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ORIGINAL RESEARCH

Stabilisation in distribution by delay feedback control for stochastic differential equations with Markovian switching and Lévy noise

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Abstract
This paper is devoted to the stability in distribution of stochastic differential equations with Markovian switching and Lévy noise by delay feedback control. By constructing efficient Lyapunov functional and linear delay feedback controls, the stability in distribution of stochastic differential equations with Markovian switching and Lévy noise is accomplished with the coefficients satisfying globally Lipschitz continuous. Moreover, the design methods of feedback control under two structures of state feedback and output injection are discussed. Finally, a numerical experiment and new algorithm are provided to sustain the new results.

1 | INTRODUCTION

Over the past years, following the development of stochastic differential equations with Markovian switching (SDEs-MS), many authors have turned their attentions to the studies of SDEs-MS-LN. This class of systems is often used to model many realistic systems, such as mathematical finance, biology and so on, see refs. [1–5]. As an important aspect of the study of SDEs-MS-LN, asymptotic stability analysis has been broadly studied in refs. [6–11] and references therein. In ref. [12], some sufficient conditions were established to get almost surely exponential stability of neural networks with Markovian switching and Lévy noise. Using a continuously differentiable Lyapunov function, Li et al. [13] showed the moment exponential input-to-state stability for nonlinear switched stochastic differential equations. As can be followed from the literature mentioned above, the majority of research in this field has concentrated on the stability of the trivial solutions in probability or moment and so on.

However, many hybrid systems under realistic backdrops do not have an equilibrium state or their solutions do not converge to zero (such as stochastic two-species competitive

Lotka–Volterra system in biology [14], flexible manufacturing systems in engineering [15]). It is unsuitable to investigate the asymptotic stability of trivial solutions in this case. From another point of view, this kind of stability is excessively mighty and powerful to some extent while in reality, it is sufficient to recognise if the probability distribution of the solution will converge to some distribution or not (but not necessary to zero). This property is called asymptotic stability in distribution. Stability in distribution of SDEs-MS has been attracted much attention and some research results appeared, for example, Yuan et al. [16], Yuan and Mao [17], Bo and Yuan [18], Dua et al. [19], Li and Zhang [20]. It is rather remarkable that You [21] and several collaborators recently made a groundbreaking work: stabilisation in distribution for SDEs-MS with Brownian motion by linear delay feedback controls (DFC) when coefficients are globally Lipschitz continuous. As a continuous stochastic process, Brownian motion has been widely used in SDEs-MS. However, due to some random jump type instantaneous disturbance, the classical Brownian motion cannot always describe this kind of phenomenon well. In this case, Lévy noise as a discontinuous process is more appropriate for modelling these systems. Unfortunately, as far as we know, the stability in distribution of SDEs-MS-LN by feedback control has not been considered.

Consider the following unstable SDEs-MS-LN

$$\begin{aligned} dY(t) =& f(Y(t), q(t))dt + g(Y(t), q(t))dW(t) \\ &+ \int_{\mathbb{R}_0^n} h(Y(t^-), q(t^-), x)\tilde{N}(dt, dx), \end{aligned} \tag{1}$$

where $Y(t) \in \mathbb{R}^d$ and $q(t)$ is a Markov chain, $W(t)$ is a Brownian motion, $\tilde{N}(dt, dx)$ is a compensated Poisson random measure and $\mathbb{R}_0^n = \mathbb{R}^n - \{0\}$. (More precise notion of SDEs-MS-LN will be given in the next section.) Can we stabilise it by DFC $u(Y(t-\tau), q(t))$ so that the controlled system

$$\begin{aligned} dY(t) =& [f(Y(t), q(t)) + u(Y(t-\tau), q(t))]dt \\ &+ g(Y(t), q(t))dW(t) \\ &+ \int_{\mathbb{R}_0^n} h(Y(t^-), q(t^-), x)\tilde{N}(dt, dx), \end{aligned} \tag{2}$$

becomes stable in distribution?

In this article we are interested in investigating how to use a DFC to stabilise a given unstable SDEs-MS-LN in the sense of asymptotic stability in distribution. The key works of this article are highlighted below.

- Based on linear delay feedback control, the stability in distribution is investigated for SDEs-MS-LN in functional space $\mathcal{D}_\tau$.
- We obtain a bound on τ^* in order that the linear DFC works if only $\tau \le \tau^*$.
- For state feedback and output injection two structure case, the corresponding feedback controls of SDEs-MS-LN are designed.
- We develop new algorithm to verify the convergence of the distribution of two segment processes.

To close this introduction, we would like to compare our work with You et al. [21]. Their paper was the first to design feedback control to stabilise a given unstable SDEs-MS in the sense of asymptotic stability in distribution. It is a very significant and profound contribution to study the stabilisation issue of the practical applications. In ref. [21], the authors assumed that the SDEs-MS were driven by Brownian motions. However, in general, many hybrid systems may be subject to some random jump type instantaneous disturbance (e.g. hurricanes, sandstorms and earthquakes). Lévy noise is therefore more useful in real world and this is what we will establish in this paper. To investigate the SDEs-MS-LN, we adopt the functional space $\mathcal{D}_\tau$ (please see Section 2 for the formal definitions) endowed with the Skorokhod topology rather $\mathcal{C}_\tau$ in ref. [21] and different Lyapunov functionals are also established in our proofs.

This article is organised as follows. We first presents some preliminaries and assumptions concerning Equation (1) in Section 2. In Section 3, in view of Lyapunov functional and Itô formula, we investigate the stable in distribution of the solution to Equation (2). In Section 4, we give some important Corollaries and discuss how to design linear DFC matrix. While the numerical simulations to support the theoretical results are given in Section 5. Finally, Section 6 states the main conclusions.

2 | PRELIMINARIES

Let $|\cdot|$ denote the Euclidean norm or the matric trace norm, respectively. $\mathbb{R}^d$ denote the d-dimensional Euclidean space and $\mathcal{B}(\mathbb{R}^d)$ represents the family of all Borel measurable sets in $\mathbb{R}^d$. If A is a matrix, its trace norm is denoted by $|A| = \sqrt{\text{trace}(A^T A)}$ while its operator norm is denoted by $\|A\| = \sup |Ay| : |y| = 1$. If A is a symmetric matrix, denote by $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ its largest and smallest eigenvalue, respectively.

$(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge 0}, \mathbb{P})$ denotes a complete probability space, in which a filtration $\{\mathcal{F}_t\}_{t\ge 0}$ satisfying the usual conditions. Let $\mathcal{D}_\tau$(or $D([-\tau, 0]; \mathbb{R}^d)(\tau > 0)$) be the family of all càdlàg (i.e. right continuous with left limits) function $\eta : [-\tau, 0] \to \mathbb{R}^d$ endowed with the Skorokhod topology and metric given by $d_S(\eta_1, \eta_2) = \inf_{\lambda\in\Lambda}\{\|\lambda\|^\circ \vee \|\eta_1 - \eta_2\circ\lambda\|_\tau\}$, where Λ denote the class of strictly increasing, continuous mappings of $[-\tau, 0]$ onto itself, $\eta_2\circ\lambda$ denotes the composition of two functions η_2 and λ, $\|\lambda\|^\circ = \sup_{-\tau\le s<t\le 0} |\log \frac{\lambda(t)-\lambda(s)}{t-s}|$ and $\|\eta\|_\tau = \sup_{-\tau\le u\le 0} |\eta(u)|$. Under the Skorohod metric d_S, $D([-\tau, 0]; \mathbb{R}^d)$ is complete and separable (ref. [22], Theorem 12.2, p. 128). Also, $\mathcal{B}(\mathcal{D}_\tau)$ represents the family of all Borel measurable sets in $\mathcal{D}_\tau$. Let $W(t) = (W_1, \ldots, W_m)$ be an m-dimensional Brownian motions and $N(t, x)$ be an

n-dimensional Poisson process and denote the compensated Poisson random measure by

$$\begin{aligned}\tilde{N}(\mathrm{d}t,\mathrm{d}x)^T &= N(\mathrm{d}t,\mathrm{d}x)-\nu(\mathrm{d}x)\mathrm{d}t\\ &= (N_1(\mathrm{d}t,\mathrm{d}x_1)-\nu_1(\mathrm{d}x_1)\mathrm{d}t,\dots,N_n(\mathrm{d}t,\mathrm{d}x_n)-\nu_n(\mathrm{d}x_n)\mathrm{d}t),\end{aligned}$$

where $\{N_k, k=1,\dots,n\}$ are independent 1-dimensional Poisson random measures with characteristic measure $\{\nu_k, k=1,\dots,n\}$ coming from n independent 1-dimensional Poisson point processes.

Let $q(t)$, $t\ge 0$, be a right-continuous Markov chain with finite state space $\mathbb{S}=\{1,2,\dots,N\}$ on the probability space. The generator of $\{q(t)\}_{t\ge 0}$ is defined by $\Gamma=(\gamma_{ij})_{N\times N}$ given by

$$\mathbb{P}\{q(t+\delta)=j|q(t)=i\}=\begin{cases}\gamma_{ij}\delta+o(\delta) & \text{if } i\neq j,\\ 1+\gamma_{ij}\delta+o(\delta) & \text{if } i=j,\end{cases}$$

where $\delta>0$ satisfies $\lim_{\delta\to 0}\frac{o(\delta)}{\delta}=0$ and γ_{ij} is the transition rate from i to j satisfying $\gamma_{ij}>0$ if $i\neq j$ while $\gamma_{ii}=-\sum_{i\neq j}\gamma_{ij}$. We assume that $W(t)$, $N(t,x)$, $q(t)$ are mutually independent.

Consider a d-dimension SDEs-MS-LN (1), where $f:\mathbb{R}^d\times\mathbb{S}\to\mathbb{R}^d$, $g:\mathbb{R}^d\times\mathbb{S}\to\mathbb{R}^{d\times m}$ and $h:\mathbb{R}^d\times\mathbb{S}\times\mathbb{R}_0^n\to\mathbb{R}^{d\times n}$ are Borel measurable functions, $Y(t^-)=\lim_{s\uparrow t}Y(s)$. We note that each column $h^{(k)}$ of the $d\times n$ matrix $h=[h_{lj}]$ depend on x only through the kth coordinate x_k, that is

$$h^{(k)}(Y,i,x)=h^{(k)}(Y,i,x_k);\quad x=(x_1,\dots,x_n)\in\mathbb{R}_0^n,$$

by means of refs. [23,24], we researched this type of dependence of SDEs-MS-LN. The component of $Y(t)=(Y_l(t))_{l\le d}=(Y_l(t),\dots,Y_d(t))$ in system (1) has the following form:

$$\begin{aligned}\mathrm{d}Y_l(t)=&f_l(Y(t),q(t))\mathrm{d}t+\sum_{j=1}^m g_{lj}(Y(t),q(t))\mathrm{d}W_j(t)\\ &+\sum_{k=1}^n\int_{\mathbb{R}\backslash\{0\}}h_{lk}(Y(t^-),q(t^-),x_k)\tilde{N}_k(\mathrm{d}t,\mathrm{d}x_k).\end{aligned}$$

To study the stabilisation in distribution of system (1), we impose the following assumption.

