H}_{\mathrm{odd},l_\nu}({}^\perp\mathfrak{A}_{\mathrm{dg}} \otimes_k k_\nu)$. On the one hand, thanks to Lemma 7.7, this implies that if λ is an eigenvalue of the automorphism $\pi_\nu \otimes_{\mathbb{Q}_{l_\nu}} \mathbb{C}$ of (7.11), then $|\lambda| = 1$, resp. $|\lambda| = N_\nu^{\frac{1}{2}}$. On the other hand, thanks to Lemma 7.2, this implies that $\chi_{(+,\nu)} \leq \sum_{w\,\mathrm{even}} \beta_w$, resp. $\chi_{(-,\nu)} \leq \sum_{w\,\mathrm{odd}} \beta_w$. As a consequence, we obtain the following inequalities:

$$|\#_{(+,\nu,n)}| = |\lambda^n_{(+,\nu,1)} + \cdots + \lambda^n_{(+,\nu,\chi_{(+,\nu)})}| \leq |\lambda_{(+,\nu,1)}|^n + \cdots + |\lambda_{(+,\nu,\chi_{(+,\nu)})}|^n = \chi_{(+,\nu)} \leq \sum_{w\,\mathrm{even}} \beta_w$$

$$|\#_{(-,\nu,n)}| = |\lambda^n_{(-,\nu,1)} + \cdots + \lambda^n_{(-,\nu,\chi_{(+,\nu)})}| \leq |\lambda_{(-,\nu,1)}|^n + \cdots + |\lambda_{(-,\nu,\chi_{(+,\nu)})}|^n = \chi_{(-,\nu)} \cdot N_\nu^{\frac{n}{2}} \leq (\sum_{w\,\mathrm{odd}} \beta_w) \cdot N_\nu^{\frac{n}{2}}\,.$$

This concludes the proof of Proposition 7.10. □

The following general result, whose proof is a simple linear algebra exercise, is well-known:

Lemma 7.12. *Given an endomorphism $f\colon V \to V$ of a finite-dimensional $\mathbb{C}$-linear vector space, we have the equality of formal power series* $\log(\frac{1}{\det(\mathrm{id} - tf|V)}) = \sum_{n\geq 1} \mathrm{trace}(f^{\circ n})\frac{t^n}{n}$, *where* $\log(t) := \sum_{n\geq 1} \frac{(-1)^{n+1}}{n}(t-1)^n$.

Given a non-archimeadean place $\nu \in \Sigma_X$, consider the formal power series and their exponentiations:

$$\begin{aligned} \phi_{(+,\nu)}(t) &:= \sum_{n\geq 1} \#_{(+,\nu,n)} \frac{t^n}{n} \qquad \varphi_{(+,\nu)}(t) := \exp(\phi_{(+,\nu)}(t)) = \sum_{n\geq 0} a_{(+,\nu,n)} t^n \\ \phi_{(-,\nu)}(t) &:= \sum_{n\geq 1} \#_{(-,\nu,n)} \frac{t^n}{n} \qquad \varphi_{(-,\nu)}(t) := \exp(\phi_{(-,\nu)}(t)) = \sum_{n\geq 0} a_{(-,\nu,n)} t^n\,. \end{aligned}$$

Note that, thanks to Lemma 7.12, we have $\varphi_{(+,\nu)}(N_\nu^{-s}) = L_{\mathrm{even},\nu}(\mathcal{A}; s)$ and $\varphi_{(-,\nu)}(N_\nu^{-s}) = L_{\mathrm{odd},\nu}(\mathcal{A}; s)$ for every non-archimedean place $\nu \in \Sigma_X$.

Definition 7.13. Consider the following multiplicative Dirichlet series

$$\varphi_+(s) := \sum_I \frac{b_{(+,I)}}{N(I)^s} \qquad \varphi_-(s) := \sum_I \frac{b_{(-,I)}}{N(I)^s}\,,$$

where $I \lhd \mathcal{O}_k[S_X^{-1}]$ is an ideal, $b_{(+,I)} := a_{(+,\nu_{\mathcal{P}_1},r_1)} \cdots a_{(+,\nu_{\mathcal{P}_m},r_m)}$ and $b_{(-,I)} := a_{(-,\nu_{\mathcal{P}_1},r_1)} \cdots a_{(-,\nu_{\mathcal{P}_m},r_m)}$ are the products associated to the (unique) prime decomposition $I = \mathcal{P}_1^{r_1} \cdots \mathcal{P}_m^{r_m}$, and $N(I)$ is the norm of I.

