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THE HYPERBOLIC UMBILIC SINGULARITY IN FAST-SLOW SYSTEMS

HILDEBERTO JARDÓN KOJAKHMETOV¹, CHRISTIAN KUEHN²,
AND MAXIMILIAN STEINERT^{2,*}

ABSTRACT. Fast-slow systems with three slow variables and gradient structure in the fast variables have, generically, hyperbolic umbilic, elliptic umbilic or swallowtail singularities. In this article we provide a detailed local analysis of a fast-slow system near a hyperbolic umbilic singularity. In particular, we show that under some appropriate non-degeneracy conditions on the slow flow, the attracting slow manifolds jump onto the fast regime and fan out as they cross the hyperbolic umbilic singularity. The analysis is based on the blow-up technique, in which the hyperbolic umbilic point is blown up to a 5-dimensional sphere. Moreover, the reduced slow flow is also blown up and embedded into the blown-up fast formulation. Further, we describe how our analysis is related to classical theories such as catastrophe theory and constrained differential equations.

KEYWORDS. Fast-slow system; Geometric desingularization; Blow-up method; Catastrophe theory.

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1. INTRODUCTION AND MAIN RESULT

In this article we analyze a fast-slow system, also called a singularly perturbed differential equation, with two fast and three slow variables of the form

$$\begin{aligned}
 \varepsilon \dot{x} &= x^2 + ay + b + O(\varepsilon) \\
 \varepsilon \dot{y} &= y^2 + ax + c + O(\varepsilon) \\
 \dot{a} &= g_a(x, y, a, b, c, \varepsilon) \\
 \dot{b} &= g_b(x, y, a, b, c, \varepsilon) \\
 \dot{c} &= g_c(x, y, a, b, c, \varepsilon),
 \end{aligned} \tag{1.1}$$

or equivalently, after suitable time change onto the fast time scale,

$$\begin{aligned}
 x' &= x^2 + ay + b + O(\varepsilon) \\
 y' &= y^2 + ax + c + O(\varepsilon) \\
 a' &= \varepsilon g_a(x, y, a, b, c, \varepsilon) \\
 b' &= \varepsilon g_b(x, y, a, b, c, \varepsilon) \\
 c' &= \varepsilon g_c(x, y, a, b, c, \varepsilon),
 \end{aligned} \tag{1.2}$$

for sufficiently small time scale separation parameter $0 < \varepsilon \ll 1$. More precisely, we focus on the dynamics of (1.2) near the origin, which is the most degenerate singularity of (1.2). We refer to this singularity as hyperbolic umbilic singularity, since it arises from catastrophe theory, as will be shown below. For basic fast-slow systems terminology we refer to section (2.1). For more details and recent results, the reader is referred to, e.g. [20, 14, 32, 7].

Let us outline our motivation to study a hyperbolic umbilic singularity and in particular system (1.2). The fast-slow system (1.2) arises from the general class of fast-slow systems of the form

$$\begin{aligned}
 z' &= \nabla_z V(z, \alpha) + O(\varepsilon) \\
 \alpha' &= \varepsilon g(z, \alpha, \varepsilon),
 \end{aligned} \tag{1.3}$$

where $z \in \mathbb{R}^n$, $\alpha \in \mathbb{R}^r$ and V is a sufficiently smooth function. The formulation (1.3) gives a (parameter-dependent) gradient field in the limit $\varepsilon \rightarrow 0$. Note that the class of systems of the form (1.3) contains any fast-slow system having only one fast variable. In particular, (1.3) contains the fast-slow systems, which exhibit fold or cusp singularities, after suitable center manifold reduction and transformations [19, 5, 31, 13, 27, 18, 20]. In this article, we are in general interested in a fast-slow system of the form (1.3) near a singularity, for which at least two fast variables are necessary. More precisely, we are interested in a fast-slow system (1.3) near a hyperbolic umbilic singularity. This singularity arises from elementary catastrophe theory [1, 22] and needs at least two fast variables in (1.3) for its occurrence. Additionally, this singularity occurs generically for at least three slow variables in (1.3). For a detailed

explanation and the definition of this singularity in the setting of (1.3) we refer to section 2, in particular to Definition 2.3. The hyperbolic umbilic singularity, among others, already appeared in Takens' study of constrained differential equations [29], which are related to the singular limit of (1.3). In fact, our main motivation is a list of "normal forms" provided by Takens [29, (4.10)], in which the hyperbolic umbilic singularity is contained. For detailed statements we refer to section 2. Roughly speaking, we choose the parameter-dependent potential

$$V(x, y, a, b, c) = \frac{1}{3}x^3 + \frac{1}{3}y^3 + axy + bx + cy \quad (1.4)$$

in (1.3), where $z = (x, y) \in \mathbb{R}^2$ are the fast variables and $\alpha = (a, b, c) \in \mathbb{R}^3$ are the slow variables. This choice leads to system (1.1). The potential V , given by (1.4), is a universal unfolding of the hyperbolic umbilic germ $\frac{1}{3}(x^3 + y^3)$ in the sense of catastrophe theory [22], see also section 2.3 below. The precise argument for our derivation of the fast-slow system (1.1) from the general form (1.3) is then given in section 2.5.

The main objective of this article is to provide a detailed analysis of the dynamics of the fast-slow system (1.1) near the hyperbolic umbilic singularity at the origin, in a generic case. This analysis faces several challenges as such singularity has not been studied before in the context of fast-slow systems. To accomplish our objective, we employ geometric techniques from dynamical systems theory and especially from the theory of fast-slow systems [20]. Our analysis is based on the blow-up technique, which has been used in numerous cases to desingularize fast-slow systems, e.g. [19, 27, 18, 5, 13, 14, 7].

For the dynamics of (1.1) near the hyperbolic umbilic, we focus on the continuation of attracting slow manifolds, denoted by $\mathcal{S}_\varepsilon^a$, under the flow of the system. Attracting slow manifolds are obtained by Fenichel theory as "perturbations" from the singular limit $\varepsilon = 0$ of the attracting region of the critical manifold $\mathcal{S}_0 = \{\nabla_z V(z, \alpha) = 0\}$; for Fenichel theory we refer the reader to [9, 33, 15, 20]. The aim is to analyze how attracting slow manifolds $\mathcal{S}_\varepsilon^a$ of (1.1) evolve near the hyperbolic umbilic singularity. Therefore, our analysis of (1.1) will be of local nature. For sufficiently small $\nu > 0$, we define an entry- and exit section by

$$\Delta^{\text{en}} = \{(x, y, a, b, c) \in \mathbb{R}^5 \mid y = -\nu\}, \quad \Delta^{\text{ex}} = \{(x, y, a, b, c) \in \mathbb{R}^5 \mid x + y = 2\nu\}. \quad (1.5)$$

Let J denote the closed line segment $\{(x, y, a, b, c) \in \mathbb{R}^5 \mid x \geq 0, y \geq 0, a = b = c = 0\} \cap \Delta^{\text{ex}}$ and denote by j_x and j_y the singletons containing the intersection of J with the x -axis, respectively y -axis. Refer to figure 1 for a sketch. Then, we track slow flow trajectories contained in (a choice of) an attracting slow manifold $\mathcal{S}_\varepsilon^a$, which start at Δ^{en} and come close to the hyperbolic umbilic singularity. These trajectories correspond to initial conditions in a subset $I_\varepsilon \subset \Delta^{\text{en}} \cap \mathcal{S}_\varepsilon^a$ in the context of the following main result, which shows that these trajectories jump onto the fast regime and fan out. This fanning-out behavior is observed at the exit section Δ^{ex} . A sketch of this result is given in figure 1 and the singular limit $\varepsilon = 0$ is illustrated in figure 2.

Theorem 1.1 (Attracting slow manifolds jump and fan out at the hyperbolic umbilic). *Consider the fast-slow system (1.1) and assume $g_b(0) > 0$ and $g_c(0) > 0$. For sufficiently small $\varepsilon > 0$, there exist open sets $I_\varepsilon \subset \Delta^{\text{en}} \cap \mathcal{S}_\varepsilon^a$ such that the transition maps $\Pi: I_\varepsilon \rightarrow \Delta^{\text{ex}}$ induced by the flow of (1.1) are well-defined and satisfy the following assertions:*

- (1) I_ε and $\Pi(I_\varepsilon)$ are 2-dimensional manifolds and I_ε converges in Hausdorff distance to a single point p for $\varepsilon \rightarrow 0$.
- (2) There exist open sets $L_\varepsilon^x, R_\varepsilon, L_\varepsilon^y$ such that $I_\varepsilon = L_\varepsilon^x \cup R_\varepsilon \cup L_\varepsilon^y$ and $\Pi(R_\varepsilon) \rightarrow J$, $\Pi(L_\varepsilon^x) \rightarrow j_x$, $\Pi(L_\varepsilon^y) \rightarrow j_y$ in Hausdorff distance for $\varepsilon \rightarrow 0$.
- (3) For $(x, y, a, b, c) \in \Pi(I_\varepsilon)$ it holds that $a = O(\varepsilon^{1/3})$, $b = O(\varepsilon^{2/3})$, $c = O(\varepsilon^{2/3})$.
- (4) If $g_a(x, y, a, b, c, 0) = a \cdot h(x, y, a, b, c)$ for sufficiently smooth h , then $a_i = O(\varepsilon^{1/3})$ for $(x_i, -\nu, a_i, b_i, c_i) \in I_\varepsilon$.

At this point we provide a brief sketch of the proof to give an overview of the upcoming steps.

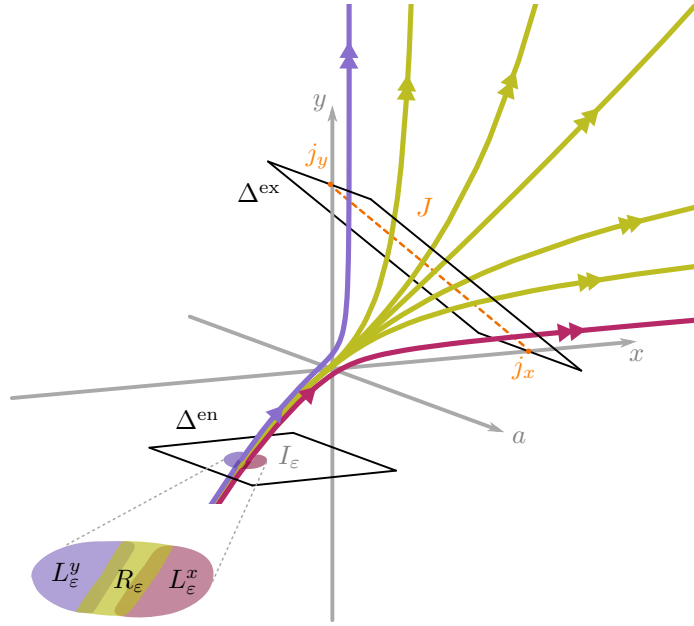


FIGURE 1. Sketch of Theorem 1.1, describing the transition from a small region of initial conditions $I_\varepsilon = L_\varepsilon^y \cup R_\varepsilon \cup L_\varepsilon^x \subset \Delta^{\text{en}}$ to Δ^{ex} : initial conditions in L_ε^y evolve close to the violet trajectory, L_ε^x evolves close to the red trajectory, whereas R_ε fans out in-between along the yellow trajectories. The trajectories were obtained by numerical integration of (1.1) with $\varepsilon = 10^{-3}$, $g_a = -1$, $g_b = 2$, $g_c = 1$ and suitably chosen initial conditions, such that they approach the hyperbolic umbilic at the origin and fan out. Most randomly sampled initial conditions near I_ε escape close to the x - or y -axis and only a few fan out in-between. This indicates that R_ε is small. The variables b, c are suppressed in this sketch.

Sketch of the proof of Theorem 1.1. The first step is to understand the slow subsystem, which is the content of section 3. In particular, we study the so-called desingularized slow flow, which was already studied to some extent by Takens in [29]. The desingularized slow flow generically has a non-hyperbolic equilibrium at the hyperbolic umbilic point. Our study is then continued by blowing up the hyperbolic umbilic point in the fast formulation (1.2) to a 5-d unit sphere \mathbb{S}^5 . The desingularized slow flow (3.8), which determines singular limit dynamics on the critical manifold, will be embedded into the blow-up analysis, which is carried out in section 5.3. The transition map Π and its properties are obtained by combining the results from an entry chart κ_{-y} , the rescaling chart κ_ε and an exit chart κ_{ex} . The charts and their notation will be set up in the beginning of section 5. In the entry chart, section 5.2, the main work is to obtain the flow nearby the hyperbolic umbilic in attracting center manifolds, which contain attracting slow manifolds. This will be accomplished by a center manifold reduction close to a point called ζ_2 , which arises from the desingularized slow flow, see Lemma 5.13. In the rescaling chart κ_ε , the key is to identify the relevant trajectories on the blow-up sphere, that lead attracting slow manifolds from ζ_2 through the blow-up space. By perturbation arguments, this reduces to the analysis of two decoupled Riccati equations, which is a similar situation to the non-degenerate fold [19]. This gives the flow close to the hyperbolic umbilic singularity, i.e. in the corresponding region in the rescaling chart. To enter this region, the transition Π_1 in section 5.4, precisely the structure of (5.19), requires that we limit the initial conditions suitably, which will eventually lead to I_ε shrinking to a point as $\varepsilon \rightarrow 0$. In Proposition 5.18 we obtain an invariant 2-manifold Γ on the blow-up sphere, leading the relevant trajectories in attracting slow manifolds $\mathcal{S}_\varepsilon^a$ towards the fast regime in the exit chart κ_{ex} . In the exit chart κ_{ex} , we encounter three hyperbolic equilibria $q_{4,5,6}$, to which trajectories in Γ connect. These three equilibria give three distinct forward asymptotic behavior of trajectories in Γ , which lead to the division of I_ε into three regions

$L_\varepsilon^x, R_\varepsilon, L_\varepsilon^y$. Further, the three equilibria $q_{4,5,6}$ organize the transition onto the fast regime and lead to fanning-out, see Proposition 5.28. To obtain the fanning-out, the blown-up vector field is integrated in the invariant plane corresponding to the fast subsystem. Essentially, a 45° rotation is included in κ_{ex} , to have all escape directions visible in a single chart. The scalings of $\Pi(I_\varepsilon)$ originate from the convergence of I_ε to a point as $\varepsilon \rightarrow 0$. The detailed proof is finished in section 5.7. For an overview of the blow-up analysis, we refer to figure 17, in which all ingredients to the proof are visible. \square

Remark 1.2. *Theorem 1.1 does not cover the fast approach of trajectories towards attracting slow manifolds. However, from the fast subsystem one can roughly infer the basin of attraction of attracting slow manifolds. Compare to figure 2. Taking the stable manifold of an attracting slow manifold into account, we remark that Theorem 1.1 does in this case only apply to suitable “fast initial conditions” located in fibers, whose base passes through I_ε .*

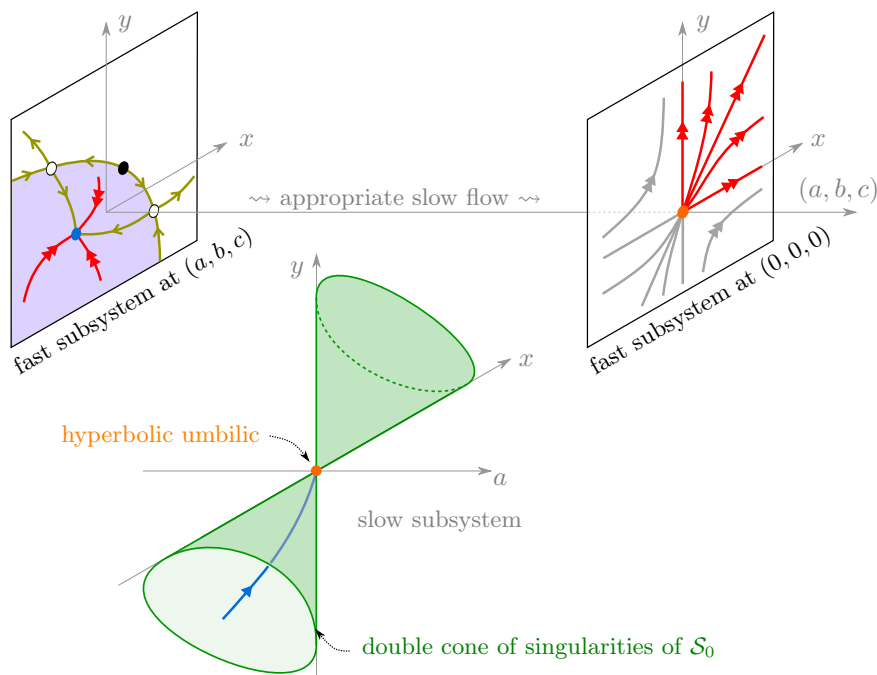


FIGURE 2. Singular limit sketch of (1.1). The upper two planar systems are fast subsystems for particular fixed choices of slow variables, i.e. “parameters” (a, b, c) . The right one is the fast subsystem at the hyperbolic umbilic, i.e. $(a, b, c) = (0, 0, 0)$. The slow subsystem on the critical manifold can be parametrized by (x, y, a) and the green double cone represents the singularities. Fast segments of candidate trajectories for (1.1) are shown in red. The blue trajectory is a particular slow candidate trajectory in the attracting part of the critical manifold, which approaches the hyperbolic umbilic singularity, depicted in orange. The blue segment can be concatenated with any of the red segments emanating from the hyperbolic umbilic, which corresponds to Theorem 1.1 in the singular limit. The violet region depicts a basin of attraction for the blue sink, which is an instance of the attracting region of the critical manifold. Here, one can think of (a, b, c) chosen such that this sink corresponds to the starting point of the blue slow candidate trajectory. Further, three fast candidate trajectories approaching the attracting critical manifold are depicted.

To motivate and illustrate the behavior of the system (1.1) and in particular Theorem 1.1, we present a singular limit sketch of (1.1) in figure 2, in which two particular fast subsystems and a particular slow flow in the slow subsystem are depicted. Note that (1.1) has a 3-dimensional critical manifold and 2-dimensional fast subsystems. A detailed analysis of the slow and fast subsystems is carried out in sections 3 and 4. Figure 2 illustrates

an appropriate way of combining the fast- and slow subsystem, i.e. the concatenation of candidate trajectories.

Remark 1.3. *The assumption $g_b(0) > 0, g_c(0) > 0$ of Theorem 1.1 will be motivated in the singular limit by Lemma 3.4, which shows that there exists a candidate trajectory σ approaching the hyperbolic umbilic point from inside the attracting critical manifold. This trajectory σ is depicted in blue in the lower cone in figure 2. In fact, the idea of the blow-up analysis is to track trajectories close to σ through the blow-up space.*

The following remark points out an interesting observation, which will be apparent from the upcoming blow-up analysis. It is related to the behavior of slow flow trajectories jumping at folds, away from the hyperbolic umbilic singularity.

Remark 1.4 (Fast-slow systems with folded singularities appear in the blow-up of (1.2)). *In the blow-up analysis of (1.2) in section 5, folded saddles or folded centers appear generically on the blow-up sphere. This observation is made in section 5.3 and analyzed further in section 6. These folded singularities seem to organize the transition of attracting slow manifolds at fold singularities of (1.2). Their further analysis does not lie in the scope of this article. However, these folded singularities, especially the folded centers, deserve future attention. In the folded saddle scenario, existing results can be applied, e.g. singular canards perturb to maximal canard solutions [27].*

The article is structured as follows. In section 2 we will briefly recall fast-slow terminology, formalize what we mean by a hyperbolic umbilic singularity in a fast-slow system and derive our system (1.1). This derivation is motivated by ‘‘Takens’ program’’ [29]. For convenience we provide Takens’ definition of constrained differential equations in appendix A.1. Sections 3 and 4 contain an analysis of the slow and fast subsystem of (1.1). The analysis of the ‘‘desingularized’’ or ‘‘reduced’’ slow flow Y in section 3.2 is fundamental for the upcoming blow-up analysis of system (1.1). The blow-up analysis is presented in section 5, in which the origin of (1.2) is blown up to a 5-d sphere. We remark that also the reduced slow flow Y is blown-up and embedded into the blow-up analysis, which is carried out in section 5.3. The combined analysis from three charts leads to the proof of Theorem 1.1 in section 5.7. Eventually, in section 6 we present further observations made in the blow-up analysis, concerning the folded singularities on the blow-up sphere and the blown-up desingularized slow flow.

2. THE HYPERBOLIC UMBILIC SINGULARITY AND TAKENS’ PROGRAM

In this section we provide the basic terminology and the context of our study. We begin by recalling some basic notion of fast-slow systems. For context, we then describe what we call ‘‘the Takens’ program’’ and its relationship with catastrophe theory and fast-slow systems. Ultimately this explains the reason to choose (1.1) for our analysis.

2.1. Fast-slow systems terminology. Throughout this document, we restrict ourselves to gradient-like fast-slow systems. Consider a general fast-slow system of the form

$$\begin{aligned} \varepsilon \dot{z} &= \nabla_z V(z, \alpha) + O(\varepsilon) \\ \dot{\alpha} &= g(z, \alpha, \varepsilon), \end{aligned} \tag{2.1}$$

where $V : \mathbb{R}^n \times \mathbb{R}^r \rightarrow \mathbb{R}$ is smooth, $0 < \varepsilon \ll 1$ is a small parameter, and $g : \mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R} \rightarrow \mathbb{R}^r$ is smooth. Later on we shall restrict to the case $n = 2$, but for now this is not necessary.

We briefly recall terminology from the theory of fast-slow systems [20, 15]. The following two subsystems of (2.1) are fundamental:

$$\begin{aligned} 0 &= \nabla_z V(z, \alpha) & z' &= \nabla_z V(z, \alpha) \\ \dot{\alpha} &= g(z, \alpha, 0), & \alpha' &= 0. \end{aligned} \tag{2.2} \tag{2.3}$$

System (2.2) is the slow subsystem (also called reduced problem), obtained from (2.1) by putting $\varepsilon = 0$. System (2.3) is the fast subsystem (also called layer problem). Let τ

denote the time in (2.1), also called the slow time scale. Then (2.3) is obtained by a time change in (2.1) onto the fast timescale $t = \tau/\varepsilon$ and putting $\varepsilon = 0$. The critical manifold $\mathcal{S}_0 = \{\nabla_z V = 0\}$ appears as equilibria of the fast subsystem and as constrained phase space of the slow subsystem. Of central importance are the normally hyperbolic points of the critical manifold \mathcal{S}_0 . In our setting, a point $p \in \mathcal{S}_0$ is normally hyperbolic if all eigenvalues of the Hessian $D_z^2 V$ have nonzero real part. By Schwarz's theorem the Hessian $D_z^2 V$ is symmetric, so its eigenvalues are real. The eigenvalues determine the stability type with respect to the fast subsystem. In this way, we shall denote the attracting (resp. repelling, resp. saddle-type) region of the critical manifold as \mathcal{S}_0^a (resp. \mathcal{S}_0^r , resp. \mathcal{S}_0^s). Accordingly $p \in \mathcal{S}_0$ is non-hyperbolic, if at least one eigenvalue of $D_z^2 V$ is zero.

Away from non-hyperbolic points of \mathcal{S}_0 , one can employ Fenichel theory [9, 33, 15, 20] to analyze (2.1). Roughly speaking, Fenichel theory allows to conclude that normally hyperbolic regions of the critical manifold \mathcal{S}_0 persist, for sufficiently small $\varepsilon > 0$, in the form of slow manifolds with the corresponding stability type. The analysis of (2.1) close to non-hyperbolic points of \mathcal{S}_0 is considerably more challenging, since Fenichel theory is not applicable. In our context, non-hyperbolic points of \mathcal{S}_0 will be referred to as singularities of (2.1). It is precisely the singularities, which in general give rise to certain phenomena, for example jumps, canards and relaxation oscillation, see for example [20, 14, 7, 32] and references therein.

By the implicit function theorem, at a normally hyperbolic point, \mathcal{S}_0 is locally given as a graph over the slow α -variables. This determines a flow in the normally hyperbolic regions of \mathcal{S}_0 . This corresponds to the flow arising from the so called “reduced” or “desingularized” slow flow, also see section 3.2. The fast subsystem gives dynamics off the critical manifold. In the singular limit $\varepsilon = 0$, the fast- and slow subsystem (2.3),(2.2) are combined by concatenation of trajectories: At singularities of \mathcal{S}_0 one allows to concatenate fast and slow trajectory segments. This singular limit analysis is usually the guiding starting point for further analysis. This is the content of section 3 and 4 in this article. We now come to the hyperbolic umbilic singularity, the choice of (1.1) and its motivation.

2.2. Takens' program. As pointed out above, one is interested in singularities of the critical manifold $\mathcal{S}_0 = \{\nabla_z V(z, \alpha) = 0\}$. The question which “types” of singularities of \mathcal{S}_0 generically occur is in fact answered by catastrophe theory. For (2.1) with $\alpha \in \mathbb{R}^r$, $r \leq 5$, we generically have that the singularities of \mathcal{S}_0 are Thom's seven elementary catastrophes plus four additional catastrophes of codimension 5, see [22]. In the article [29], Takens introduced the notion of “constrained differential equations” and analyzed their behavior. We refer to appendix A.1 for Takens' precise definition. In fast-slow terminology, Takens roughly studied the singular limit of (2.1), in particular (2.2), and analyzed the effect of singularities of \mathcal{S}_0 on the slow flow restricted to the attracting part of the critical manifold. Takens carried out a local classification of the slow flow near singularities up to topological equivalence for $r \leq 2$. Furthermore, Takens provided a list of “normal forms” for the singular limit of (2.1) near singularities for $r \leq 3$. We refer to this as “Takens' program”, which is in fact based on catastrophe theory, and serves as motivation for our analysis.

2.3. Catastrophe theory framework. It will be convenient to briefly recall notions from catastrophe theory, for details we refer to [22, 8, 1].

We begin with some notations: Let \mathcal{E}_n denote the local ring of germs (of smooth functions) at $0 \in \mathbb{R}^n$. Let \mathfrak{m}_n denote the ideal of germs vanishing at 0. It follows that \mathfrak{m}_n is the maximal ideal in \mathcal{E}_n , which means that there does not exist a proper ideal of \mathcal{E}_n strictly containing \mathfrak{m}_n . Let \mathfrak{m}_n^k denote the set of germs $f \in \mathfrak{m}_n$ such that the $(k-1)$ -jet of f vanishes, that is $\mathfrak{m}_n^k = \{f \in \mathfrak{m}_n \mid j^{k-1} f = 0\}$. Let $\Delta(f)$ denote the Jacobi ideal of $f \in \mathfrak{m}_n^2$, which is the ideal generated by the partial derivatives of f over \mathcal{E}_n , formally $\Delta(f) = \langle \partial_1 f, \dots, \partial_n f \rangle_{\mathcal{E}_n}$.

Next, let the codimension of f be defined by the dimension of $\mathfrak{m}_n/\Delta(f)$ as an \mathbb{R} -vector space. Further, two germs $f, g \in \mathfrak{m}_n$ are right-equivalent, if they agree up to a local coordinate change, that is, there exists a local diffeomorphism ψ with $\psi(0) = 0$ such that $f = g \circ \psi$.

Finally, let us recall the notion of unfoldings. An r -parameter unfolding F of $f \in \mathfrak{m}_n$ is a germ of a smooth function $F: \mathbb{R}^n \times \mathbb{R}^r \rightarrow \mathbb{R}$, for which $F(\cdot, 0) = f$. An r -parameter unfolding F of f is *versal*, if any other s -parameter unfolding G of f can be expressed in terms of F (“ G is induced by F ”), that is, there exist smooth germs $\phi: (\mathbb{R}^{n+s}, 0) \rightarrow (\mathbb{R}^{n+r}, 0)$, $\bar{\phi}: (\mathbb{R}^s, 0) \rightarrow (\mathbb{R}^r, 0)$, $\mu: (\mathbb{R}^s, 0) \rightarrow (\mathbb{R}, 0)$ such that $\phi(\cdot, 0) = \text{id}_{\mathbb{R}^n}$, $\text{pr}_2 \circ \phi = \bar{\phi} \circ \text{pr}_2$ and $G = F \circ \phi + \mu \circ \text{pr}_2$. Here we denote by pr_2 the natural projection onto the unfolding parameters, which discards the spatial variables. Two r -parameter unfoldings F, G of f are isomorphic, if G is induced by F and ϕ and $\bar{\phi}$ are local diffeomorphisms. In fact, a germ $f \in \mathfrak{m}_n^2$ has a versal unfolding if and only if its codimension is finite [22]. A versal unfolding with r -parameters is universal, if the number r is minimal. In fact, the minimal r is given by the codimension of f . We point out that universal unfoldings of a fixed germ f are not unique, for an example refer to appendix A.3. Further, note that universal unfoldings are also called miniversal unfoldings in slightly different contexts.