Assumption 2.1. Assume that f,g and h are globally Lipschitz continuous. That is, there exists positive constants e_1, e_2 and e_3 such that

$$|f(y,i)-f(z,i)|^2\le e_1|y-z|^2,\ |g(y,i)-g(z,i)|^2\le e_2|y-z|^2$$

and

$$\sum_{k=1}^n\int_{\mathbb{R}\backslash\{0\}}|h^{(k)}(y,i,x_k)-h^{(k)}(z,i,x_k)|^2\nu_k(\mathrm{d}x_k)\le e_3|y-z|^2$$

for all $y,z\in\mathbb{R}^d$ and $i\in\mathbb{S}$.

From Assumption 2.1, it can be directly obtained that

$$|f(y,i)|^2\le 2e_1|y|^2+e_0,\qquad |g(y,i)|^2\le 2e_2|y|^2+e_0 \tag{3}$$

and

$$\sum_{k=1}^n\int_{\mathbb{R}\backslash\{0\}}|h^{(k)}(y,i,x_k)|^2\nu_k(\mathrm{d}x_k)\le 2e_3|y|^2+e_0 \tag{4}$$

for each $(y,i)\in\mathbb{R}^d\times\mathbb{S}$, where $e_0=2\max_{i\in\mathbb{S}}(|f(0,i)|^2\vee|g(0,i)|^2)\vee\sum_{k=1}^n\int_{\mathbb{R}\backslash\{0\}}|h^{(k)}(0,i,x_k)|^2\nu_k(dx_k)$.

Under Assumption 2.1, it is proved in Wei et al. [10] that the solution $Y(t)$ of SDEs-MS-LN (1) exists and is unique for all $t\ge 0$. Suppose that the original SDEs-MS-LN (1) is not stable in distribution. To make it stable, we construct a feedback control in the drift part to stabilise the original SDEs-MS-LN (1). We use the linear form of DFC to make the design more concise and simple, that is $u(Y(t-\tau),q(t))=U(q(t))Y(t-\tau)$, where $U(i)\equiv U_i\in\mathbb{R}^{d\times d}$ $(1\le i\le N)$. Thus the controlled system (2) can be rewritten as

$$\begin{aligned}\mathrm{d}Y(t)=&[f(Y(t),q(t))+U(q(t))Y(t-\tau)]\mathrm{d}t\\ &+g(Y(t),q(t))dW(t)\\ &+\int_{\mathbb{R}_0^n}h(Y(t^-),q(t^-),x)\tilde{N}(\mathrm{d}t,\mathrm{d}x).\end{aligned} \tag{5}$$

Actually, the controlled system (5) is a stochastic delay differential equations with Markovian switching and Lévy noise (SDDEs-MS-LN) and so we naturally choose the corresponding initial data as

$$\begin{cases}\{Y(s):-\tau\le s\le 0\}=\eta\in\mathcal{D}_\tau,\\ q(0)=i\in\mathbb{S}.\end{cases} \tag{6}$$

As is known to all (see, e.g. refs. [24], [25], and [28]), for any initial data (6), Assumption 2.1 guarantees the existence of global solution for the SDDEs-MS-LN. Define $Y_t=\{Y(t+s):-\tau\le s\le 0\}$ for $t\ge 0$. Denote by $Y^{\eta,i}(t)$ the solution to SDDEs-MS-LN (5) with initial data (6) and denote by $q^i(t)$ the Markov chain starting from i. In addition, as far as we known that (see, e.g. ref. [28])

$$\mathbb{E}[\|Y_t^{\eta,i}\|_\tau^2]<C_t(1+\|\eta\|_\tau^2)\ \ \forall t\ge 0, \tag{7}$$

where C_t represents a positive number depending on t but not on (η,i). For $t\ge 0$, we can easily get that the joint process $(Y_t,q(t))$ is a time-homogeneous Markov process with transition probability $p(t,\eta,i;d\xi\times\{j\})$ (see, e.g. refs. [24] and [29]), where $p(t,\eta,i;d\xi\times\{j\})$ denotes transition probability measure on $\mathcal{D}_\tau \times \mathbb{S}$, namely

$$\mathbb{P}((Y_t^{\eta,i}, q^i(t)) \in O \times G) = \sum_{j\in G} \int_O p(t,\eta,i;d\xi \times \{j\}) \tag{8}$$

for any $O \in \mathcal{B}(\mathcal{D}_\tau)$ and $G \in \mathbb{S}$.

Let $\mathcal{P}(\mathcal{D}_\tau)$ be the family of probability measures on the measurable space $(\mathcal{D}_\tau, \mathcal{B}(\mathcal{D}_\tau))$. For $\mathbb{P}_1, \mathbb{P}_2 \in \mathcal{P}(\mathcal{D}_\tau)$, define metric $d_{\mathbb{L}}$ by

$$d_{\mathbb{L}}(\mathbb{P}_1, \mathbb{P}_2) = \sup_{F\in\mathbb{L}} \left| \int_{\mathcal{D}_\tau} F(\eta)\mathbb{P}_1(\mathrm{d}\eta) - \int_{\mathcal{D}_\tau} F(\eta)\mathbb{P}_2(\mathrm{d}\eta) \right|, \tag{9}$$

where $\mathbb{L} = \{F : \mathcal{D}_\tau \to \mathbb{R}$ satisfying $|F(\eta) - F(\xi)| \le d_{\mathbb{S}}(\eta, \xi)$ and $|F(\eta)| \le 1$ for $\eta, \xi \in \mathcal{D}_\tau\}$. Besides, let $\mathcal{L}(Y_t)$ represent the probability measure generated by Y_t on $(\mathcal{D}_\tau, \mathcal{B}(\mathcal{D}_\tau))$.

Definition 2.2. The SDDEs-MS-LN (5) is said to be stable in distribution if there exists a probability measure $\omega_\tau \in \mathcal{P}(\mathcal{D}_\tau)$ such that

$$\lim_{t\to\infty} d_{\mathbb{L}}(\mathcal{L}(Y_t^{\eta,i}), \omega_\tau) = 0 \tag{10}$$

for all $(\eta, i) \in \mathcal{D}_\tau \times \mathbb{S}$.

Remark 2.3. From Yuan and Mao [24], we know that the stability in distribution is defined on the joint process $(Y_t^{\eta,i}, q^i(t))$, that is, the transition probability measure $p(t,\eta,i;d\xi \times \{j\})$ converges weakly to a probability measure on $\mathcal{D}_\tau \times \mathbb{S}$ as $t \to \infty$. Clearly, it is known (see, e.g. ref. [26]) that the Markov chain $q^i(t)$ has a unique stationary distribution.

3 | STABILISATION IN DISTRIBUTION

Denote by $C^2(\mathbb{R}^d \times \mathbb{S}; \mathbb{R}_+)$ the family of all non-negative continuous functions $\Phi(y,i)$ defined on $\mathbb{R}^d \times \mathbb{S}$, such that for all $i \in \mathbb{S}$, they are twice continuously differentiable in y. Suppose that there exist one $\Phi \in C^2(\mathbb{R}^d \times \mathbb{S}; \mathbb{R}_+)$, define an operator $L\Phi$ from $\mathbb{R}^d \times \mathbb{S}$ to $\mathbb{R}$ by

$$\begin{aligned} &L\Phi(y,i) \\ =&\Phi_y(y,i)[f(y,i) + U_i y] \\ &+ \frac{1}{2}\mathrm{trace}[g(y,i)^T \Phi_{yy}(y,i) g(y,i)] \\ &+ \int_{\mathbb{R}\setminus\{0\}} \sum_{k=1}^n [\Phi(y + h^{(k)}(y,i,x_k), i) - \Phi(y,i) \\ &- \Phi_y(y,i) h^{(k)}(y,i,x_k)]\nu_k(\mathrm{d}x_k) + \sum_{j=1}^N \gamma_{ij}\Phi(y,j), \end{aligned} \tag{11}$$

where $\Phi_y(y,i) = \left(\frac{\partial\Phi(y,i)}{\partial y_1}, \frac{\partial\Phi(y,i)}{\partial y_2}, \dots, \frac{\partial\Phi(y,i)}{\partial y_d}\right)$, $\Phi_{yy}(y,i) = \left(\frac{\partial^2\Phi(y,i)}{\partial y_i \partial y_j}\right)_{d\times d}$.

In what follows we also need to consider the difference between two solutions of the system (5) starting from different initial data, that is,

$$\begin{aligned} &Y^{\eta,i}(t) - Y^{\xi,i}(t) \\ =&\eta - \xi + \int_0^t [f(Y^{\eta,i}(s), q_i(s)) - f(Y^{\xi,i}(s), q_i(s)) \\ &+ U(q_i(s))(Y^{\eta,i}(s-\tau) - Y^{\xi,i}(s-\tau))]\mathrm{d}s \\ &+ \int_0^t [g(Y^{\eta,i}(s), q_i(s)) - g(Y^{\xi,i}(s), q_i(s))]\mathrm{d}W(s) \\ &+ \int_0^t \int_{\mathbb{R}_0^n} [h(Y^{\eta,i}(s^-), q_i(s^-), x) \\ &- h(Y^{\xi,i}(s^-), q_i(s^-), x)]\tilde{N}(\mathrm{d}s, \mathrm{d}x). \end{aligned} \tag{12}$$

Let $\Psi \in C^2(\mathbb{R}^d \times \mathbb{S}; \mathbb{R}_+)$, we define an operator $\mathbb{L}\Psi : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S} \to \mathbb{R}$ related to Equation (12) by

$$\begin{aligned} &\mathbb{L}\Psi(y,z,i) \\ =&\Psi_y(y-z,i)[f(y,i) - f(z,i) + U_i(y-z)] \\ &+ \frac{1}{2}\mathrm{trace}[(g(y,i) - g(z,i))^T \Psi_{yy}(y-z,i)(g(y,i) - g(z,i))] \\ &+ \int_{\mathbb{R}\setminus\{0\}} \sum_{k=1}^n [\Psi(y - z + h^{(k)}(y,i,x_k) - h^{(k)}(z,i,x_k), i), i) \\ &- \Psi(y-z,i) - \Psi_y(y-z,i)(h^{(k)}(y,i,x_k) \\ &- h^{(k)}(z,i,x_k))]\nu_k(\mathrm{d}x_k) + \sum_{j=1}^N \gamma_{ij}\Psi(y-z,j). \end{aligned} \tag{13}$$

Throughout this paper, we will set $e_4 = \max_{i\in\mathbb{S}} \|U_i\|^2$. In order to establish criterion on asymptotic stability in distribution of the SDDEs-MS-LN (5), we make the following assumptions.