The following general result, concerning the absolute convergence of Dirichlet series, is well-known:

Lemma 7.14. *We have the following equivalences*

$$\sum_I \frac{|b_{(+,I)}|}{|N(I)^s|} < \infty \Leftrightarrow \prod_{\mathcal{P}} \sum_{n\geq 0} \frac{|b_{(+,\mathcal{P}^n)}|}{|N(\mathcal{P}^n)^s|} < \infty \qquad \sum_I \frac{|b_{(-,I)}|}{|N(I)^s|} < \infty \Leftrightarrow \prod_{\mathcal{P}} \sum_{n\geq 0} \frac{|b_{(-,\mathcal{P}^n)}|}{|N(\mathcal{P}^n)^s|} < \infty\,, \tag{7.15}$$

where $\mathcal{P}\triangleleft\mathcal{O}_k[S_X^{-1}]$ is a prime ideal. Moreover, if the left-hand side, resp. right-hand side, of (7.15) converges, then we obtain the equality $\sum_I \frac{b_{(+,I)}}{N(I)^s} = \prod_{\mathcal{P}} \sum_{n\geq 0} \frac{b_{(+,\mathcal{P}^n)}}{N(\mathcal{P}^n)^s}$, resp. $\sum_I \frac{b_{(-,I)}}{N(I)^s} = \prod_{\mathcal{P}} \sum_{n\geq 0} \frac{b_{(-,\mathcal{P}^n)}}{N(\mathcal{P}^n)^s}$.

Given a prime ideal $\mathcal{P}\triangleleft\mathcal{O}_k[S_X^{-1}]$, recall that its norm $N(\mathcal{P})$ is defined as the cardinality of the quotient field $\mathcal{O}_k[S_X^{-1}]/\mathcal{P}$. Since $\mathcal{O}_k[S_X^{-1}]/\mathcal{P}$ is isomorphic to the residue field $\kappa_{\nu_{\mathcal{P}}}$, we hence obtain the formal equalities:

$$\sum_{n\geq 0} \frac{b_{(+,\mathcal{P}^n)}}{N(\mathcal{P}^n)^s} = \sum_{n\geq 0} \frac{a_{(+,\nu_{\mathcal{P}},n)}}{N(\mathcal{P})^{ns}} = \sum_{n\geq 0} \frac{a_{(+,\nu_{\mathcal{P}},n)}}{N_{\nu_{\mathcal{P}}}^{ns}} = \varphi_{(+,\nu_{\mathcal{P}})}(N_{\nu_{\mathcal{P}}}^{-s}) = L_{\mathrm{even},\nu_{\mathcal{P}}}(\mathcal{A};s)$$

$$\sum_{n\geq 0} \frac{b_{(-,\mathcal{P}^n)}}{N(\mathcal{P}^n)^s} = \sum_{n\geq 0} \frac{a_{(-,\nu_{\mathcal{P}},n)}}{N(\mathcal{P})^{ns}} = \sum_{n\geq 0} \frac{a_{(-,\nu_{\mathcal{P}},n)}}{N_{\nu_{\mathcal{P}}}^{ns}} = \varphi_{(-,\nu_{\mathcal{P}})}(N_{\nu_{\mathcal{P}}}^{-s}) = L_{\mathrm{odd},\nu_{\mathcal{P}}}(\mathcal{A};s)\,.$$

Therefore, thanks to Lemma 7.14 and to classical properties of Dirichlet series, in order to prove Theorem 1.2, it suffices to show the following claim: we have $\prod_{\mathcal{P}} \sum_{n\geq 0} \frac{|a_{(+,\nu_{\mathcal{P}},n)}|}{N(\mathcal{P})^{nz}} < \infty$, resp. $\prod_{\mathcal{P}} \sum_{n\geq 0} \frac{|a_{(-,\nu_{\mathcal{P}},n)}|}{N(\mathcal{P})^{nz}} < \infty$, for every real number $z>1$, resp. $z>\frac{3}{2}$. Note that, by construction, we have the following inequalities:

$$\sum_{n\geq 0} \frac{|a_{(+,\nu_{\mathcal{P}},n)}|}{N(\mathcal{P})^{nz}} \leq \exp\Big(\sum_{n\geq 1} \frac{|\#_{(+,\nu_{\mathcal{P}},n)}|}{nN(\mathcal{P})^{nz}}\Big) \qquad \sum_{n\geq 0} \frac{|a_{(-,\nu_{\mathcal{P}},n)}|}{N(\mathcal{P})^{nz}} \leq \exp\Big(\sum_{n\geq 1} \frac{|\#_{(-,\nu_{\mathcal{P}},n)}|}{nN(\mathcal{P})^{nz}}\Big)\,. \tag{7.16}$$

Moreover, by taking $\exp(-)$ to Lemma 7.17 below, we observe that $\prod_{\mathcal{P}} \exp(\sum_{n\geq 1} \frac{|\#_{(+,\nu_{\mathcal{P}},n)}|}{nN(\mathcal{P})^{nz}}) < \infty$, resp. $\prod_{\mathcal{P}} \exp(\sum_{n\geq 1} \frac{|\#_{(-,\nu_{\mathcal{P}},n)}|}{nN(\mathcal{P})^{nz}}) < \infty$, for every real number $z>1$, resp. $z>\frac{3}{2}$. Consequently, by taking the product, over all the prime ideals $\mathcal{P}\triangleleft\mathcal{O}_k[S_X^{-1}]$, of the inequalities (7.16), we obtain the aforementioned claim.