A fundamental result in catastrophe theory is the classification up to right-equivalence of germs in \mathfrak{m}_n^2 with low codimension. Here we present a version of this result up to codimension 4, which is also known under the name “Thom’s seven elementary catastrophes”. Recall that a germ f has a non-degenerate critical point at the origin if $\nabla f(0) = 0$ and the Hessian $D^2 f(0)$ is invertible. Recall as well that such f with non-degenerate critical point can locally be written in simple form by the Morse Lemma.

Theorem 2.1 (Thom’s seven elementary catastrophes [22, 1, 4]). *Let $f \in \mathfrak{m}_n^2$ with codimension ≤ 4 . Then f is either non-degenerate (i.e. Morse) or right-equivalent to one of the following germs, up to sign and up to addition of a non-degenerate quadratic form in other variables.*

<i>catastrophe</i>	<i>germ</i>	<i>universal unfolding</i>	<i>codimension</i>
<i>fold</i>	x^3	$x^3 + ax$	1
<i>cuspl</i>	x^4	$x^4 + ax^2 + bx$	2
<i>swallowtail</i>	x^5	$x^5 + ax^3 + bx^2 + cx$	3
<i>hyperbolic umbilic</i>	$x^3 + y^3$	$x^3 + y^3 + axy + bx + cy$	3
<i>elliptic umbilic</i>	$x^3 - xy^2$	$x^3 - xy^2 + a(x^2 + y^2) + bx + cy$	3
<i>butterfly</i>	x^6	$x^6 + ax^4 + bx^3 + cx^2 + dx$	4
<i>parabolic umbilic</i>	$x^2y + y^4$	$x^2y + y^4 + ax^2 + by^2 + cx + dy$	4

TABLE 1. The seven elementary catastrophes. The third column contains a choice of universal unfolding for each corresponding germ in the second column.

Remark 2.2. *Note that Theorem 2.1 says that for any germ f with $f(0) = 0, \nabla f(0) = 0$ and codimension ≤ 4 , we can find local coordinates (x_1, \dots, x_n) centered at the origin such that f is of the form*

$$\pm h(x_1, \dots, x_k) + \sum_{i>k} \pm x_i^2, \quad (2.4)$$

where either h is (up to right-equivalence) a germ appearing in the above list with $k \in \{1, 2\}$, or $h = 0$ and $k = 0$.

2.4. Singularities of (2.1). In view of the above discussion, we propose the following natural definition to classify singularities of (2.1). More precisely, we want to classify the singularities of the critical manifold $\mathcal{S}_0 = \{\nabla_z V(z, \alpha) = 0\}$ of (2.1) without considering any dynamics. Due to the scope of this paper, we are particularly interested in the catastrophes of codimension 3, especially the hyperbolic umbilic singularity.

Definition 2.3 (Singularities of the critical manifold of (2.1)). *Consider the fast-slow system (2.1) with $z \in \mathbb{R}^n, \alpha \in \mathbb{R}^r$. We say that (2.1) has a $*$ singularity at $(\bar{z}, \bar{\alpha}) \in \mathcal{S}_0$, if \mathcal{S}_0 is*

locally an r -dimensional manifold at $(\bar{z}, \bar{\alpha})$ and the germ $f_{\bar{z}, \bar{\alpha}} := V(\cdot + \bar{z}, \bar{\alpha}) - V(\bar{z}, \bar{\alpha}) \in \mathfrak{m}_n^2$ is right-equivalent to

$$\pm h(z_1, \dots, z_k) + \sum_{i>k} \pm z_i^2, \quad (2.5)$$

where h is a germ for the $*$ catastrophe in table 1, and $k \in \{1, 2\}$ accordingly.

To this definition, we shall give the following remarks.

Remark 2.4.

- (a) Definition 2.3 does not concern dynamics on or nearby \mathcal{S}_0 .
- (b) Normally hyperbolic points of \mathcal{S}_0 correspond to germs $f_{\bar{z}, \bar{\alpha}}$ that are non-degenerate (i.e. Morse).
- (c) If $r \leq 4$, for generic V the critical manifold is indeed r -dimensional and the only singularities that occur are the catastrophes in table 1 [22]. A similar statement holds for $r \leq 5$ by extending table 1 to codimension 5 catastrophes [1, 22].
- (d) The number k is given by the corank of $f_{\bar{z}, \bar{\alpha}}$, which is the corank of its Hessian at the origin.
- (e) Definition 2.3 does not cover non-degeneracy conditions for the singularity. An example for this is the 1-d choice $V(z) = z^3$, which leads to the set $\{z = 0\}$ of fold singularities.
- (f) In the case that a choice of V leads to a transcritical singularity or a pitchfork singularity, the critical manifold fails to be a manifold near the singularities.
- (g) Finite determinacy of germs can lead to conditions for singularities on the level of jets. An example for this is the hyperbolic umbilic germ $f = x^3 + y^3$. One can check that this germ is 3-determined, which implies that any choice of V , such that $f_{\bar{z}, \bar{\alpha}}$ has the same 3-jet as f and the critical manifold is indeed locally a manifold, gives a hyperbolic umbilic singularity at $(\bar{z}, \bar{\alpha})$. Note that one can also take the 3-determined germ $f = x^3 + xy^2$ to represent the hyperbolic umbilic singularity, among many other possible choices.

2.5. The choice of (1.1). We are interested in the hyperbolic umbilic singularity, which is the first catastrophe in table 1, for which two fast directions are needed. Our choice is motivated by Takens' work [29, (4.10)] and we will argue similarly to Takens by giving the following "derivation" for the fast-slow system (1.1):

Let us call $V(z, \alpha)$ from (2.1) a (parameter-dependent) potential. The catastrophes of codimension 3 from Theorem 2.1 occur at isolated points for generic 3-parameter potentials. Thus we restrict to (2.1) with $\alpha = (a, b, c) \in \mathbb{R}^3$. There exists an open and dense set of potentials \mathcal{J} , such that $V \in \mathcal{J}$ in fact gives a versal unfolding of the singularity it unfolds and the critical manifold is indeed a 3-d critical manifold [22, (9.17)]. Assume that such generic potential V unfolds a hyperbolic umbilic singularity at the origin. Then the germ $f := V(\cdot, 0) - V(0, 0) \in \mathfrak{m}_n^2$ is right-equivalent to $g := x^3 + y^3 + \sum_{2 < j \leq n} \pm z_j^2$, where z_j are, if $n > 2$, the remaining coordinates of \mathbb{R}^n . In other words, we have $g = f \circ \psi$ for a local diffeomorphism ψ . We remark that at this point we make a choice, namely, we choose the germ $x^3 + y^3$ to represent the hyperbolic umbilic. This choice is not canonical and it influences the family of fast subsystems we eventually obtain. For now we neglect the fast dynamics, but we will argue for our choice after Remark 2.6.

The potential V is (up to shift) a versal unfolding of f . A (uni-)versal unfolding of g is given by $G := x^3 + y^3 + axy + bx + cy + \sum_{2 < j \leq n} \pm z_j^2$. Then one can check that $V(\psi(\cdot), \cdot)$ is a versal unfolding of g and any two versal unfoldings with the same number of parameters are isomorphic [22]. Thus there exists a local diffeomorphism $\phi: (\mathbb{R}^{n+3}, 0) \rightarrow (\mathbb{R}^{n+3}, 0)$ with $\phi(\cdot, 0) = \psi$ and $\text{pr}_2 \circ \phi = \bar{\phi} \circ \text{pr}_2$ for a local diffeomorphism $\bar{\phi}: (\mathbb{R}^3, 0) \rightarrow (\mathbb{R}^3, 0)$, such that $V = G \circ \phi$ up to addition of a constant and a smooth function in the parameters. Hence there are local coordinates $(x, y, z_3, \dots, z_n, a, b, c)$ centered at 0, such that the critical manifold

$\{\nabla_z V = 0\} \subset \mathbb{R}^{n+3}$ of (2.1) near the origin can be written as the set satisfying:

$$\begin{aligned} 0 &= 3x^2 + ay + b \\ 0 &= 3y^2 + ax + c \\ 0 &= \pm 2z_3 \\ &\vdots \\ 0 &= \pm 2z_n. \end{aligned} \tag{2.6}$$

For the slow subsystem the z_j directions are irrelevant. In the family of fast subsystems the z_j directions are hyperbolic directions. Thus, by a center manifold argument, we discard the z_j variables. This leads to the slow subsystem we are interested in. Again, we point out that this slow subsystem choice is carried out similarly in Takens' program [29]. Finally, we take the fast-slow system of the form (2.1), which contains the above constructed slow subsystem.

For convenience we scale the germ $x^3 + y^3$ to $x^3/3 + y^3/3$ and hence consider a universal unfolding given by $V := x^3/3 + y^3/3 + axy + bx + cy$. This leads to a fast-slow system (1.1), i.e. a fast-slow system of the form

$$\begin{aligned} \varepsilon \dot{x} &= x^2 + ay + b + O(\varepsilon) \\ \varepsilon \dot{y} &= y^2 + ax + c + O(\varepsilon) \\ \dot{a} &= g_a(x, y, a, b, c, \varepsilon) \\ \dot{b} &= g_b(x, y, a, b, c, \varepsilon) \\ \dot{c} &= g_c(x, y, a, b, c, \varepsilon), \end{aligned} \tag{2.7}$$

which precisely has, at the origin, a hyperbolic umbilic singularity in the sense of Definition 2.3.

Remark 2.5. *By replacing the potential V by $-V$, the attracting and repelling regions of the critical manifold swap their stability types. Therefore, on the level of fast-slow systems of the form (2.1), the sign in front of V cannot be neglected in general. An example for this is the cusp [5, 13]. However, for the hyperbolic umbilic singularity, the setting of $-V$ is analogous to V , as will be apparent from the upcoming analysis.*

We remark that so far we neglected the effect of the above constructed local diffeomorphism on the fast dynamics, i.e. the family of fast subsystems (2.3), which is a family of gradient vector fields. First, an important remark about the fast subsystems of (2.7) seems to be in order. This remark addresses the fact that versality in the sense of catastrophe theory on the level of potential functions does in general not give versality in the sense of unfoldings of vector fields, i.e. on the level of flows of the associated gradient fields.

Remark 2.6 (Versality in the sense of unfoldings of vector fields). *The family of gradient systems $Z_{a,b,c}$ given by the family of fast subsystems of (2.7), which is precisely the planar family (4.1), does not give a versal unfolding of the vector field $Z_{0,0,0} = x^2\partial_x + y^2\partial_y$ in the space of (gradient) vector fields. Here, we mean by versal unfolding of a vector field the following: Let X_0 be a vector field, then we call a family of vector fields X_μ containing X_0 versal if for any other family Y_λ containing X_0 , there exist a homeomorphism $h(\lambda) = \mu$ between the parameter spaces and a family of homeomorphisms H_λ , not necessarily depending continuously on λ , such that Y_λ is topologically equivalent to $X_{h(\lambda)}$ via H_λ . The latter is also known as fiber topological equivalence [21]. The proof that $Z_{a,b,c}$ does not give a versal unfolding of $Z_{0,0,0}$ is given by Guckenheimer in [10]: small perturbations of $Z_{0,0,0}$ in the space of gradient vector fields produce phase portraits, which contain a saddle connection. However, the family $Z_{a,b,c}$ does not contain phase portraits with saddle connections, refer to the analysis of the fast subsystem in section 4. Thus $Z_{a,b,c}$ cannot be versal in the above sense. In [10] Guckenheimer uses the name ‘‘universality’’ instead of versality. The minimal number of parameters for a versal unfolding of $Z_{0,0,0}$ in the above sense is 4, and a proof of*

this claim is contained in [30]. The notion of a versal unfolding of a vector field given above is called an almost universal unfolding in [30], where “almost” refers to the fact that in this case H_λ does not depend continuously on λ in a neighborhood of the origin.

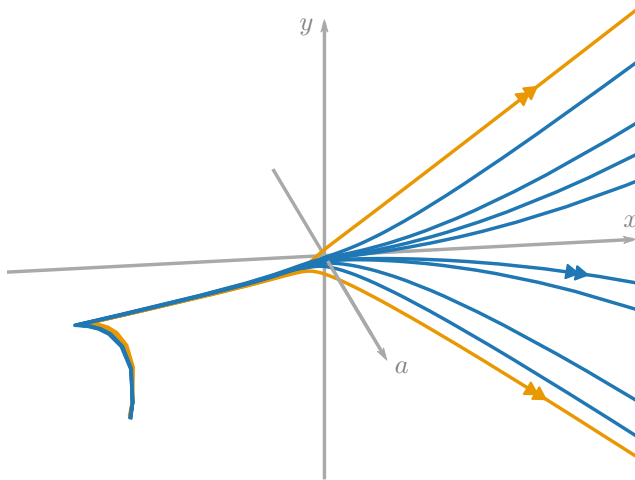


FIGURE 3. Numerical integration of (1.3) with $z = (x, y)$, $\alpha = (a, b, c)$, $V = y^3/3 + yx^2 + a(y^2 - x^2) + bx + cy$ and $g = (-1, 1, 2)$. According to Definition 2.3, this system has a hyperbolic umbilic singularity at the origin, but corresponds to a different choice of germ representing a hyperbolic umbilic catastrophe, i.e. $y^3/3 + yx^2$, and an associated universal unfolding (see appendix A.3). Ten initial conditions close to the point $(x, y, a, b, c) = (-1.5, -2, 1, -2, -1)$ were chosen, which fan out near the origin. Similar to figure 1, most randomly sampled initial conditions close to this point would escape close to the two orange trajectories. Besides that, the fast approach towards attracting slow manifolds is visible here: the transition from fast to slow dynamics happens at the kink on the left.

In the above derivation in section 2.5, we have chosen the germ $x^3 + y^3$ (up to scaling) to represent the hyperbolic umbilic catastrophe, which leads to (2.7), which is identical to (1.1). As already mentioned, in general this choice might affect the fast dynamics, more precisely the fast subsystems. For example, we could have chosen the germ $y^3 + yx^2$, which is right-equivalent to $x^3 + y^3$, see e.g. appendix A.3. However, the associated gradient fields to these germs are not topologically equivalent: $x^2\partial_x + y^2\partial_y$ has two hyperbolic sectors and two parabolic sectors, whereas $2xy\partial_x + (3y^2 + x^2)\partial_y$ only has two hyperbolic sectors and no parabolic sectors [28, 17]. However, by a result from Khesin [17, 16], the family of fast subsystems of our chosen system (2.7) is generic in the following sense:

In [17] it is shown that there are seven three-parameter families of gradient fields, whose fiber topological equivalence classes (up to orientation of the orbits) form an open and dense set in the space of three-parameter families of gradient fields, which have a degenerate critical point at the origin. Three out of these seven families unfold a hyperbolic umbilic singularity in the sense of catastrophe theory. From the upcoming analysis in section 4 it follows that the family of fast subsystems of (2.7) is equivalent to family number 5 of Theorem 1 in Khesin’s article [17], since they topologically have the same bifurcation set and do not show saddle connections. Also see Corollary 2 in [16]. The main point is that the fast subsystem of (2.7) is generic in the set of three-parameter families of gradient fields, which have a degenerate critical point at the origin. Further, the phase portraits in Khesin’s family number 6 and family number 5 in [17] differ only at parameter values zero. At vanishing parameters, the phase portraits of these two families differ precisely in the way the phase portraits of the gradients of the germs $x^3 + y^3$ and $y^3 + yx^2$ above differ. Hence, the analysis of the fast subsystem in section 4 for parameters $(a, b, c) \neq (0, 0, 0)$ does also apply to Khesin’s family number 6. Moreover, we conjecture that the choice of family number 6 (i.e. choosing the

germ $y^3 + yx^2$ and a corresponding universal unfolding for V) would still result in a fanning-out behavior similar to Theorem 1.1. This conjecture is supported by a numerical simulation of (1.3) with potential $V = y^3/3 + yx^2 + a(y^2 - x^2) + bx + cy$, shown in figure 3. Also see appendix A.3 for this potential.

In summary, in this section we have argued and justified the choice of (1.1) as an appropriate representative of any fast-slow system of the form (1.3) near an isolated hyperbolic umbilic singularity. We now proceed with the singular limit analysis of the fast-slow system (1.1), or equivalently (1.2).

3. CRITICAL MANIFOLD AND SLOW FLOW

The slow subsystem of (1.1) is given by

$$\begin{aligned} 0 &= x^2 + ay + b \\ 0 &= y^2 + ax + c \\ \dot{a} &= g_a(x, y, a, b, c, 0) \\ \dot{b} &= g_b(x, y, a, b, c, 0) \\ \dot{c} &= g_c(x, y, a, b, c, 0). \end{aligned} \tag{3.1}$$

3.1. The critical manifold and its singularities. The 3-dimensional critical manifold \mathcal{S}_0 of (1.1) is given by

$$\mathcal{S}_0 = \{x^2 + ay + b = 0, y^2 + ax + c = 0\}. \tag{3.2}$$

Since the fast variables of (1.1) are of gradient type, the singular points of \mathcal{S}_0 are precisely the points, where $D^2 V$ is not invertible. Explicitly, these are all points on \mathcal{S}_0 , where

$$\det D^2 V = \det \begin{bmatrix} 2x & a \\ a & 2y \end{bmatrix} = 4xy - a^2 = 0. \tag{3.3}$$

The projection of the 3-manifold \mathcal{S}_0 onto the ambient (x, y, a) -coordinates is a bijection. This gives a parametrization for \mathcal{S}_0 , given by

$$\Psi: \mathbb{R}^3 \rightarrow \mathbb{R}^5, (x, y, a) \mapsto (x, y, a, b = -x^2 - ay, c = -y^2 - ax). \tag{3.4}$$

Via Ψ the singular points (3.3) appear as a double-cone, see figure 4. The following Lemma establishes the different stability regions $\mathcal{S}_0^a, \mathcal{S}_0^r, \mathcal{S}_0^s$ of \mathcal{S}_0 with respect to the fast subsystem.

Lemma 3.1. *The critical manifold \mathcal{S}_0 has three connected normally hyperbolic parts, each having different stability properties with respect to the fast subsystem: An attracting part \mathcal{S}_0^a , a saddle-type part \mathcal{S}_0^s and a repelling part \mathcal{S}_0^r . Via Ψ , such regions are given by $\mathcal{S}_0^a = \{4xy - a^2 > 0, x + y < 0\}$, $\mathcal{S}_0^r = \{4xy - a^2 < 0\}$, and $\mathcal{S}_0^s = \{4xy - a^2 > 0, x + y > 0\}$.*

Proof. Recall that $D^2 V$ can be viewed as linearization of the fast subsystem. The regions are determined by the signs of the eigenvalues of $D^2 V$, which are determined by the signs of $\det D^2 V = 4xy - a^2$ and $\text{tr} D^2 V = 2(x + y)$. This directly leads to the above regions. \square

We now aim to classify the singular points of \mathcal{S}_0 . Let π be the natural projection $\pi: (x, y, a, b, c) \mapsto (a, b, c)$ onto the parameter subspace. Then $\tilde{\pi} = \pi \circ \Psi$,

$$\tilde{\pi}: (x, y, a) \mapsto (a, -x^2 - ay, -y^2 - ax) \tag{3.5}$$

gives a smooth map $\mathcal{S}_0 \rightarrow \mathbb{R}^3$ of 3-manifolds, which is the restriction of π to \mathcal{S}_0 in the coordinates Ψ from (3.4). In catastrophe theory $\tilde{\pi}$ is called the catastrophe map. The singular points of \mathcal{S}_0 correspond to singularities of $\tilde{\pi}$, that is, points at which the linearization is singular. Further, the set of singular points is generically stratified into submanifolds of singularities of the same type [2, 3, 34, 12]. In our case, the singular points are stratified as follows:

Lemma 3.2. *Besides the hyperbolic umbilic singularity at the origin, there is a line of cusp points given by $x = y = a/2$, and the remaining surface $4xy - a^2 = 0$ consists of fold points.*

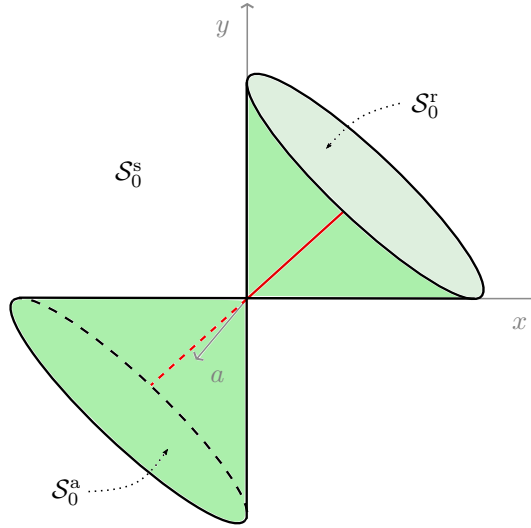


FIGURE 4. Stratification of \mathcal{S}_0 : The critical manifold itself corresponds to the ambient (x, y, a) -space. The double cone determined by (3.3) represents the singular points of the critical manifold \mathcal{S}_0 via Ψ , compare to figure 2. The hyperbolic umbilic singularity is located at the origin. The 2-manifold given by the double cone away from the origin consists of a surface of fold points (green) and a line of cusp points (red). The attracting region \mathcal{S}_0^a is the region inside the lower cone, the repelling region \mathcal{S}_0^r is the region inside the upper cone.

Proof. Let $\Sigma^1(\tilde{\pi})$ be the set of points, where the kernel of $d\tilde{\pi}$ has dimension 1. Then $\Sigma^1(\tilde{\pi})$ is precisely the 2-manifold determined by $4xy - a^2 = 0$ and $(x, y) \neq 0$. Folds and cusps are contained in there. Restrict $\tilde{\pi}$ to $\Sigma^1(\tilde{\pi})$. Then $d\tilde{\pi}|_{\Sigma^1(\tilde{\pi})}$ has a one dimensional kernel along $x = y = a/2$, which gives the cusp line. The situation is depicted in figure 4. The notation $\Sigma^1(\cdot)$ refers to the (Thom-Boardman) symbols Σ^I , refer to [2, 3]. \square

3.2. The slow flow and its desingularization. The right hand side g of the slow subsystem (3.1) determines dynamics on the critical manifold \mathcal{S}_0 through the equations (3.2) of the critical manifold. This dynamics is referred to as slow flow on \mathcal{S}_0 . The goal of this section is to analyze this slow flow, more precisely its interaction with the hyperbolic umbilic point. Since we are intuitively dealing with a flow on the manifold \mathcal{S}_0 , the natural approach would be to use charts and push forward this flow to euclidean space. However, we point out that the slow flow is in general not a flow in the usual sense of vector fields on manifolds. In (3.1), the behavior of the slow flow is linked to the geometry of \mathcal{S}_0 . More precisely, away from the singular points of \mathcal{S}_0 , the slow flow behaves like a usual flow of a vector field on \mathcal{S}_0 . But this behavior breaks down at the singular points of \mathcal{S}_0 . For example, existence and uniqueness of integral curves might fail at singular points. The procedure to obtain integral curves for the slow flow on \mathcal{S}_0 was carried out by Takens in [29]. Let us outline this important procedure.

Recall that g is the right hand side of (3.1). A natural idea is to “transform” g via $\tilde{\pi}: \mathcal{S}_0 \rightarrow \mathbb{R}^3$, which maps points in \mathcal{S}_0 parametrized by (x, y, a) via Ψ to their parameters (a, b, c) . Note that $\tilde{\pi}$ is a local diffeomorphism away from the singular points, which allows to pullback g . In fact, one can “lift” the slow flow g to a vector field Y on \mathcal{S}_0 formally given by

$$Y = \det d\tilde{\pi} \cdot (d\tilde{\pi})^{-1}g(x, y, \tilde{\pi}(x, y, a)), \quad (3.6)$$

although $d\tilde{\pi}$ is not invertible everywhere. The reason for this is that by Cramer’s rule

$$(d\tilde{\pi})^{-1} = \frac{1}{\det d\tilde{\pi}} \text{adj}(d\tilde{\pi}), \quad (3.7)$$

where $\text{adj}(\cdot)$ denotes the transpose of the cofactor matrix. Therefore (3.6) reduces to $Y = \text{adj}(d\tilde{\pi})g(x, y, \tilde{\pi}(x, y, a))$. Away from singular points of \mathcal{S}_0 , Y is just a rescaling of the vector

field $(d\tilde{\pi})^{-1}g$. The factor $\det d\tilde{\pi} = 4xy - a^2$ serves as desingularization, which leads to the term “desingularized slow flow” for Y . Note that Y is smooth if g is. It is important to note that by the desingularization factor $\det d\tilde{\pi}$ in (3.6), the orientation of trajectories is reversed in the region $\{\det d\tilde{\pi} < 0\}$. This time reversion in the region $\{\det d\tilde{\pi} < 0\}$ on \mathcal{S}_0 must not be neglected, since only after this we obtain the correct flow on the critical manifold away from singular points. For illustrating examples of the desingularization procedure (3.6) we refer to [12].

In our case of the hyperbolic umbilic, (3.6) leads to the following desingularized slow flow:

Lemma 3.3. *The desingularized slow flow Y for (1.1) is given by*

$$\begin{aligned}\dot{x} &= a\tilde{g}_c(x, y, a) - 2y\tilde{g}_b(x, y, a) + (xa - 2y^2)\tilde{g}_a(x, y, a) \\ \dot{y} &= a\tilde{g}_b(x, y, a) - 2x\tilde{g}_c(x, y, a) + (ya - 2x^2)\tilde{g}_a(x, y, a) \\ \dot{a} &= (4xy - a^2)\tilde{g}_a(x, y, a),\end{aligned}\tag{3.8}$$

where the time orientation of trajectories needs to be reversed in the saddle-type region $\mathcal{S}_0^s = \{4xy - a^2 < 0\}$. The notation $\tilde{g}_\bullet(x, y, a) = g_\bullet(x, y, \tilde{\pi}(x, y, a), 0) = g_\bullet(x, y, a, -x^2 - ay, -y^2 - ax, 0)$ is used.

Proof. Computation of $\text{adj}(d\tilde{\pi})g(x, y, \tilde{\pi}(x, y, a), 0)$. The region $\{\det d\tilde{\pi} < 0\}$ where the orientation of trajectories must be reversed coincides with the saddle-type region \mathcal{S}_0^s . \square

Recall that Y induces dynamics on \mathcal{S}_0 . For the remainder of this article we let Y denote the vector field (3.8), where the orientation of integral curves in \mathcal{S}_0^s , i.e. outside of the double cone of singularities, is reversed.

We are interested in the dynamics close to the hyperbolic umbilic singularity, located at the origin. From now on, let

$$A_0 := g_a(0), B_0 := g_b(0), C_0 := g_c(0).\tag{3.9}$$

Note that Y (3.8) has an equilibrium at the origin. We can classify this equilibrium of Y according to the following lemma. This is similar to [12].

Lemma 3.4. *For any $g = (g_a, g_b, g_c)$ in (1.1), the origin is a non-hyperbolic equilibrium of the desingularized slow flow Y (3.8). The linearization at the origin has eigenvalues 0 and $\pm 2\sqrt{B_0C_0}$. Taking the time reversion in \mathcal{S}_0^s into account, the local behavior near the origin is characterized by the following arrangement of invariant manifolds according to the values of B_0 and C_0 :*

$B_0, C_0 > 0$: two 1-d stable manifolds and 1-d center manifold

$B_0, C_0 < 0$: two 1-d unstable manifolds and 1-d center manifold

B_0, C_0 have opposite sign : 1-d center manifold and rotational behavior around it

$B_0 = 0$ or $C_0 = 0$: degenerate case

The first three cases occur generically.