Assumption 3.1. There are positive numbers b_0, θ_1, b_2, c_1, function $\Phi(y,i) \in C^2(\mathbb{R}^d \times \mathbb{S}; \mathbb{R}_+)$ and $J_1(y) \in C(\mathbb{R}^d; \mathbb{R}_+)$ such that

$$\begin{aligned} &c_1|y|^2 \le \Phi(y,i) \le J_1(y), \\ &L\Phi(y,i) + \theta_1|\Phi_y(y,i)|^2 \le -b_0 J_1(y) + b_2 \end{aligned} \tag{14}$$

for all $(y,i) \in \mathbb{R}^d \times \mathbb{S}$.

Assumption 3.2. There are positive numbers c_2, θ_2, b_1, function $\Psi(y,i) \in C^2(\mathbb{R}^d \times \mathbb{S}; \mathbb{R}_+)$ and $J_2(y) \in C(\mathbb{R}^d; \mathbb{R}_+)$ such that

$$
\begin{aligned}
&c_2|y|^2 \le \Psi(y,i) \le J_2(y),\\
&\mathbb{L}\Psi(y,z,i) + \theta_2|\Psi_y(y-z,i)|^2 \le -b_1 J_2(y-z)
\end{aligned}
\tag{15}
$$

for all $(y,z,i) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}$.

3.1 | Lyapunov functionals

In this section, we aim to deal with the asymptotic stability in distribution of the SDDEs-MS-LN (5). Unless otherwise specified, all hypothetical positive numbers below are independent of the initial data (6). To achieve our purpose, we first establish the Lyapunov functional on the segments $\hat{Y}_t := \{Y(t+u) : -2\tau \le u \le 0\}$ and $\hat{q}_t = \{q(t+u) : -2\tau \le u \le 0\}$ for $t \ge \tau$. Let $q(s) = q(0)$ for $-2\tau \le s \le 0$. Obviously $\hat{Y}_t$ is $D([-2\tau,0];\mathbb{R}^d)$-valued which is not same as Y_t. We design a Lyapunov functional as follow.

$$
V(\hat{Y}_t,\hat{q}_t,t) := \Phi(Y(t),q(t)) + \hat{V}(\hat{Y}_t,\hat{q}_t,t), \quad \text{for } t \ge -\tau \tag{16}
$$

where

$$
\begin{aligned}
\hat{V}(\hat{Y}_t,\hat{q}_t,t) =& \alpha \int_{t-\tau}^{t}\int_{s}^{t} \Big[\tau|f(Y(v),q(v)) + U_{q(v)}Y(v-\tau)|^2\\
&+ \int_{\mathbb{R}_0^n} |h(Y(v^-),q(v^-),x)|^2 \nu(\mathrm{d}x)\\
&+ |g(Y(v),q(v))|^2\Big]\,\mathrm{d}v\mathrm{d}s
\end{aligned}
\tag{17}
$$

and α is a positive number to be chosen later.

One can see that

$$
c_1|Y(t)|^2 \le V(\hat{Y}_t,\hat{q}_t,t) \le J_1(Y(t)) + \hat{V}(\hat{Y}_t,\hat{r}_t,t). \tag{18}
$$

For simplicity, let $Y^{\eta,i}(t) = Y(t)$ and fix the initial data (η,i) arbitrarily. Applying the functional Itô formula (It can be obtained immediately by combining the ideas from refs. [30] and [31]) to Equation (16) yields, for $t \ge \tau$,

$$
\mathrm{d}V(\hat{Y}_t,\hat{q}_t,t) = \mathrm{L}V(\hat{Y}_t,\hat{q}_t,t)\,\mathrm{d}t + \mathrm{d}I(t) \tag{19}
$$

where $I(t)$ is a martingale with $I(0) = 0$, and

$$
\begin{aligned}
&\mathrm{L}V(\hat{Y}_t,\hat{q}_t,t)\\
&= L\Phi(Y(t),q(t)) - \Phi_Y(Y(t),q(t))U_{q(t)}(Y(t)-Y(t-\tau))\\
&\quad + \alpha\tau\Big[\tau|f(Y(t),q(t)) + U_{q(t)}Y(t-\tau)|^2 + |g(Y(t),q(t))|^2\\
&\quad + \int_{\mathbb{R}_0^n} |h(Y(s^-),q(s^-),x)|^2\nu(\mathrm{d}x)\Big]\\
&\quad - \alpha\int_{t-\tau}^{t}\Big[\tau|f(Y(s),q(s)) + U_{q(s)}Y(s-\tau)|^2\\
&\quad + |g(Y(s),q(s))|^2 + \int_{\mathbb{R}_0^n} |h(Y(s^-),q(s^-),x)|^2\nu(\mathrm{d}x)\Big]\,\mathrm{d}s\\
&\le L\Phi(Y(t),q(t)) + \theta_1|\Phi_Y(Y(t),q(t))|^2\\
&\quad + \frac{1}{4\theta_1}\|U_{q(t)}\|^2|Y(t)-Y(t-\tau)|^2\\
&\quad + \alpha\tau\Big[\tau|f(Y(t),q(t)) + U_{q(t)}Y(t-\tau)|^2 + |g(Y(t),q(t))|^2\\
&\quad + \int_{\mathbb{R}_0^n} |h(Y(s^-),q(s^-),x)|^2\nu(\mathrm{d}x)\Big]\\
&\quad - \alpha\int_{t-\tau}^{t}\Big[\tau|f(Y(s),q(s)) + U_{q(s)}Y(s-\tau)|^2\\
&\quad + |g(Y(s),q(s))|^2 + \int_{\mathbb{R}_0^n} |h(Y(s^-),q(s^-),x)|^2\nu(\mathrm{d}x)\Big]\,\mathrm{d}s.
\end{aligned}
\tag{20}
$$

Using Assumption 2.1 that one gains

$$
\begin{aligned}
&\alpha\tau\Big[\tau|f(Y(t),q(t)) + U_{q(t)}Y(t-\tau)|^2\\
&+|g(Y(t),q(t))|^2 + \int_{\mathbb{R}_0^n} |h(Y(t^-),q(t^-),x)|^2\nu(\mathrm{d}x)\Big]\\
\le&\alpha\tau\Big[4e_1\tau|Y(t)|^2 + 2e_0\tau + 2e_4\tau|Y(t-\tau)|^2\\
&+2e_2|Y(t)|^2 + e_0 + 2e_3|Y(t)|^2 + e_0\Big]\\
\le&\alpha\tau\Big[2(2e_1\tau + e_2 + e_3)|Y(t)|^2\\
&+2e_0(\tau+1) + 2e_4\tau|Y(t-\tau)|^2\Big]\\
\le&\alpha\tau\Big[2(2e_1\tau + e_2 + e_3 + 2e_4\tau)|Y(t)|^2\\
&+2e_0(\tau+1) + 4e_4\tau|Y(t)-Y(t-\tau)|^2\Big].
\end{aligned}
\tag{21}
$$

In view of Assumption 3.1, we get from Equations (20) and (21) that

$$
\begin{aligned}
&\mathrm{L}V(\hat{Y}_t,\hat{q}_t,t)\\
&\le -b_0J_1(Y(t)) + b_2 + \frac{e_4}{4\theta_1}|Y(t)-Y(t-\tau)|^2\\
&\quad + \alpha\tau[2(2e_1\tau + e_2 + e_3 + 2e_4\tau)|Y(t)|^2\\
&\quad + 2e_0(\tau+1) + 4e_4\tau|Y(t)-Y(t-\tau)|^2]\\
&\quad - \alpha\int_{t-\tau}^{t}\Big[\tau|f(Y(s),q(s)) + U_{q(s)}Y(s-\tau)|^2\\
&\quad +|g(Y(s),q(s))|^2 + \int_{\mathbb{R}_0^n} |h(Y(s^-),q(s^-),x)|^2\nu(\mathrm{d}x)\Big]\,\mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&\le -bJ_1(Y(t)) + b_2 + 2e_0\alpha\tau(\tau+1) \\
&\quad + \left(\frac{e_4}{4\theta_1} + 4e_4\alpha\tau^2\right)|Y(t) - Y(t-\tau)|^2 \\
&\quad - \alpha\int_{t-\tau}^{t}\Big[\tau|f(Y(s),q(s)) + U_{q(s)}Y(s-\tau)|^2 \\
&\quad + |g(Y(s),q(s))|^2 + \int_{\mathbb{R}_0^n}|h(Y(s^-),q(s^-),x)|^2\nu(\mathrm{d}x)\Big]\mathrm{d}s
\end{aligned}
\tag{22}
$$

for $t \ge \tau$, where $b = b_0 - 2\alpha\tau(2e_1\tau + e_2 + e_3 + 2e_4\tau)/c_1$. Note that, it follows from Equation (5) along with Equations (3) and (4) that one gains

$$
\begin{aligned}
&\mathbb{E}|Y(t) - Y(t-\tau)|^2 \\
=&\mathbb{E}|\int_{t-\tau}^{t}[f(Y(s),q(s)) + U(q(s))Y(s-\tau)]\mathrm{d}s \\
&+ \int_{t-\tau}^{t} g(Y(s),q(s))\mathrm{d}W(s) \\
&+ \int_{t-\tau}^{t}\int_{\mathbb{R}_0^n} h(Y(s^-),q(s^-),x)\tilde{N}(\mathrm{d}s,\mathrm{d}x)|^2 \\
\le&3\tau\mathbb{E}\int_{t-\tau}^{t}|f(Y(s),q(s)) + U(q(s))Y(s-\tau)|^2\mathrm{d}s \\
&+ 3\mathbb{E}|\int_{t-\tau}^{t} g(Y(s),q(s))\mathrm{d}W(s)|^2 \\
&+ 3\mathbb{E}|\int_{t-\tau}^{t}\int_{\mathbb{R}_0^n} h(Y(s^-),q(s^-),x)\tilde{N}(\mathrm{d}s,\mathrm{d}x)|^2.
\end{aligned}
\tag{23}
$$

By Itô isometry,

$$
\begin{aligned}
&\mathbb{E}|Y(t) - Y(t-\tau)|^2 \\
\le&3\tau\mathbb{E}\int_{t-\tau}^{t}|f(Y(s),q(s)) + U(q(s))Y(s-\tau)|^2\mathrm{d}s \\
&+ 3\mathbb{E}\int_{t-\tau}^{t}|g(Y(s),q(s))|^2\mathrm{d}s \\
&+ 3\mathbb{E}\int_{t-\tau}^{t}\int_{\mathbb{R}_0^n}|h(Y(s^-),q(s^-),x)|^2\nu(\mathrm{d}x)\mathrm{d}s.
\end{aligned}
\tag{24}
$$

Choose $\alpha = \frac{3e_4}{\theta_1}$ and $\tau \le \frac{1}{4\sqrt{e_4}}$. It then follows from Equations (22)–(24) that

$$
\begin{aligned}
\mathbb{E}(\mathrm{L}V(\hat{Y}_t,\hat{q}_t,t)) \le& -b\mathbb{E}(J_1(Y(t))) + b_2 \\
&+ \frac{6e_4e_0}{\theta_1}\tau(\tau+1)
\end{aligned}
\tag{25}
$$

for $t \ge \tau$.