Lemma 7.17. *We have $\sum_{\mathcal{P}} \sum_{n\geq 1} \frac{|\#_{(+,\nu_{\mathcal{P}},n)}|}{nN(\mathcal{P})^{nz}} < \infty$, resp. $\sum_{\mathcal{P}} \sum_{n\geq 1} \frac{|\#_{(-,\nu_{\mathcal{P}},n)}|}{nN(\mathcal{P})^{nz}} < \infty$, for every real number $z>1$, resp. $z>\frac{3}{2}$.*

Proof. Note that, thanks to Proposition 7.10 and to the equality $N(\mathcal{P}) = N_{\nu_{\mathcal{P}}}$, it suffices to show that $\sum_{\mathcal{P}} \sum_{n\geq 1} \frac{1}{N(\mathcal{P})^{nz}} < \infty$, resp. $\sum_{\mathcal{P}} \sum_{n\geq 1} \frac{1}{N(\mathcal{P})^{nz}} < \infty$, for every real number $z>1$, resp. $z>\frac{3}{2}$. Let us assume first that $\mathrm{char}(k)=0$. Recall that in this case k is a finite field extension of $\mathbb{Q}$. In what concerns $\sum_{\mathcal{P}} \sum_{n\geq 1} \frac{1}{N(\mathcal{P})^{nz}} < \infty$, with $z>1$, we have the following (in)equalities ($p\in\mathbb{Z}$ stands for a prime number)

$$\sum_{\mathcal{P}} \sum_{n\geq 1} \frac{1}{N(\mathcal{P})^{nz}} \leq \sum_p \sum_{\mathcal{P}|p} \sum_{n\geq 1} \frac{1}{p^{nz}} \tag{7.18}$$

$$\leq [k:\mathbb{Q}]\cdot \sum_p \sum_{n\geq 1} \frac{1}{p^{nz}} \tag{7.19}$$

$$= [k:\mathbb{Q}]\cdot \sum_p \frac{1}{p^z-1} \tag{7.20}$$

$$\leq 2\cdot[k:\mathbb{Q}]\cdot \sum_p \frac{1}{p^z} \tag{7.21}$$

$$\leq 2\cdot[k:\mathbb{Q}]\cdot \sum_{n\geq 1} \frac{1}{n^z} < \infty\,, \tag{7.22}$$

where (7.18) follows from the fact that the norm $N(\mathcal{P})$ is a power of p whenever $\mathcal{P}$ divides p, (7.19) from the fact that the number of prime ideals $\mathcal{P}$ which divide p is always bounded by the degree $[k:\mathbb{Q}]$, (7.20) from the (convergent) geometric series $\sum_{n\geq 0} \frac{1}{(p^z)^n} = \frac{1}{1-\frac{1}{p^z}}$, (7.21) from a simple inspection, and (7.22) from the fact that the Riemann zeta function $\zeta(s) := \sum_{n\geq 1}\frac{1}{n^s}$ is convergent when $\mathrm{Re}(s) > 1$. Similarly, in what concerns $\sum_{\mathcal{P}}\sum_{n\geq 1}\frac{1}{N(\mathcal{P})^{nz}} < \infty$, with $z > \frac{3}{2}$, we have the (in)equalities:

$$\begin{aligned}
\sum_{\mathcal{P}}\sum_{n\geq 1}\frac{1}{N(\mathcal{P})^{n(z-\frac{1}{2})}} &\leq \sum_{p}\sum_{\mathcal{P}|p}\sum_{n\geq 1}\frac{1}{p^{n(z-\frac{1}{2})}}\\
&\leq [k:\mathbb{Q}]\cdot\sum_{p}\sum_{n\geq 1}\frac{1}{p^{n(z-\frac{1}{2})}}\\
&= [k:\mathbb{Q}]\cdot\sum_{p}\frac{1}{p^{(z-\frac{1}{2})}-1}\\
&\leq 2\cdot[k:\mathbb{Q}]\cdot\sum_{p}\frac{1}{p^{(z-\frac{1}{2})}}\\
&\leq 2\cdot[k:\mathbb{Q}]\cdot\sum_{n\geq 1}\frac{1}{n^{(z-\frac{1}{2})}} < \infty\,.
\end{aligned}$$

Let us now assume that $\mathrm{char}(k) > 0$. Recall that in this case k is a finite field extension of $\mathbb{F}_q(t)$. In what concerns $\sum_{\mathcal{P}}\sum_{n\geq 1}\frac{1}{N(\mathcal{P})^{nz}} < \infty$, with $z > 1$, we have the following (in)equalities ($\langle p(t)\rangle$, resp. $\langle q(t)\rangle$, stands for a prime ideal, resp. ideal, of the ring $\mathbb{F}_q[t]$):

$$\sum_{\mathcal{P}}\sum_{n\geq 1}\frac{1}{N(\mathcal{P})^{nz}} \leq \sum_{\langle p(t)\rangle}\sum_{\mathcal{P}|\langle p(t)\rangle}\sum_{n\geq 1}\frac{1}{N(\langle p(t)\rangle)^{nz}} \tag{7.23}$$