Proof. The linearization at the origin is given by

$$\begin{bmatrix} 0 & -2B_0 & C_0 \\ -2C_0 & 0 & B_0 \\ 0 & 0 & 0 \end{bmatrix},\tag{3.10}$$

which has the eigenvalues $0, \pm 2\sqrt{B_0C_0}$. In the case $B_0, C_0 > 0$ the eigenvalues and corresponding eigenvectors of the linearization are given by

$$0: \left(\frac{B_0}{2C_0}, \frac{C_0}{2B_0}, 1 \right), -2\sqrt{B_0C_0}: \left(\sqrt{B_0/C_0}, 1, 0 \right), 2\sqrt{B_0C_0}: \left(-\sqrt{B_0/C_0}, 1, 0 \right).\tag{3.11}$$

In the case $B_0, C_0 < 0$ we have

$$0: \left(\frac{B_0}{2C_0}, \frac{C_0}{2B_0}, 1 \right), -2\sqrt{B_0C_0}: \left(-\sqrt{B_0/C_0}, 1, 0 \right), 2\sqrt{B_0C_0}: \left(\sqrt{B_0/C_0}, 1, 0 \right).\tag{3.12}$$

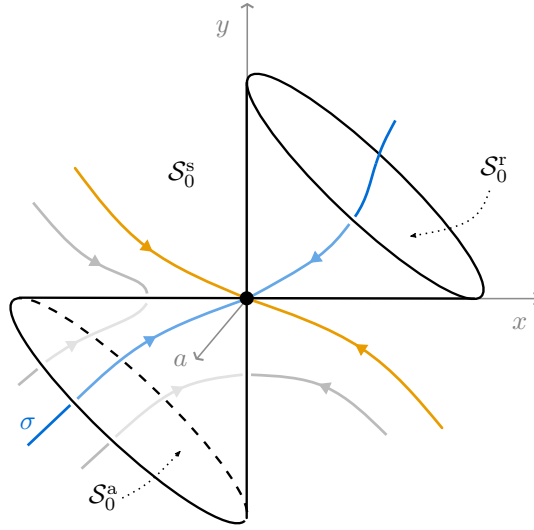


FIGURE 5. Four trajectories of the desingularized slow flow Y (3.8) approach the origin in the case $B_0, C_0 > 0$, which correspond to the 1-d stable/unstable manifolds at the origin before desingularization. In the vicinity of the origin one of such manifolds is contained in \mathcal{S}_0^a (denoted by σ) and \mathcal{S}_0^s , and the other in \mathcal{S}_0^r . Also compare to figure 2, where σ appears as blue trajectory. A 1-d center manifold, not shown, also exists and is tangent to the cone at the origin. In particular, the hyperbolic umbilic is not a funnel in the sense of Takens [29], and thus we shall concentrate on tracking σ and its perturbations. This property of the hyperbolic umbilic is evident in the blow-up analysis. Away from the center manifold, almost all trajectories of the slow flow leave \mathcal{S}_0^a via folds (and cusps), as depicted by means of two trajectories in gray.

Note the swapped eigenvectors corresponding to the latter two nonzero eigenvalues. This is due to the fact that $C_0\sqrt{B_0/C_0} = -\sqrt{B_0C_0}$ if $B_0, C_0 < 0$. In the case $B_0, C_0 > 0$, there exists a 1-d stable manifold contained in $\mathcal{S}_0^a \cup \mathcal{S}_0^s$ close to the origin. Further, there exists a 1-d unstable manifold contained in \mathcal{S}_0^r close to the origin. Due to the time reversion in \mathcal{S}_0^s we obtain four trajectories approaching the origin. See figure 5. Similarly, we obtain four trajectories emanating from the origin if $B_0, C_0 < 0$. Note that in both cases the 1-d center manifolds are “tangent” to the double cone surface at the origin. \square

Lemma 3.4 topologically determines the slow flow dynamics away from the 1-d center manifold(s) in case of $B_0C_0 \neq 0$, close to the origin. Motivated by Lemma 3.4, we shall restrict the forthcoming analysis to the generic sub-case $B_0, C_0 > 0$. The simple reason for this is that the flow on the stable manifolds is directed towards the hyperbolic umbilic point in this case. In contrast, for $B_0, C_0 < 0$ there are no trajectories approaching the hyperbolic umbilic point, except possibly on the center manifold. In the case of B_0, C_0 having opposite sign (i.e. $B_0C_0 < 0$), we obtain a pair of conjugated purely imaginary eigenvalues at the hyperbolic umbilic. To desingularize this case, the approach of [11] might be applicable.

Remark 3.5 (The non-hyperbolic equilibrium of Y at the origin). *For any $g = (g_a, g_b, g_c)$ in (1.1), respectively (3.1), the vector field Y (3.8) has a non-hyperbolic equilibrium at the origin. Roughly speaking, this occurs due to two fast variables in (1.1) with appropriate non-linear right hand side for $\varepsilon = 0$, in our case second order polynomials. The equilibrium at the origin is due to the presence of $\text{adj}(d\bar{\pi})$ in (3.6) and the non-hyperbolicity is due to the quadratic factor $\det d\bar{\pi} = 4xy - a^2$ in the third equation of (3.8), coming from the desingularization process. This is in contrast to the case of folds, cusps and the swallowtails, where the desingularized slow flow is generically nonzero at the most degenerate singularity [29]. In the case of folded singularities [27], the desingularized slow flow generically has a hyperbolic equilibrium at the folded singularity.*

From now on, we assume $B_0, C_0 > 0$, if not stated otherwise. Note that by swapping the coordinate labels $b \leftrightarrow c$ and $x \leftrightarrow y$ in (1.1), we obtain (1.1) with swapped $g_b \leftrightarrow g_c$, $f_x \leftrightarrow f_y$ and permuted first two arguments in these functions. This suggests that in the singular limit the case $g_b = g_c$ might play a special role. Indeed, if $g_b = g_c$ then $\{x = y\}$ is invariant for (3.8). We will see later that the case $B_0 = C_0$ will play a special role on the blow-up sphere.

Let σ denote the unique trajectory, which approaches the origin off the center manifold in forward time from inside \mathcal{S}_0^a sufficiently close to the origin. Refer to figure 5. This trajectory will play an important role in the forthcoming analysis, as we aim to track it through the hyperbolic umbilic.

4. FAST SUBSYSTEM

In this section we analyze the fast subsystem of (1.1): its equilibria, their bifurcations and heteroclinic connections. This analysis is naturally the next step to analyze the singular limit of a fast-slow system. The analysis will show the interplay of the fast fibers with the critical manifold \mathcal{S}_0 , since equilibria of the fast subsystem are instances of \mathcal{S}_0 . Considering (2.3), we have the parameter space (a, b, c) and the planar gradient system

$$\begin{aligned} x' &= x^2 + ay + b \\ y' &= y^2 + ax + c. \end{aligned} \tag{4.1}$$

Let $Z_{a,b,c}(x, y)$ denote the vector field determined by (4.1). This system was already analyzed to certain extent in [10, 30, 17].

4.1. Equilibria and their bifurcations. The vector field $Z_{a,b,c}$ has at most four equilibria, since, for $a \neq 0$, they correspond to intersections of the two parabolas

$$\begin{aligned} 0 &= x^2 + ay + b \\ 0 &= y^2 + ax + c. \end{aligned} \tag{4.2}$$

One parabola is symmetric with respect to the y -axis, the other symmetric with respect to the x -axis. Depending on the sign of a , the parabolas either open right and up or open left and down. For $a = 0$, each of the parabolas degenerates into a pair of parallel lines.

Linearization of $Z_{a,b,c}(x, y)$ gives $\det DZ_{a,b,c}(x, y) = 4xy - a^2$, $\text{tr} DZ_{a,b,c}(x, y) = 2(x + y)$. Note that for each a , these two expressions determine the regions, in which the phase space (x, y) of the vector field $Z_{a,b,c}(x, y)$ contains sinks, sources, saddles or fold/cusp bifurcations. These regions appear precisely as in taking a section $a = \text{const}$ through figure 4. Also refer to figure 6, in which these regions are separated by the green dashed curves.

Thus, there are four generic configurations of the equilibria of $Z_{a,b,c}(x, y)$, which are shown in figure 6: (A) no equilibria, (B) a saddle and a sink, (C) a saddle and a source and (D) two saddles, a source and a sink. Note that (B) can only occur for $a > 0$, while (C) can only occur for $a < 0$. The fold/cusp bifurcations, which happen upon variation of the parameters a, b, c , visually correspond to transitions between the configurations in figure 6. That is, by suitably shifting the parabolas along their symmetry-axes by varying b, c and changing the parabolas' "width" by varying a .

The following symmetries allow to simplify the study of the family of vector fields $Z_{a,b,c}$ and its bifurcations.

Lemma 4.1. *The family of vector fields Z , as defined in (4.1), has the following symmetries:*

- (S1) $Z_{a,c,b} \circ A = A \circ Z_{a,b,c}$, where A is reflection along $x = y$. That is, $Z_{a,b,c}$ and $Z_{a,c,b}$ are conjugate via A . In particular, the stability types of the reflected equilibria remain invariant.
- (S2) $Z_{-a,b,c} \circ R = Z_{a,b,c}$, where R is rotation by 180° . Integral curves of $Z_{-a,b,c}$ are obtained from integral curves of $Z_{a,b,c}$ by rotation via R and time reversion. In particular, under $a \mapsto -a$, the positions of equilibria of Z are rotated by R and saddles remain saddles, but sinks become sources and vice versa.

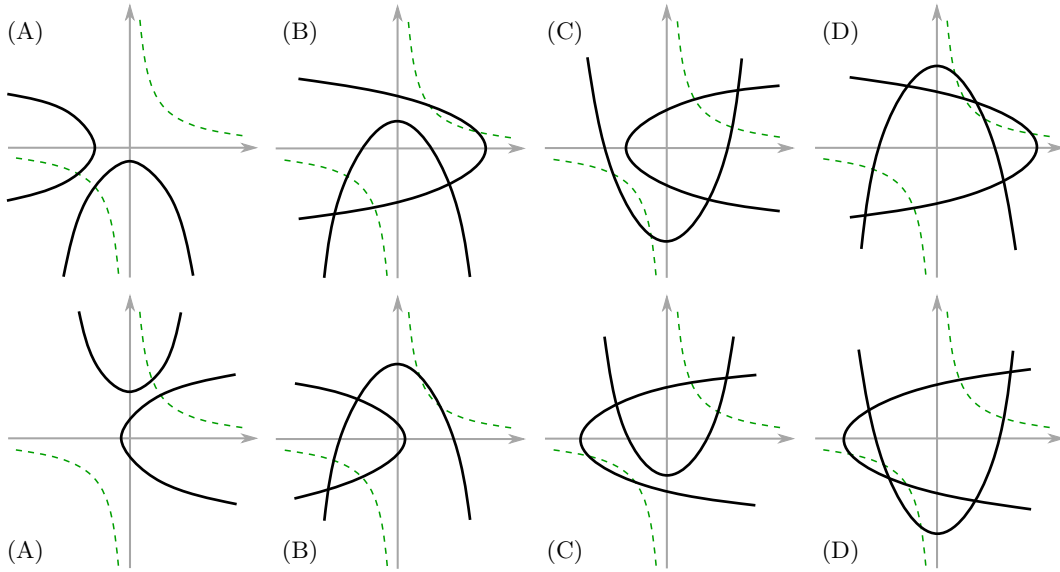


FIGURE 6. The generic intersections of the two parabolas (4.2) give the generic configurations (A),(B),(C),(D) of equilibria of Z . The dashed green curves indicate the fold curves for fixed $|a| > 0$, which divide the plane in three regions: sinks occur in the lower left region, sources in the upper right region, saddles in between.

Proof. (S1) follows by direct computation using $A(x, y) = (y, x)$. For (S2), observe that since R amounts to multiplication by (-1) , $Z_{-a,b,c}$ and $-Z_{a,b,c}$ are conjugate via R , which implies the statements. \square

We will simply refer to the properties (S1) and (S2) from Lemma 4.1 by (S1) and (S2) in the remainder of this section. Note how the symmetries appear in figure 6: switching from the upper configuration (A) to the lower configuration (A) amounts to (S2), similarly for (D). Switching from the upper row to the lower row in case of (B) and (C) amounts to (S1).

Recall the catastrophe set or bifurcation set, which is given by projection of the set of singularities (represented by the double cone) onto the parameter space (a, b, c) . By the symmetries (S1) and (S2) from Lemma 4.1, the bifurcation set in the parameter space is symmetric to the planes $a = 0$ and $b = c$. Furthermore, the bifurcation set is qualitatively well known, see e.g. [26, 4]. This leads to the qualitative bifurcation diagram depicted in figure 7. The bifurcation set divides the parameter space into four connected regions (A),(B),(C) and (D), which correspond to the generic configurations in figure 6. Topological equivalence of the associated vector fields Z in these regions follows from the absence of saddle connections:

Lemma 4.2. *In each of the open and connected regions (A),(B),(C),(D) the family of vector fields $Z_{a,b,c}(x, y)$ has topologically equivalent flows.*

Proof. The vector field $Z_{a,b,c}(x, y)$ is a gradient. By the Andronov-Pontryagin criterion for planar structural stability we only need to consider the region (D), where a saddle to saddle connection is possible. But the vector fields $Z_{a,b,c}(x, y)$ in (D) do not show a saddle connection. This was already shown implicitly in [10]. Here we give an alternative argument, similar to [17]: by [6] saddle connections in quadratic planar gradient systems can only occur along straight lines. Assume that $Z_{a,b,c}(x, y)$ has an invariant straight line $y = mx + d$ connecting the two saddles. Note that necessarily $m < 0$, see figure 6. Then the zero level set of $p(x, y) = y' - mx' = y^2 - mx^2 + a(x - my) + c - mb$ needs to contain $y = mx + d$ for all x -values between the saddles. The gradient ∇p vanishes in a single point, so away from this point $p(x, y) = 0$ determines a smooth curve. If the straight line and $p(x, y) = 0$ locally agree, there would exist a real valued function $\lambda(x)$, such that $\nabla p(x, mx + d) = \lambda(x)[-m \ 1]$,

the latter being the normal vector to the invariant line. This implies $\lambda(x) = 2x - a/m$ and $\lambda(x) = m(2x - a) + 2d$. But due to $m < 0$ this cannot hold, even locally. \square

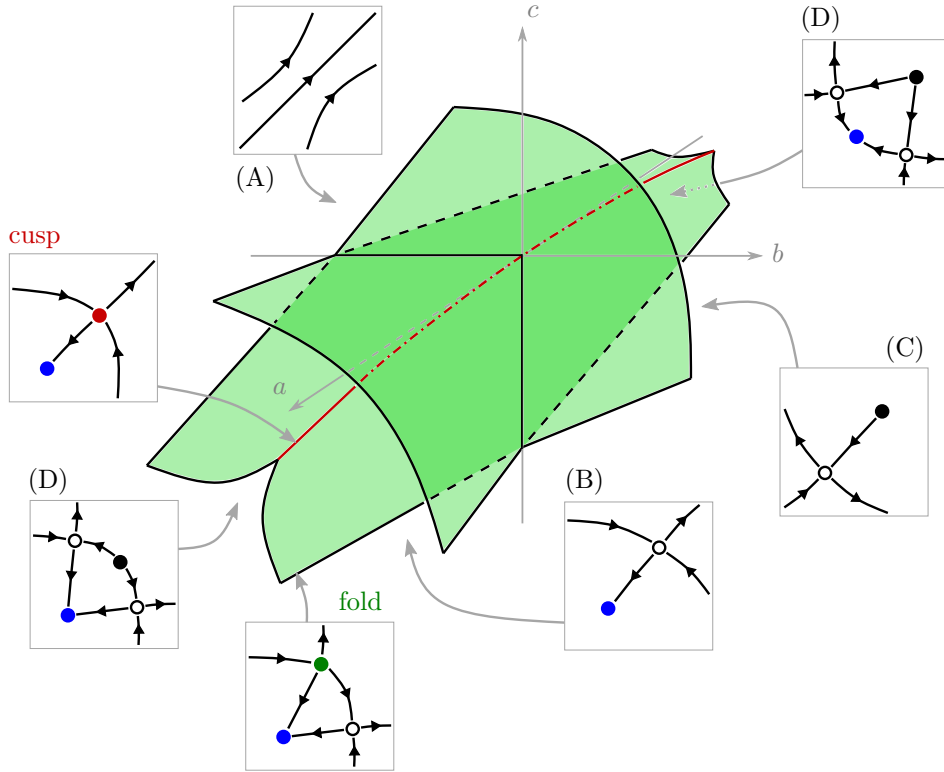


FIGURE 7. Qualitative bifurcation diagram of $Z_{a,b,c}$ (4.1) in parameter space: The bifurcation set is given by projection of the singular points of \mathcal{S}_0 onto (a, b, c) , and corresponds to the hyperbolic umbilic point at the origin, a curve of cusp points (red) and two surfaces of fold points (green), which intersect along the negative b - and c -axes. The phase portraits show the qualitative behavior of the fast subsystem at bifurcations and between them. Folds are shown in green, cusps in red, sinks in blue, sources in black and saddles in white. Naturally, sinks/sources and saddles correspond to points in $\mathcal{S}_0^a/\mathcal{S}_0^r$ and \mathcal{S}_0^s , respectively.

4.2. Feasible jumps. We now aim to combine the analysis of the fast and slow subsystem away from the hyperbolic umbilic singularity. The goal is to see which concatenations of singular candidate trajectories are possible at all, due to the geometry of \mathcal{S}_0 and the fast subsystems. In the following we will focus on the possibilities at jump points, i.e. singular points, where singular candidate trajectories on the critical manifold “jump” onto the fast subsystem. As in lower dimensions, one expects that generic fold points indeed serve as jump points: that is, if the slow flow is transverse to the surface of fold points. Refer to the “normal switching condition” in [20]. In fact, at the fold or cusp points, we can locally employ a center manifold reduction to reduce to one fast variable. This allows to directly apply the existing theory, but only locally. We now aim to obtain global insight. Since the fast subsystem is 2-dimensional and the critical manifold is 3-dimensional, a visualization of feasible jumps as in, e.g. [13], is not directly applicable. Moreover, in case there is only one fast variable present, the effect of the fast subsystem for candidate trajectories is usually straight forward, since it amounts to projection along the fast fibers. Here, the fast fibers are 2-d with dynamics determined by a gradient vector field. The essential question for jumps is: *To which equilibria can the fast subsystem lead candidate trajectories, emanating from a fold or cusp point?*

To answer this question, we need to do consider $Z_{a,b,c}(x, y)$ at parameter values (a, b, c) that correspond to a fold or a cusp bifurcation. We then need to check which candidate trajectories, emanating from these “static” bifurcating equilibria are feasible. That is, we need to check the forward asymptotic behavior of trajectories, which are backward asymptotic to these bifurcation equilibria. Consider the generic configurations (A),(B),(C),(D), which correspond to regions in parameter space as indicated in the bifurcation diagram in figure 7. These regions are separated by the bifurcation set. We call a fold bifurcation of type (A) \leftrightarrow (B), if it occurs in $Z_{a,b,c}$ for parameter variation (a, b, c) from region (A) to region (B) or vice versa. Similarly, we use this notation for all other bifurcations according to the bifurcation diagram in figure 7. Note that if $a = 0$ we can have two simultaneous folds and two “overlapping” cusps at $a = b = c = 0$, i.e. the hyperbolic umbilic.

In the following we will encounter fast candidate trajectories, which “escape” towards infinity. To treat this behavior, let us setup a suitable “exit” section, which solutions pass before escaping. For the vector field $Z_{a,b,c}$ consider the section

$$\Delta^{\text{ex}} = \{x + y = 2\nu\}, \quad (4.3)$$

for $\nu > 0$. Then there is an open neighbourhood K of the origin in parameter (a, b, c) -space, such that all equilibria of $Z_{a,b,c}$ for $(a, b, c) \in K$ are located below Δ^{ex} . This assumption is merely to simplify the following statements. In the scope of the current section $\nu > 0$ can be chosen arbitrarily large. In this setting, we state the following Lemma, which constructs positively invariant regions, in which jumping candidate trajectories need to be contained. Eventually, this leads to feasible jumps along the fast subsystem.

Lemma 4.3. *Consider $Z_{a,b,c}$ at parameter values $(a, b, c) \in K$ contained in the bifurcation set.*

- (a) *If $(a \neq 0, b, c)$ does **not** correspond to a fold or cusp bifurcation (B) \leftrightarrow (D), then any trajectory emanating from the bifurcation equilibrium arrives at the section Δ^{ex} .*
- (b) *If $(a \neq 0, b, c)$ corresponds to a fold or cusp bifurcation (B) \leftrightarrow (D), then every trajectory emanating from the bifurcation equilibrium either arrives at Δ^{ex} or is forward asymptotic to a sink or a saddle.*
- (c) *If $(a = 0, b, c)$: Any trajectory emanating from a fold equilibrium located on the negative x - or y -axis arrives at Δ^{ex} . Any trajectory emanating from a fold equilibrium located on the positive x - or y -axis is forward asymptotic to the simultaneous fold on the negative axis or arrives at Δ^{ex} . Any trajectory of $Z_{0,0,0}$ emanating from the origin arrives at Δ^{ex} .*
- (d) *From Δ^{ex} any trajectory escapes towards infinity and cannot connect to equilibria.*

Proof. This follows by looking at the nullclines of $Z_{a,b,c}$, i.e. the configuration of the parabolas (4.2), for parameters in the bifurcation set. The proof is given by means of figure 8, which contains four qualitative phase portraits, to which all cases except $Z_{0,0,0}$ can be reduced. We explain the reduction and figure 8: Assume $a \neq 0$. Consider the phase portrait of $Z_{a,b,c}$ at parameter values as mentioned above. Due to (S1) from Lemma 4.1, the argument for the folds reduces to four qualitatively distinct phase portraits corresponding to folds (A) \leftrightarrow (B), (A) \leftrightarrow (C), (B) \leftrightarrow (D) and (C) \leftrightarrow (D). By (S2), we can take the phase portrait for a fold (B) \leftrightarrow (D) and obtain the phase portrait for a fold (C) \leftrightarrow (D) by 180° rotation and time reversion. Similarly for folds (A) \leftrightarrow (B) and (A) \leftrightarrow (C). Hence only two fold phase portraits contain all the information we need for the proof. In figure 8, these are the upper two phase portraits: Trajectories being backward asymptotic to the folds (green) must be contained in the violet regions, which are positively invariant. Trajectories being forward asymptotic to the folds must be contained in the orange regions, which are negatively invariant. After time reversion and 180° rotation the orange region corresponds to a positively invariant region containing trajectories emanating from folds in reverse time.

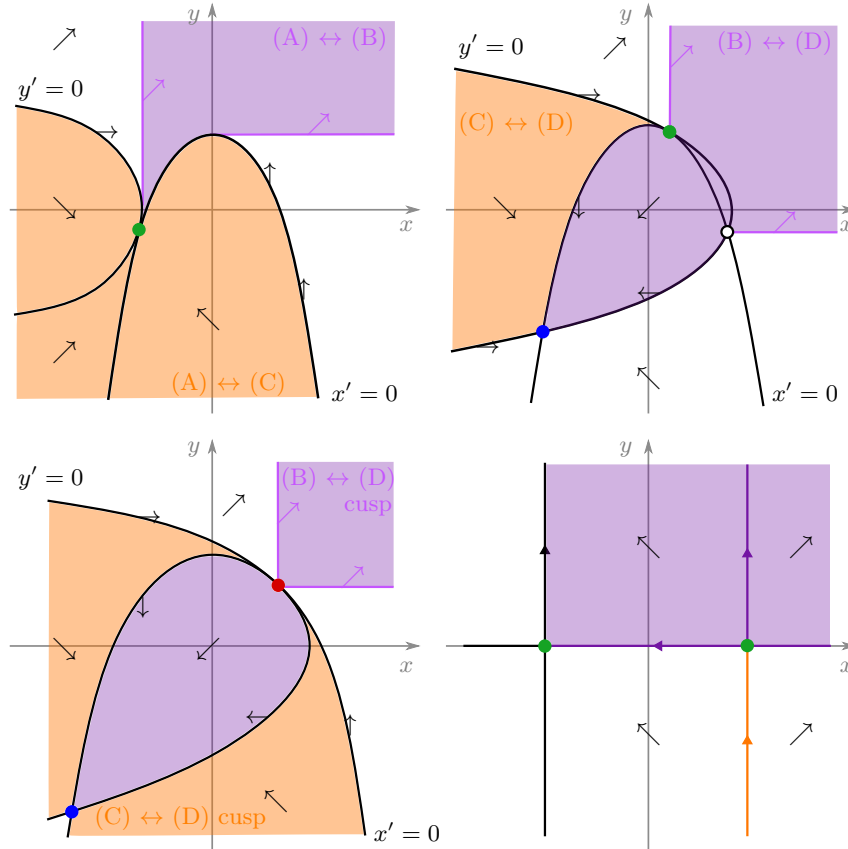


FIGURE 8. Qualitative phase portraits for the proof of Lemma 4.3: Up to (S1) and (S2), these four phase portraits qualitatively contain all relevant phase portraits of $Z_{a,b,c}$ at parameter values contained in the bifurcation set. The black parabolas (which degenerate into straight lines for $a = 0$) are the nullclines, the arrows indicate the signs of x', y' , i.e. the direction of the flow. The violet regions define positively invariant regions, in which trajectories emanating from the bifurcating equilibrium (fold in green or cusp in red) are contained. After time reversion the orange regions play a similar role. Folds are shown in green, cusps in red, sinks in blue, saddles in white. Recall that $\Delta^{\text{ex}} = \{x + y = 2\nu\}$.

The upper left portrait treats folds (A) ↔ (B) with the violet positively invariant region and, via time reversion and 180° rotation (S2), folds of type (A) ↔ (C) with orange. The upper right portrait treats (B) ↔ (D) with the violet positively invariant region and (C) ↔ (D) with the orange negatively invariant region, where again time reversion and 180° rotation (S2) needs to be employed. Recall that (S2) turns a sink into a source. Two additional configurations of the nullclines lead to cusps, which can be reduced again by (S2) to a single phase portrait containing all the necessary information. This is the lower left phase portrait in figure 8. It treats cusps (B) ↔ (D) with the violet region, cusps (C) ↔ (D) with the orange region via (S2). The case $a = 0$ follows similarly, the nullcline parabolas just degenerate into (pairs of) straight lines. The case $Z_{0,0,0}$ follows easily. By (S1), the remaining cases with $a = 0$ can be reduced to a single phase portrait. This is the lower right portrait in figure 8, which does only show the violet and orange regions corresponding to the fold on the right hand side. Note that in $a = 0$ away from the origin two folds happen simultaneously. The case of $Z_{0,0,0}$, which is the fast subsystem at the hyperbolic umbilic itself, follows easily by looking at its phase portrait. Assertion (d) follows from Remark 4.5. \square

Remark 4.4 (The geometry near folds and cusps). *Near the singularly perturbed cusp in dimension three [5, 13] the critical manifold is “S-shaped” due to two intersecting fold curves. Further, along the fold curves, an attracting branch is “folded over” a repelling branch.*

Combined with the “S-shaped” critical manifold, this can lead to singular limit relaxation oscillation type behavior near the cusp. In our case, however, the above analysis shows that at folds a branch of \mathcal{S}_0^s is folded over either a branch of \mathcal{S}_0^a or \mathcal{S}_0^r , given that $a \neq 0$. See for example figure 7. Moreover, at folds near the cusp curves, where \mathcal{S}_0^s is folded over \mathcal{S}_0^a , there are only instances of \mathcal{S}_0^s and \mathcal{S}_0^r present in the fast subsystem, except for the fold itself. Hence, at such folds we can in principle only jump to \mathcal{S}_0^s . These are folds of type (C) \leftrightarrow (D), and Lemma 4.3 shows that the fast subsystem does not allow jumps to \mathcal{S}_0^s . Thus, a direct jump back to \mathcal{S}_0 is impossible, and therefore a singular limit relaxation oscillation behavior as in the 3-d cusp is impossible. In particular, a singular limit return mechanism from \mathcal{S}_0^a back to \mathcal{S}_0^a must go through the hyperbolic umbilic. At folds near the cusp curves, where \mathcal{S}_0^s is folded over \mathcal{S}_0^r , there are only instances of \mathcal{S}_0^s and \mathcal{S}_0^a present in the fast subsystem, except the fold itself. These are folds of type (B) \leftrightarrow (D). Lemma 4.3 does not exclude jumps from these folds to \mathcal{S}_0^s or \mathcal{S}_0^a .

Remark 4.5 (The exit section Δ^{ex} and escape towards infinity). The vector field $Z_{a,b,c}(x, y)$ might not be transverse to Δ^{ex} at isolated points in the regions $x' < 0, y' > 0$ and $x' > 0, y' < 0$. However, this section still works as suitable “exit” section: Any section of the form $\{x = \text{const}\}$ or $\{y = \text{const}\}$ is transverse to the flow in these regions, which we can place “before” Δ^{ex} . Together with Δ^{ex} this then gives “triangular”-like compact regions, which solutions need to exit in finite time through Δ^{ex} by Poincaré-Bendixson. Further, figure 8 shows that in this case trajectories can only escape towards infinity and cannot connect to any equilibrium.