3.2 | Lemmas

To show our main results on the stabilisation in distribution, we first present a couple of lemmas.

Lemma 3.3. *Let Assumption 2.1 and Assumption 3.1 hold. If $\tau > 0$ is small enough for*

$$
c_1b_0 - \frac{6e_4}{\theta_1}\tau(2e_1\tau + e_2 + e_3 + 2e_4\tau) > 0 \text{ and } \tau \le \frac{1}{4\sqrt{e_4}},
$$

then for any given initial data (6), the solution of Equation (5) has the properties that

$$
\mathbb{E}\|Y_t^{\eta,i}\|_\tau^2 \le C(1 + \|\eta\|_\tau^2) \tag{26}
$$

for all $t \ge 0$, where C represents a positive constant.

Proof. From Equation (25), one can see that

$$
\mathbb{E}(\mathrm{L}V(\hat{Y}_t,\hat{q}_t,t)) \le -b\mathbb{E}(J_1(Y(t))) + \beta_0 \tag{27}
$$

for $t \ge \tau$, where $\beta_0 = b_2 + \frac{6e_4e_0}{\theta_1}\tau(\tau+1)$. Let β_3 be the unique root to the equation

$$
b - \beta_3 - 2\beta_3M_1\tau e^{2\tau\beta_3} = 0,
$$

where M_1 is a positive number to be determined later. In view of the functional Itô formula to $e^{\beta_3t}V(\hat{Y}_t,\hat{q}_t,t)$, we have

$$
\begin{aligned}
&e^{\beta_3t}\mathbb{E}(V(\hat{Y}_t,\hat{q}_t,t)) - e^{\beta_3\tau}\mathbb{E}(V(\hat{Y}_\tau,\hat{q}_\tau,\tau)) \\
=&\mathbb{E}\int_{\tau}^{t} e^{\beta_3s}(\beta_3V(\hat{Y}_s,\hat{q}_s,s) + \mathrm{L}V(\hat{Y}_s,\hat{q}_s,s))\mathrm{d}s,
\end{aligned}
\tag{28}
$$

for $t \ge \tau$. Using Equations (7) and (18), we hence obtain that

$$
\begin{aligned}
&c_1e^{\beta_3t}\mathbb{E}|Y(t)|^2 - \beta_4(1 + \|\eta\|_\tau^2) \\
\le&\mathbb{E}\int_{\tau}^{t} e^{\beta_3s}[\beta_3(J_1(Y(s)) + \hat{V}(\hat{Y}_s,\hat{q}_s,s)) \\
&+ \mathrm{L}V(\hat{Y}_s,\hat{q}_s,s)]\mathrm{d}s,
\end{aligned}
\tag{29}
$$

where β_4 is a positive number. Moreover, by Equations (3) and (4), we derive that

$$
\begin{aligned}
&\mathbb{E}(\hat{V}(\hat{Y}_s,\hat{q}_s,s)) \\
\le&\alpha\tau\mathbb{E}\int_{s-\tau}^{s}\Big[\tau|f(Y(v),q(v)) + U_{q(v)}Y(v-\tau)|^2 \\
&+ |g(Y(v),q(v))|^2 + \int_{\mathbb{R}_0^n}|h(Y(v^-),q(v^-),x)|^2\nu(\mathrm{d}x)\Big]\mathrm{d}v
\end{aligned}
$$

$$\leq \alpha\tau \int_{s-\tau}^{s} \left[2(2e_1\tau + e_2 + e_3)\mathbb{E}|Y(v)|^2 + 2e_0(\tau+1) + 2e_4\tau\mathbb{E}|Y(v-\tau)|^2\right] \mathrm{d}v$$

$$\leq \alpha\tau \int_{s-\tau}^{s} 2(2e_1\tau + e_2 + e_3)\mathbb{E}|Y(v)|^2 dv + 2\alpha\tau^2 e_0(\tau+1) + 2e_4\alpha\tau \int_{s-2\tau}^{s} \mathbb{E}|Y(v)|^2 \mathrm{d}v$$

$$\leq \alpha\tau \int_{s-2\tau}^{s} 2(2e_1\tau + e_2 + e_3 + e_4)\mathbb{E}|Y(v)|^2 \mathrm{d}v + 2\alpha\tau^2 e_0(\tau+1)$$

$$\leq M_1 \int_{s-2\tau}^{s} \mathbb{E}(J_1(Y(v)))\mathrm{d}v + M_2 \tag{30}$$

where $M_1 = \frac{1}{c_1}2\alpha\tau(2e_1\tau + e_2 + e_3 + e_4)$ and $M_2 = 2\alpha\tau^2 e_0(\tau+1)$. Substituting Equation (30) into Equation (29), we can get

$$c_1 e^{\beta_3 t}\mathbb{E}|Y(t)|^2 - \beta_4(1 + \|\eta\|_\tau^2) \leq \int_\tau^t e^{\beta_3 s}\beta_3\left(M_1 \int_{s-2\tau}^{s} \mathbb{E}(J_1(Y(v)))\mathrm{d}v + M_2 + \mathbb{E}(J_1(Y(s)))\right)\mathrm{d}s + \int_\tau^t e^{\beta_3 s}\left(-b\mathbb{E}(J_1(Y(s))) + \beta_0\right)\mathrm{d}s. \tag{31}$$

Noting that

$$\int_{3\tau}^{t} e^{\beta_3 s}\left(\int_{s-2\tau}^{s} \mathbb{E}(J_1(Y(v)))\mathrm{d}v\right)\mathrm{d}s \leq \int_\tau^t \mathbb{E}(J_1(Y(v)))\left(\int_v^{v+2\tau} e^{\beta_3 s}\mathrm{d}s\right)\mathrm{d}v \leq 2\tau e^{2\tau\beta_3}\int_\tau^t e^{\beta_3 v}\mathbb{E}(J_1(Y(v)))\mathrm{d}v. \tag{32}$$

Then it follows from Equation (31) that

$$c_1 e^{\beta_3 t}\mathbb{E}|Y(t)|^2 - \beta_4(1 + \|\eta\|_\tau^2)$$
$$\leq \beta_3 M_1 \int_\tau^{3\tau} e^{\beta_3 s}\int_{s-2\tau}^{s} \mathbb{E}(J_1(Y(v)))\mathrm{d}v\mathrm{d}s + (2\beta_3 M_1\tau e^{2\tau\beta_3} + \beta_3)\int_\tau^t e^{\beta_3 s}\mathbb{E}(J_1(Y(s)))\mathrm{d}s + (M_2\beta_3 + \beta_0)\int_\tau^t e^{\beta_3 s}\mathrm{d}s - b\int_\tau^t e^{\beta_3 s}\mathbb{E}(J_1(Y(s)))\mathrm{d}s$$
$$\leq \beta_3 M_1 \int_\tau^{3\tau} e^{\beta_3 s}\int_{s-2\tau}^{s} \mathbb{E}(J_1(Y(v)))\mathrm{d}v)\mathrm{d}s - (b - \beta_3 - 2\beta_3 M_1\tau e^{2\tau\beta_3})\int_\tau^t e^{\beta_3 s}\mathbb{E}(J_1(Y(s)))\mathrm{d}s + (M_2\beta_3 + \beta_0)\int_\tau^t e^{\beta_3 s}\mathrm{d}s$$
$$\leq \beta_3 M_1 \int_\tau^{3\tau} e^{\beta_3 s}\int_{s-2\tau}^{s} \mathbb{E}(J_1(Y(v)))\mathrm{d}v)\mathrm{d}s + (M_2\beta_3 + \beta_0)\int_\tau^t e^{\beta_3 s}\mathrm{d}s \leq e^{\beta_3 t}\beta_7, \tag{33}$$

where β_7 and following β_8 etc. are all positive numbers. This implies that

$$\mathbb{E}|Y(t)|^2 \leq \beta_8(1 + \|\eta\|_\tau^2), \quad t \geq \tau. \tag{34}$$

Next, we shall estimate the segment process Y_t. Let $t \geq 2\tau$ and $\theta \in [0,\tau]$. In view of the Itô formula and Equation (5), we have

$$|Y(t-\theta)|^2 = |Y(t-\tau)|^2 + 2\int_{t-\tau}^{t-\theta} Y^T(s)[f(Y(s),q(s)) + U_{q(s)}Y(s-\tau)]\mathrm{d}s + 2\int_{t-\tau}^{t-\theta} Y^T(s)g(Y(s),q(s))\mathrm{d}W(s) + \int_{t-\tau}^{t-\theta} |g(Y(s),q(s))|^2\mathrm{d}s$$
$$+ \int_{t-\tau}^{t-\theta}\int_{\mathbb{R}\setminus\{0\}}\sum_{k=1}^{n}[|Y(s) + h^{(k)}(Y(s^-),q(s^-),x_k)|^2 - |Y(s)|^2 - 2Y^T(s)h^{(k)}(Y(s^-),q(s^-),x_k)]\nu_k(\mathrm{d}x_k)\mathrm{d}s$$
$$+ \sum_{k=1}^{n}\int_{t-\tau}^{t-\theta}\int_{\mathbb{R}\setminus\{0\}}[|Y(s^-) + h^{(k)}(Y(s^-),q(s^-),x_k)|^2 - |Y(s^-)|^2]\tilde{N}(\mathrm{d}s,\mathrm{d}x_k). \tag{35}$$