$$\leq [k:\mathbb{F}_q(t)]\cdot\sum_{\langle p(t)\rangle}\sum_{n\geq 1}\frac{1}{N(\langle p(t)\rangle)^{nz}} \tag{7.24}$$

$$= [k:\mathbb{F}_q(t)]\cdot\sum_{\langle p(t)\rangle}\frac{1}{N(\langle p(t)\rangle)^{z}-1} \tag{7.25}$$

$$\leq 2\cdot[k;\mathbb{F}_q(t)]\cdot\sum_{\langle p(t)\rangle}\frac{1}{N(\langle p(t)\rangle)^{z}} \tag{7.26}$$

$$\leq 2\cdot[k;\mathbb{F}_q(t)]\cdot\sum_{\langle q(t)\rangle}\frac{1}{N(\langle q(t)\rangle)^{z}} < \infty\,, \tag{7.27}$$

where (7.23) follows from the fact that the norm $N(\mathcal{P})$ is a power of $N(\langle p(t)\rangle)$ whenever $\mathcal{P}$ divides $\langle p(t)\rangle$, (7.24) from the fact that the number of prime ideals $\mathcal{P}$ which divide $\langle p(t)\rangle$ is always bounded by the degree $[k:\mathbb{F}_q(t)]$, (7.25) from the (convergent) geometric series $\sum_{n\geq 0}\frac{1}{(N(\langle p(t)\rangle)^z)^n} = \frac{1}{1-\frac{1}{N(\langle p(t)\rangle)^z}}$, (7.26) from a simple inspection, and (7.27) from the fact that the classical zeta function $\sum_{\langle q(t)\rangle}\frac{1}{N(\langle q(t)\rangle)^s}$ is convergent when $\mathrm{Re}(s) > 1$. Similarly, in what concerns $\sum_{\mathcal{P}}\sum_{n\geq 1}\frac{1}{N(\mathcal{P})^{nz}} < \infty$, with $z > \frac{3}{2}$, we have the (in)equalities:

$$\begin{aligned}
\sum_{\mathcal{P}}\sum_{n\geq 1}\frac{1}{N(\mathcal{P})^{n(z-\frac{1}{2})}} &\leq \sum_{\langle p(t)\rangle}\sum_{\mathcal{P}|\langle p(t)\rangle}\sum_{n\geq 1}\frac{1}{N(\langle p(t)\rangle)^{n(z-\frac{1}{2})}}\\
&\leq [k:\mathbb{F}_q(t)]\cdot\sum_{\langle p(t)\rangle}\sum_{n\geq 1}\frac{1}{N(\langle p(t)\rangle)^{n(z-\frac{1}{2})}}\\
&= [k:\mathbb{F}_q(t)]\cdot\sum_{\langle p(t)\rangle}\frac{1}{N(\langle p(t)\rangle)^{(z-\frac{1}{2})}-1}\\
&\leq 2\cdot[k:\mathbb{F}_q(t)]\cdot\sum_{\langle p(t)\rangle}\frac{1}{N(\langle p(t)\rangle)^{(z-\frac{1}{2})}}\\
&\leq 2\cdot[k:\mathbb{F}_q(t)]\cdot\sum_{\langle q(t)\rangle}\frac{1}{N(\langle q(t)\rangle)^{(z-\frac{1}{2})}} < \infty\,.
\end{aligned}$$

This concludes the proof of Lemma 7.17. ∎

8. Proof of Theorem 1.3

Note first that if the L-functions $\{L_w(X;s)\}_{w\,\text{even}}$, resp. $\{L_w(X;s)\}_{w\,\text{odd}}$, satisfy condition (C1), then the shifted L-functions $\{L_w(X;s+\frac{w}{2})\}_{w\,\text{even}}$, resp. $\{L_w(X;s+\frac{w-1}{2})\}_{w\,\text{odd}}$, also satisfy condition (C1). Thanks to Proposition 6.6, this hence implies that the noncommutative L-function $L_{\text{even}}(\text{perf}_{\text{dg}}(X);s)$, resp. $L_{\text{odd}}(\text{perf}_{\text{dg}}(X);s)$, admits a (unique) meromorphic continuation to the entire complex plane. Now, Proposition 6.6 implies moreover the implications (1.4).

Let us assume now that $\text{char}(k)>0$ and that the conjecture $\text{R}_{\text{even}}(\text{perf}_{\text{dg}}(X))$, resp. $\text{R}_{\text{odd}}(\text{perf}_{\text{dg}}(X))$, holds. It follows from the work of Grothendieck [5] and Deligne [3, 4] that the L-function $L_w(X;s)$, with $0\leq w\leq 2\text{dim}(X)$, does not have a pole in the critical strip $\frac{w}{2}<\text{Re}(s)<\frac{w}{2}+1$. Consequently, thanks to Proposition 6.6, we conclude that all the zeros of the L-function $L_w(X;s)$ that are contained in the critical strip $\frac{w}{2}<\text{Re}(s)<\frac{w}{2}+1$ lie necessarily in the vertical line $\text{Re}(s)=\frac{w+1}{2}$ (otherwise, the noncommutative L-function $L_{\text{even}}(\text{perf}_{\text{dg}}(X);s)$, resp. $L_{\text{odd}}(\text{perf}_{\text{dg}}(X);s)$, would have a zero outside the vertical line $\text{Re}(s)=\frac{1}{2}$, resp. $\text{Re}(s)=1$). In other words, the converse implications of (1.4) hold.