Remark 4.6 (Asymptotic estimates from the positively/negatively invariant regions). By employing the nullclines and the direction of $Z_{a,b,c}(x, y)$ along the diagonal $x = y$, it is easy to obtain forward asymptotic estimates for trajectories, which jump onto the fast subsystem at folds. At folds (C) \leftrightarrow (D), any solution (x, y) emanating from the corresponding fold equilibrium satisfies the following: if $b < c$: $x = O(\sqrt{y})$ while $y \rightarrow +\infty$, if $c < b$: $y = O(\sqrt{x})$ while $x \rightarrow +\infty$. At folds (A) \leftrightarrow (B), any solution (x, y) emanating from the fold equilibrium satisfies: if $b < c$: $y > x$ and $y \rightarrow +\infty$, if $c < b$: $x < y$ and $x \rightarrow +\infty$. In the upcoming blow-up analysis, these estimates can be employed to connect the corresponding orbits on the equator of the blow-up sphere to their “exit equilibria” in the charts $\kappa_{\pm a}$.

Now, the jump possibilities given by Lemma 4.3 can be visualized by attaching the qualitative fast subsystems to the corresponding regions on the double cone from figure 4, which represent the singularities of \mathcal{S}_0 . This is shown in figure 9. In principle, this allows to concatenate candidate trajectories of the singular limit of system (1.1), if we add the desingularized slow flow Y to the figure 9 and respect the statements from Lemma 4.3. An example how to read figure 9 and concatenate trajectories is given in the proof of Proposition 4.7.

Roughly speaking, the feasible jumps from Lemma 4.3 and figure 9 are summarized in the following proposition.

Proposition 4.7 (Feasible singular limit jumps). The fast-slow system (1.1) allows the following singular limit jump behavior via fast segments:

- (a) It is impossible to jump via a fast candidate trajectory from folds or cusps enclosing \mathcal{S}_0^a back onto the critical manifold \mathcal{S}_0 .
- (b) From folds enclosing \mathcal{S}_0^r for $a > 0$, there exist fast candidate trajectories jumping to \mathcal{S}_0^a and \mathcal{S}_0^s .
- (c) From cusps next to \mathcal{S}_0^r for $a > 0$, there exist fast candidate trajectories jumping to \mathcal{S}_0^a .
- (d) From folds enclosing \mathcal{S}_0^i for $a = 0$, there exists a single fast candidate trajectory jumping to a fold enclosing \mathcal{S}_0^a .
- (e) Any other jumping candidate trajectory escapes towards infinity and cannot arrive at \mathcal{S}_0 .

Proof. (a): From figure 6 one can deduce that folds and in particular cusps next to \mathcal{S}_0^a are of type (C) \leftrightarrow (D), in particular for $a < 0$, compare to figure 7. Then Lemma 4.3 implies

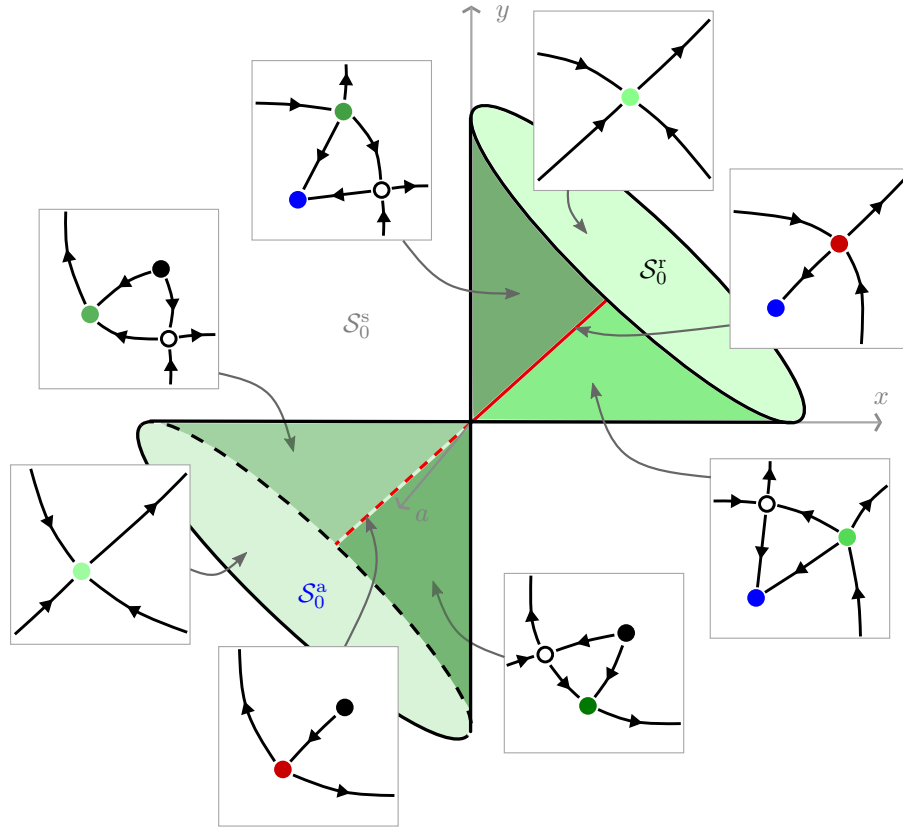


FIGURE 9. Qualitative fast subsystems corresponding to the singularities on the double cone for $a \neq 0$, each fast subsystem phase portrait is attached to a set of singularities on the cone. The a -axis is directed towards the observer. Cusp points are depicted in red, fold points in green, sinks in blue, sources in black and saddles in white. Recall that equilibria appearing in the fast subsystem correspond to instances of $\mathcal{S}_0^a, \mathcal{S}_0^r, \mathcal{S}_0^s$ or singular points of \mathcal{S}_0 , as indicated by the color coding. The phase portraits only contain few heteroclinic connections and “escaping” trajectories, but give a qualitatively complete picture according to Lemma 4.3. For future reference, it is worth noting that the only trajectories that can jump towards \mathcal{S}_0^a are those jumping at the upper cone.

assertion (a), which is clarified by Remark 4.4. In figure 9 this is visualized as follows: Starting inside \mathcal{S}_0^a , we assume that a suitable slow flow brings us to a fold surrounding the attracting region. Then we look to the four attached fast subsystems, which qualitatively show the fast dynamics at these folds or cusps. In each of these portraits we need to follow trajectories emanating from the fold equilibrium (green) or cusp (red), which all escape towards infinity.

(b) and (c): Similarly, this follows from Lemma 4.3 by checking that a heteroclinic connection from the fold equilibrium to the saddle indeed exists. In figure 9, this is visualized by considering the four fast subsystems attached to the upper cone. We see that in this case connections to the sink (blue) and saddle (white) exist for $a > 0$.

(d): This occurs qualitatively in the setting of the lower right phase portrait of figure 8, which is (d) in Lemma 4.3.

(e): In the language of Lemma 4.3 this corresponds to arrival at the exit section $\Delta^{\text{ex}} = \{x + y = 2\nu\}$. In the scope of this section $\nu > 0$ can be chosen arbitrarily. \square

The fact that there cannot be jumps from fold points enclosing \mathcal{S}_0^a back to \mathcal{S}_0^a was already proven in [12]. However, note that the latter did not assume any dynamics in the fast fibers, since there Takens’ setting of constrained differential equations was used. Proposition 4.7

shows that a return mechanism from \mathcal{S}_0^a back to \mathcal{S}_0^a solely via folds or cusps is impossible. However, via a singular canard through the hyperbolic umbilic point one can get a candidate trajectory from \mathcal{S}_0^a to \mathcal{S}_0^r (or \mathcal{S}_0^s) and possibly jump back to \mathcal{S}_0^a from the upper cone of singularities. This was already observed in Remark 4.4. By Lemma 3.4 a singular canard through the hyperbolic umbilic cannot occur for $B_0C_0 > 0$.

Moreover, the above results give the idea that generically the hyperbolic umbilic point will behave “like a catapult”. In particular, any trajectory jumping at a singularity seems to escape generically along the positive x - or y -direction. Note that for trajectories approaching folds on the upper cone from the saddle-type region \mathcal{S}_0^s there is the possibility for a relaxation oscillation in the singular limit: Hitting a fold on the upper cone in $a > 0$, the trajectory might then jump to the saddle-type region “on the other side” of the cone, from where, upon hitting another fold on the front of the upper cone, the trajectory might jump back to the starting point. A rigorous analysis of these relaxation oscillations, i.e. the question whether they possibly persist for $\varepsilon > 0$ is beyond the scope of this document. Once an orbit jumps onto the attracting part upon hitting a fold point, we observe the “catapult behavior” again, if we assume that the singularities serve indeed as jump points. Generically, a fold point is indeed a jump point. In general, there possibly exist folded singularities on the double cone, which can give rise to canard trajectories passing to nearby regions of \mathcal{S}_0 with distinct stability properties, see e.g. [27, 20].

Next, we are going to investigate the dynamics in the vicinity of the hyperbolic umbilic point by means of a blow-up, which is the main part of the present article.

5. BLOW-UP ANALYSIS

We are interested in the behavior of attracting slow manifolds of (1.1) in a vicinity of the hyperbolic umbilic point for small $\varepsilon > 0$. We investigate (1.1) by means of the following vector field X , which is of the form

$$\begin{aligned} x' &= x^2 + ay + b + \varepsilon f_x(x, y, a, b, c, \varepsilon) \\ y' &= y^2 + ax + c + \varepsilon f_y(x, y, a, b, c, \varepsilon) \\ a' &= \varepsilon g_a(x, y, a, b, c, \varepsilon) \\ b' &= \varepsilon g_b(x, y, a, b, c, \varepsilon) \\ c' &= \varepsilon g_c(x, y, a, b, c, \varepsilon) \\ \varepsilon' &= 0. \end{aligned} \tag{5.1}$$

We assume that the functions f_x, f_y, g_a, g_b, g_c are sufficiently smooth. Recall that we use the notation $A_0 = g_a(0), B_0 = g_b(0), C_0 = g_c(0)$. According to the hypothesis of Theorem 1.1, and motivated by Lemma 3.4, we shall be concerned with the case $B_0, C_0 > 0$.

5.1. Preparations for blowing up. In the following we blow-up the vector field X at the origin. That is, we “replace” the origin by the 5-d sphere \mathbb{S}^5 . First, we set up the notation that is heavily used in the following analysis. With these we also briefly recall the blow-up method, see [14, 20] for further details. More precisely, we transform the vector field X (5.1) by the weighted (quasi-homogeneous) blow-up transformation

$$\begin{aligned} \Phi: B := \mathbb{S}^5 \times [0, \bar{r}] \subset \mathbb{R}^7 &\rightarrow \mathbb{R}^6 \\ (\bar{x}, \bar{y}, \bar{a}, \bar{b}, \bar{c}, \bar{\varepsilon}, \bar{r}) &\mapsto (\bar{r}\bar{x}, \bar{r}\bar{y}, \bar{r}\bar{a}, \bar{r}^2\bar{b}, \bar{r}^2\bar{c}, \bar{r}^3\bar{\varepsilon}), \end{aligned} \tag{5.2}$$

where $\bar{x}^2 + \bar{y}^2 + \bar{a}^2 + \bar{b}^2 + \bar{c}^2 + \bar{\varepsilon}^2 = 1$. The map Φ is a diffeomorphism for $\bar{r} > 0$, to where we can pullback the vector field X . There exists a sufficiently smooth extension \bar{X} on B , such that the pushforward of \bar{X} is X , i.e. $\Phi_*\bar{X} = X$. Note that by this procedure, we have “blown up” the hyperbolic umbilic singularity at the origin to a sphere $\mathbb{S}^5 \times \{0\}$. Typically, by blowing up a sufficiently degenerate singularity, \bar{X} vanishes on the blow-up sphere $\{\bar{r} = 0\} \simeq \mathbb{S}^5$. The key point of the blow-up method is that one aims to rescale \bar{X} (dividing out a common factor) in such a way, that distinct (semi-)hyperbolic equilibria on

$\{\bar{r} = 0\}$ appear. By this, the orbit structure of \bar{X} in $\{\bar{r} > 0\}$ remains unchanged. Often this rescaling is taken out locally, i.e. by studying \bar{X} locally in charts. The general goal is then to infer the dynamics in $\{\bar{r} > 0\}$ close to the sphere from the dynamics on the invariant blow-up sphere $\{\bar{r} = 0\}$.

In the following we refer to \mathbb{S}^5 as the blow-up sphere and identify it with $\mathbb{S}^5 \times \{0\} \subset \mathbb{S}^5 \times [0, \bar{r}] = B$, where we call B the blow-up space. Note that the blow-up space and the blow-up sphere are (sub-)manifolds. To work on them in a convenient way one usually introduces charts, in which Φ takes the form of a “directional blow-up”.

Let us denote these “usual” charts by $\kappa_{\pm\bullet}$, where \bullet corresponds to a coordinate direction. Then

$$\Phi_{\pm\bullet} := \Phi \circ \kappa_{\pm\bullet}^{-1} \quad (5.3)$$

is simply given by putting $\bar{\bullet} = \pm 1$ in (5.2) and renaming the variables accordingly. This notation comes from the fact that in $\kappa_{\pm\bullet}$ the blow-up space $B|_{\pm\bar{\bullet} > 0}$ is visible. Let us give a brief example to clarify this notation: The chart κ_{-y} is precisely designed in such a way, that the blow-up Φ in chart κ_{-y} is given by

$$\Phi_{-y}: (x_1, r_1, a_1, b_1, c_1, \varepsilon_1) \mapsto (r_1 x_1, -r_1, r_1 a_1, r_1^2 b_1, r_1^2 c_1, r_1^3 \varepsilon_1), \quad (5.4)$$

where $(x_1, r_1, a_1, b_1, c_1, \varepsilon_1)$ are the local coordinates determined by κ_{-y} . Note how Φ_{-y} is obtained from Φ (5.2) by putting $\bar{y} = -1$ and relabelling the coordinates.

Remark 5.1. *The blow-up exponents in (5.2) are chosen in such a way, that they match the quasi-homogeneity of X in the fast variables and allow for rescaling in the slow variables. For example, in the chart κ_{-y} , all terms in the first component in the pullback of X via Φ_{-y} (i.e. the equation for x_1') share the same power of r_1 as factor. This allows to rescale and obtain nontrivial dynamics on $\{\bar{r} = 0\}$. Moreover, with this exponent choice, the critical manifold of X appears as 2-submanifold on the blow-up sphere $\{\bar{r} = 0\}$.*

We aim to study \bar{X} by means of the directional blow-ups $\Phi_{\pm\bullet}$ in the charts $\kappa_{\pm\bullet}$. For reference, we state all blow-up maps Φ_{\bullet} that are to be used in the forthcoming analysis. This also fixes the naming conventions for the coordinates determined by the charts κ_{\bullet} . The blow-up maps we are going to use explicitly are:

$$\begin{aligned} \Phi_{-y}: x &= r_1 x_1, & y &= -r_1, & a &= r_1 a_1, & b &= r_1^2 b_1, & c &= r_1^2 c_1, & \varepsilon &= r_1^3 \varepsilon_1; \\ \Phi_{\varepsilon}: x &= r_2 x_2, & y &= r_2 y_2, & a &= r_2 a_2, & b &= r_2^2 b_2, & c &= r_2^2 c_2, & \varepsilon &= r_2^3; \\ \Phi_a: x &= r_3 x_3, & y &= r_3 y_3, & a &= r_3, & b &= r_3^2 b_3, & c &= r_3^2 c_3, & \varepsilon &= r_3^3 \varepsilon_3; \\ \Phi_{-a}: x &= r_4 x_4, & y &= r_4 y_4, & a &= -r_4, & b &= r_4^2 b_4, & c &= r_4^2 c_4, & \varepsilon &= r_4^3 \varepsilon_4; \\ \Phi_{\text{ex}}: x &= r_5(1 + x_5), & y &= r_5(1 - x_5), & a &= r_5 a_5, & b &= r_5^2 b_5, & c &= r_5^2 c_5, & \varepsilon &= r_5^3 \varepsilon_5. \end{aligned} \quad (5.5)$$

We emphasize the slightly “unusual” blow-up Φ_{ex} , which is roughly obtained by including a 45° degree rotation in the fast variables. For further details on Φ_{ex} refer to the exit chart analysis in section 5.6.

We now give an overview of the forthcoming blow-up analysis, which also describes which charts/blow-ups we will use. The goal is to understand the dynamics of the fast-slow vector field X close to the origin. Notice that X has a manifold of equilibria $\mathcal{S}_0 \times \{\varepsilon = 0\}$, i.e., the critical manifold. By Fenichel theory [9, 15] compact submanifolds of the normally hyperbolic regions of \mathcal{S}_0 perturb for sufficiently small $\varepsilon > 0$ to (non-unique) invariant slow manifolds, on which the flow converges to the slow flow on the critical manifold for $\varepsilon \rightarrow 0$. These invariant slow manifolds appear in sections $\varepsilon = \text{constant}$ of 4-dimensional center manifolds along $\mathcal{S}_0^a \times \{\varepsilon = 0\}$, $\mathcal{S}_0^r \times \{\varepsilon = 0\}$ and $\mathcal{S}_0^s \times \{\varepsilon = 0\}$. Trajectories, which approach slow manifolds, will generically approach slow manifolds obtained as perturbations of the attracting region \mathcal{S}_0^a . Refer to figure 4 for the region \mathcal{S}_0^a . In the blow-up of X , the hyperbolic umbilic singularity is “replaced” by the 5-d sphere $\mathbb{S}^5 \simeq B|_{\{\bar{r}=0\}}$. In all of the charts, we will see certain regions of phase space and away from $\{\bar{r} = 0\}$ all objects, e.g. invariant manifolds or the critical manifold, appear in a locally diffeomorphic manner. In the singular

limit $\varepsilon = 0$, the blow-up space is restricted to $\{\bar{\varepsilon} = 0\}$, and the blow-up sphere \mathbb{S}^5 reduces to $B|_{\{\bar{\varepsilon}=0, \bar{r}=0\}} \simeq \mathbb{S}^4$, to which we will refer as equator of the blow-up sphere.

Remark 5.2. *The following notational convention will be used: objects in original coordinates, which are visible in a blow-up Φ_\bullet , will be indexed by the corresponding index used for the new coordinates in Φ_\bullet . For example, \mathcal{S}_0^a seen in Φ_{-y} (i.e. the blow-up in κ_{-y}) will be denoted by $\mathcal{S}_{0,1}^a$. Similarly, the section Δ^{ex} seen in Φ_{ex} will be denoted by Δ_5^{ex} . However, we will always simply use the variable t to denote time (i.e. the variable to parametrize integral curves) in each of the blown-up vector fields and we remark that the meaning of t will be clear from the context.*

In the following we present the analysis in the entry chart κ_{-y} , the rescaling chart (also called central chart) κ_ε and the exit chart κ_{ex} . The charts $\kappa_{\pm a}$ are only relevant in section 6 and are not used to prove Theorem 1.1. However, in (5.5), $\Phi_{\pm a}$ appear before the exit chart blow-up Φ_{ex} , since in principle one would study transitions from the entry chart κ_{-y} to $\kappa_{\pm a}$ in order to analyze the dynamics away from the hyperbolic umbilic singularity. To cover all transitions onto the fast regime in a single exit chart, we use a slightly “unusual” chart κ_{ex} , which includes, up to scaling, a 45° rotation of the fast variables of X . In our notation, κ_{ex} intuitively corresponds to “ κ_{x+y} ”. The corresponding blow-up map Φ_{ex} can then also be roughly understood as blowing up X “in direction of $x+y$ ”. This choice is motivated by (4.3) and Lemma 4.3. After the analysis in the three charts κ_{-y} , κ_ε , κ_{ex} we are in position to prove Theorem 1.1. Finally, we also present a brief analysis of the dynamics on the blow-up sphere in the two charts $\kappa_{\pm a}$ in section 6, which are useful to understand the desingularized slow flow near the hyperbolic umbilic singularity and the flow on the blow-up sphere close to the equator $\{\bar{\varepsilon} = \bar{r} = 0\}$.

5.2. Analysis in the entry chart κ_{-y} . Blowing up the vector field X (5.1) by Φ_{-y} , as defined in (5.5), and desingularizing accordingly by dividing out the common factor r_1 , we obtain the blow-up in κ_{-y} :

$$\begin{aligned} x'_1 &= x_1^2 - a_1 + b_1 + x_1(1 + a_1x_1 + c_1) + r_1\varepsilon_1(f_x + x_1f_y) \\ r'_1 &= -r_1(1 + a_1x_1 + c_1) - r_1^2\varepsilon_1f_y \\ a'_1 &= a_1(1 + a_1x_1 + c_1) + r_1\varepsilon_1(g_a + a_1f_y) \\ b'_1 &= 2b_1(1 + a_1x_1 + c_1) + \varepsilon_1g_b + 2r_1\varepsilon_1b_1f_y \\ c'_1 &= 2c_1(1 + a_1x_1 + c_1) + \varepsilon_1g_c + 2r_1\varepsilon_1c_1f_y \\ \varepsilon'_1 &= 3\varepsilon_1(1 + a_1x_1 + c_1) + 3r_1\varepsilon_1^2f_y. \end{aligned} \tag{5.6}$$

In (5.6) we use the shorthand f_x for $f_x \circ \Phi_{-y}$, similarly for f_y, g_a, g_b, g_c . Recall that dividing out the common factor r_1 amounts to a time reparametrization of trajectories away from the blow-up sphere. The system (5.6) has invariant subspaces $\{r_1 = 0\}, \{\varepsilon_1 = 0\}$ and $\{r_1 = 0, \varepsilon_1 = 0\}$. The subspace $\{r_1 = 0\}$ corresponds to dynamics on the blowup sphere \mathbb{S}^5 . Analogously, the subspace $\{\varepsilon_1 = 0\}$ corresponds to the singular limit and the subspace $\{r_1 = 0, \varepsilon_1 = 0\}$ to the part of the 4-dimensional equator of \mathbb{S}^5 , where $\bar{y} < 0$ holds. Note the constant of motion $r_1^3\varepsilon_1$, induced by the blown-up invariant sets $\{\varepsilon = \text{const}\}$ of X .

In $\{r_1 = 0, \varepsilon_1 = 0\}$ the system (5.6) reduces to

$$\begin{aligned} x'_1 &= x_1^2 + (1 + a_1x_1 + c_1)x_1 - a_1 + b_1 \\ a'_1 &= a_1(1 + a_1x_1 + c_1) \\ b'_1 &= 2b_1(1 + a_1x_1 + c_1) \\ c'_1 &= 2c_1(1 + a_1x_1 + c_1), \end{aligned} \tag{5.7}$$

which has two isolated equilibria

$$\begin{aligned} q_1 &:= (x_1 = 0, a_1 = 0, b_1 = 0, c_1 = 0) \\ q_2 &:= (x_1 = -1, a_1 = 0, b_1 = 0, c_1 = 0) \end{aligned} \tag{5.8}$$

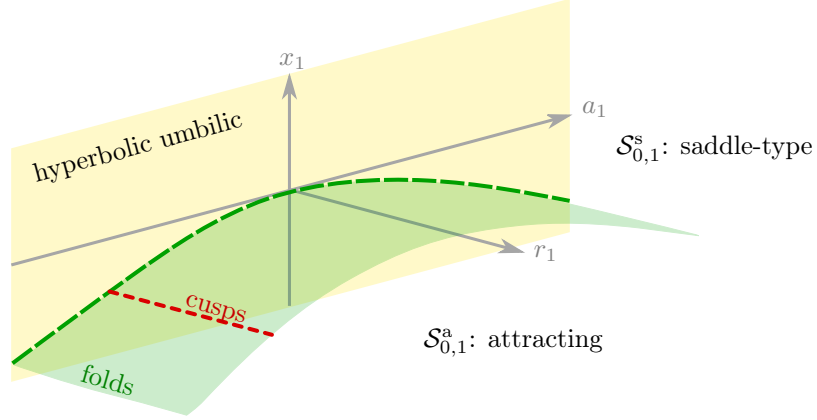


FIGURE 10. Stratification of the blown-up critical manifold in chart κ_{-y} , given by Lemma 5.3.

and a 2-manifold of equilibria determined by

$$\begin{aligned} x_1^2 + b_1 - a_1 &= 0 \\ 1 + a_1 x_1 + c_1 &= 0. \end{aligned} \quad (5.9)$$

In $\{\varepsilon_1 = 0\}$ the system (5.6) simplifies to

$$\begin{aligned} x_1' &= x_1^2 + (1 + a_1 x_1 + c_1)x_1 - a_1 + b_1 \\ r_1' &= -r_1(1 + a_1 x_1 + c_1) \\ a_1' &= a_1(1 + a_1 x_1 + c_1) \\ b_1' &= 2b_1(1 + a_1 x_1 + c_1) \\ c_1' &= 2c_1(1 + a_1 x_1 + c_1). \end{aligned} \quad (5.10)$$

In addition to the isolated equilibria q_1, q_2 on the blow-up sphere, system (5.10) has a 3-manifold of equilibria

$$\mathcal{S}_{0,1} = \{x_1^2 + b_1 - a_1 = 0, 1 + a_1 x_1 + c_1 = 0\}, \quad (5.11)$$

where the 2-manifold determined by (5.9) appears as slice $\mathcal{S}_{0,1}|_{r_1=0}$. Via Φ_{-y} the manifold $\mathcal{S}_{0,1}|_{r_1>0}$ corresponds to the critical manifold $\mathcal{S}_0|_{y<0}$. One should recall that the hyperbolic umbilic point is an object of codimension 3 in the critical manifold. In blown-up coordinates, it corresponds to the intersection of the 3-dimensional critical manifold with the 5-dimensional blow-up sphere \mathbb{S}^5 , which gives a 2-dimensional manifold. This 2-manifold will appear in multiple charts and is contained in the singular limit. That is, it is contained in the equator $\{\bar{\varepsilon} = 0, \bar{r} = 0\} \simeq \mathbb{S}^4$ of the blow-up sphere \mathbb{S}^5 , which precisely is given by (5.9) in the chart κ_{-y} . The manifold $\mathcal{S}_{0,1}$ is a graph over the (x_1, r_1, a_1) -coordinates.

Lemma 5.3. *The manifold $\mathcal{S}_{0,1}|_{r_1>0}$ is subdivided into an attracting region $\mathcal{S}_{0,1}^a := \{4x_1 + a_1^2 < 0, r_1 > 0\}$, a saddle-type region $\mathcal{S}_{0,1}^s := \{4x_1 + a_1^2 > 0, r_1 > 0\}$, a fold surface given by $\{4x_1 + a_1^2 = 0, r_1 > 0\}$, which contains the cusp line $\{x_1 = -1, a_1 = -2, r_1 > 0\}$. The 2-manifold $\mathcal{S}_{0,1}|_{r_1=0}$ represents the blown-up hyperbolic umbilic singularity.*

Proof. Lemma 3.2 blown up by Φ_{-y} . See figure 10. \square

For the following it is convenient to rectify the critical manifold \mathcal{S}_1 by introducing a coordinate transformation $T: x_1 = x_1, r_1 = r_1, a_1 = a_1, u = x_1^2 + b_1 - a_1, v = 1 + a_1 x_1 +$

$c_1, \varepsilon_1 = \varepsilon_1$. Applying this transformation to (5.6) leads to

$$\begin{aligned}
x'_1 &= u + x_1 v + r_1 \varepsilon_1 (f_x + x_1 f_y) \\
r'_1 &= -r_1 v - r_1^2 \varepsilon_1 f_y \\
a'_1 &= a_1 v + r_1 \varepsilon_1 (g_a + a_1 f_y) \\
u' &= 2x_1 u + a_1 v + 2uv + \varepsilon_1 g_b + O(r_1 \varepsilon_1) \\
v' &= a_1 u - 2v + 2v^2 + \varepsilon_1 g_c + O(r_1 \varepsilon_1) \\
\varepsilon'_1 &= 3\varepsilon_1 v + 3r_1 \varepsilon_1^2 f_y.
\end{aligned} \tag{5.12}$$

Again, we use the shorthand $f_x = f_x \circ \Phi_{-y} \circ T^{-1}$, similarly for f_y, g_a, g_b, g_c . Each of the $O(r_1 \varepsilon_1)$ terms is of the form $r_1 \varepsilon_1 \tilde{f}$, for a sufficiently smooth function \tilde{f} . The effect of T is that the critical manifold $\mathcal{S}_{0,1}$ in (5.12) is now given by $\{u = 0, v = 0, \varepsilon_1 = 0\}$.