Following Kunita's estimates (see ref. [27]), we obtain

$$\mathbb{E}\sup_{0\leq\theta\leq\tau}|Y(t-\theta)|^2 \leq c_3\Big\{\mathbb{E}\int_{t-\tau}^{t}[|f(Y(s),q(s))|^2 + |U_{q(s)}Y(s-\tau)|^2]\mathrm{d}s + \mathbb{E}|Y(t-\tau)|^2 + \mathbb{E}\int_{t-\tau}^{t}|g(Y(s),q(s))|^2\mathrm{d}s$$
$$+ \mathbb{E}\int_{t-\tau}^{t}\int_{\mathbb{R}\setminus\{0\}}\sum_{k=1}^{n}|h^{(k)}(Y(s^-),q(s^-),x_k)|^2\nu(\mathrm{d}x_k)\mathrm{d}s\Big\}. \tag{36}$$

By Equations (3) and (4), we have

$$\mathbb{E}\sup_{0\leq\theta\leq\tau}|Y(t-\theta)|^2 \leq c_4\left(\mathbb{E}|Y(t-\tau)|^2 + \int_{t-\tau}^{t}\mathbb{E}|Y(s)|^2\mathrm{d}s\right.$$

$$+\int_{t-\tau}^{t}\mathbb{E}|Y(s-\tau)|^2\mathrm{d}s+c_5\Bigg), \tag{37}$$

where c_3–c_5 are all positive constants. This, together with Equation (34), yields,

$$\mathbb{E}\|Y_t\|_\tau^2 \le \beta_9(1+\|\eta\|_\tau^2), \tag{38}$$

where $\beta_9 = c_4(\beta_8 + 2\tau\beta_8 + c_5)$. Together with Equation (7), obtain that the assertion (26). The proof is complete. □

Lemma 3.4. *Let Assumption 2.1 and Assumption 3.2 hold. If* $\tau > 0$ *is small enough for*

$$b_1c_2 - \frac{3e_4}{\theta_2}\tau(2e_1\tau + e_2 + e_3 + 2e_4\tau) > 0 \text{ and } \tau \le \frac{1}{2\sqrt{2e_4}},$$

then for any $(\eta,\xi,i) \in \mathcal{D}_\tau \times \mathcal{D}_\tau \times \mathbb{S}$,

$$\mathbb{E}\|Y_t^{\eta,i} - Y_t^{\xi,i}\|_\tau^2 \le \alpha_1\|\eta-\xi\|_\tau^2 e^{-\alpha_2 t} \tag{39}$$

for all $t \ge 2\tau$, *where* α_1 *and* α_2 *represents positive numbers.*

Proof. Fix any $(\eta,\xi,i) \in \mathcal{D}_\tau \times \mathcal{D}_\tau \times \mathbb{S}$ and we simply write $E(t) = Y^{\eta,i}(t) - Y^{\xi,i}(t)$. So $E_t = \{E(t+s) : -\tau \le s \le 0\}$ for $t \ge 0$ and $\hat{E}_t = \{E(t+s) : -2\tau \le s \le 0\}$ for $t \ge \tau$. Next, apply another Lyapunov functional $\tilde{V}(\hat{E}_t, \hat{q}_t, t)$

$$\begin{aligned}
&\tilde{V}(\hat{E}_t,\hat{q}_t,t)\\
:=&\Psi(Y^{\eta,i}(t),Y^{\xi,i}(t),q(t)) + \alpha\int_{t-\tau}^{t}\int_{s}^{t}\big[\tau|f(Y^{\eta,i}(v),q(v))\\
&- f(Y^{\xi,i}(v),q(v)) + U_{q(v)}E(v-\tau)|^2\\
&+ |g(Y^{\eta,i}(v),q(v)) - g(Y^{\xi,i}(v),q(v))|^2\\
&+ \int_{\mathbb{R}_0^n}|h(Y^{\eta,i}(v^-),q(v^-),x)\\
&\quad -h(Y^{\xi,i}(v^-),q(v^-),x)|^2\nu(\mathrm{d}x)\big]\,\mathrm{d}v\mathrm{d}s,
\end{aligned} \tag{40}$$

for $t \ge \tau$. Following the functional Itô formula, one get

$$\mathrm{d}\tilde{V}(\hat{E}_t,\hat{q}_t,t) = \mathrm{L}\tilde{V}(\hat{E}_t,\hat{q}_t,t)\mathrm{d}t + \mathrm{d}\tilde{I}(t) \tag{41}$$

for $t \ge \tau$, where $\tilde{I}(t)$ is a martingale with $\tilde{I}(0) = 0$, and

$$\begin{aligned}
&\mathrm{L}\tilde{V}(\hat{E}_t,\hat{q}_t,t)\\
=&\mathbb{L}\Psi(Y^{\eta,i}(t),Y^{\xi,i}(t),q(t))\\
&-\Psi_Y(Y^{\eta,i}(t),Y^{\xi,i}(t),q(t))U_{q(t)}(E(t)-E(t-\tau))\\
&+\alpha\tau\big[\tau|f(Y^{\eta,i}(t),q(t)) - f(Y^{\xi,i}(t),q(t))\\
&+U_{q(t)}E(t-\tau)|^2 + |g(Y^{\eta,i}(t),q(t)) - g(Y^{\xi,i}(t),q(t))|^2\\
&+\int_{\mathbb{R}_0^n}|h(Y^{\eta,i}(t^-),q(t^-),x) - h(Y^{\xi,i}(t^-),q(t^-),x)|^2\nu(\mathrm{d}x)\big]\\
&-\alpha\int_{t-\tau}^{t}\big[\tau|f(Y^{\eta,i}(s),q(s)) - f(Y^{\xi,i}(s),q(s))\\
&+U_{q(s)}E(s-\tau)|^2 + |g(Y^{\eta,i}(s),q(s)) - g(Y^{\xi,i}(s),q(s))|^2\\
&+\int_{\mathbb{R}_0^n}|h(Y^{\eta,i}(s^-),q(s^-),x)\\
&\quad -h(Y^{\xi,i}(s^-),q(s^-),x)|^2\nu(\mathrm{d}x)\big]\,\mathrm{d}s.
\end{aligned} \tag{42}$$

We can derive by Assumption 2.1 and Assumption 3.2 that

$$\begin{aligned}
&\mathrm{L}\tilde{V}(\hat{E}_t,\hat{q}_t,t)\\
\le&-\bar{b}J_2(E(t)) + (e_4/4\theta_2 + 2e_4\alpha\tau^2)|E(t)-E(t-\tau)|^2\\
&-\alpha\int_{t-\tau}^{t}\big[\tau|f(Y^{\eta,i}(s),q(s)) - f(Y^{\xi,i}(s),q(s))\\
&+U_{q(s)}E(s-\tau)|^2 + |g(Y^{\eta,i}(s),q(s)) - g(Y^{\xi,i}(s),q(s))|^2\\
&+\int_{\mathbb{R}_0^n}|h(Y^{\eta,i}(s^-),q(s^-),x)\\
&\quad -h(Y^{\xi,i}(s^-),q(s^-),x)|^2\nu(\mathrm{d}x)\big]\,\mathrm{d}s
\end{aligned} \tag{43}$$

for $t \ge \tau$, where $\bar{b} = b_1 - \frac{1}{c_2}\alpha\tau(2e_1\tau + e_2 + e_3 + 2e_4\tau)$.

On the other hand, we can compute that

$$\begin{aligned}
&\mathbb{E}|E(t)-E(t-\tau)|^2\\
\le&\mathbb{E}\Big|\int_{t-\tau}^{t}[f(Y^{\eta,i}(s),q(s)) - f(Y^{\xi,i}(s),q(s))\\
&+U(q(s))E(s-\tau)]\mathrm{d}s\\
&+\int_{t-\tau}^{t}[g(Y^{\eta,i}(s),q(s)) - g(Y^{\xi,i}(s),q(s))]\mathrm{d}W(s)\\
&+\int_{t-\tau}^{t}\int_{\mathbb{R}_0^n}[h(Y^{\eta,i}(s^-),q(s^-),x)\\
&\quad -h(Y^{\xi,i}(s^-),q(s^-),x)]\tilde{N}(\mathrm{d}s,\mathrm{d}x)\Big|^2\\
\le&3\tau\mathbb{E}\int_{t-\tau}^{t}|f(Y^{\eta,i}(s),q(s)) - f(Y^{\xi,i}(s),q(s))\\
&+U(q(s))E(s-\tau)|^2\mathrm{d}s\\
&+3\mathbb{E}\int_{t-\tau}^{t}|g(Y^{\eta,i}(s),q(s)) - g(Y^{\xi,i}(s),q(s))|^2\mathrm{d}s
\end{aligned}$$

$$+ 3\mathbb{E}\int_{t-\tau}^{t}\int_{\mathbb{R}_0^n}|h(Y^{\eta,i}(s^-),q(s^-),x) - h(Y^{\xi,i}(s^-),q(s^-),x)|^2\nu(\mathrm{d}x)\mathrm{d}s. \tag{44}$$

Choosing $\alpha = \frac{3e_4}{\theta_2}$ and $\tau \le \frac{1}{2\sqrt{2e_4}}$. Therefore, we have

$$\mathbb{E}(\mathrm{L}\tilde{V}(\hat{E}_t,\hat{q}_t,t)) \le -\bar{b}\mathbb{E}(J_2(E(t))) \tag{45}$$

for $t \ge \tau$. The following proof is similar to Lemma 3.3, thus we have

$$\mathbb{E}|E(t)|^2 \le \alpha_3\|\eta-\xi\|_\tau^2 e^{-\alpha_2 t} \tag{46}$$

for $t \ge \tau$, where α_3 and α_2 represents positive constants. Nevertheless, for $t \ge 2\tau$, it is easily obtain that

$$\begin{aligned}&\mathbb{E}\sup_{0\le\theta\le\tau}|E(t-\theta)|^2\\ \le& c_6\Big\{\mathbb{E}|E(t-\tau)|^2 + \mathbb{E}\int_{t-\tau}^{t}[|f(Y^{\eta,i}(s),q(s)) - f(Y^{\xi,i}(s),q(s))|^2 + |U_{q(s)}E(s-\tau)|^2]\mathrm{d}s\\ &+ \mathbb{E}\int_{t-\tau}^{t}|g(Y^{\eta,i}(s),q(s)) - g(Y^{\xi,i}(s),q(s))|^2\mathrm{d}s\\ &+ \mathbb{E}\int_{t-\tau}^{t}\int_{\mathbb{R}\setminus\{0\}}\sum_{k=1}^{n}|h^{(k)}(Y^{\eta,i}(s^-),q(s^-),x_k) - h^{(k)}(Y^{\xi,i}(s^-),q(s^-),x_k)|^2\nu(\mathrm{d}x_k)\mathrm{d}s\Big\}.\end{aligned} \tag{47}$$