Let us assume now that $\text{char}(k)=0$, that the L-functions $\{L_w(X;s)\}_{0\leq w\leq 2\text{dim}(X)}$ satisfy condition (C2), and that the conjecture $\text{R}_{\text{even}}(\text{perf}_{\text{dg}}(X))$, resp. $\text{R}_{\text{odd}}(\text{perf}_{\text{dg}}(X))$, holds. Since the L-function $L_w(X;s)$, with $0\leq w\leq 2\text{dim}(X)$, does not have a pole in the critical strip $\frac{w}{2}<\text{Re}(s)<\frac{w}{2}+1$, we hence conclude from Proposition 6.6 that all the zeros of $L_w(X;s)$ that are contained in the critical strip $\frac{w}{2}<\text{Re}(s)<\frac{w}{2}+1$ lie necessarily in the vertical line $\text{Re}(s)=\frac{w+1}{2}$ (otherwise, the noncommutative L-function $L_{\text{even}}(\text{perf}_{\text{dg}}(X);s)$, resp. $L_{\text{odd}}(\text{perf}_{\text{dg}}(X);s)$, would have a zero outside the vertical line $\text{Re}(s)=\frac{1}{2}$, resp. $\text{Re}(s)=1$). In other words, the converse implications of (1.4) also hold.

9. Proof of Theorem 2.3

By definition of the Lefschetz decomposition $\text{perf}(X)=\langle\mathbb{A}_0,\mathbb{A}_1(1),\ldots,\mathbb{A}_{i-1}(i-1)\rangle$, we have a chain of admissible triangulated subcategories $\mathbb{A}_{i-1}\subseteq\cdots\subseteq\mathbb{A}_1\subseteq\mathbb{A}_0$ with $\mathbb{A}_r(r):=\mathbb{A}_r\otimes\mathcal{L}_X(r)$; note that $\mathbb{A}_r(r)$ is equivalent to $\mathbb{A}_r$. Let us write $\mathfrak{a}_r$ for the right-orthogonal complement of $\mathbb{A}_{r+1}$ in $\mathbb{A}_r$; these are called the *primitive subcategories* in [13, §4]. By construction, we have the following semi-orthogonal decompositions:

$$\mathbb{A}_r=\langle\mathfrak{a}_r,\mathfrak{a}_{r+1},\ldots,\mathfrak{a}_{i-1}\rangle\qquad 0\leq r\leq i-1\,. \tag{9.1}$$

Following [13, Thm. 6.3], we have a chain of admissible triangulated subcategories $\mathbb{B}_{j-1}\subseteq\cdots\subseteq\mathbb{B}_1\subseteq\mathbb{B}_0$ and an associated HP-dual Lefschetz decomposition $\text{perf}(Y)=\langle\mathbb{B}_{j-1}(1-j),\ldots,\mathbb{B}_1(-1),\mathbb{B}_0\rangle$ with respect to $\mathcal{L}_Y(1)$. Moreover, we have the following semi-orthgonal decompositions:

$$\mathbb{B}_r=\langle\mathfrak{a}_0,\mathfrak{a}_1,\ldots,\mathfrak{a}_{\text{dim}(V)-r-2}\rangle\qquad 0\leq r\leq j-1\,. \tag{9.2}$$

Furthermore, since the linear subspace $L\subset V^\vee$ is generic, we can assume without loss of generality that the linear sections X_L and Y_L are not only smooth but also that they have the *expected dimensions*, i.e., $\text{dim}(X_L)=\text{dim}(X)-\text{dim}(L)$ and $\text{dim}(Y_L)=\text{dim}(Y)-\text{dim}(L^\perp)$. As explained in [13, Thm. 6.3], this yields the following semi-orthogonal decompositions

$$\text{perf}(X_L)=\langle\mathcal{C}_L,\mathbb{A}_{\text{dim}(L)}(1),\ldots,\mathbb{A}_{i-1}(i-\text{dim}(L))\rangle \tag{9.3}$$

$$\text{perf}(Y_L)=\langle\mathbb{B}_{j-1}(\text{dim}(L^\perp)-j),\ldots,\mathbb{B}_{\text{dim}(L^\perp)}(-1),\mathcal{C}_L\rangle\,, \tag{9.4}$$