Lemma 5.4. *Along $\mathcal{S}_{0,1}^a$ there exist 4-dimensional attracting center manifolds $W^c(\mathcal{S}_{0,1}^a)$, which can be extended to $r_1 = 0$. Via Φ_{-y} the center manifolds along $\mathcal{S}_0^a \times \{\varepsilon = 0\}$ (in original coordinates) correspond to instances of $W^c(\mathcal{S}_{0,1}^a)|_{r_1 > 0}$. The attracting slow manifolds obtained by Fenichel theory coincide with intersections of the form $W^c(\mathcal{S}_{0,1}^a) \cap \{r_1^3 \varepsilon_1 = \text{const}\}$.*

Proof. Linearization of (5.12) along $\mathcal{S}_{0,1} \simeq \{\varepsilon_1 = 0, u = 0, v = 0\}$ is given by the matrix

$$\begin{bmatrix}
0 & 0 & 0 & 1 & x_1 & O(r_1) \\
0 & 0 & 0 & 0 & -r_1 & O(r_1^2) \\
0 & 0 & 0 & 0 & a_1 & O(r_1) \\
0 & 0 & 0 & 2x_1 & a_1 & B_0 + O(r_1) \\
0 & 0 & 0 & a_1 & -2 & C_0 + O(r_1) \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}. \tag{5.13}$$

The matrix (5.13) has a quadruple zero eigenvalue and two further eigenvalues $\lambda_{1,2} = x_1 - 1 \pm \sqrt{(1-x_1)^2 + 4x_1 + a_1^2}$. Notice that $\lambda_{1,2} < 0$ along $\mathcal{S}_{0,1}^a$ and $\lambda_1 > 0 > \lambda_2$ along $\mathcal{S}_{0,1}^s$. Note that along the fold surface given by $\{4x_1 + a_1^2 = 0\}$ we have $\lambda_1 = 0, \lambda_2 = 2(x_1 - 1)$, while $x_1 \leq 0$. Since $\varepsilon = r_1^3 \varepsilon_1$ is a constant of motion in κ_{-y} , the (blown-up) attracting slow manifolds are contained in intersections of the form $W^c(\mathcal{S}_{0,1}^a) \cap \{r_1^3 \varepsilon_1 = \text{const}\}$. \square

Remark 5.5. *The equilibria q_1, q_2 from (5.8) arise from interaction with the fast subsystem. For further illustration, refer to figure 14 and the explanation next to (5.50).*

5.3. Blow-up of the desingularized slow flow Y . The desingularized slow flow Y (3.8) constitutes a singular limit slow flow on the critical manifold \mathcal{S}_0 . It is worth recalling the necessary time reversion for Y in the saddle region \mathcal{S}_0^s due to the desingularization process. A natural idea is to incorporate Y into our blow-up analysis, i.e. the blow-up of the fast formulation X (5.1). By this we aim to study the singular limit interaction of Y with the blown-up hyperbolic umbilic singularity. That is, we will “embed” Y into the blow-up of X . Note that this step was also implicitly carried out in the analysis of a fold singularity or cusp singularity [19, 5, 13]. However, in these cases, the desingularized slow flow was regular and therefore considerably simpler to embed into the blow-up analysis. In contrast to this, we are dealing with a desingularized slow flow Y , which has a non-hyperbolic equilibrium at the hyperbolic umbilic singularity. Even in the studies of folded singularities [27, 24, 7], these appear as hyperbolic singularities of the desingularized slow flow.

Recall the blow-up transformation Φ given by (5.2). Let $\Theta: \mathbb{S}^2 \times [0, \bar{r}] \rightarrow \mathbb{R}^3, (\bar{x}, \bar{y}, \bar{a}, \bar{r}) \mapsto (\bar{r}\bar{x}, \bar{r}\bar{y}, \bar{r}\bar{a})$ be the blow-up arising from Φ restricted to the (x, y, a) -coordinates. Define the blown-up \bar{Y} of the desingularized slow flow Y by

$$\Theta_* \bar{Y} = Y. \tag{5.14}$$

In fact, \bar{Y} has nontrivial dynamics on the blow-up sphere \mathbb{S}^2 without employing a time rescaling. Note that one can use the charts $\kappa_{\pm x, \pm y, \pm a}$ to study \bar{Y} , as this just amounts

to blowing up Y by the corresponding directional blow-ups induced by Θ . We denote the directional blow-ups induced by Θ with the same subscripts, i.e. Θ_\bullet .

Since Y determines the slow flow on the critical manifold \mathcal{S}_0 , in blown-up coordinates \bar{Y} determines the slow flow on the blown-up critical manifold in each of the charts $\kappa_{\pm x, \pm y, \pm a}$. By this we mean the following: Looking at the blow-up of X in chart κ_\bullet , i.e. blowing up X with Φ_\bullet , the critical manifold appears as a manifold of equilibria. The slow flow on this manifold of equilibria is then determined by the blow-up of Y by Θ_\bullet . Note that since the critical manifold is parametrized by (x, y, a) -coordinates, the coordinates used for parametrizing the critical manifold in the chart κ_\bullet match with the coordinates used for blowing up Y by Θ_\bullet . This directly allows to “embed” the blown-up slow flow \bar{Y} in each of the above charts into the blow-up of X . For example, in the chart κ_{-y} , we have that the critical manifold appears as the manifold $\mathcal{S}_{0,1}$, which is parametrized by the coordinates (x_1, r_1, a_1) . These are precisely the coordinates, which are used for blowing up Y by Θ_{-y} . We will start with the analysis of \bar{Y} in κ_{-y} in this section. Similarly, one can employ the other charts to complete the analysis of \bar{Y} . Eventually, this leads to a blown-up version of figure 5, i.e. a “usual” blow-up analysis of the vector field \bar{Y} , with time reversion in \mathcal{S}_0^s . A sketch of the flow of \bar{Y} is shown in figure 12.

To embed Y into \bar{X} in chart κ_{-y} , we take the desingularized slow flow Y (3.8) and transform it by

$$\Theta_{-y}: x = r_1 x_1, y = -r_1, a = r_1 a_1. \quad (5.15)$$

This gives

$$\begin{aligned} \dot{x}_1 &= a_1 \tilde{g}_c - 2\tilde{g}_b + r_1(a_1 x_1 - 2)\tilde{g}_a + x_1 J \\ \dot{r}_1 &= -r_1 J \\ \dot{a}_1 &= a_1 J - r_1(4x_1 + a_1^2)\tilde{g}_a, \end{aligned} \quad (5.16)$$

where $J = J(x_1, r_1, a_1) = a_1 \tilde{g}_b - 2x_1 \tilde{g}_c - r_1(a_1 + 2x_1^2)\tilde{g}_a$ and the shorthand \tilde{g}_\bullet for $\tilde{g}_\bullet \circ \Theta_{-y}$ is used with \tilde{g}_\bullet defined in (3.8). System (5.16) represents a vector field on the manifold $\mathcal{S}_{0,1}$ in the coordinates (x_1, r_1, a_1) and, simultaneously, a “usual” blow-up of Y in negative y -direction. The invariant set $\{r_1 = 0\}$ of (5.16) corresponds to the blown-up hyperbolic umbilic point. Refer to figure 10, in which (5.16) now induces a flow. Recall that for Y we need to reverse the time orientation of trajectories in the saddle-type region. This we have to take into account for \bar{Y} as well: In (5.16) we reverse the time-orientation of orbits in the saddle-type region $\{4x_1 + a_1^2 > 0\}$, now including $r_1 = 0$, in order to obtain a consistent flow.

Remark 5.6. *Another way to motivate the time reversion in $\{r_1 = 0, 4x_1 + a_1^2 > 0\}$ will be apparent from the charts $\kappa_{\pm a}$ in section 6. In these charts, the regions $\{r_1 = 0, a_1 > 0, 4x_1 + a_1^2 > 0\}$ and $\{r_1 = 0, a_1 < 0, 4x_1 + a_1^2 > 0\}$ appear as saddle-type regions of 2-d critical manifolds. Hence, its desingularized slow flow undergoes a time reversion in such regions.*

Recall from our notation that $\tilde{g}_a(0) = A_0$, $\tilde{g}_b(0) = B_0$ and $\tilde{g}_c(0) = C_0$. The vector field (5.16) has three equilibria in $\{r_1 = 0\}$, which are located at

$$\begin{aligned} \zeta_1 &:= (x_1 = -B_0^2/C_0^2, a_1 = -2B_0/C_0), \\ \zeta_{2,3} &:= (x_1 = \mp\sqrt{B_0/C_0}, a_1 = 0). \end{aligned} \quad (5.17)$$

Note that ζ_2 and ζ_3 only exist if B_0, C_0 have identical sign. The equilibrium point ζ_1 lies on $\{4x_1 + a_1^2 = 0\}$, which is the extension of the fold surface from Lemma 5.3 to $\{r_1 = 0\}$. In case of $B_0 = C_0 \neq 0$, ζ_1 extends the cusp line to $\{r_1 = 0\}$. In the vicinity of ζ_2 lies the attracting part of the (blown-up) critical manifold $\mathcal{S}_{0,1}^a$, while ζ_3 lies next to the saddle-type part $\mathcal{S}_{0,1}^s$. See figure 11. We now analyze in further detail the equilibria $\zeta_{1,2,3}$, since they organize the flow.

Lemma 5.7. *Let $B_0, C_0 > 0$. With reversed time orientation in $\{4x_1 + a_1^2 > 0\}$, the points $\zeta_{2,3}$ are hyperbolic equilibria with the following stability: In the invariant plane $\{r_1 = 0\}$ $\zeta_{2,3}$ are sources. At each of $\zeta_{2,3}$ there exist unique 1-d stable manifolds transverse to the plane $\{r_1 = 0\}$. If the time reversion is neglected, the x_1 -axis is invariant and establishes a heteroclinic connection from ζ_2 to ζ_3 .*

Proof. Linearization of (5.16) at a point $(x_1, r_1 = 0, a_1)$ leads to

$$\begin{bmatrix} B_0 a_1 - 4C_0 x_1 & * & C_0 + x_1 B_0 \\ 0 & -B_0 a_1 + 2C_0 x_1 & 0 \\ -2C_0 a_1 & * & 2B_0 a_1 - 2C_0 x_1 \end{bmatrix}. \quad (5.18)$$

The stars * denote unspecified entries. At ζ_2 this leads to eigenvalues $4\sqrt{B_0 C_0}, 2\sqrt{B_0 C_0}$ in the invariant plane $\{r_1 = 0\}$ and $-2\sqrt{B_0 C_0}$ with eigenvector transverse to $\{r_1 = 0\}$. At ζ_3 there are eigenvalues $-4\sqrt{B_0 C_0}, -2\sqrt{B_0 C_0}$ in the invariant plane $\{r_1 = 0\}$ and $2\sqrt{B_0 C_0}$ with eigenvector transverse to it. Taking into account the reversed time orientation for $4x_1 + a_1^2 > 0$ implies the assertions. The heteroclinic connection along the invariant x_1 -axis to ζ_3 is clear. \square

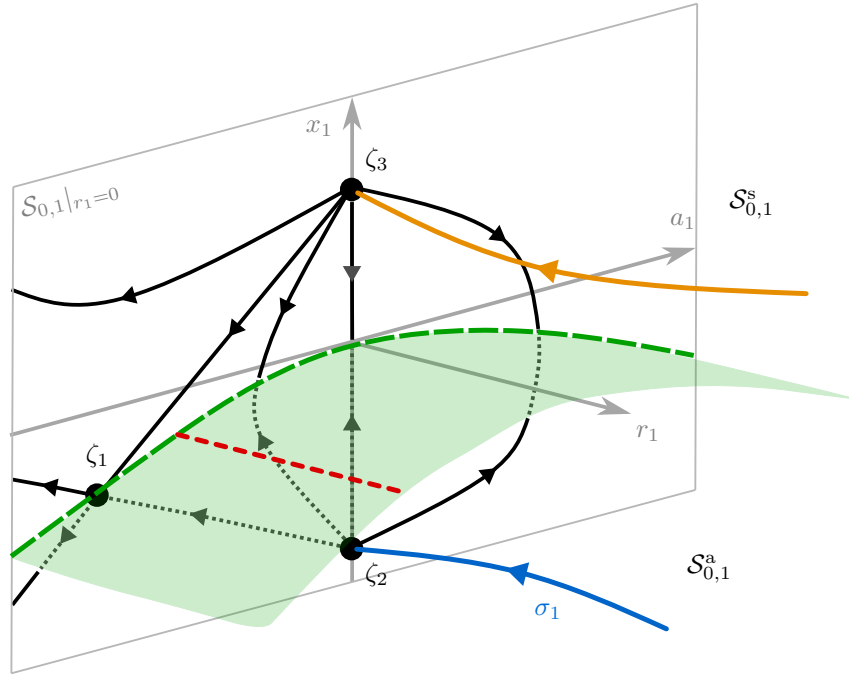


FIGURE 11. Sketch of the blown-up desingularized slow flow (5.16) for $B_0 > C_0 > 0$ close to $\mathcal{S}_{0,1}|_{r_1=0}$. Time reversion is taken into account above the green surface of fold points, that is in $\mathcal{S}_{0,1}^s$. The black trajectories indicate the flow on the invariant plane $\{r_1 = 0\} \simeq \mathcal{S}_{0,1}|_{r_1=0}$, containing $\zeta_{1,2,3}$. The trajectory σ_1 (the blown-up version of σ) approaches ζ_2 from inside $\mathcal{S}_{0,1}^a$ and the orange trajectory reaches ζ_3 from inside $\mathcal{S}_{0,1}^s$, see Lemma 3.4. The center manifold based at ζ_1 is not depicted.

Recall the trajectory σ from Lemma 3.4, which represents the stable manifold of the origin in the attracting part of \mathcal{S}_0 . Let σ_1 denote σ in the present blown-up coordinates.

Remark 5.8. *Recall figure 5. Via Θ_{-y} the equilibrium ζ_2 serves as arrival point for the trajectory σ at the blown-up hyperbolic umbilic. The equilibrium ζ_3 serves as arrival point for the unique trajectory, which approaches the hyperbolic umbilic point from the saddle-type region $y < 0$. Similarly, there will exist two more equilibria ζ_4 and ζ_5 of \bar{Y} on the blow-up sphere, which correspond to the two other trajectories, which arrive the hyperbolic umbilic from $y > 0$.*

Refer to figure 11 for a sketch of the flow of (5.16) close to the invariant plane $\{r_1 = 0\}$. Note that Lemma 5.7 shows that the trajectory σ_1 is transverse to the plane $\{r_1 = 0\}$. This implies that σ_1 is transverse to $\{r_1 = \nu\}$, for sufficiently small $\nu > 0$.

There exists the third equilibrium ζ_1 in $\{r_1 = 0\}$, which lies on $4x_1 + a_1^2 = 0$, the extension of the fold surface to $r_1 = 0$. Employing the linearization (5.18) gives

Lemma 5.9. *Let $B_0C_0 \neq 0$. In $\{r_1 = 0\}$, without taking the time reversion into account, ζ_1 has eigenvalues $\pm 2\sqrt{B_0C_0}$. Hence ζ_1 is a saddle if $B_0C_0 > 0$ and ζ_1 is a center if $B_0C_0 < 0$. There exists a 1-d center manifold at ζ_1 , transverse to $\{r_1 = 0\}$.*

In fact, in the charts $\kappa_{\mp a}$, there appear fast-slow systems on the blow-up sphere, in which ζ_1 is a folded saddle, respectively a folded center. For more details on the folded singularities arising in the blow-up analysis we refer to section 6.

Remark 5.10. *For the following analysis we are interested in the flow close to ζ_2 , since the trajectory σ_1 arrives there. In particular, we will not investigate the role of the center manifold at ζ_1 further.*

Until this point, we have analyzed the blow-up \bar{Y} in the chart induced by κ_{-y} . For a full picture of \bar{Y} it remains to study \bar{Y} in other charts to obtain the complete dynamics of \bar{Y} close to the blow-up sphere \mathbb{S}^2 . In κ_{-x} (and similarly in κ_x or κ_y) the findings are similar to κ_{-y} and we omit the presentation here.

Remark 5.11. *The heteroclinic connections from $\zeta_{2,3}$ to ζ_1 as shown in figure 11 are implied by the analysis in chart κ_{-a} by Lemma 6.4. In particular, the stable/unstable manifolds of ζ_1 in $\{r_1 = 0\}$ are transverse to the extended fold surface for $B_0 \neq C_0$.*

Employing several charts to analyze \bar{Y} , we obtain the complete blow-up of \bar{Y} , as visualized in figure 12. Once again, we emphasize the time reversion in the saddle-type region, which needs to be taken into account. The qualitative flow in figure 12 is implied by the analysis in the charts $\kappa_{\pm a}$ in section 6, in particular 6.2. On the blow-up sphere there exist three further equilibria which we denote by ζ_4 , ζ_5 and ζ_6 . The equilibria $\zeta_{5,6}$ are analogous to $\zeta_{2,1}$, but are located next to the repelling region of the critical manifold. The role of the equilibrium ζ_4 is analogous to ζ_3 . We remark that figure 12 is the blown-up version of figure 5.

Remark 5.12. *The idea of the following analysis is to track small perturbations of the trajectory $\sigma \simeq \sigma_1$ for small $\varepsilon > 0$ through the blow-up. The importance of $\sigma \simeq \sigma_1$ is that in the case $B_0, C_0 > 0$ it approaches the blow-up sphere at ζ_2 from inside \mathcal{S}_0^a , see figures 11 and 12. Consequently, we will focus on a neighborhood of ζ_2 for the transition into the rescaling chart κ_ε . Further, we remark that in principle trajectories might arrive at ζ_1 through a center manifold from inside \mathcal{S}_0^a . Since we focus on a neighborhood of ζ_2 this has no effect on our analysis and we do not investigate a center manifold at ζ_1 further. However, we note that Lemma 6.1 shows that there do not exist trajectories on the blow-up sphere emanating from ζ_1 into the rescaling chart.*

5.4. Transition from κ_{-y} to κ_ε . Recall that we are interested in the evolution of trajectories in attracting slow manifolds close to the hyperbolic umbilic point. From Lemma 5.4, these trajectories are contained in the center manifold $W^c(\mathcal{S}_{0,1}^a)$. In the singular limit, the only slow flow trajectory reaching the hyperbolic umbilic is σ , which we denote by σ_1 in blown-up coordinates κ_{-y} . From the previous section we know that σ_1 reaches the blow-up sphere at ζ_2 . Thus, ζ_2 will be our “base point” to study transitions inside $W^c(\mathcal{S}_{0,1}^a)$ into the rescaling chart κ_ε . In other words, we aim to track the flow close to ζ_2 in $W^c(\mathcal{S}_{0,1}^a)$.

Lemma 5.13. *Let $\tilde{x}_1 = x_1 + \sqrt{B_0/C_0}$, such that ζ_2 is translated to the origin. Then the flow in $W^c(\mathcal{S}_{0,1}^a)|_{\varepsilon_1 > 0}$ close to $\zeta_2 = (\tilde{x}_1 = 0, a_1 = 0, r_1 = 0, \varepsilon_1 = 0)$ is determined up to*

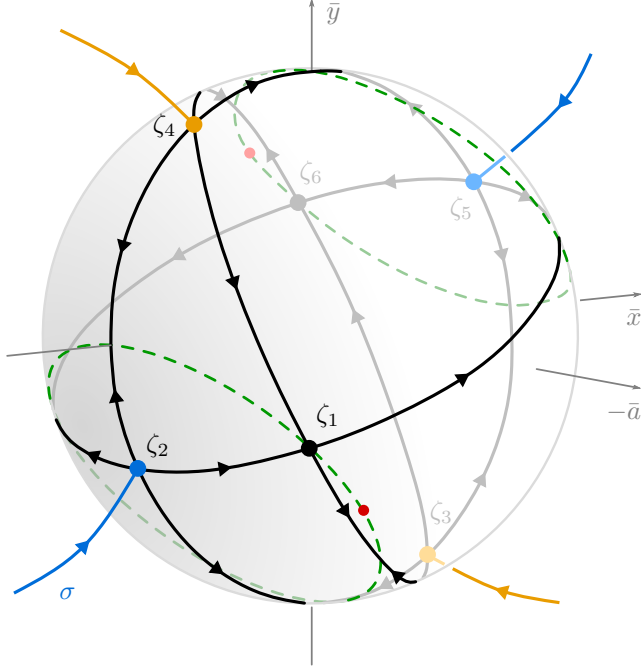


FIGURE 12. Complete blow-up \bar{Y} of the desingularized slow flow Y (3.8) for $B_0 > C_0 > 0$ at the origin, with time reversion in the saddle-type region taken into account. The green dashed curves depict the fold curves, which bound the attracting/repelling/saddle-type regions on the blow-up sphere. We remark that this figure shows a blown-up version of figure 5, but for visualization reasons the negative a -direction points here towards the observer. On the blow-up sphere \mathbb{S}^2 , only trajectories contained in the stable/unstable manifolds of the folded saddles ζ_1, ζ_6 and trajectories on the equator $\bar{a} = 0$ are depicted. These organize the flow on \mathbb{S}^2 . For further details refer to section 6, in particular 6.2. The red points depict the cusp points, i.e. where the cusp curves touch the blow-up sphere.

orbital equivalence by

$$\begin{aligned} \tilde{x}'_1 &= 2\tilde{x}_1 + \Lambda \\ a'_1 &= a_1 + r_1 \frac{\tilde{g}_a}{G} + r_1^3 \varepsilon_1 f_1 \\ r'_1 &= -r_1 \\ \varepsilon'_1 &= 3\varepsilon_1, \end{aligned} \tag{5.19}$$

where $\Lambda = \Lambda(\tilde{x}_1, a_1, r_1, \varepsilon_1)$ satisfies $\Lambda(0) = \partial_{\tilde{x}_1} \Lambda(0) = 0$, \tilde{g}_a is a shorthand recycled notation for $\tilde{g}_a \circ \Theta_{-y}$, where \tilde{g}_a is defined in (3.8), and G is a sufficiently smooth function with $G(0) > 0$.

Proof. Consider the system (5.12) with its linearization (5.13). The critical manifold is given by $\{u = v = \varepsilon_1 = 0\}$. Close to ζ_2 the center manifold $W^c(\mathcal{S}_{0,1}^a)$ is given as a graph $(u, v) = h(\tilde{x}_1, a_1, r_1, \varepsilon_1) = (h_1, h_2)$. It follows that the map h factors ε_1 . Further h is tangent to the center eigenspace E^c of the linearization (5.13) at ζ_2 . The eigenspace E^c is given as the graph of the linear function $(u, v) = (\varepsilon_1 \sqrt{B_0 C_0}/2, \varepsilon_1 C_0/2)$. This implies that h_2 is of the form $h_2 = \varepsilon_1 \tilde{h}_2$, with $\tilde{h}_2(\zeta_2) = C_0/2 > 0$. We now plug $(u, v) = h$ into (5.12), where we first translate ζ_2 to the origin, to obtain the flow in $W^c(\mathcal{S}_{0,1}^a)$ close to ζ_2 . This leads to a

vector field of the form

$$\begin{aligned}\tilde{x}'_1 &= (\tilde{x}_1 - \sqrt{B_0/C_0})(\varepsilon_1 \tilde{h}_2 + r_1 \varepsilon_1 f_y) + \varepsilon_1 \tilde{h}_1 + r_1 \varepsilon_1 f_x \\ a'_1 &= a_1(\varepsilon_1 \tilde{h}_2 + r_1 \varepsilon_1 f_y) + r_1 \varepsilon_1 g_a \\ r'_1 &= -r_1(\varepsilon_1 \tilde{h}_2 + r_1 \varepsilon_1 f_y) \\ \varepsilon'_1 &= 3\varepsilon_1(\varepsilon_1 \tilde{h}_2 + r_1 \varepsilon_1 f_y).\end{aligned}\tag{5.20}$$

For clarity we omit all function arguments. For $\varepsilon_1 > 0$ the factor $\varepsilon_1(\tilde{h}_2 + r_1 f_y)$ is positive sufficiently close to ζ_2 . Therefore we divide (5.20) by $\varepsilon_1(\tilde{h}_2 + r_1 f_y)$.

$$\begin{aligned}\tilde{x}'_1 &= (\tilde{x}_1 - \sqrt{B_0/C_0}) + \frac{\tilde{h}_1}{\tilde{h}_2 + r_1 f_y} + r_1 \frac{f_x}{\tilde{h}_2 + r_1 f_y} \\ a'_1 &= a_1 + r_1 \frac{g_a}{\tilde{h}_2 + r_1 f_y} \\ r'_1 &= -r_1 \\ \varepsilon'_1 &= 3\varepsilon_1.\end{aligned}\tag{5.21}$$

Note that this uncovers a flow in the critical manifold $W^c(\mathcal{S}_{0,1}^a)|_{\varepsilon_1=0}$ close to ζ_2 . In fact, we must recover, up to orbital equivalence, the desingularized slow flow (5.16) near ζ_2 for $\varepsilon_1 = 0$. We want to simplify (5.21) further. First, note that by the shorthand g_a we actually mean the function $(g_a \circ \Phi_{-y} \circ T^{-1})(\tilde{x}_1 - \sqrt{B_0/C_0}, r_1, a_1, u, v, \varepsilon_1)$, where further $u = \varepsilon_1 \tilde{h}_1$ and $v = \varepsilon_1 \tilde{h}_2$ are plugged in. Written out, this equals $g_a(r_1 x_1, -r_1, r_1 a_1, r_1^2(\varepsilon_1 \tilde{h}_1 - x_1^2 + a_1), r_1^2(\varepsilon_1 \tilde{h}_2 - 1 - a_1 x_1), r_1^3 \varepsilon_1)$. By splitting off $g_a|_{\varepsilon=0}$ in original coordinates by a Taylor argument, this can be rewritten as $(\tilde{g}_a \circ \Theta_{-y})(x_1, r_1, a_1) + r_1^3 \varepsilon_1 f_1$, where f_1 is a sufficiently regular function (which is bounded in a neighborhood of the origin in original coordinates) and $\tilde{g}_a(x, y, a) = g_a(x, y, a, -x^2 - ay, -y^2 - ax, 0)$. Let $G = \tilde{h}_2 + r_1 f_y$, where f_y is a shorthand for $(f_y \circ \Phi_{-y} \circ T^{-1})(x_1, r_1, a_1, \varepsilon_1 \tilde{h}_1, \varepsilon_1 \tilde{h}_2, \varepsilon_1)$. For the first equation, the constant term cancels out with $\tilde{h}_1(0)/\tilde{h}_2(0)$ and employing the invariance equation for the graph of the center manifold with the ansatz $\tilde{h}_1 = \sqrt{B_0 C_0}/2 + \xi_1 \tilde{x}_1 + \dots$ and $\tilde{h}_2 = C_0/2 + \xi_2 \tilde{x}_1 + \dots$ implies the assertion. \square

Note that the linearization of (5.19) at ζ_2 has resonant eigenvalues $2, 1, -1, 3$. Let $W^c(\zeta_2)$ denote the intersection of (any instance of) $W^c(\mathcal{S}_{0,1}^a)$ with the blow-up sphere $\{r_1 = 0\}$ in κ_{-y} .

Lemma 5.14. *The branch $\varepsilon_1 > 0$ of the 3-dimensional attracting center manifold $W^c(\zeta_2)$ is unique.*

Proof. By Lemma 5.13, the equilibrium ζ_2 is a source in $W^c(\zeta_2)|_{\varepsilon_1 \geq 0}$. Moreover, $W^c(\zeta_2)$ is attracting. Then, the asymptotic rate fibration with base being the center manifold implies uniqueness of the branch for $\varepsilon_1 > 0$. \square

$W^c(\zeta_2)$ contains the ‘‘singular continuations’’ of σ_1 , which start at ζ_2 and emanate on the blow-up sphere into the rescaling chart κ_ε . In general, we are interested in the transition from trajectories in attracting slow manifolds starting at Δ^{en} close to the hyperbolic umbilic point. The section Δ^{en} appears in κ_{-y} as

$$\Delta_1^{\text{en}} = \{r_1 = \nu\}.\tag{5.22}$$

The attracting slow manifolds are contained in $W^c(\mathcal{S}_{0,1}^a)$ and their evolution close to ζ_2 is determined by (5.19). To enter the rescaling chart κ_ε we set up a section

$$\Delta_1^{\text{en} \rightarrow \varepsilon} = \{\varepsilon_1 = \delta\},\tag{5.23}$$

for small $\delta > 0$, and aim to study a transition of the form $\Delta_1^{\text{en}} \cap W^c(\mathcal{S}_{0,1}^a)|_{\varepsilon_1 > 0} \rightarrow \Delta_1^{\text{en} \rightarrow \varepsilon}$ near ζ_2 via the flow of (5.19). Note that ζ_2 is a source on the blow-up sphere and the r_1 -direction is decreasing. The eigenvalue of the linearization of (5.19) on the blow-up sphere transverse to the equator is 3, so the expansion is strongest in this direction. Since the 1-d stable

manifold of ζ_2 is given by $\sigma_1 \subset \{\varepsilon_1 = 0\}$, we expect that initial conditions close to σ_1 will lead to solution trajectories with least blow-up. Recall that we want to see the evolution of slow manifolds *near* the hyperbolic umbilic, which is located in $a = 0$ in original coordinates. In fact, also from the analysis in the rescaling chart it will be clear that we need control on the variable a_1 .