By use of Assumption 1, one get that

$$\mathbb{E}\sup_{0\le\theta\le\tau}|E(t-\theta)|^2 \le c_7\left(\mathbb{E}|E(t-\tau)|^2 + \int_{t-\tau}^{t}\mathbb{E}|E(s)|^2\mathrm{d}s + \int_{t-\tau}^{t}\mathbb{E}|E(s-\tau)|^2\mathrm{d}s\right), \tag{48}$$

where c_6 and c_7 are all positive numbers. This, together with Equation (46), yields,

$$\mathbb{E}\|E_t\|_\tau^2 \le \alpha_1\|\eta-\xi\|_\tau^2 e^{-\alpha_2 t},\ \forall t \ge 2\tau, \tag{49}$$

where $\alpha_1 = c_7(\alpha_3 + 2\alpha_3\tau)$. This imply the assertion (39) hold. The proof is hence complete. □

3.3 | Key theorems

Theorem 3.5. *Let Assumption 2.1 and Assumption 3.1 hold. Let $\tau_1^*,\tau_2^*,\tau_3^*,\tau_4^*$ be the unique positive roots to the following equations*

$$\begin{aligned}b_0c &= \frac{3e_4}{\theta}2\tau_1^*(2e_1\tau_1^* + e_2 + e_3 + 2e_4\tau_1^*),\ \tau_2^* = \frac{1}{4\sqrt{e_4}},\\ b_1c &= \frac{3e_4}{\theta}\tau_3^*(2e_1\tau_3^* + e_2 + e_3 + 2e_4\tau_3^*),\ \tau_4^* = \frac{1}{2\sqrt{2e_4}},\end{aligned} \tag{50}$$

respectively, and set $\tau^* = \tau_1^*\wedge\tau_2^*\wedge\tau_3^*\wedge\tau_4^*$. Then for each $\tau < \tau^*$, there exists a unique probability measure $\omega(\tau) \in \mathcal{P}(\mathcal{D}_\tau)$ such that

$$\lim_{t\to\infty} d_{\mathbb{L}}(\mathcal{L}(Y_t^{\eta,i}),\omega_\tau) = 0 \tag{51}$$

for all $(\eta,i) \in \mathcal{D}_\tau\times\mathbb{S}$.

Proof. At first, we show that for any compact set $\mathcal{J} \subset \mathcal{D}_\tau$,

$$\lim_{t\to\infty} d_{\mathbb{L}}(\mathcal{L}(Y_t^{\eta,i}),\mathcal{L}(Y_t^{\xi,j})) = 0 \tag{52}$$

uniformly in $(\eta,\xi,i,j) \in \mathcal{J}\times\mathcal{J}\times\mathbb{S}\times\mathbb{S}$. Define sequence of the stopping time $o_{ij} = \inf\{t : q^i(t) = q^j(t), t \ge 0\}$. Obviously through the ergodicity of the Markov chain, we can know that $o_{ij} < \infty$ a.s. As a result, for each positive $\varepsilon \in (0,1)$, there exists an $K_1 > 0$ such that

$$\mathbb{P}(o_{ij} \le K_1) > 1 - \frac{\varepsilon}{6}. \tag{53}$$

Recalling a known result that

$$\sup_{(\eta,i)\in\mathcal{J}\times\mathbb{S}}\mathbb{E}(\sup_{-\tau\le t\le K_1}|Y^{\eta,i}(t)|) < \infty, \tag{54}$$

this implies that we can find enough large $K_2 > 0$ satisfying

$$\mathbb{P}(\Omega_{\eta,i}) > 1 - \frac{\varepsilon}{12}\quad \forall(\eta,i) \in \mathcal{J}\times\mathbb{S}, \tag{55}$$

where $\Omega_{\eta,i} = \{\omega \in \Omega : \sup_{-\tau\le t\le K_1}|Y^{\eta,i}(t,\omega)| \le K_2\}$. For any $F \in \mathbb{L}$ and $t \ge K_1$, we yield

$$|\mathbb{E}F(Y_t^{\eta,i}) - \mathbb{E}F(Y_t^{\xi,j})| \le \frac{\varepsilon}{3} + H_1(t), \tag{56}$$

where $H_1(t) := \mathbb{E}(I_{\{o_{ij}\le K_1\}}|F(Y_t^{\eta,i}) - F(Y_t^{\xi,j})|)$. Set $\Omega_1 = \Omega_{\eta,i}\cap\Omega_{\xi,i}\cap\{o_{ij} \le K_1\}$. By the Markov property of joint process $(Y_t^{\eta,i},q^i(t))$ and the property of conditional expectation, we derive

$$\begin{aligned}&H_1(t)\\ =&\mathbb{E}\Big(I_{\{o_{ij}\le K_1\}}\mathbb{E}(|F(Y_t^{\eta,i}) - F(Y_t^{\xi,j})||\mathcal{F}_{o_{ij}})\Big)\\ =&\mathbb{E}\Big(I_{\{o_{ij}\le K_1\}}\end{aligned}$$

$$\times\mathbb{E}(|F(Y^{\tilde{\eta},l}_{t-o_{ij}})-F(Y^{\tilde{\xi},l}_{t-o_{ij}})|)\ |_{\tilde{\eta}=Y^{\eta,i}_{o_{ij}},\tilde{\xi}=Y^{\xi,j}_{o_{ij}},l=q^{i}_{o_{ij}}=q^{j}_{o_{ij}}}\Big)$$

$$\leq\frac{\varepsilon}{3}+\mathbb{E}\Big(I_{\Omega_1} \tag{57}$$

$$\times\mathbb{E}(|F(Y^{\tilde{\eta},l}_{t-o_{ij}})-F(Y^{\tilde{\xi},l}_{t-o_{ij}})|)\ |_{\tilde{\eta}=Y^{\eta,i}_{o_{ij}},\tilde{\xi}=Y^{\xi,j}_{o_{ij}},l=q^{i}_{o_{ij}}=q^{j}_{o_{ij}}}\Big)$$

$$\leq\frac{\varepsilon}{3}+\mathbb{E}\Big(I_{\Omega_1}\mathbb{E}d_S(Y^{\tilde{\eta},l}_{t-o_{ij}},Y^{\tilde{\xi},l}_{t-o_{ij}})\ |_{\tilde{\eta}=Y^{\eta,i}_{o_{ij}},\tilde{\xi}=Y^{\xi,j}_{o_{ij}},l=q^{i}_{o_{ij}}=q^{j}_{o_{ij}}}\Big).$$

It follows from the Proposition 1.17 in ref. [32] and $d_S(\eta_1,\eta_2)\leq\|\eta_1-\eta_2\|$, for any $\eta_1,\eta_2\in\mathcal{D}_\tau$ (see, e.g. ref. [33, p. 19]) that

$$H_1(t)\leq\frac{\varepsilon}{3}+\mathbb{E}\Big(I_{\Omega_1}\mathbb{E}(\|Y^{\tilde{\eta},l}_{t-o_{ij}}-Y^{\tilde{\xi},l}_{t-o_{ij}}\|_\tau)\Big). \tag{58}$$

For any $\omega\in\Omega_1$, we note that $\|\tilde{\eta}\|\vee\|\tilde{\xi}\|\leq h$. By apply Lemma 3.4, we can find positive number K_3 satisfying

$$\mathbb{E}(\|Y^{\tilde{\eta},l}_{t-o_{ij}}-Y^{\tilde{\xi},l}_{t-o_{ij}}\|_\tau)\leq\frac{\varepsilon}{3},\qquad\forall t\geq K_1+K_3 \tag{59}$$

whenever $\omega\in\Omega_1$. we obtain that

$$|\mathbb{E}F(Y^{\eta,i}_t)-\mathbb{E}F(Y^{\xi,j}_t)|\leq\varepsilon,\qquad\forall t\geq K_1+K_3. \tag{60}$$

Thanks to the arbitrariness of F, for all $(\eta,\xi,i,j)\in\mathcal{J}\times\mathcal{J}\times\mathbb{S}\times\mathbb{S}$, we must have

$$d_{\mathbb{L}}(\mathcal{L}(Y^{\eta,i}_t),\mathcal{L}(Y^{\xi,j}_t))\leq\varepsilon,\qquad\forall t\geq K_1+K_3. \tag{61}$$

This proves Equation (52).

We next prove that $\{\mathcal{L}(Y^{\eta,i}_t)\}_{t\geq0}$ is a Cauchy sequence in $\mathcal{P}(\mathcal{D}_\tau)$ with metric $d_{\mathbb{L}}$ for any $(\eta,i)\in\mathcal{D}_\tau\times\mathbb{S}$, i.e. we need to show that there exists a positive number K_4 satisfying, for any $\varepsilon>0$, $u\geq K_4$ and $v>0$,

$$d_{\mathbb{L}}(\mathcal{L}(Y^{\eta,i}_{u+v}),\mathcal{L}(Y^{\eta,i}_u))\leq\varepsilon. \tag{62}$$

The proof of Equation (62) is standard, so we omit it here (For more details, see ref. [21]).