where $\mathcal{C}_L$ is a common (triangulated) category. Let us denote by $\mathbb{A}_{r,\text{dg}}$, by $\mathfrak{a}_{r,\text{dg}}$, and by $\mathcal{C}_{L,\text{dg}}$, the dg enhancements of $\mathbb{A}_r$, $\mathfrak{a}_r$, and $\mathcal{C}_L$, induced from the dg category $\text{perf}_{\text{dg}}(X_L)$. Similarly, let us denote by $\mathbb{B}_{r,\text{dg}}$ and by $\mathcal{C}'_{L,\text{dg}}$ the dg enhancements of $\mathbb{B}_r$ and $\mathcal{C}_L$ induced from the dg category $\text{perf}_{\text{dg}}(Y_L)$. Since the functor $\text{perf}(X_L)\to\mathcal{C}_L\to\text{perf}(Y_L)$ is of Fourier-Mukai type, the dg categories $\mathcal{C}_{L,\text{dg}}$ and $\mathcal{C}'_{L,\text{dg}}$ are Morita equivalent. Note that, by construction, all the aforementioned dg categories are geometric noncommutative k-schemes. Note that by combining the above semi-orthogonal decompositions (9.1)-(9.4) with Lemma 9.8 below and with an iterated application of Proposition 6.4, we obtain the following equalities

$$L_{\text{even}}(\text{perf}_{\text{dg}}(X_L);s)=L_{\text{even}}(\mathcal{C}_{L,\text{dg}};s)\cdot\zeta_k(s)\cdots\zeta_k(s)\qquad L_{\text{odd}}(\text{perf}_{\text{dg}}(X_L);s)=L_{\text{odd}}(\mathcal{C}_{L,\text{dg}};s) \tag{9.5}$$

$$L_{\text{even}}(\text{perf}_{\text{dg}}(Y_L);s)=\zeta_k(s)\cdots\zeta_k(s)\cdot L_{\text{even}}(\mathcal{C}_{L,\text{dg}};s)\qquad L_{\text{odd}}(\text{perf}_{\text{dg}}(Y_L);s)=L_{\text{odd}}(\mathcal{C}_{L,\text{dg}};s)\,, \tag{9.6}$$

where the number of copies of the Dedekind zeta function $\zeta_k(s)$ in (9.5), resp. in (9.6), is equal to the sum of the ranks of the (free) Grothendieck groups $K_0(\mathbb{A}_{\text{dim}(L)}),\ldots,K_0(\mathbb{A}_{i-1})$, resp. $K_0(\mathbb{B}_{j-1}),\ldots,K_0(\mathbb{B}_{\text{dim}(L^\perp)})$.

Consequently, since the Dedekind zeta function $\zeta_k(s)$ satisfies conditions (C1)-(C2), the following holds:

$$\text{(9.7)}\quad \mathrm{ERH}_k \Rightarrow \big(\mathrm{R}_{\mathrm{even}}(\mathrm{perf}_{\mathrm{dg}}(X_L)) \Leftrightarrow \mathrm{R}_{\mathrm{odd}}(\mathrm{perf}_{\mathrm{dg}}(Y_L))\big) \qquad \mathrm{R}_{\mathrm{odd}}(\mathrm{perf}_{\mathrm{dg}}(X_L)) \Leftrightarrow \mathrm{R}_{\mathrm{odd}}(\mathrm{perf}_{\mathrm{dg}}(Y_L))\,.$$

The proof follows now from the combination of (9.7) with Theorem 1.3.

Lemma 9.8. *We have the following computations (with $0 \leq r \leq i-1$)*

$$L_{\mathrm{even}}(\mathfrak{a}_{r,\mathrm{dg}}; s) = \underbrace{\zeta_k(s)\cdots\zeta_k(s)}_{n_r} \qquad L_{\mathrm{odd}}(\mathfrak{a}_{r,\mathrm{dg}}; s) = 1\,, \tag{9.9}$$

where n_r stands for the rank of the (free) Grothendieck group $K_0(\mathfrak{a}_r)$.

Proof. Recall from §4.2 the definition of the universal additive invariant $U\colon \mathrm{dgcat}(k) \to \mathrm{Hmo}_0(k)$. By assumption, we have a full exceptional collection $\mathbb{A}_0 = \langle \mathcal{E}_1, \ldots, \mathcal{E}_n\rangle$. As explained in [21, §2.4.2], this implies that $U(\mathbb{A}_{0,\mathrm{dg}}) \simeq U(k)^{\oplus n}$ in the additive category $\mathrm{Hmo}_0(k)$. Moreover, since $\mathfrak{a}_r$ is an admissible triangulated subcategory of $\mathbb{A}_0$, $U(\mathfrak{a}_{r,\mathrm{dg}})$ becomes a direct summand of $U(\mathbb{A}_{0,\mathrm{dg}})$. Thanks to the computation