Consider a transition map $\Pi_1: I_1 \subset \Delta_1^{\text{en}} \cap W^c(\mathcal{S}_{0,1}^a)|_{\varepsilon_1 > 0} \rightarrow \Delta_1^{\text{en} \rightarrow \varepsilon}$. Take an initial condition $(\tilde{x}_i, a_i, \nu, \varepsilon_i)$. From (5.19) we have that $r_1(t) = \nu e^{-t}$, $\varepsilon_1(t) = \varepsilon_i e^{3t}$, where we use t to indicate the time parametrization in the current chart. This gives the transition time $T = \log(\delta/\varepsilon_i)/3$ for Π_1 . Assume for a moment that the function $g_a|_{\varepsilon=0}$ factors the variable a . Then we have

Lemma 5.15. *Assume that $g_a|_{\varepsilon=0} = a \cdot \bar{g}_a$ for some sufficiently smooth \bar{g}_a . Then the transition map $\Pi_1: I_1 \subset \Delta_1^{\text{en}} \cap W^c(\mathcal{S}_{0,1}^a)|_{\varepsilon_1 > 0} \rightarrow \Delta_1^{\text{en} \rightarrow \varepsilon}$ is of the form*

$$\Pi_1: \begin{bmatrix} x_i \\ a_i \\ \nu \\ \varepsilon_i \end{bmatrix} \mapsto \begin{bmatrix} \Pi_{1x_1}(x_i, a_i, \nu, \varepsilon_i) \\ \left(\frac{\delta}{\varepsilon_i}\right)^{1/3} O(a_i) + O\left(\varepsilon_i^{2/3} \log \varepsilon_i\right) \\ \nu \left(\frac{\varepsilon_i}{\delta}\right)^{1/3} \\ \delta \end{bmatrix} \quad (5.24)$$

Proof. Let $a_1(t) = e^t(a_i + z_a(t))$, such that $z_a(0) = 0$. It follows from (5.19) that $z'_a = e^{-t} r_1^2 a_1 \bar{g}_a + r_1^3 \varepsilon_1 f_1$, for sufficiently smooth functions \bar{g}_a, f_1 . This implies

$$z_a(T) = \int_0^T \nu^2 e^{-2t} a_i \bar{g}_a dt + \int_0^T \nu^2 e^{-2t} z_a \bar{g}_a dt + \int_0^T \varepsilon_i \nu^3 f_1 dt, \quad (5.25)$$

where the functions are evaluated along the corresponding integral curve. Recall that \bar{g}_a, f_1 are bounded close to the blow-up sphere and that $T = \log(\delta/\varepsilon_i)/3$. The last integral is of order $O(\varepsilon_i \log \varepsilon_i)$. The first integral is of order $O(a_i)$. Application of the Grönwall inequality gives that the second integral is also $O(a_i)$. Hence $a_1(T)$ is of the form $a_1(T) = \left(\frac{\delta}{\varepsilon_i}\right)^{1/3} (a_i + O(a_i) + O(\varepsilon_i \log \varepsilon_i))$. \square

In the general case that $g_a|_{\varepsilon=0}$ does not factor the variable a , we have the rough estimate $a_1(T) = \left(\frac{\delta}{\varepsilon_i}\right)^{1/3} (a_i + O(\nu))$ in context of the above Lemma.

Remark 5.16. *The assumption that $g_a|_{\varepsilon=0} = a \cdot \bar{g}_a$ allows to control $a_1(T)$ solely in terms of a_i and ε_i . In particular, Lemma 5.15 implies that the domain I_1 of Π_1 must be of width $O(\varepsilon_i^{1/3})$ in a_1 -direction, to have $a_1(T)$ small. That is, for small, fixed ε_i , the domain $I_1|_{\varepsilon_i}$ is a thin (but bounded) strip of width $O(\varepsilon_i^{1/3})$. In the singular limit, it contains the trajectory σ_1 , which lies in the invariant plane $\{a_1 = 0\}$ in this case. In the general setting, i.e. without the assumption $g_a|_{\varepsilon=0} = a \cdot \bar{g}_a$, the domain I_1 needs to be chosen sufficiently small in order to guarantee that $a_1(T)$ is small. We do not have an estimate of the size of the domain with respect to the initial conditions in this case. We shall choose I_1 sufficiently small in section 5.7 by a time reversion argument. In the singular limit, I_1 coincides with the single point $p := \sigma_1 \cap \Delta_1^{\text{en}}$.*

5.5. Analysis in the rescaling chart κ_ε . Blowing up X via Φ_ε and rescaling time accordingly leads to

$$\begin{aligned} x'_2 &= x_2^2 + a_2 y_2 + b_2 + O(r_2) \\ y'_2 &= y_2^2 + a_2 x_2 + c_2 + O(r_2) \\ a'_2 &= A_0 r_2 + O(r_2^2) \\ b'_2 &= B_0 + O(r_2) \\ c'_2 &= C_0 + O(r_2) \\ r'_2 &= 0. \end{aligned} \quad (5.26)$$

In $\{r_2 = 0\}$ the vector field (5.26) reduces to a coupled system of two Riccati-type equations

$$\begin{aligned} x_2' &= x_2^2 + a_2 y_2 + b_2 \\ y_2' &= y_2^2 + a_2 x_2 + c_2 \\ a_2' &= 0 \\ b_2' &= B_0 \\ c_2' &= C_0, \end{aligned} \tag{5.27}$$

whose phase space is foliated into invariant sets $\{a_2 = \text{const}\}$. Note that $a_2 = a_1 \varepsilon_1^{-1/3}$, which gives the invariant sets on the blow-up sphere in κ_{-y} , also visible in (5.6). Further note that the sets $\{C_0 b_2 - B_0 c_2 = \text{const}\}$ are invariant. Recall the attracting center manifold $W^c(\zeta_2)$ from the previous section. We aim to follow trajectories in $W^c(\zeta_2)|_{\varepsilon_1 > 0}$, which emanate from ζ_2 on the blow-up sphere into the rescaling chart. We expect that these orbits will transition onto the fast regime, that is, connect to the appropriate equilibria “on the other side” of the blow-up sphere. In $\{a_2 = 0\}$ the coupling of (5.27) is absent. For a_2 close to 0 the flow of (5.27) is a regular perturbation of the flow in $\{a_2 = 0\}$. The full flow (5.26) is in turn a regular perturbation of (5.27). Therefore, we begin with the analysis of the invariant set $\{a_2 = 0\}$ of (5.27). Note that this essentially leads to a pair of decoupled Riccati equations, as they appear in the rescaling chart for the non-degenerate fold [19]. The task is now to identify the trajectories being backward asymptotic to ζ_2 and to determine their forward asymptotics.

For $a_2 = 0$ the vector field (5.27) is of product structure (Z_1, Z_2) , where both Z_i are vector fields on \mathbb{R}^2 . In fact, each Z_i has a unique “dividing solution”, denote it by γ_i for the moment, which is an integral curve dividing the phase plane in two classes of solutions with different asymptotic behavior, itself having distinct behavior [19, 23]. Only combinations of these γ_i lead to trajectories being backward asymptotic to ζ_2 , as shall be apparent from the proof of Proposition 5.18 below.

Remark 5.17. *To state the following result in a convenient way, we anticipate the main objects visible in the exit chart κ_{ex} . These are three equilibria denoted by q_4, q_5, q_6 , which are located on the equator of the blow-up sphere. As will turn out later, they arise from the interaction of the fast subsystem with the hyperbolic umbilic singularity and will serve as transition points onto the fast regime. Refer to section 5.6, in particular figure 14.*

Proposition 5.18. *Let $B_0, C_0 > 0$. In $\{a_2 = 0\}$ the vector field (5.27) has a 1-parameter family of integral curves γ_s , $s \in \mathbb{R}$, given by*

$$\gamma_s(t) = (\gamma_{x_2}(t), \gamma_{y_2}(t+s), a_2 = 0, B_0 t, C_0(t+s)), \tag{5.28}$$

which are the only trajectories in $\{a_2 = 0\}$ backward asymptotic to ζ_2 . In (5.28), γ_{x_2} is given by (5.34) and γ_{y_2} by (5.36). The family γ_s determines an invariant 2-manifold Γ in $\{r_2 = a_2 = 0\}$. The trajectories γ_s have the following forward asymptotic behavior:

- For $s > s_0$, γ_s is forward asymptotic to q_4 ,
- γ_{s_0} is forward asymptotic to q_5 ,
- for $s < s_0$, γ_s is forward asymptotic to q_6 ,

where q_4, q_5 , and q_6 are equilibria visible in the chart κ_{ex} (see Remark 5.17), $s_0 = (B_0^{-1/3} - C_0^{-1/3})z_0$, and $z_0 < 0$ is the first zero of the Airy function Ai of the first kind, see e.g. [25, 9.9(i)].

Proof. In $\{a_2 = 0\}$, system (5.27) reduces to two decoupled Riccati equations, as they appear in the rescaling chart of the regular fold [19]. First, consider

$$\begin{aligned} x_2' &= x_2^2 + b_2 \\ b_2' &= B_0. \end{aligned} \tag{5.29}$$

In [23, Chapter II, 9.] the asymptotic behavior of a Riccati equation is studied by reduction to a Bessel equation. Here, we choose to reduce to an Airy equation. Letting $x_2 = -\frac{u'}{u}$ and $b_2 = B_0 t$ transforms (5.29) into

$$\begin{aligned} u'' &= -B_0 t u \\ t' &= 1, \end{aligned} \quad (5.30)$$

which has the two linearly independent solutions $\text{Ai}(-B_0^{1/3}t), \text{Bi}(-B_0^{1/3}t)$. Hence, we obtain the general solution for (5.29) given by

$$x_2(t) = \frac{B_0^{1/3} \omega_1 \text{Ai}'(-B_0^{1/3}t) + B_0^{1/3} \omega_2 \text{Bi}'(-B_0^{1/3}t)}{\omega_1 \text{Ai}(-B_0^{1/3}t) + \omega_2 \text{Bi}(-B_0^{1/3}t)}, \quad (5.31)$$

where ω_1 and ω_2 are constants that do not vanish simultaneously. It is convenient to re-write the solutions (5.31) in terms of a single parameter $d \in (-\infty, \infty]$ as

$$x_2(t) = \begin{cases} \tilde{B} \frac{d \text{Ai}'(-\tilde{B}t) + \text{Bi}'(-\tilde{B}t)}{d \text{Ai}(-\tilde{B}t) + \text{Bi}(-\tilde{B}t)}, & d < \infty, \\ \tilde{B} \frac{\text{Ai}'(-\tilde{B}t)}{\text{Ai}(-\tilde{B}t)}, & d = +\infty, \end{cases} \quad (5.32)$$

where $d = \frac{\omega_1}{\omega_2}$, and $\tilde{B} = B_0^{1/3}$. Each choice of d defines a curve with asymptotes at certain values of t , due to the zeros of the denominator. The solutions (5.32) are sketched in figure 13. The case $d = +\infty$ corresponds to the so called “dividing solution”. By Lemma A.3, we

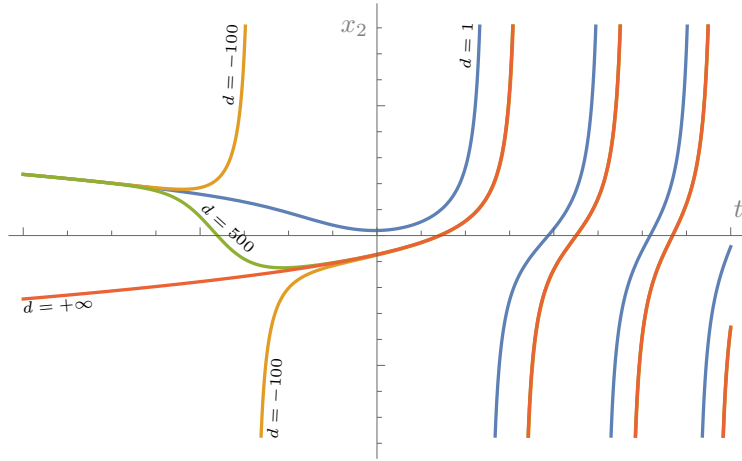


FIGURE 13. Qualitative plot of the solutions (5.32) to the Riccati equation (5.29) with $t = b_2/B_0$ and $d = -100, 1, 500, +\infty$. Any solution is defined for almost every $t \in \mathbb{R}$, countably many vertical asymptotes arise from the properties of the Airy functions. Note the appearance of the parabola $B_0 t = -x_2^2$, which opens to the left and which all solutions approach as $t \rightarrow -\infty$.

write the equations of the critical manifold from κ_{-y} (and similarly in κ_{-x}) in κ_ε -coordinates, which yields

$$\begin{aligned} b_2 &= -a_2 y_2 - x_2^2 \\ c_2 &= -a_2 x_2 - y_2^2. \end{aligned} \quad (5.33)$$

Again by Lemma A.3, solutions of (5.27) are backward asymptotic to $\mathcal{S}_{0,1}|_{r_1=0}$ in κ_{-y} if and only if they asymptotically satisfy (5.33) for $y_2 \rightarrow -\infty$. We seek for solutions in $\{a_2 = 0\}$ being backward asymptotic to ζ_2 . Hence, we seek solutions of (5.27) satisfying $b_2 = -x_2^2$ as $x_2 \rightarrow -\infty$ and $c_2 = -y_2^2$ as $y_2 \rightarrow -\infty$ asymptotically in backward time. The requirement $b_2 = -x_2^2$ as $x_2 \rightarrow -\infty$ is only satisfied by the “dividing solution”, i.e. (5.32) with $d = +\infty$: The Airy functions Ai, Bi do not have zeros on the positive half-line and have asymptotic

expansions [25, 9.7(ii)]. Employing such asymptotic expansions, it follows that $x_2 \rightarrow -\infty$ in (5.32) as $t \rightarrow -\infty$ only in the case $d = +\infty$: for $t \rightarrow -\infty$ one has

$$\gamma_{x_2}(t) := \tilde{B} \frac{\text{Ai}'(-\tilde{B}t)}{\text{Ai}(-\tilde{B}t)} \sim -\sqrt{-B_0 t}, \quad (5.34)$$

so γ_{x_2} is backward asymptotic to the branch $x_2 < 0$ of the parabola $b_2 = B_0 t = -x_2^2$, see figure 13. Note that solutions obtained from (5.32), which are defined on finite time intervals (t_1, t_2) , may also show $x_2 \rightarrow -\infty$ for $t \downarrow t_1$. However, in this case, $b_2 \not\rightarrow -\infty$, so these solutions cannot satisfy $b_2 = -x_2^2$ as $x_2 \rightarrow -\infty$.

An identical analysis as above applies to the second half of (5.27) in $\{a_2 = 0\}$, which is the system

$$\begin{aligned} y_2' &= y_2^2 + c_2 \\ c_2' &= C_0. \end{aligned} \quad (5.35)$$

The “dividing solution” for (5.35) is given by

$$\gamma_{y_2}(t) := \tilde{C} \frac{\text{Ai}'(-\tilde{C}t)}{\text{Ai}(-\tilde{C}t)}, \quad (5.36)$$

where $\tilde{C} = C_0^{1/3}$. By the product structure of (5.27) in $\{a_2 = 0\}$ we obtain all integral curves built from the “dividing solutions” γ_i essentially by putting $(\gamma_1(t), \gamma_2(t+s))$ for a free parameter $s \in \mathbb{R}$. Thus, we obtain a 1-parameter family of solutions in $\{a_2 = 0\}$, determined by

$$\gamma_s(t) = (\gamma_{x_2}(t), \gamma_{y_2}(t+s), a_2 = 0, B_0 t, C_0(t+s)), \quad (5.37)$$

where $s \in \mathbb{R}$. The above argument then implies that the trajectories γ_s are precisely the integral curves, which are backward asymptotic to the critical manifold. Then, employing asymptotic expansions of the Airy functions [25, 9.7(ii)] and Lemma A.3, one checks that

$$\frac{\gamma_{x_2}(t)}{-\gamma_{y_2}(t+s)} \rightarrow -\sqrt{B_0/C_0}, \quad (5.38)$$

as $t \rightarrow -\infty$. This implies that the trajectories γ_s are precisely the solutions of (5.27) in $\{a_2 = 0\}$, which are backward asymptotic to ζ_2 .

What remains to be checked is the forward asymptotics of the trajectories γ_s . We will employ $\kappa_{\varepsilon \rightarrow \text{ex}}$ from Lemma A.3 and the fact that the Airy function Ai does have a largest zero [25, 9.9(i)], which we denote by z_0 . Let M, N be the first zeros of the denominators of γ_{x_2} and γ_{y_2} , in the sense that $\text{Ai}(-B_0^{1/3}t)$ is defined on $(-\infty, M)$ and $\text{Ai}(-C_0^{1/3}t)$ is defined on $(-\infty, N)$, both domains maximal. Thus $z_0 = -B_0^{1/3}M = -C_0^{1/3}N$. Then γ_{x_2} is defined on $(-\infty, M)$ and $\gamma_{y_2}(\cdot + s)$ is defined on $(-\infty, N_s)$, where $N_s = N - s$. Since Ai is positive before its first zero z_0 and $\text{Ai}'(z_0) > 0$, it follows that $\gamma_{x_2}(t) \rightarrow +\infty$ for $t \uparrow M$ and $\gamma_{y_2}(t+s) \rightarrow +\infty$ for $t \uparrow N_s$. Each $\gamma_s(t)$ is defined for all $t < \min\{M, N_s\}$. The ratio of M and N_s determines the forward asymptotics of γ_s . The relevant component of the coordinate change $\kappa_{\varepsilon \rightarrow \text{ex}}$ is

$$x_5 = \frac{x_2 - y_2}{x_2 + y_2}, \quad (5.39)$$

since by Lemma A.3 all other components converge to zero along every γ_s in forward time. By (5.49) we have that q_4 corresponds to $x_5 = -1$, q_5 to $x_5 = 0$ and q_6 to $x_5 = 1$. Consider the three cases $M > N_s$, $M < N_s$ and $M = N_s$:

If $M > N_s$ then $\gamma_{y_2}(t+s)$ blows up as $t \uparrow N_s$, so

$$x_5 = \left(\frac{\gamma_{x_2}(t)}{\gamma_{y_2}(t+s)} - 1 \right) \left(\frac{\gamma_{x_2}(t)}{\gamma_{y_2}(t+s)} + 1 \right)^{-1} \rightarrow -1 \quad \text{for } t \uparrow N_s < M. \quad (5.40)$$

If $M < N_s$ then $\gamma_{x_2}(t)$ blows up as $t \uparrow M$, so

$$x_5 = \left(1 - \frac{\gamma_{y_2}(t+s)}{\gamma_{x_2}(t)}\right) \left(1 + \frac{\gamma_{y_2}(t+s)}{\gamma_{x_2}(t)}\right)^{-1} \rightarrow 1 \quad \text{for } t \uparrow M < N_s. \quad (5.41)$$

If $M = N_s \iff s = N - M$, both $\gamma_{x_2}(t) \rightarrow +\infty$ and $\gamma_{y_2}(t+s) \rightarrow +\infty$ as $t \uparrow M$. In the case of $B_0 = C_0$, and therefore $s = 0$, we directly see from (5.39) that γ_0 is forward asymptotic to q_5 . To treat the general case $s = N - M$, consider

$$x_5 = \left(\frac{\gamma_{x_2}(t)}{\gamma_{y_2}(t+N-M)} - 1\right) \left(\frac{\gamma_{x_2}(t)}{\gamma_{y_2}(t+N-M)} + 1\right)^{-1}. \quad (5.42)$$

To find the limit of $\frac{\gamma_{x_2}(t)}{\gamma_{y_2}(t+N-M)}$ as $t \uparrow M$, note that a first order Taylor expansion gives $\text{Ai}(-\tilde{B}t) \sim -\tilde{B} \text{Ai}'(z_0)$ as $t \rightarrow M$. Similarly $\text{Ai}(-\tilde{C}(t+N-M)) \sim -\tilde{C} \text{Ai}'(z_0)$ as $t \rightarrow M$. This implies that $\frac{\gamma_{x_2}(t)}{\gamma_{y_2}(t+N-M)} \rightarrow 1$. Hence, γ_{N-M} is forward asymptotic to q_5 , and we let $s_0 = N - M$. \square

Remark 5.19. *Solutions (x_2, y_2, b_2, c_2) in $\{r_2 = 0\}$ need to asymptotically satisfy $b_2 = -a_2 y_2 - x_2^2$, $c_2 = -a_2 x_2 - y_2^2$ in order to be asymptotic to the critical manifold. Further they need to show $y_2 \rightarrow -\infty$ in order to arrive at the equator in κ_{-y} , $x_2 \rightarrow -\infty$ to arrive at the equator in κ_{-x} , et cetera. Suppose that such solutions are asymptotic to slow flow equilibria $\zeta_{2,3,4,5}$ or fast subsystem equilibria $q_{1,2,3,4,5,6}$ (in fact, there do not exist solutions of (5.27) being asymptotic to $\zeta_{1,6}$, see Remark 5.20). It is then possible to determine the forward/backward limits of all solutions in $\{r_2 = 0, a_2 = 0\}$ by proceeding similarly to the proof of Proposition 5.18.*

Remark 5.20. *At this point we remark that the charts $\kappa_{\pm a}$ can be used to supply an analysis of the flow of (5.27) for sufficiently large $|a_2|$. More precisely, the fast-slow systems (6.3) and (6.4) in section 6 are instances of (5.27) for large $|a_2|$. In fact, in the singular limit analysis of (6.3) and (6.4), Remark 4.6 can be employed to show that the relevant candidate trajectories coming from ζ_2 are jumping and arriving to q_4 or q_6 , depending on the location of the jump. Moreover, in $\kappa_{\pm a}$ one sees that there do not exist trajectories of (5.27) being forward/backward asymptotic to ζ_1 , see e.g. Lemma 6.1. An analogous statement can be obtained for ζ_6 .*

The family γ_s from Proposition 5.18 gives all trajectories in $W^c(\zeta_2)|_{a_2=0}$, which enter the rescaling chart κ_ε and are backward asymptotic to ζ_2 . Recall that $W^c(\zeta_2)$ is contained in the blow-up sphere. The section $\Delta_1^{-y \rightarrow \varepsilon}$, which appears in κ_ε as $\Delta_2^{-y \rightarrow \varepsilon} = \{y_2 = -\delta^{-1/3}\}$, has transverse intersection with Γ , where Γ is the invariant manifold determined by the family γ_s .

For a_2 sufficiently close to zero, the flow of (5.27) is a regular perturbation of the flow in $\{a_2 = 0\}$. Perturbations of γ_s give a 2-parameter family of integral curves following Γ closely through the rescaling chart. Furthermore, the a_2 -perturbations of any γ_s are backward asymptotic to ζ_2 : Since these regular perturbations are $O(a_2)$ -uniformly close to the unperturbed solutions on any finite time interval, the a_2 -perturbations must also be contained in $W^c(\zeta_2)$, because in backward time they cannot get exponentially repelled from $W^c(\zeta_2)$. Moreover, by Lemma (5.13), the equilibrium ζ_2 appears in $W^c(\zeta_2)$ as source, which implies the backward asymptotic behavior of the perturbations. Let $\hat{\Gamma}$ denote the invariant 3-manifold obtained by regular a_2 -perturbations of trajectories in Γ and note that $\hat{\Gamma} = \{r_2 = 0, |a_2| < L\}$ for some small $L > 0$. For trajectories in the submanifold $\Gamma \subset \hat{\Gamma}$ we have established the forward asymptotic behavior. The forward asymptotic behavior of trajectories in $\hat{\Gamma}$ will be established in the exit chart κ_{ex} after Lemma 5.27.

For small $r_2 > 0$ the full system (5.26) is again a regular perturbation of the flow in $\{r_2 = 0\}$, i.e. (5.27). In particular, trajectories in $\hat{\Gamma}$ get regularly perturbed for small $r_2 > 0$. These are the trajectories in attracting slow manifolds close to the hyperbolic umbilic. We will track these further in the exit chart κ_{ex} . Recall from Lemma A.3 that we can change coordinates to κ_{ex} , as soon as $x_2 + y_2 > 0$.

We conclude this section with a simple statement on the structure of a regular transition map Π_2 from a suitable subset of $\Delta_2^{-y \rightarrow \varepsilon}$ to (one choice of) suitable sections close to the equilibria $q_{4,5,6}$, which will be specified later.

Lemma 5.21. *Let $V \subset \Delta_2^{-y \rightarrow \varepsilon}$ be an open set corresponding to initial conditions obtained from r_2 -perturbations of $\hat{\Gamma}$ and let Ω be any hypersurface, which intersects all trajectories with initial condition in V . Then the transition $\Pi_2: V \rightarrow \Omega$ is of the form*

$$\Pi_2: \begin{bmatrix} x_j \\ -\delta^{-1/3} \\ a_j \\ b_j \\ c_j \\ r_j \end{bmatrix} \mapsto \begin{bmatrix} x_e \\ y_e \\ 0 \\ b_e \\ c_e \\ 0 \end{bmatrix} + \begin{bmatrix} O(a_j) \\ O(a_j) \\ a_j \\ O(a_j) \\ O(a_j) \\ 0 \end{bmatrix} + \begin{bmatrix} O(r_j) \\ O(r_j) \\ O(r_j) \\ O(r_j) \\ O(r_j) \\ r_j \end{bmatrix}, \quad (5.43)$$

where a_j and r_j are sufficiently small and the map $(x_j, -\delta^{-1/3}, b_j, c_j) \mapsto (x_e, y_e, b_e, c_e)$ is determined by the flow in Γ , its integral curves are given by (5.28).

Proof. By the above considerations and regular perturbation theory. \square

Remark 5.22. $C_0 b_2 - B_0 c_2$ is a constant of motion for (5.27). In fact, for $\gamma_s \in \Gamma$ we have $C_0 b_2 - B_0 c_2 = -B_0 C_0 s$. That is, the b_2 - and c_2 -component of the initial condition of γ_s determine s and therefore its forward limit. Hence b_j and c_j in (5.43) determine the family member γ_s , from which the transition (5.43) is a regular perturbation.

5.6. Analysis in the exit chart κ_{ex} . Let us briefly explain how the chart κ_{ex} and its blow-up Φ_{ex} are obtained. Recall that Φ_{ex} is defined by

$$\Phi_{\text{ex}}: x = r_5(1 + x_5), y = r_5(1 - x_5), a = r_5 a_5, b = r_5^2 b_5, c = r_5^2 c_5, \varepsilon = r_5^3 \varepsilon_5. \quad (5.44)$$

Let $\tilde{x} = \frac{x-y}{2}$ and $\tilde{y} = \frac{x+y}{2}$. Up to scaling the coordinates, this is a 45° rotation. In these new coordinates \tilde{x}, \tilde{y} , the map Φ_{ex} is a usual weighted blow-up in direction of \tilde{y} . Further, note that (5.44) implies $\frac{x-y}{2} = r_5 x_5$ and $\frac{x+y}{2} = r_5$. We can also transform the original vector field X (5.1) according to $\tilde{x} = \frac{x-y}{2}$ and $\tilde{y} = \frac{x+y}{2}$. The first two components of X then transform to

$$\begin{aligned} \tilde{x}' &= 2\tilde{x}\tilde{y} - a\tilde{x} + \frac{b-c}{2} + O(\varepsilon) \\ \tilde{y}' &= \tilde{x}^2 + \tilde{y}^2 + a\tilde{y} + \frac{b+c}{2} + O(\varepsilon). \end{aligned} \quad (5.45)$$

For $\varepsilon = 0$, this is a gradient vector field in the fast variables (\tilde{x}, \tilde{y}) arising from the potential

$$\tilde{V} = \frac{1}{3}\tilde{y}^3 + \tilde{y}\tilde{x}^2 + \tilde{a}(\tilde{y}^2 - \tilde{x}^2) + \tilde{b}\tilde{x} + \tilde{c}\tilde{y}, \quad (5.46)$$

with parameters $\tilde{a} = \frac{a}{2}$, $\tilde{b} = \frac{b-c}{2}$ and $\tilde{c} = \frac{b+c}{2}$. In fact, the potential \tilde{V} is another universal unfolding of the hyperbolic umbilic catastrophe. This is shown in appendix A.3. With this universal unfolding \tilde{V} , the double cone of singularities is given by $\tilde{y}^2 = \tilde{a}^2 + \tilde{x}^2$, i.e. the rotational axis of the double cone is the \tilde{y} -axis.