It can be seen from Equation (62) that there exists $\omega_\tau\in\mathcal{P}(\mathcal{D}_\tau)$ such that

$$\lim_{t\to\infty}d_{\mathbb{L}}(\mathcal{L}(Y^{0,1}_t),\omega_\tau)=0. \tag{63}$$

By the triangle inequality, together with Equation (52), we derive that

$$\lim_{t\to\infty}d_{\mathbb{L}}(\mathcal{L}(Y^{\eta,i}_t),\omega_\tau)\leq\lim_{t\to\infty}d_{\mathbb{L}}(\mathcal{L}(Y^{\eta,i}_t),\mathcal{L}(Y^{0,1}_t))$$
$$+\lim_{t\to\infty}d_{\mathbb{L}}(\mathcal{L}(Y^{0,1}_t),\omega_\tau)$$
$$=0$$

for all $(\eta,i)\in\mathcal{D}_\tau\times\mathbb{S}$. The require assertion (51) hold. The proof is complete. □

4 | DESIGN OF MATRICES U_i

In order to facilitate the calculation and design of Matrices, we here adopt the most common form of function as follows

$$\Phi(y,i)=\Psi(y,i)=y^TQ_iy \tag{64}$$

for some N symmetric positive definite matrices Q_i $(i\in\mathbb{S})$. It can be obtained directly from Equations (11) and (13)

$$\begin{aligned}&L\Phi(y,i)\\&=2y^TQ_i[f(y,i)+U_i(y)]+\mathrm{trace}[g(y,i)^TQ_ig(y,i)]\\&+\int_{\mathbb{R}\backslash\{0\}}\sum_{k=1}^n[(h^{(k)}(y,i,x_k))^TQ_ih^{(k)}(y,i,x_k)\\&+(h^{(k)}(y,i,x_k))^TQ_iy-y^TQ_ih^{(k)}(y,i,x_k)]\nu_k(\mathrm{d}x_k)\\&+\sum_{j=1}^N\gamma_{ij}y^TQ_jy\end{aligned} \tag{65}$$

and

$$\begin{aligned}&\mathbb{L}\Psi(y,z,i)\\&=2(y-z)^TQ_i[f(y,i)-f(z,i)+U_i(y-z)]\\&+\mathrm{trace}[(g(y,i)-g(z,i))^TQ_i(g(y,i)-g(z,i))]\\&+\int_{\mathbb{R}\backslash\{0\}}\sum_{k=1}^n[(h^{(k)}(y,i,x_k)-h^{(k)}(z,i,x_k))^T\\&\times Q_i(h^{(k)}(y,i,x_k)-h^{(k)}(z,i,x_k))\\&+(h^{(k)}(y,i,x_k)-h^{(k)}(z,i,x_k))^TQ_i(y-z)\\&-(y-z)^TQ_i(h^{(k)}(y,i,x_k)-h^{(k)}(z,i,x_k))]\nu_k(\mathrm{d}x_k)\\&+\sum_{j=1}^N\gamma_{ij}(y-z)^TQ_j(y-z).\end{aligned} \tag{66}$$

Then we get the following useful corollary directly from Lemma 3.3, Lemma 3.4 and Theorem 3.5.

Corollary 4.1. *Let Assumption 2.1 hold. If there exist positive numbers $\bar{b}_0$, $\bar{b}_1$ and $\bar{b}_2$ and positive definite matrices Q_i such that*

$$\begin{aligned}&L\Phi(y,i)\leq-\bar{b}_0|y|^2+\bar{b}_2,\\&\mathbb{L}\Psi(y,z,i)\leq-\bar{b}_1|y-z|^2\end{aligned} \tag{67}$$

for all $y, z \in \mathbb{R}^d$ *and* $i \in \mathbb{S}$. *If* $\tau > 0$ *is small enough for*

$$\bar{b}_1 - \check{c}^2\bar{\theta} - \alpha\tau(2e_1\tau + e_2 + e_3 + 2e_4\tau) > 0,\ \tau \le \frac{1}{2\sqrt{2e_4}},$$

$$\bar{b}_0 - \check{c}^2\bar{\theta} - 2\alpha\tau(2e_1\tau + e_2 + e_3 + 2e_4\tau) > 0,\ \tau \le \frac{1}{4\sqrt{e_4}},$$

where $\alpha = \frac{3e_4}{\bar{\theta}}$, $\bar{\theta} \in (0, \frac{\bar{b}_1}{\check{c}^2} \wedge \frac{\bar{b}_0}{\check{c}^2})$ *and* $\check{c} = \max_{i\in\mathbb{S}} \|Q_i\|$. *Then the SDDEs-MS-LN (5) is stable in distribution.*

The stability in distribution of SDDEs-MS-LN (5) largely relies on the matrices U_i. Here we will expound how to design these matrices. Namely, for some integer $l > 0$, we will find the matrices in the structure form of $U_i = A_iB_i$ with $A_i \in \mathbb{R}^{d\times l}$ and $B_i \in \mathbb{R}^{l\times d}$. For the convenience of calculations, we will discuss how to design these matrices to make Corollary 4.1 hold but the advantages of our new results will be explained clearly. We will research state feedback and output injection (ref. [34]) as follows.

(i) State feedback

In this case, B_i*'s* are given so our goal is to find A_i*'s* so that SDDEs-MS-LN (5) is stability in distribution. We will complete the matrix design in two steps.

Step 1: Based on Assumption 2.1, we can seek N couple of positive-definite symmetric matrices $(Q_i, \hat{Q}_i)$ in order for

$$\begin{aligned}
&2(y-z)^TQ_i[f(y,i)-f(z,i)]\\
&+\mathrm{trace}[(g(y,i)-g(z,i))^TQ_i(g(y,i)-g(z,i))]\\
&+\int_{\mathbb{R}\setminus\{0\}}\sum_{k=1}^n[(h^{(k)}(y,i,x_k)-h^{(k)}(z,i,x_k))^T\\
&\times Q_i(h^{(k)}(y,i,x_k)-h^{(k)}(z,i,x_k))\\
&+(h^{(k)}(y,i,x_k)-h^{(k)}(z,i,x_k))^TQ_i(y-z)\\
&-(y-z)^TQ_i(h^{(k)}(y,i,x_k)-h^{(k)}(z,i,x_k)]\nu_k(\mathrm{d}x_k)\\
\le&(y-z)^T\hat{Q}_i(y-z).
\end{aligned}$$

It is straightforward to show from Assumption 2.1 and Equation (68) that

$$\begin{aligned}
&2y^TQ_i[f(y,i)+U_i(y)]+\mathrm{trace}[g(y,i)^TQ_ig(y,i)]\\
&+\int_{\mathbb{R}\setminus\{0\}}\sum_{k=1}^n[(h^{(k)}(y,i,x_k))^TQ_ih^{(k)}(y,i,x_k)\\
&+(h^{(k)}(y,i,x_k))^TQ_iy-y^TQ_ih^{(k)}(y,i,x_k)]\nu_k(\mathrm{d}x_k)\\
&+\sum_{j=1}^N\gamma_{ij}y^TQ_jy\le y^T\hat{Q}_iy+\bar{b}_2.
\end{aligned}$$

Step 2: By linear matrix inequalities (LMIs, see, e.g. ref. [35]), we can find a solution of matrices A_i, that is

$$\hat{Q}_i + A_iQ_iB_i + B_i^TQ_iA_i^T + \sum_{j=1}^N \gamma_{ij}Q_j < 0. \tag{68}$$

Corollary 4.2. *Let Assumption 2.1 hold and moreover we can seek matrices* A_i *as shown in Steps 1 and 2. Then the conditions of Corollary 4.1 are satisfied with* $U_i = A_iB_i$ *and*

$$\bar{b}_1 = -\max_{i\in\mathbb{S}} \lambda_{\max}\left(\hat{Q}_i + A_iQ_iB_i + A_i^TQ_iB_i^T + \sum_{j=1}^N \gamma_{ij}Q_j\right).$$

(ii) Output injection

In this case, A_i*'s* are given so our goal is to design B_i*'s* so that SDDEs-MS-LN (5) is stability in distribution. This situation is similar to state feedback, and the following corollary can be drawn.

Corollary 4.3. *Let Assumption 2.1 hold and moreover we can seek matrices* Q_i, $\hat{Q}_i$, *and* B_i *as shown in Steps 1 and 2. Then the conditions of Corollary 4.1 are satisfied with* $U_i = A_iB_i$ *and*

$$\bar{b}_1 = -\max_{i\in\mathbb{S}} \lambda_{\max}\left(\hat{Q}_i + A_iQ_iB_i + B_i^TQ_iA_i^T + \sum_{j=1}^N \gamma_{ij}Q_j\right).$$

5 | EXAMPLE

A simple example is provide to illustrate the numerical results of our theory.

Example 5.1. Consider an unstable SDEs-MS-LN:

$$\begin{aligned} dY(t) &= (\mu(q(t)) + Y(t))dt + \sigma(q(t))Y(t)dW(t)\\ &\quad + \gamma(q(t^-))Y(t^-)d\tilde{N}(t), \end{aligned} \tag{69}$$

with initial value $Y_0 \in \mathbb{R}$. Here, $W(t)$ is a scalar Brownian motion, $\tilde{N}(t)$ is a compensated Poisson random measure which means that $N(t)$ is a scalar Poisson process with intensify λ, $q(t) \in \mathbb{S} = \{1, 2\}$ and

$$\Gamma = \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix}.$$

Set $\lambda = 1$, $\mu(1) = \mu(2) = 1.5$, $\sigma(1) = 0.2$, $\sigma(2) = 0.5$, $\gamma(1) = 0.8$, and $\gamma(2) = 1.2$. From ref. [36], we can easily get the mean of the solution to system (69) as follows

$$\mathbb{E}Y(t) = -1.5 + e^t(Y(0) + 1.5).$$

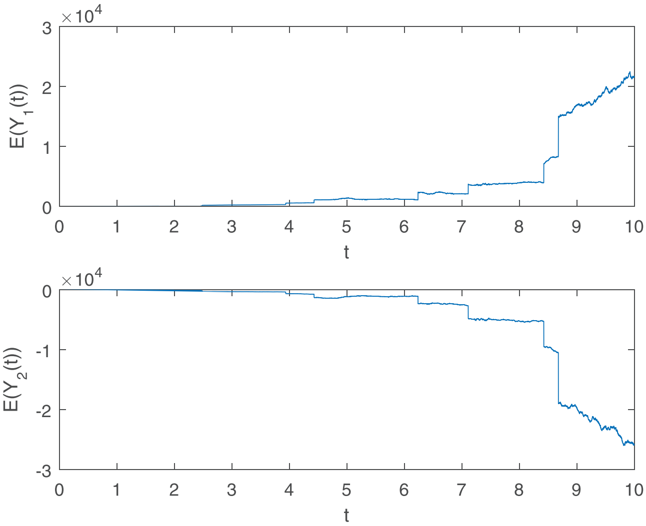


FIGURE 1 Up: the trajectory of $\mathbb{E}Y_1(t)$ with initial value $Y_0 = 1$. Down: the trajectory of $\mathbb{E}Y_2(t)$ with initial value $Y_0 = -4$

Moreover, such as $\lim_{t\to+\infty} \mathbb{E}Y(t) \to +\infty$ when $Y_0 = 1$ while $\lim_{t\to+\infty} \mathbb{E}Y(t) \to -\infty$ when $Y_0 = -4$. These show that the system (69) is not stable in distribution. Computer simulation is also performed to show that the system (69) is not stable in distribution, as shown in Figure 1.