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{Hmo}_0(k)}(U(\mathbb{A}_0), U(\mathbb{A}_0)) &\simeq \mathrm{Hom}_{\mathrm{Hmo}_0(k)}(U(k)^{\oplus n}, U(k)^{\oplus n}) \\
&\simeq \mathrm{Hom}_{\mathrm{Hmo}_0(k)}(U(k), U(k))^{\oplus(n\times n)} \\
&= K_0\mathrm{rep}(k,k)^{\oplus(n\times n)} \simeq K_0\mathcal{D}_c(k)^{\oplus(n\times n)} \simeq \mathbb{Z}^{\oplus(n\times n)}\,,
\end{aligned}$$

we observe that the endomorphism ring of $U(\mathbb{A}_{0,\mathrm{dg}})$ is isomorphic to the ring of $(n\times n)$-matrices with $\mathbb{Z}$-coefficients. This implies that $U(\mathfrak{a}_{r,\mathrm{dg}})$ is then necessarily isomorphic to $U(k)^{\oplus n_r}$ for a certain integer $n_r \leq n$. Moreover, this integer n_r is equal to the rank of the (free) Grothendieck group $K_0(\mathfrak{a}_r)$ because

$$\mathrm{Hom}_{\mathrm{Hmo}_0(k)}(U(k), U(\mathfrak{a}_{r,\mathrm{dg}})) := K_0\mathrm{rep}(k, \mathfrak{a}_{r,\mathrm{dg}}) \simeq K_0(\mathfrak{a}_r) \quad \text{and} \quad \mathrm{Hom}_{\mathrm{Hmo}_0(k)}(U(k), U(k)^{\oplus n_r}) \simeq \mathbb{Z}^{\oplus n_r}\,.$$

Now, recall from Proposition 5.5 and Lemma 4.3 that, given any non-archimedean place $\nu \in \Sigma_k$, the associated functors $\mathrm{H}_{\mathrm{even},l_\nu}(-\otimes_k k_\nu)$ and $\mathrm{H}_{\mathrm{odd},l_\nu}(-\otimes_k k_\nu)$ are additive invariants. Consequently, since these additive invariants factor through the additive category $\mathrm{Hmo}_0(k)$ (consult the factorization (4.4)), we obtain an isomorphism of $\mathbb{Q}_{l_\nu}$-linear $\mathrm{Gal}(\overline{k_\nu}/k_\nu)$-modules between $\mathrm{H}_{\mathrm{even},l_\nu}(\mathfrak{a}_{r,\mathrm{dg}} \otimes_k k_\nu)$ and the direct sum $\mathrm{H}_{\mathrm{even},l_\nu}(k\otimes_k k_\nu)^{\oplus n_r}$ and between $\mathrm{H}_{\mathrm{odd},l_\nu}(\mathfrak{a}_{r,\mathrm{dg}} \otimes_k k_\nu)$ and the direct sum $\mathrm{H}_{\mathrm{odd},l_\nu}(k\otimes_k k_\nu)^{\oplus n_r}$. Therefore, making use of the canonical Morita equivalence $k \to \mathrm{perf}_{\mathrm{dg}}(\mathrm{Spec}(k))$, of Proposition 5.6, and of Definition 6.3, we obtain the above computations (9.9). □

10. Proof of Theorems 3.1, 3.2, 3.5, and 3.7

Proof of Theorem 3.1. By construction of the noncommutative gluing $X \odot_{\mathrm{B}} Y$, we have the semi-orthogonal decomposition $H^0(X\odot_{\mathrm{B}} Y) = \langle H^0(\mathrm{perf}_{\mathrm{dg}}(X)), H^0(\mathrm{perf}_{\mathrm{dg}}(Y))\rangle$. Consequently, thanks to Proposition 6.4, we conclude that $L_{\mathrm{even}}(X\odot_{\mathrm{B}} Y; s) = L_{\mathrm{even}}(\mathrm{perf}_{\mathrm{dg}}(X); s)\cdot L_{\mathrm{even}}(\mathrm{perf}_{\mathrm{dg}}(Y); s)$. This yields the implication $\mathrm{R}_{\mathrm{even}}(\mathrm{perf}_{\mathrm{dg}}(X)) \wedge \mathrm{R}_{\mathrm{even}}(\mathrm{perf}_{\mathrm{dg}}(X)) \Rightarrow \mathrm{R}_{\mathrm{even}}(X\odot_{\mathrm{B}} Y)$. Therefore, the proof follows now from Theorem 1.3. The proof of the odd case is similar: simply replace the word "even" by the word "odd".

Proof of Theorem 3.2. Thanks to the decomposition $\mathrm{perf}(X) = \langle \mathcal{T}, \mathcal{O}_X, \ldots, \mathcal{O}_X(n-\deg(X))\rangle$, an iterated application of Proposition 6.4 yields the following equalities:

$$L_{\mathrm{even}}(\mathrm{perf}_{\mathrm{dg}}(X); s) = L_{\mathrm{even}}(\mathcal{T}_{\mathrm{dg}}; s)\cdot \underbrace{\zeta_k(s)\cdots\zeta_k(s)}_{n-\deg(X)+1} \qquad L_{\mathrm{odd}}(\mathrm{perf}_{\mathrm{dg}}(X); s) = L_{\mathrm{odd}}(\mathcal{T}_{\mathrm{dg}}; s)\,. \tag{10.1}$$

On the one hand, since the Dedekind zeta function $\zeta_k(s)$ satisfies conditions (C1)-(C2), the left-hand side of (10.1) implies that $\mathrm{R}_{\mathrm{even}}(\mathrm{perf}_{\mathrm{dg}}(X)) \Rightarrow \mathrm{R}_{\mathrm{even}}(\mathcal{T}_{\mathrm{dg}})$. On the other hand, the right-hand side of (10.1) implies that $\mathrm{R}_{\mathrm{odd}}(\mathrm{perf}_{\mathrm{dg}}(X)) \Leftrightarrow \mathrm{R}_{\mathrm{odd}}(\mathcal{T}_{\mathrm{dg}})$. Consequently, the proof follows now from Theorem 1.3.