Remark 5.23. *For the prior analysis we have chosen the universal unfolding V leading to (1.1), since the expressions of the desingularized slow flow and the vector field in the rescaling chart are more convenient to analyze.*

Blowing up X with Φ_{ex} and dividing out a common factor r_5 leads to

$$\begin{aligned} x'_5 &= 2x_5 - a_5x_5 + \frac{b_5 - c_5}{2} - x_5F + r_5\varepsilon_5 \frac{f_x - f_y}{2} \\ r'_5 &= r_5F \\ a'_5 &= -a_5F + r_5\varepsilon_5g_a \\ b'_5 &= -2b_5F + \varepsilon_5g_b \\ c'_5 &= -2c_5F + \varepsilon_5g_c \\ \varepsilon'_5 &= -3\varepsilon_5F, \end{aligned} \tag{5.47}$$

where $F = F(x_5, r_5, a_5, b_5, c_5, \varepsilon_5) = 1 + x_5^2 + a_5 + \frac{b_5 + c_5}{2} + r_5\varepsilon_5 \frac{f_x + f_y}{2}$ and the shorthand f_x for $f_x \circ \Phi_{\text{ex}}$ is used, similarly for f_y, g_a, g_b, g_c .

The subspaces $\{r_5 = 0\}$ and $\{\varepsilon_5 = 0\}$ are invariant. Furthermore, $\{r_5 = 0, a_5 = 0\}$, $\{\varepsilon_5 = 0, a_5 = 0\}$, $\{\varepsilon_5 = 0, b_5 = 0\}$ and $\{\varepsilon_5 = 0, c_5 = 0\}$ (and any of their intersections) are also invariant.

Consider (5.47) restricted to the invariant subspace $\{r_5 = 0, \varepsilon_5 = 0\}$. Similarly to the entry charts, this leads to a 3-manifold of equilibria determined by

$$\begin{aligned} 1 + x_5^2 + a_5 + \frac{b_5 + c_5}{2} &= 0 \\ 2x_5 - a_5x_5 + \frac{b_5 - c_5}{2} &= 0, \end{aligned} \tag{5.48}$$

which is the critical manifold appearing in the exit chart. Besides that, there are three isolated equilibria

$$\begin{aligned} q_4 &= (x_5 = -1, a_5 = 0, b_5 = 0, c_5 = 0), \\ q_5 &= (x_5 = 0, a_5 = 0, b_5 = 0, c_5 = 0), \\ q_6 &= (x_5 = 1, a_5 = 0, b_5 = 0, c_5 = 0), \end{aligned} \tag{5.49}$$

which are precisely the equilibria on the blow-up sphere arising from the fast subsystem.

We now further motivate the role of the equilibria $q_{4,5,6}$. Recall the fast subsystem obtained from X (5.1) by letting $\varepsilon = 0$. Trajectories in the fast subsystem can only approach the origin in forward or backward time if $a = b = c = 0$. Hence, in the singular limit, the fast dynamics interacting with the hyperbolic umbilic singularity are given by the system

$$\begin{aligned} x' &= x^2 \\ y' &= y^2, \end{aligned} \tag{5.50}$$

where the hyperbolic umbilic point appears as nilpotent equilibrium. Also refer to figure 2, which contains the phase portrait of (5.50). The blow-up of (5.50) is contained in the blow up of the vector field X (5.1) by restricting to the subspace $\{\bar{a} = \bar{b} = \bar{c} = \bar{\varepsilon} = 0\}$. In κ_{ex} this corresponds to the invariant subspace $\{\varepsilon_5 = 0, a_5 = 0, b_5 = 0, c_5 = 0\}$, which we will analyze shortly in the following.

The blow-up of (5.50) is visualized in figure 14. From Proposition 5.18 we know that the family of integral curves γ_s in $\{r_2 = a_2 = 0\} \subset \kappa_\varepsilon$ connects in forward time to $q_{4,5,6}$. More precisely, only γ_{s_0} connects to q_5 , whereas the remaining members of the family γ_s connect to either q_4 or q_6 . In the singular limit, from $q_{4,5,6}$, such trajectories continue onto the fast regime and away from the blow-up sphere. This motivates that trajectories of X , which are contained in attracting slow manifolds and come close to the hyperbolic umbilic, will transition onto the fast regime near the equilibria $q_{4,5,6}$. The saddle q_5 leads to trajectories fanning out in-between the positive x - and y -axis. The main goal of this section is to investigate this transition away from the blow-up sphere.

Remark 5.24. *Due to the rotated choice of Φ_{ex} (5.44), the three equilibria $q_{4,5,6}$ and all “fast escape directions” are visible in the single chart κ_{ex} .*

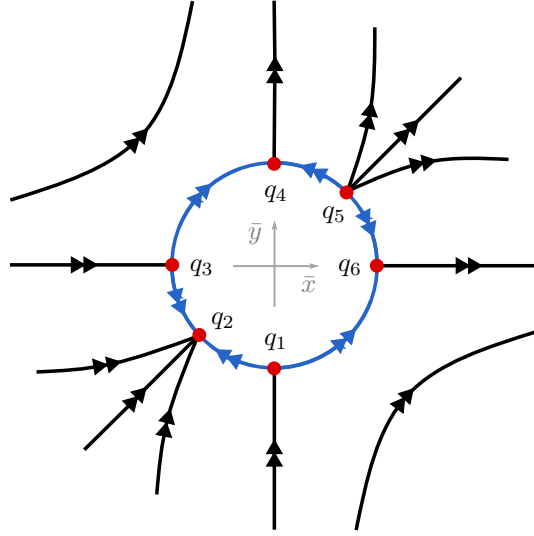


FIGURE 14. Blow-up of (5.50), i.e. the fast subsystem interacting with the hyperbolic umbilic point. The blow-up sphere \mathbb{S}^1 restricted to $\bar{a} = \bar{b} = \bar{c} = \bar{\varepsilon} = 0$ appears as \mathbb{S}^1 (blue), with six equilibria $q_{1,2,3,4,5,6}$ (red), where $q_{4,5,6}$ are visible in κ_{ex} , $q_{1,2}$ appear in κ_{-y} and $q_{2,3}$ in κ_{-x} . Note that the unstable manifolds at $q_{4,5,6}$ also appear in this figure. As we show in this section, in the singular limit the trajectories coming from the rescaling chart κ_ε escape towards the fast regime along the aforementioned unstable manifolds.

We briefly look at the linearization of (5.47) at $q_{4,5,6}$. This directly leads to the following result.

Lemma 5.25. *At q_4, q_5, q_6 , the resonant linearizations of (5.47) are of the following upper-triangular form:*

$$\text{At } q_4, q_6: \begin{bmatrix} -2 & 0 & & & & \\ & 2 & & * & & \\ & & -2 & & & \\ & & & -4 & & \\ & & & & -4 & \\ & & & & & -6 \end{bmatrix}, \text{ at } q_5: \begin{bmatrix} 1 & 0 & & & & \\ & 1 & & * & & \\ & & -1 & & & \\ & & & -2 & & \\ & & & & -2 & \\ & & & & & -3 \end{bmatrix}.$$

The equilibria $q_{4,5,6}$ are hyperbolic, but have resonant eigenvalues. Note that in the invariant subspace $\{a_5 = b_5 = c_5 = \varepsilon_5 = 0\}$ the system (5.47) reduces to

$$\begin{aligned} x'_5 &= x_5(1 - x_5^2) \\ r'_5 &= r_5(1 + x_5^2). \end{aligned} \tag{5.51}$$

The vector field (5.51) is precisely a directional blow-up of the fast subsystem, in fact blown up by Φ_{ex} . Its flow is visualized in figure 16. Note that the flow of (5.51) appears also in figure 14 in the region $\{\bar{x} + \bar{y} > 0\}$.

An immediate consequence is the following.

Lemma 5.26. *The unique 2-d unstable manifold at q_5 is given by the plane $\{x_5 \in (-1, 1), a_5 = b_5 = c_5 = \varepsilon_5 = 0\}$. The unstable manifold at q_4 is given by the line $\{x_5 = -1, a_5 = b_5 = c_5 = \varepsilon_5 = 0\}$, and the unstable manifold at q_6 is given by the line $\{x_5 = 1, a_5 = b_5 = c_5 = \varepsilon_5 = 0\}$. The parts of these unstable manifolds with $r_5 > 0$ correspond, via Φ_{ex} , to the positive y -axis and to the positive x -axis. The equilibria $q_{4,6}$ appear in $\{r_5 = 0\}$ as sinks. The 4-d stable manifold $W^s(q_5)$ of q_5 is contained in $\{r_5 = 0\}$ and is transverse to the x_5 -axis in $\{r_5 = 0\}$. The equilibria q_4 and q_6 are sinks in $\{r_5 = 0\}$.*

Proof. The unstable manifolds follow from (5.51). The identification of the axes follows from (5.44). Lemma 5.25 gives $W^s(q_5)$ and its transversality. \square

We now briefly look at the singular limit dynamics on the blow-up sphere in the exit chart. Restrict the blown-up vector field (5.47) to $\{r_5 = 0, \varepsilon_5 > 0\}$. There we can identify the objects from the analysis in κ_ε . Recall that the family of integral curves γ_s in the rescaling chart forms an invariant 2-manifold Γ on the blow-up sphere. The manifold Γ is contained in $\{r_2 = 0, a_2 = 0\}$, which corresponds to $\{r_5 = 0, a_5 = 0, \varepsilon_5 > 0\}$ in the exit chart. In the following, we focus on the dynamics close to $q_{4,5,6}$ and Γ .

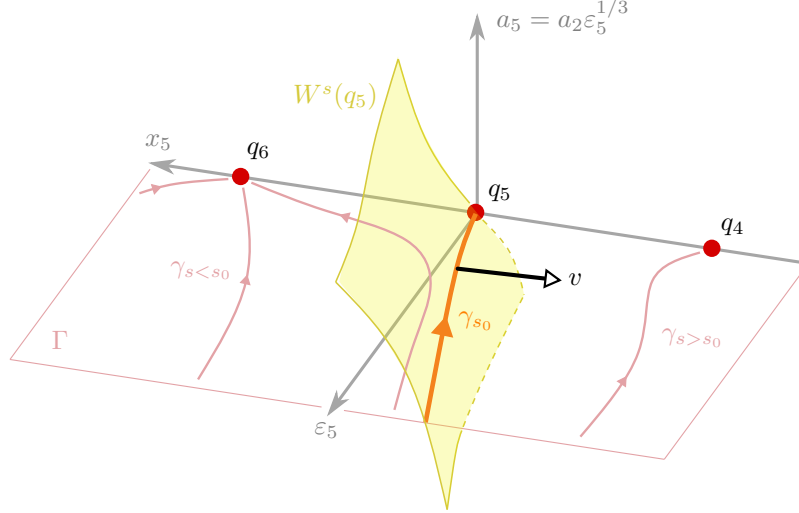


FIGURE 15. Sketch of the transverse intersection of Γ and $W^s(q_5)$ inside $\{r_5 = 0, \varepsilon_5 > 0\}$ close to q_5 . The vector $v(t)$ is a tangent vector of Γ located along γ_{s_0} , being a derivative of the family γ_s in direction of the family parameter s .

The x_5 -axis, which is invariant under the flow of (5.47), serves up to $q_{4,6}$ as unstable manifold for q_5 . The stable manifold $W^s(q_5)$ at q_5 is 4-dimensional and transverse to the x_5 -axis in $\{r_5 = 0\}$. The manifold $W^s(q_5)$ is a hypersurface in $\{r_5 = 0\}$ and separates the dynamics between the sinks q_4 and q_6 . The intersection of Γ and $W^s(q_5)$ is nonempty, due to the existence of γ_{s_0} . More precisely, we have the following lemma.

Lemma 5.27. *The manifolds Γ and $W^s(q_5)$ intersect transversely in $\{r_5 = 0, \varepsilon_5 > 0\}$ near q_5 .*

Proof. Consider the tangent space of Γ along γ_{s_0} , denoted by $T_{\gamma_{s_0}} \Gamma$. Let $v(t) = \partial_s|_{(t,s_0)} \gamma_s \in T_{\gamma_{s_0}(t)} \Gamma$, which is the tangent vector of Γ at $\gamma_{s_0}(t)$ given by the derivative of γ_s in direction of the family parameter s . For simplicity, denote the κ_ε -components of $\gamma_{s_0}(t)$ by $(x_2, y_2, 0, b_2, c_2)$. Transforming v into κ_{ex} -coordinates results in

$$\left(-2x_2 \frac{y_2^2 + c_2}{(x_2 + y_2)^2}, 0, -8b_2 \frac{y_2^2 + c_2}{(x_2 + y_2)^3}, -8c_2 \frac{y_2^2 + c_2}{(x_2 + y_2)^3} + \frac{4C_0}{(x_2 + y_2)^2}, -24 \frac{y_2^2 + c_2}{(x_2 + y_2)^4} \right). \quad (5.52)$$

We know that γ_{s_0} converges to q_5 for $t \uparrow M$. As $t \uparrow M$ we have $x_2, y_2 \rightarrow +\infty$, $x_2/y_2 \rightarrow 1$ and b_2, c_2 converge to finite values. Scaling v by the scalar $1/x_2$ does not affect the direction of v close to q_5 . Employing $x_2, y_2 \rightarrow +\infty$ and $x_2/y_2 \rightarrow 1$ implies that $x_2^{-1}v \rightarrow (-1/2, 0, 0, 0, 0)$ for $t \uparrow M$, which is a vector along the x_5 -axis. In $\{r_5 = 0\}$, the manifold $W^s(q_5)$ is of codimension 1 and transverse to the x_5 -axis. Thus, sufficiently close to q_5 , the sum of the tangent spaces of Γ and $W^s(q_5)$ span $\{r_5 = 0\}$. \square

Recall that the invariant 3-manifold $\hat{\Gamma}$ is obtained by perturbation of any γ_s to small $|a_2|$. By transversality, the intersection of Γ and $W^s(q_5)$ perturbs to the leaves $\{r_2 = 0, a_2 = \text{const}\}$ for a_2 close to zero, that is, to $\hat{\Gamma}$. Since q_4 and q_6 are hyperbolic sinks in $\{r_5 = 0\}$, the perturbations of γ_s for $s \neq s_0$ in $\hat{\Gamma}$ are also forward asymptotic to q_4 or q_6 , respectively. The perturbations of γ_{s_0} must be contained in $W^s(q_5)$, due to transversality. In particular,

the intersection of $\hat{\Gamma}$ and $W^s(q_5)$ is an invariant 2-manifold, which we denote by $\hat{\Gamma}_{s_0}$. In summary, the perturbed trajectories in $\hat{\Gamma}$ inherit their forward asymptotic behavior from Γ .

Recall from the rescaling chart that for small $r_2 > 0$, we want to track perturbations of trajectories in $\hat{\Gamma}$. The manifold $\hat{\Gamma}$ is foliated by trajectories that emanate from ζ_2 , and converge to the three equilibria $q_{4,5,6}$. The evolution of the perturbations of $\hat{\Gamma}$ through the chart κ_{ex} will clearly be affected by the hyperbolic, resonant equilibria $q_{4,5,6}$.

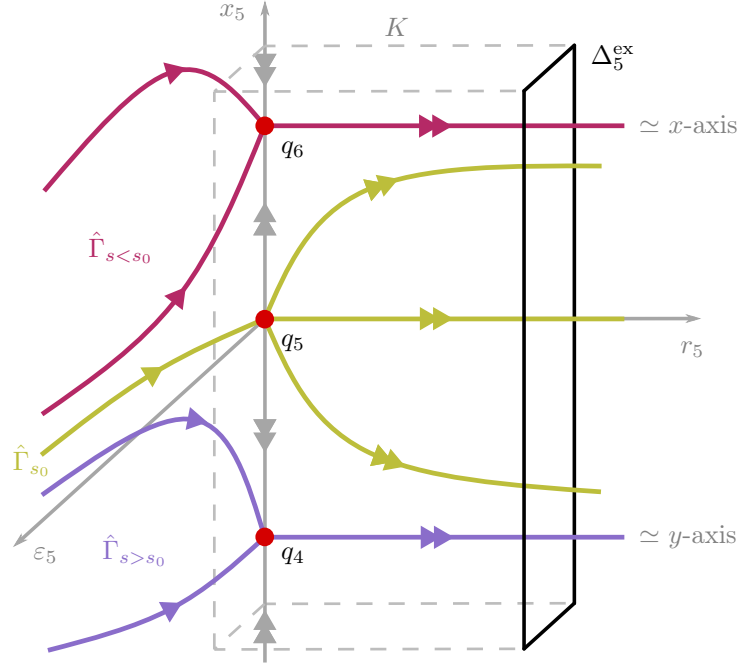


FIGURE 16. Sketch of the dynamics near $q_{4,5,6}$, in which the coordinates a_5, b_5, c_5 are suppressed. In the (r_5, x_5) -plane the flow (5.51) is shown, i.e. a_5, b_5, c_5 are set to zero. The (ε_5, x_5) -plane represents the 5-d invariant set $\{r_5 = 0\}$, in which trajectories in $\hat{\Gamma}$ approach $q_{4,5,6}$, as sketched.

The setup of the equilibria $q_{4,5,6}$ is shown in figure 16. We are interested in the transition of perturbed trajectories from $\hat{\Gamma}$ away from the blow-up sphere to the section

$$\Delta_5^{\text{ex}} = \{r_5 = \nu\} \quad (5.53)$$

close to $q_{4,5,6}$ via the flow of (5.47). Due to hyperbolicity of $q_{4,5,6}$ the qualitative behavior of the flow close to these equilibria is organized by the stable and unstable manifolds. In particular, we expect trajectories starting with small $r_5 > 0$ to escape from the saddles $q_{4,5,6}$ close to the unstable manifolds contained in the (x_5, r_5) -plane. That is, close to the singular limit flow (5.51). The key difference between q_5 and $q_{4,6}$ is that at q_5 a second unstable direction is present, namely x_5 . Therefore, we expect q_5 to have the effect of fanning out trajectories in-between $x_5 = \pm 1$, i.e. the positive x - and y -axis. Furthermore, we expect that most trajectories will transition to the fast regime close to $q_{4,6}$ and escape along the positive x - and y -axis, since on the blow-up sphere $\hat{\Gamma}_{s_0}$ is of codimension one in $\hat{\Gamma}$. Also refer to figure 16.

The relevant vector field for the transition is (5.47). The value of the function $f_x + f_y$ is constant in $\{r_5 = 0\}$. Hence, the function $F = 1 + x_5^2 + a_5 + (b_5 + c_5)/2 + r_5 \varepsilon_5 (f_x + f_y)/2$ is positive close to the x_5 -axis. Dividing out F gives, close to the x_5 -axis, an orbitally

equivalent vector field of the form

$$\begin{aligned}
x'_5 &= -x_5 + \frac{2x_5 - a_5x_5 + (b_5 - c_5)/2}{F} + r_5\varepsilon_5f_1 \\
r'_5 &= r_5 \\
a'_5 &= -a_5 + r_5\varepsilon_5g_a/F \\
b'_5 &= -2b_5 + \varepsilon_5g_b/F \\
c'_5 &= -2c_5 + \varepsilon_5g_c/F \\
\varepsilon'_5 &= -3\varepsilon_5,
\end{aligned} \tag{5.54}$$

for a sufficiently regular function f_1 . As we have seen above, trajectories in $\hat{\Gamma} \subset \{r_5 = 0\}$ converge to one of $q_{4,5,6}$. We now aim to track r_2 -perturbations of trajectories in $\hat{\Gamma}$ via the flow of (5.54).

Let us choose a compact box \tilde{K} about the x_5 -axis, such that $q_{4,5,6}$ are contained in its interior. On \tilde{K} the functions $g_a/F, g_b/F, g_c/F$ are bounded. Inside \tilde{K} , consider another box of the form

$$K = \{x_5 \in [-\Xi, \Xi], r_5 \in [0, \nu], \varepsilon_5 \in [0, \delta], a_5 \in [-a_M, a_M], b_5 \in [-b_M, b_M], c_5 \in [-c_M, c_M]\} \tag{5.55}$$

We can find sufficiently small $\nu, \delta, a_M, b_M, c_M > 0$ such that integral curves of (5.54) can exit K only through $\{r_5 = \nu\}$ or $\{x_5 = \pm\Xi\}$. Note that the x_5 -axis is invariant and the flow on it is directed towards $q_{4,6}$ for $|x_5| > 1$. Hence we can even find small $\nu, \delta, a_M, b_M, c_M > 0$ and $\Xi > 1$ such that the flow can exit K only through $\{r_5 = \nu\}$.

Next, consider a trajectory with initial condition $(x_k, r_k, a_k, b_k, c_k, \varepsilon_k) \in K$. This trajectory needs to exit K through $\Delta_5^{\text{ex}} = \{r_5 = \nu\}$. Assume $r_k \in (0, \nu)$. We have from (5.54) that $r_5(t) = r_k e^t$, $\varepsilon_5(t) = \varepsilon_k e^{-3t}$. This implies a travel duration $T = \log(\frac{\nu}{r_k})$ to Δ_5^{ex} . Let us now propose the ansatz

$$\begin{aligned}
a_5(t) &= e^{-t}(a_i + z_a(t)), \\
b_5(t) &= e^{-2t}(b_i + z_b(t)), \\
c_5(t) &= e^{-2t}(c_i + z_c(t)),
\end{aligned} \tag{5.56}$$

with $z_a(0) = z_b(0) = z_c(0) = 0$. It follows that

$$\begin{aligned}
z'_a &= e^{-t}r_k\varepsilon_k\frac{g_a}{F}, \\
z'_b &= e^{-t}\varepsilon_k\frac{g_b}{F}, \\
z'_c &= e^{-t}\varepsilon_k\frac{g_c}{F},
\end{aligned} \tag{5.57}$$

where the functions are evaluated along the corresponding trajectory. Since

$$z_a(T) = r_k\varepsilon_k \int_0^T \frac{g_a}{F} e^{-t} dt, \tag{5.58}$$

we directly obtain $z_a(T) = O(r_k\varepsilon_k)$. Similarly

$$z_b(T) = \varepsilon_k \int_0^T \frac{g_b}{F} e^{-t} dt, \tag{5.59}$$

which is bounded as $r_k \downarrow 0$ and its bound scales with factor ε_k . In fact, we can write $g_b/F = B_0/F_1 + r_5f_2$, where $F_1 = F|_{r_5=0}$ and f_2 is a sufficiently regular function. This leads, for $r_k \rightarrow 0$, to

$$z_b(T) = \varepsilon_k \int_0^T \frac{B_0}{F_1} e^{-t} dt + \varepsilon_k r_k \int_0^T \frac{f_2}{F} dt = O(1) + O(r_k \log r_k). \tag{5.60}$$

A similar calculation holds for $z_c(T)$. Note that by the above considerations, we obtain the structure of *any* transition in K to Δ_5^{ex} in the variables $r_5, a_5, b_5, c_5, \varepsilon_5$. We are still missing

the x_5 -direction, so we proceed as follows: In $\{a_5 = b_5 = c_0 = \varepsilon_5 = 0\}$ the vector field (5.54) reduces to

$$x'_5 = \frac{x_5(1 - x_5^2)}{1 + x_5^2}, \quad (5.61)$$

which is separable and can be integrated by employing a partial fraction decomposition. Let $\bar{x}(t, x_k)$ denote the integral curve of (5.61) with $\bar{x}(0, x_k) = x_k$. Clearly $\bar{x}(t, -x_k) = -\bar{x}(t, x_k)$, and \bar{x} is explicitly given by

$$\bar{x}(t, x_k) = \begin{cases} \frac{|1 - x_k^2|}{2|x_k|} e^{-t} + \sqrt{\frac{|1 - x_k^2|^2}{4|x_k|^2} e^{-2t} + 1}, & x_k > 1 \\ 1, & x_k = 1 \\ -\frac{|1 - x_k^2|}{2|x_k|} e^{-t} + \sqrt{\frac{|1 - x_k^2|^2}{4|x_k|^2} e^{-2t} + 1}, & x_k \in (0, 1) \\ 0, & x_k = 0. \end{cases} \quad (5.62)$$

For the x'_5 equation in (5.54) we take the ansatz $x_5(t) = \bar{x}(t) + z_x(t)$ with $z_x(0) = 0$. Rewriting the first equation of (5.54) to

$$x'_5 = -x_5 + \frac{2x_5 - a_5x_5 + (b_5 - c_5)/2}{1 + x_5^2 + a_5 + (b_5 + c_5)/2} + r_5\varepsilon_5 f_2, \quad (5.63)$$

for some sufficiently regular f_2 , and employing a Taylor expansion at $(x_5, 0, 0, 0, 0, 0)$ in $\{r_5 = \varepsilon_5 = 0\}$ leads to the equation $z'_x = a_5 f_3 + b_5 f_4 + c_5 f_5 + r_5 \varepsilon_5 f_2$, where f_3, f_4, f_5 are sufficiently regular functions. Hence

$$z_x(T) = \int_0^T a_5 f_3 dt + \int_0^T b_5 f_4 dt + \int_0^T c_5 f_5 dt + \int_0^T r_5 \varepsilon_5 f_2 dt, \quad (5.64)$$

where the functions are evaluated along the corresponding integral curve. Bounding the functions, plugging in $a_5(t) = e^{-t}(a_k + z_a(t))$, $b_5(t)$, $c_5(t)$, $r_5(t)$, $\varepsilon_5(t)$ and, where necessary, integrating by parts, shows that $z_x(T) = O(r_k \varepsilon_k, a_k, b_k, c_k, \varepsilon_k)$. Thus, with the analysis performed above, we obtain the following result that summarizes the transition in the present chart κ_{ex} .

Proposition 5.28. *Any transition $\Pi_5: K \rightarrow \Delta_5^{\text{ex}} = \{r_5 = \nu\}$ via the flow of (5.54) in K is of the form*

$$\begin{bmatrix} x_k \\ r_k \\ a_k \\ b_k \\ c_k \\ \varepsilon_k \end{bmatrix} \mapsto \begin{bmatrix} \Pi_{x_5}(x_k) + O(r_k \varepsilon_k, a_k, b_k, c_k, \varepsilon_k) \\ \nu \\ \frac{r_k}{\nu} a_k + O(r_k^2 \varepsilon_k) \\ \left(\frac{r_k}{\nu}\right)^2 (b_k + O(\varepsilon_k)) \\ \left(\frac{r_k}{\nu}\right)^2 (c_k + O(\varepsilon_k)) \\ \left(\frac{r_k}{\nu}\right)^3 \varepsilon_k \end{bmatrix}, \quad (5.65)$$

where the $O(\varepsilon_k)$ terms are bounded as $r_k \rightarrow 0$, $\Pi_{x_5}(-x_k) := -\Pi_{x_5}(x_k)$ and

$$\Pi_{x_5}(x_k) = \begin{cases} \frac{|1 - x_k^2|}{2|x_k|} \frac{r_k}{\nu} + \sqrt{\frac{|1 - x_k^2|^2}{4|x_k|^2} \left(\frac{r_k}{\nu}\right)^2 + 1}, & x_k > 1 \\ 1, & x_k = 1 \\ -\frac{|1 - x_k^2|}{2|x_k|} \frac{r_k}{\nu} + \sqrt{\frac{|1 - x_k^2|^2}{4|x_k|^2} \left(\frac{r_k}{\nu}\right)^2 + 1}, & x_k \in (0, 1) \\ 0, & x_k = 0. \end{cases} \quad (5.66)$$

5.7. Proof of Theorem 1.1. In the following proof we collect the results from sections 5.2 to 5.6 and “glue” them together. Recall that we essentially work inside the continuation of $W^c(\mathcal{S}_{0,1}^a)$, which is a 4-dimensional invariant manifold containing the 3-d attracting slow manifolds.