The Figure 1 shows the trajectories of $\mathbb{E}Y(t)$ with different initial value $Y_0 = 1$ and $Y_0 = -4$ (i.e. the trajectories of $\mathbb{E}Y_1(t)$ and $\mathbb{E}Y_2(t)$), which show that $\mathbb{E}Y_1(t)$ tend to $+\infty$ and $\mathbb{E}Y_2(t)$ tend to $-\infty$.

Next, we will apply our theory to design a linear DFC to stabilise the SDEs-MS-LN. Assume that the linear DFC has the form $-k(i)Y(t-\tau)$, where $k(1)$ and $k(2)$ to be determined. Then the controlled system can be written as

$$\begin{aligned} dY(t) =& [\mu(q(t)) + Y(t) - k(q(t))Y(t-\tau)]dt \\ &+ \sigma(q(t))Y(t)dW(t) + \gamma(q(t^-))Y(t^-)d\tilde{N}(t). \end{aligned} \tag{70}$$

Let Q_i $(i \in 1, 2)$ be the identity matrix, then the conditions of Corollary 4.1 holds as long as

$$-\bar{b}_1 = \max_{i\in\mathbb{S}} \left(2 + \sigma^2(i) - 2k(i) + \gamma^2(i)\right) < 0. \tag{71}$$

Setting $2 + \sigma^2(i) - 2k(i) + \gamma^2(i) = -11$ for $i = 1, 2$, namely

$$2 + 0.25 - 2k(1) + 0.64 = -11, \ \ 2 + 2.25 - 2k(2) + 1 = -11,$$

we get $k(1) = 6.84$ and $k(2) = 7.345$. Consequently, the conditions of Corollary 4.1 holds with $\bar{b}_1 = \bar{b}_0 = 11$. It is easy to show that Assumption 2.1 is satisfied provided $e_1 = 1$, $e_2 = 0.25$, and $e_3 = 1.44$. Moreover, by $e_4 = \max_{i\in\mathbb{S}} \|k(i)\|^2$ and Equation (18), we compute $e_4 = 53.949$, and $\check{c} = 1$. Therefore, choosing $\bar{\theta} = 0.1$, we can compute

$$\tau_1^* = 0.003284, \ \tau_2^* = 0.048135, \ \tau_3^* = 0.001786, \ \tau_4^* = 0.034038$$

and hence $\tau^* = 0.001786$. By Corollary 4.1, we can obtain that for all $\tau < 0.001786$ so that the controlled system (70) is stable in distribution.

The numerical example discussed in ref. [21] does not illustrate how to verify the convergence of the distribution of the two segment processes, we therefore develop new algorithm to tackle such problem.

5.1 | Algorithm and simulation

It is not hard to verify that there exists a unique probability measure ω_τ such that $\lim_{t\to\infty} d_{\mathbb{L}}(\mathcal{L}(Y_t^{\eta,i}), \omega_\tau) = 0$, for all $(\eta, i) \in \mathcal{D}_\tau \times \mathbb{S}$. However, the explicit form of ω_τ is rarely found. In order to illustrate the stability in distribution of system (70), we provide an algorithm as follows. Define $Y_t^{\eta,i} = \{Y^{\eta,i}(t+u) : -\tau \le u \le 0\}$ with initial data (η, i) and $Y_{\bar{t}}^{\xi,\bar{i}} = \{Y^{\xi,\bar{i}}(\bar{t}+u) : -\tau \le u \le 0\}$ with initial data $(\xi, \bar{i})$. Let $\Delta = \frac{\tau}{m}$, where Δ is the stepsize and m is a positive integer. For any $t, \bar{t} > 0$, there are two positive integers k and $\bar{k}$ such that $t \in [(k-1)\Delta, k\Delta)$ and $\bar{t} \in [(\bar{k}-1)\Delta, \bar{k}\Delta)$. The discrete approximation of segment process can be expressed as

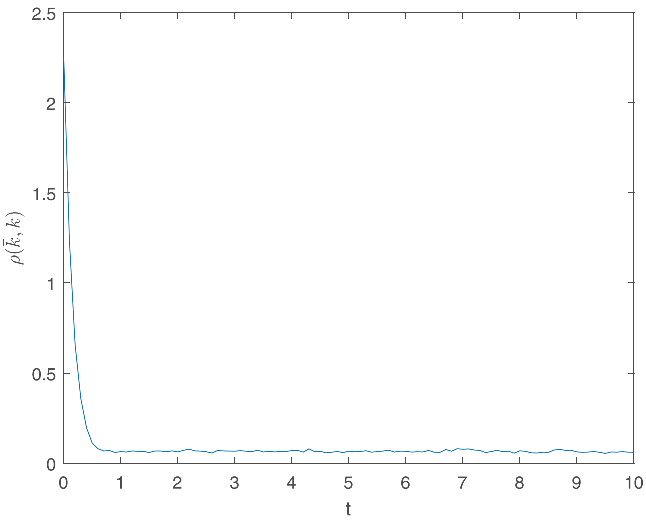
$$\begin{aligned} Y_{lj}^{\eta,i}(k) &= Y^{\eta,i}(k\Delta - l\Delta, \varpi_j), \\ Y_{lj}^{\xi,\bar{i}}(\bar{k}) &= Y^{\xi,\bar{i}}(\bar{k}\Delta - l\Delta, \varpi_j), \end{aligned}$$

where $l = 0, 1, \ldots, m$, $j = 1, \ldots, J$, $k = 1, 2, \ldots$, and $\bar{k} = 1, 2, \ldots$. We have noticed that

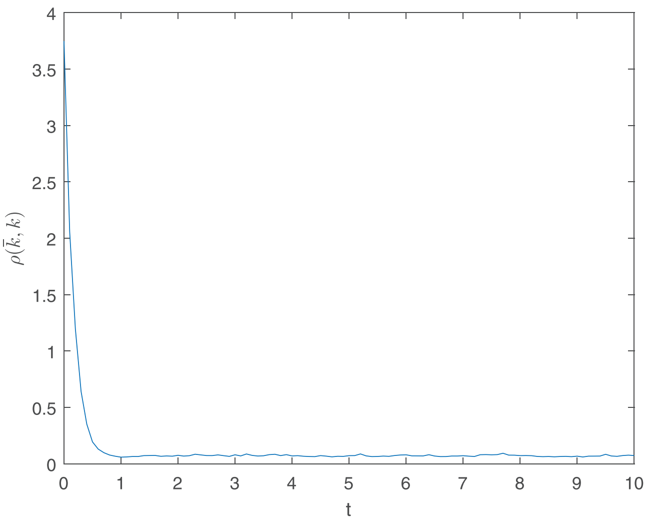
$$\begin{aligned} d_{\mathbb{L}}(\mathcal{L}(Y_{k\Delta}^{\eta,i}), \mathcal{L}(Y_{\bar{k}\Delta}^{\xi,\bar{i}})) =& \sup_{\phi\in\mathbb{L}} |\mathbb{E}\phi(Y_{k\Delta}^{\eta,i}) - \mathbb{E}\phi(Y_{\bar{k}\Delta}^{\xi,\bar{i}})| \\ \le& \mathbb{E}\|Y_{k\Delta}^{\eta,i} - Y_{\bar{k}\Delta}^{\xi,\bar{i}}\|_\tau \\ \approx& \frac{1}{J}\sum_{j=1}^{J} \sup_{0\le l\le m} |Y_{lj}^{\eta,i}(k) - Y_{lj}^{\xi,\bar{i}}(\bar{k})| \\ \triangleq& \rho(\bar{k}, k). \end{aligned}$$

When $\bar{k}$ takes a sufficiently large fixed value, we regard the distribution of segment process $Y_{\bar{k}\Delta}^{\xi,\bar{i}}$ as the true probability distribution. If we have $\rho(\bar{k}, k)$ tend to zero along the time line, then the distribution of $Y_{k\Delta}^{\eta,i}$ will converge to distribution of $Y_{\bar{k}\Delta}^{\xi,\bar{i}}$. Thus we claim that $\lim_{t\to\infty} d_{\mathbb{L}}(\mathcal{L}(Y_t^{\eta,i}), \omega_\tau) \to 0$, namely the controlled system (70) is stable in distribution.

To perform a computer simulation, we set $\tau = 10^{-3}$, $\Delta = 10^{-4}$, $i = 1$ and $\bar{i} = 2$. In this example we regard the distribution of segment process $Y_{\bar{k}\Delta}^{\xi,\bar{i}}$ with initial data $(\xi, \bar{i})$ at time $t = 20$ (i.e. $\bar{k} = \frac{t}{\Delta}$) generated by the Euler–Maruyama method as the



(a) started from initial data $\eta_1(t)$



(b) started from initial data $\eta_2(t)$

FIGURE 2 The computer simulation of $\rho(\bar{k}, k)$ started from different initial data ($\tau = 10^{-3}, \Delta = 10^{-4}, \xi(t) = 1.5 + 0.55\sin(t)$ on $t \in [-0.001, 0]$)

true probability distribution. We observe that the distributions of segment process $Y_{k\Delta}^{\eta,i}$ at time $t \in [0, 10]$ approximate the degree of the true distribution. Moreover, we choose two different initial data ($\eta_1(t) = -2 + 0.55\sin(t)$ and $\eta_2(t) = 3 + \cos(t)$ on $t \in [-0.001, 0]$) for the distributions of segment process $Y_{k\Delta}^{\eta,i}$. Figure 2. plots the value $\rho(\bar{k}, k)$ with different initial data along the time line. It indicates that as time advances the values of $\rho(\bar{k}, k)$ simulated by EM method with different initial data converge to 0, which indicates the SDDEs-MS-LN (70) is stable in distribution.

6 | CONCLUSION

In this paper, we have studied the stability in distribution of a given unstable SDEs-MS-LN by linear DEC $u(Y(t - \tau), q(t)) = U(q(t))Y(t - \tau)$. Under global Lipschitz condition, we have shown that a bound on τ^* is given so that the SDDEs-MS-LN is stability in distribution as long as $\tau \leq \tau^*$. Especially, we focuses on in more detail how to design the linear DFC by linear matrix inequalities. Finally, we develop a new algorithm to illustrate how to verify the convergence of the distribution of the two segment processes.

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CONFLICT OF INTEREST

The authors declare no conflict of interest.

DATA AVAILABILITY STATEMENT

Data openly available in a public repository that issues datasets with DOIs.

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