Proof of Theorem 3.5. As proved in [23, Thm. 3.5], since the quotient k-algebra A/J is separable, we have an isomorphism $U(A)_{\mathbb{Q}} \simeq U(k_1)_{\mathbb{Q}} \oplus \cdots \oplus U(k_n)_{\mathbb{Q}}$ in the $\mathbb{Q}$-linearization $\mathrm{Hmo}_0(k)_{\mathbb{Q}}$ of the additive category $\mathrm{Hmo}_0(k)$; consult §4.2. Thanks to Proposition 5.5 and Lemma 4.3, given any non-archimedean place $\nu \in \Sigma_k$, the associated functors $\mathrm{H}_{\mathrm{even},l_\nu}(-\otimes_k k_\nu)$ and $\mathrm{H}_{\mathrm{odd},l_\nu}(-\otimes_k k_\nu)$ are additive invariants. Since these additive invariants take values in the category $\mathrm{Gal}(\overline{k_\nu}/k_\nu)$-Mod, which is a $\mathbb{Q}$-linear category, the above isomorphism combined with the factorization (4.4) leads to the following isomorphisms of $\mathrm{Gal}(\overline{k_\nu}/k_\nu)$-modules:

$$\mathrm{H}_{\mathrm{even},l_\nu}(A\otimes_k k_\nu)\simeq\bigoplus_{i=1}^n \mathrm{H}_{\mathrm{even},l_\nu}(k_i\otimes_k k_\nu) \qquad \mathrm{H}_{\mathrm{odd},l_\nu}(A\otimes_k k_\nu)\simeq\bigoplus_{i=1}^n \mathrm{H}_{\mathrm{odd},l_\nu}(k_i\otimes_k k_\nu)\,.$$

Making use of the canonical Morita equivalences $k_i\to\mathrm{perf}_{\mathrm{dg}}(\mathrm{Spec}(k_i))$, of Proposition 5.6, and of Definition 6.3, we hence conclude that $L_{\mathrm{even}}(A;s)=\prod_{i=1}^n L_0(\mathrm{Spec}(k_i))$ and $L_{\mathrm{odd}}(A;s)=1$. This yields the implication $\mathrm{R}_0(\mathrm{Spec}(k_1))\wedge\cdots\wedge\mathrm{R}_0(\mathrm{Spec}(k_n))\Rightarrow\mathrm{R}_{\mathrm{even}}(A)$.

Proof of Theorem 3.7. Recall first that we have a canonical Morita equivalence $A\to\mathrm{perf}_{\mathrm{dg}}(A)$. Following Orlov [17, Cor. 3.4], there exists a smooth proper k-scheme X such that $\mathrm{perf}(A)$ is an admissible triangulated subcategory of $\mathrm{perf}(X)$. Moreover, since the quotient dg k-algebra A/J_+ is separable, we have a semi-orthogonal decomposition $\mathrm{perf}(X)=\langle\mathrm{perf}(D_1),\ldots,\mathrm{perf}(D_n)\rangle$, where $D_1,\ldots,D_n$ are separable division k-algebras. Let us denote by $k_1,\ldots,k_n$ the centers of the k-algebras $D_1,\ldots,D_n$; these are separable finite field extensions of k. Making use of [23, Thm. 2.11], we hence conclude that $U(A)_\mathbb{Q}$ becomes a direct summand of $U(\mathrm{perf}_{\mathrm{dg}}(X))_\mathbb{Q}\simeq U(k_1)_\mathbb{Q}\oplus\cdots\oplus U(k_n)_\mathbb{Q}$ in the $\mathbb{Q}$-linearization $\mathrm{Hmo}_0(k)_\mathbb{Q}$ of the additive category $\mathrm{Hmo}_0(k)$; consult §4.2. Note that, similarly to the proof of Theorem 3.5, this implies that $L_{\mathrm{odd}}(A;s)=1$. Now, recall from [21, §4.9] that the classical category of Artin motives (with $\mathbb{Q}$-coefficients) may be identified with the idempotent completion of the full subcategory of $\mathrm{Hmo}_0(k)_\mathbb{Q}$ consisting of the objects $\{U(\mathrm{perf}_{\mathrm{dg}}(X))_\mathbb{Q}\,|\,X \text{ is a 0-dimensional } k\text{-scheme}\}$. Under this identification, $U(A)_\mathbb{Q}$ corresponds to an Artin motive $\rho\colon\mathrm{Gal}(\overline{k}/k)\to\mathrm{GL}(V)$, where V is a finite dimensional $\mathbb{Q}$-vector space, and the noncommutative L-function $L_{\mathrm{even}}(A;s)$ corresponds to the classical Artin L-function $L(\rho;s)$ of ρ. Consequently, the proof follows now from Weil's work [27, §V].

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