Proof of Theorem 1.1. We begin by choosing an open set of initial conditions $V_1 \subset \Delta^{-y \rightarrow \varepsilon} \cap W^c(\mathcal{S}_{0,1}^a)|_{r_1 > 0}$. The analysis in the rescaling chart dictates, which trajectories with initial condition in V_1 we are able to track. After shrinking V_1 suitably, we can assume the following two assertions: First, the flow from Lemma 5.13 takes V_1 in backward time to Δ_1^{en} . Secondly, the set $V_2 = \kappa_{-y \rightarrow \varepsilon}(V_1) \subset \Delta_2^{-y \rightarrow \varepsilon}$ is lead by the flow of (5.26) along the invariant 3-manifold $\hat{\Gamma} \subset \{r_2 = 0\}$ through the rescaling chart close to $q_{4,5,6}$. Refer to Lemma 5.21 and before for the definition of $\hat{\Gamma}$. The trajectories in $\hat{\Gamma}$ are forward asymptotic to one of the hyperbolic saddles $q_{4,5,6}$. Hence, by choosing $r_2 > 0$ sufficiently small (i.e. shrinking V_1 accordingly), we can assume that Π_2 maps V_2 to (faces of) K . Here Π_2 is the transition map from Lemma 5.21 and K the compact box defined for Proposition 5.28 in which the transition map Π_5 is defined. Recall that V_1 lies in $W^c(\mathcal{S}_{0,1}^a)$, and hence we can let V_1 flow in backward time via (5.19) to $\Delta_1^{\text{en}} \cap W^c(\mathcal{S}_{0,1}^a)$. This gives the domain I_1 for the transition Π_1 inside the center manifold. In backward time (5.19) leads to exponential contraction towards the 1-d unstable manifold, which is given by $\sigma_1 \subset \mathcal{S}_{0,1}^a$. Notice that I_1 then degenerates to the single point $p = \sigma_1 \cap \Delta_1^{\text{en}}$ for $\varepsilon_1 \rightarrow 0$.

We now distinguish three regions in I_1 , which give qualitatively different jumping behavior. Recall that $\hat{\Gamma}_{s_0} := \hat{\Gamma} \cap W^s(q_5)$ is an invariant 2-manifold on the blow-up sphere, consisting of perturbations of γ_{s_0} . Similarly, let $\hat{\Gamma}_{s > s_0}$ and $\hat{\Gamma}_{s < s_0}$ denote the invariant sub-manifolds of the 3-manifold $\hat{\Gamma}$ consisting of perturbations of some $\gamma_{s > s_0}$ or $\gamma_{s < s_0}$, respectively. For small $r_2 > 0$ the manifold $\hat{\Gamma}_{s_0}$ gets perturbed away from the blow-up sphere to an invariant 3-manifold, let us denote it by Σ . Recall that $W^c(\mathcal{S}_{0,1}^a)$ is 4-dimensional. Then $V_2 \subset \Delta_2^{-y \rightarrow \varepsilon} \cap W^c(\mathcal{S}_{0,1}^a)$ is a 3-manifold. Consequently Σ intersects V_2 in a 2-manifold. Limiting to small $r_2 > 0$, $\Sigma \cap V_2$ splits V_2 into two disjoint open subsets U^{q_4} and U^{q_6} , where U^{q_4} contains perturbations of $\hat{\Gamma}$ that connect to q_4 . Furthermore, let U^{q_5} denote a (small) neighborhood of $\Sigma \cap V_2$ in V_2 . The sets $U^{q_4}, U^{q_5}, U^{q_6}$ are an open cover of V_2 and correspond to distinct jumping behaviors. The set U^{q_5} leads to trajectories jumping and fanning out close to q_5 , whereas U^{q_4} and U^{q_6} lead to trajectories jumping close to q_4 , respectively q_6 , with contraction towards the 1-d unstable manifolds $W^u(q_4)$ (the y -axis in original coordinates), and $W^u(q_6)$ (the x -axis in original coordinates) respectively.

Next, we let the aforementioned three regions U^{q_4}, U^{q_5} , and U^{q_6} flow in backward time to I_1 , which gives three open sets $R^{q_4}, R^{q_5}, R^{q_6}$ covering I_1 . In backward time, i.e. Π_1^{-1} , the flow contracts towards σ_1 . Hence, in the singular limit $\varepsilon_1 \rightarrow 0$, the set I_1 collapses in wedge-like fashion into the single point corresponding to $\sigma_1 \cap \Delta^{\text{en}}$: by this we mean that ε -slices through I_1 converge in Hausdorff distance to the single point, after discarding the ε -direction. Similar statements holds for any of the 3-manifolds $R^{q_4}, R^{q_5}, R^{q_6}$. We refer to figure 17 for an overview sketch of the situation. Finally, we “blow-down” by considering the transition map $\hat{\Pi}: I \rightarrow \Delta^{\text{ex}}$ defined by

$$\hat{\Pi} = \Phi_{\text{ex}} \circ \Pi_5 \circ \kappa_{\varepsilon \rightarrow \text{ex}} \circ \Pi_2 \circ \kappa_{-y \rightarrow \varepsilon} \circ \Pi_1 \circ \Phi_{-y}^{-1}, \quad (5.67)$$

where $I = \Phi_{-y}(I_1)$ is a 3-manifold. By construction the map $\hat{\Pi}$ is the transition map from Δ^{en} to Δ^{ex} along the flow of X given by (5.1). Now let I_ε denote a slice through I for $\varepsilon = \text{const}$. Then Π from Theorem 1.1 is given by $\hat{\Pi}|_{I_\varepsilon}$ and I_ε is a 2-manifold. Clearly, we discard the ε -direction when we take slices at fixed value of ε . It follows that I_ε converges in Hausdorff distance to a single point. Each I_ε also contains the corresponding ε -slices through $R^{q_4}, R^{q_5}, R^{q_6}$. Let us denote these slices by $L_\varepsilon^y, R_\varepsilon$ and L_ε^x , such that L_ε^y corresponds to an ε -slice through R^{q_5} , similarly R_ε to an ε -slice through R^{q_5} and L_ε^x to an ε -slice through R^{q_6} . We refer to L_ε^x and L_ε^y as the lateral regions, which correspond to initial conditions escaping along the x -axis, respectively the y -axis. The sets L_ε^x and L_ε^y are disjoint, and $I_\varepsilon = L_\varepsilon^x \cup R_\varepsilon \cup L_\varepsilon^y$, where all sets are open.

Recall that J is the closed line segment defined by $\Delta^{\text{ex}} \cap \{x \geq 0, y \geq 0, a = b = c = 0\}$ and let j_x , respectively j_y , denote its points of intersection with the x -axis, respectively y -axis. We need to argue that $\Pi(L_\varepsilon^x) \rightarrow j_x$, $\Pi(L_\varepsilon^y) \rightarrow j_y$ and $\Pi(R_\varepsilon) \rightarrow J$ in Hausdorff distance for $\varepsilon \rightarrow 0$. Let us assume for a moment that the scalings $a = O(\varepsilon^{1/3})$, $b = O(\varepsilon^{2/3})$, $c = O(\varepsilon^{2/3})$ for $(x, y, a, b, c) \in \Pi(I_\varepsilon)$ as $\varepsilon \rightarrow 0$ hold. Then the assertions follow from the definition of the sets U^{q_4} , U^{q_5} , U^{q_6} and from the fact that Π is continuous. Indeed, U^{q_5} is a neighborhood of $\Sigma \cap V_2$ in V_2 and the structure of the transition map Π_5 in Proposition 5.28, in particular (5.66) for $r_k \rightarrow 0$, imply the assertions.

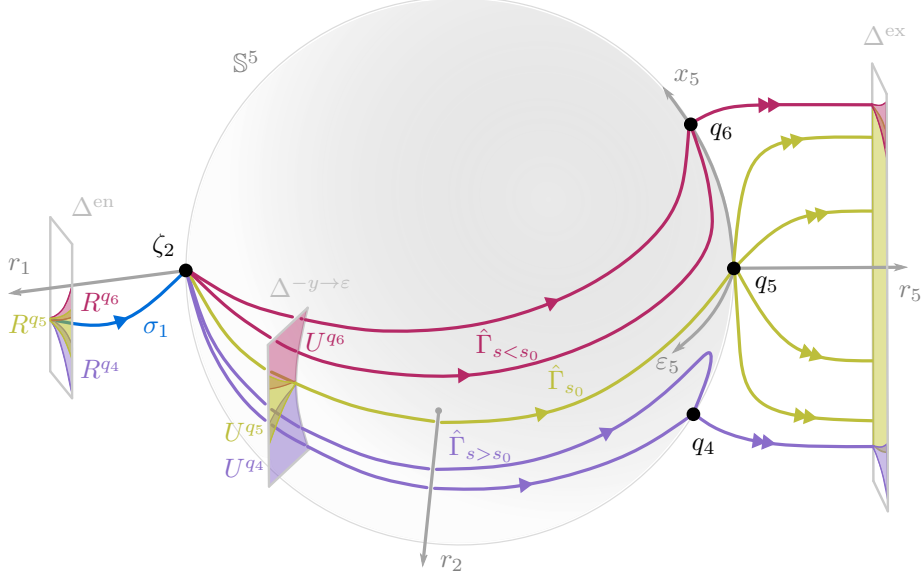


FIGURE 17. Rough sketch of the transition $\Pi_5 \circ \kappa_{\varepsilon \rightarrow \text{ex}} \circ \Pi_2 \circ \kappa_{-y \rightarrow \varepsilon} \circ \Pi_1$ for initial conditions in $I_1 = R^{q_4} \cup R^{q_5} \cup R^{q_6}$ in blow-up space “inside” the continuation of the 4-d invariant manifold $W^c(\mathcal{S}_{0,1}^3)$. We remark that the dimensions of the sketched objects have been reduced for visualization reasons. The colored regions inside the sections indicate the 3-manifolds R^{q_4} , R^{q_5} , $R^{q_6} \subset \Delta^{\text{en}}$, respectively U^{q_4} , U^{q_5} , $U^{q_6} \subset \Delta^{-y \rightarrow \varepsilon}$, and their evolution steered by singular limit candidate trajectories up to Δ^{ex} . Note that the fanning-out behavior of R^{q_5} is visible. Slices of the form $\varepsilon = \text{const}$ through R^{q_5} give R_ε in Theorem 1.1. The sketched coordinate axes correspond by abuse of notation to the euclidean coordinates transported to blow-up space via the charts.

Next, we prove the scalings $a = O(\varepsilon^{1/3})$, $b = O(\varepsilon^{2/3})$ and $c = O(\varepsilon^{2/3})$ for $(x, y, a, b, c) \in \Pi(I_\varepsilon)$. Observe that Proposition 5.28 already suggests these scalings. However, one should note at this point, that in general Π_2 from Lemma 5.21 passes a_2 unaffected to leading order through the rescaling chart. This implies that in the exit chart, the scaling of a_1 , caused by Π_1 in the entry chart, remains present. The structure of (5.19) suggests that $\Pi_{1a}(x_i, a_i, \nu, \varepsilon_i) = O(\varepsilon_i^{-1/3})$, which would cancel out with the leading order scaling of Π_5 . Thus, heuristically, we should not see any a priori scaling in the a -variable. However, in our setting of $\Pi(I_\varepsilon)$, we expect to see “artificial scalings” caused by our choice of domain I_ε , which shrinks to a single point p for $\varepsilon \rightarrow 0$. Indeed, as we will shortly show, the scalings $a = O(\varepsilon^{1/3})$, $b = O(\varepsilon^{2/3})$ and $c = O(\varepsilon^{2/3})$ appear because of our choice of initial conditions in I_ε .

We build the composition $\Pi_5 \circ \kappa_{\varepsilon \rightarrow \text{ex}} \circ \Pi_2: V \rightarrow \Delta_5^{\text{ex}}$ to see the scalings formally. From Lemma 5.21 we get $\Pi_2(x_j, -\delta^{-1/3}, a_j, b_j, c_j, r_j) = (x_e + O(a_j, r_j), y_e + O(a_j, r_j), a_j + O(r_j), b_e + O(a_j, r_j), c_e + O(a_j, r_j), r_j) \in \kappa_{\text{ex} \rightarrow \varepsilon}(\partial K \setminus \{r_5 = \nu\})$. According to Lemma A.3

we can write $x_e + y_e + O(a_j, r_j) = 2\varepsilon_k^{-1/3}$ for $\varepsilon_k \in (0, \delta]$, which implies

$$(\kappa_{\varepsilon \rightarrow \text{ex}} \circ \Pi_2)(x_j, -\delta^{-1/3}, a_j, b_j, c_j, r_j) = \begin{bmatrix} \frac{1}{2}\varepsilon_k^{1/3}(x_e - y_e + O(a_j, r_j)) \\ r_j\varepsilon_k^{-1/3} \\ \varepsilon_k^{1/3}(a_j + O(r_j)) \\ \varepsilon_k^{2/3}(b_e + O(a_j, r_j)) \\ \varepsilon_k^{2/3}(c_e + O(a_j, r_j)) \\ \varepsilon_k \end{bmatrix}. \quad (5.68)$$

Applying Proposition 5.28 leads to $\Pi_5 \circ \kappa_{\varepsilon \rightarrow \text{ex}} \circ \Pi_2: V_2 \rightarrow \Delta_5^{\text{ex}}$ given by

$$\begin{bmatrix} x_j \\ -\delta^{-1/3} \\ a_j \\ b_j \\ c_j \\ r_j \end{bmatrix} \mapsto \begin{bmatrix} * \\ \nu \\ \frac{r_j}{\nu}(a_j + O(r_j^2)) \\ (\frac{r_j}{\nu})^2(b_e + O(a_j, r_j) + O(\varepsilon_k^{2/3})) \\ (\frac{r_j}{\nu})^2(c_e + O(a_j, r_j) + O(\varepsilon_k^{2/3})) \\ (\frac{r_j}{\nu})^3 \end{bmatrix}, \quad (5.69)$$

where the $O(\varepsilon_k^{2/3})$ terms are bounded as $r_k \rightarrow 0$ (and $r_j \rightarrow 0$) and we leave the entry $*$ unspecified. First, recall that $r_j = \nu\varepsilon_i^{1/3} = \varepsilon^{1/3}$. Observe that a_j appears in the right hand side of (5.69), so in general can possibly affect the scalings if a_j blows up for $\varepsilon \rightarrow 0$. In our setting, however, we have chosen the domain I_ε in such a way, that we only encounter small, in fact bounded values of a_j as $\varepsilon \rightarrow 0$. This implies the scalings in a , b and c as claimed in Theorem 1.1.

It remains to consider the case that g_a is factored by a . For this case we have Lemma 5.15, which implies

$$a_j = \delta^{-1/3}\Pi_{1a}(x_i, a_i, \nu, \varepsilon_i) = \varepsilon_i^{-1/3}O(a_i) + O(\varepsilon_i^{2/3} \log \varepsilon_i). \quad (5.70)$$

This implies that $I_{1|\varepsilon_i}$ is of width $a_i = O(\varepsilon_i^{1/3})$ as $\varepsilon_i \rightarrow 0$. Thus, for the case of g_a being factored by a , this gives that I_ε is a thin strip of width $O(\varepsilon^{1/3})$ in a -direction, for $\varepsilon \rightarrow 0$. \square

6. FURTHER OBSERVATIONS

In this last section we briefly present further observations, obtained by analyzing the blow-up \bar{X} of (5.1) in the two charts $\kappa_{\pm a}$. The main observation is that in the blow-up \bar{X} , one generically encounters fast-slow systems on the blow-up sphere with folded saddles or folded centers. These organize the flow on the blow-up sphere close to the equator $\bar{\varepsilon} = 0$.

To formulate the observations, we look at \bar{X} in the two charts $\kappa_{\pm a}$. The blow-up of X (5.1) in the chart κ_a is, after suitable time rescaling, given by

$$\begin{aligned} x'_3 &= x_3^2 + y_3 + b_3 + O(r_3\varepsilon_3) \\ y'_3 &= y_3^2 + x_3 + c_3 + O(r_3\varepsilon_3) \\ r'_3 &= A_0r_3^2\varepsilon_3 + O(r_3^3\varepsilon_3) \\ b'_3 &= B_0\varepsilon_3 + O(r_3\varepsilon_3) \\ c'_3 &= C_0\varepsilon_3 + O(r_3\varepsilon_3) \\ \varepsilon'_3 &= -3A_0r_3\varepsilon_3^2 + O(r_3^2\varepsilon_3^2). \end{aligned} \quad (6.1)$$

Similarly, in the chart κ_{-a} , we obtain the blown-up vector field

$$\begin{aligned}
x'_4 &= x_4^2 - y_4 + b_4 + O(r_4^2 \varepsilon_4) \\
y'_4 &= y_4^2 - x_4 + c_4 + O(r_4^2 \varepsilon_4) \\
r'_4 &= -A_0 r_4^2 \varepsilon_4 + O(r_4^3 \varepsilon_4) \\
b'_4 &= B_0 \varepsilon_4 + O(r_4 \varepsilon_4) \\
c'_4 &= C_0 \varepsilon_4 + O(r_4 \varepsilon_4) \\
\varepsilon'_4 &= 3A_0 r_4 \varepsilon_4^2 + O(r_4^2 \varepsilon_4^2).
\end{aligned} \tag{6.2}$$

We restrict the analysis of (6.1) and (6.2) to the invariant blow-up sphere, i.e. we put $r_3 = 0$ and $r_4 = 0$, respectively. For brevity we give our observations mainly for (6.2) and only indicate the similar statements for (6.1). In $\{r_4 = 0\}$ (6.2) simplifies to

$$\begin{aligned}
x'_4 &= x_4^2 - y_4 + b_4 \\
y'_4 &= y_4^2 - x_4 + c_4 \\
b'_4 &= B_0 \varepsilon_4 \\
c'_4 &= C_0 \varepsilon_4 \\
\varepsilon'_4 &= 0,
\end{aligned} \tag{6.3}$$

which is a fast-slow system near the equator $\{\bar{\varepsilon} = 0\} \cap \mathbb{S}^5$, i.e. for small $\varepsilon_4 > 0$. The 2-dimensional critical manifold of (6.3) is given by $E_4 := \{x_4^2 - y_4 + b_4 = 0, y_4^2 - x_4 + c_4 = 0\}$, which is the intersection of the blown-up critical manifold of \bar{X} with the blow-up sphere in κ_{-a} . Via $\kappa_{-y \rightarrow -a}$ the manifold $\mathcal{S}_{0,1}|_{r_1=0, a_1 < 0}$ corresponds to $E_4|_{y_4 < 0}$, and so in particular, if $B_0 C_0 > 0$, the folded saddle ζ_1 from (5.17) is present in (6.3). Furthermore, we have $\varepsilon_4 = -a_2^{-3}$, so the invariant sets $\{r_2 = 0, a_2 = \text{const}\}$ of the coupled Riccati system (5.27) in κ_ε appear in form of the fast-slow system (6.3) in κ_{-a} . The invariant sets $\{\varepsilon_4 = \text{const}\}$ for (6.3) imply that there do not exist trajectories in $\{r_2 = 0\}$, which approach in forward or backward time points on the half-equator $\{\bar{\varepsilon} = 0, \bar{a} < 0\} \cap \mathbb{S}^5$. In particular, this implies

Lemma 6.1. *Let $B_0 C_0 > 0$. Then no integral curve of (5.27) is forward or backward asymptotic to ζ_1 .*

In case of $B_0 C_0 < 0$, the folded equilibrium ζ_1 is a folded center and visible in κ_a instead of κ_{-a} . Hence, to obtain a similar statement in this case, we need to employ (6.1) in $\{r_3 = 0\}$, which explicitly is the fast-slow system

$$\begin{aligned}
x'_3 &= x_3^2 + y_3 + b_3 \\
y'_3 &= y_3^2 + x_3 + c_3 \\
b'_3 &= B_0 \varepsilon_3 \\
c'_3 &= C_0 \varepsilon_3 \\
\varepsilon'_3 &= 0,
\end{aligned} \tag{6.4}$$

for small $\varepsilon_3 > 0$. A similar argument then implies that Lemma 6.1 also holds for $B_0 C_0 < 0$. In fact, in each of these cases, a second folded equilibrium is present, which we denote by ζ_6 . That is, ζ_6 is the folded singularity of (6.4) in case of $B_0 C_0 > 0$, but in case of $B_0 C_0 < 0$ ζ_6 appears as folded singularity of (6.3). Analogous results to Lemma 6.1 hold for ζ_6 instead of ζ_1 if $B_0 C_0 \neq 0$.

6.1. Folded singularities in \bar{X} on the blow-up sphere. The fast-slow systems (6.3) and (6.4) appear in the blow-up \bar{X} in the charts $\kappa_{\mp a}$ on the blow-up sphere near the equator $\{\bar{\varepsilon} = 0\} \cap \mathbb{S}^5$. In fact, the coupled Riccati system (5.27) on the blow-up sphere in κ_ε corresponds to (6.3) for $a_2 < 0$ and to (6.4) for $a_2 > 0$. As noted already, (6.3) and (6.4), have folded singularities. Here, we present a brief analysis of (6.3) and comment on the analogous results for (6.4) afterwards.

The critical manifold E_4 of (6.3) is parametrized by (x_4, y_4) and has fold points along $4x_4y_4 = 1$ and a cusp point located at $x_4 = y_4 = -1/2$, which is implied by Lemma 3.2 restricted to $\{a = -1\}$. The desingularized slow flow of (6.3) is given by

$$U: \begin{bmatrix} \dot{x}_4 \\ \dot{y}_4 \end{bmatrix} = \begin{bmatrix} 0 & -2B_0 \\ -2C_0 & 0 \end{bmatrix} \begin{bmatrix} x_4 \\ y_4 \end{bmatrix} - \begin{bmatrix} C_0 \\ B_0 \end{bmatrix} \quad (6.5)$$

with reversed time orientation of orbits in the saddle-type region $\{4x_4y_4 < 1\}$. The desingularized slow flow U is analyzed in the following Lemma. A sketch of the phase portrait is given in figure 18.

Lemma 6.2. *Assume $B_0C_0 \neq 0$. The desingularized slow flow U (6.5) of (6.3) has a single folded singularity. If $B_0C_0 > 0$, a folded saddle is located on the negative branch of $4x_4y_4 = 1$, which corresponds to ζ_1 . There exists a singular canard passing from \mathcal{S}_0^a to \mathcal{S}_0^s and a singular faux canard passing from \mathcal{S}_0^s to \mathcal{S}_0^a . In particular, if $B_0 = C_0$, ζ_1 is located at the cusp point. If $B_0C_0 < 0$, a folded center is located on the positive branch of $4x_4y_4 = 1$, which corresponds to ζ_6 in this case.*

Proof. The equilibrium of U is located at $x_4 = -\frac{B_0}{2C_0}$, $y_4 = -\frac{C_0}{2B_0}$, which is on the lower fold curve if B_0, C_0 have the same sign. If $B_0 = C_0$ it coincides with the cusp point. Eigenvalues of the linearization are $\pm 2\sqrt{B_0C_0}$, the situation partly resembles Lemma 3.4. For $B_0, C_0 > 0$, the corresponding eigenvectors are

$$-2\sqrt{B_0C_0}: (\sqrt{B_0/C_0}, 1), \quad 2\sqrt{B_0C_0}: (-\sqrt{B_0C_0}, 1). \quad (6.6)$$

For $B_0, C_0 < 0$ the eigenvectors swap, similar to Lemma 3.4. An eigenvector is tangent to the fold curves if and only if $B_0 = C_0$. This is precisely the case when the folded saddle coincides with the cusp point. If $B_0, C_0 > 0$ and $B_0 \neq C_0$, the stable/unstable manifolds of ζ_1 are transverse to the fold curve. Finally, since U is a desingularized slow flow, time reversion has to be applied in the saddle-type region $\{4x_4y_4 < 1\}$. \square

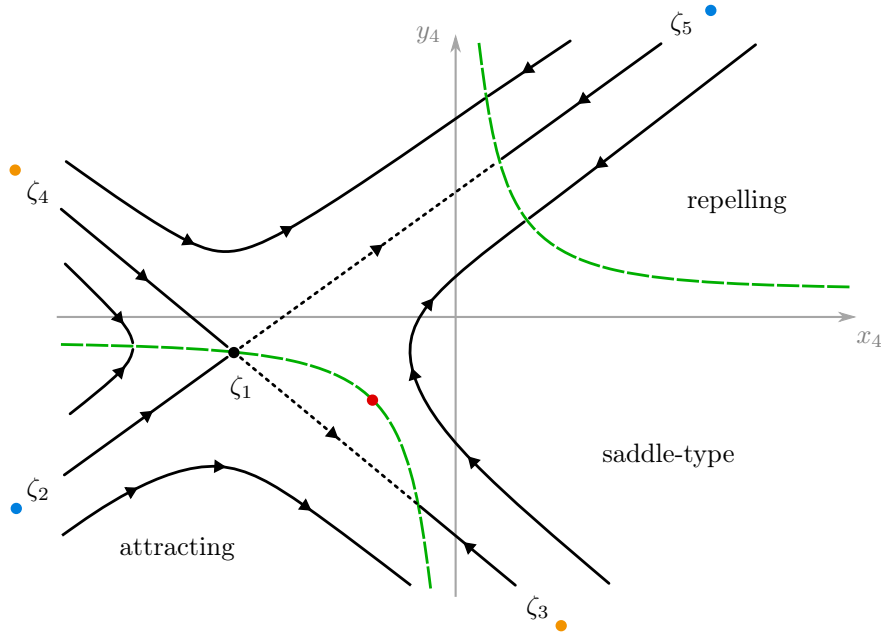


FIGURE 18. Phase portrait of the desingularized slow flow U (6.5) on the critical manifold of (6.3) with reversed time orientation in the saddle-type region and $B_0 > C_0 > 0$. The fold curves are indicated dashed green, the red point represents the cusp. In case of $C_0 > B_0 > 0$, the folded saddle ζ_1 is located on the branch “below” the cusp point.

A similar analysis as above holds for the fast-slow system (6.4). In particular, the desingularized slow flow of (6.4) on the corresponding critical manifold $E_3 = \{x_3^2 + y_3 + b_3 = 0, y_3^2 + x_3 + c_3 = 0\}$ is given by

$$Q: \begin{bmatrix} \dot{x}_3 \\ \dot{y}_3 \end{bmatrix} = \begin{bmatrix} 0 & -2B_0 \\ -2C_0 & 0 \end{bmatrix} \begin{bmatrix} x_3 \\ y_3 \end{bmatrix} + \begin{bmatrix} C_0 \\ B_0 \end{bmatrix}, \quad (6.7)$$

where the time orientation of orbits in the saddle type region $\{4x_3y_3 < 1\}$ needs to be reversed. Lemma 6.2 holds mutatis mutandis for the desingularized slow flow Q (6.7) of (6.4). In particular, if $B_0C_0 > 0$, Q has a folded saddle corresponding to ζ_6 on the upper fold curve. If $B_0C_0 < 0$, then Q has a folded center located on the lower fold curve, which corresponds to ζ_1 in this case. A phase portrait of Q for the case $C_0 > B_0 > 0$ is given in figure 19.

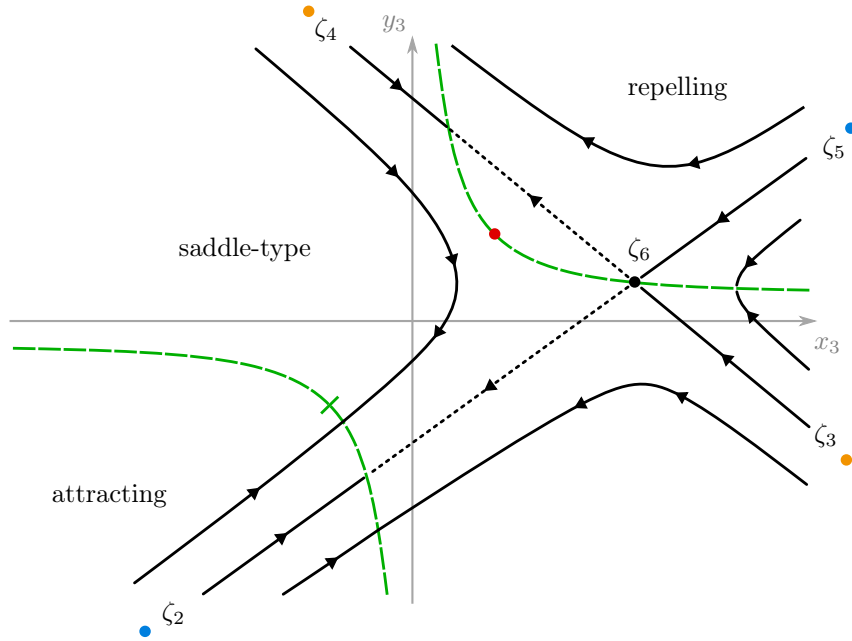


FIGURE 19. Phase portrait of the desingularized slow flow Q (6.7) on the critical manifold of (6.4) with reversed time orientation in the saddle-type region and $B_0 > C_0 > 0$. The fold curves are indicated dashed green, the red point represents the cusp. In case of $C_0 > B_0 > 0$, ζ_6 is located on the branch “above” the cusp point.

6.2. Dynamics of \bar{Y} on the blow-up sphere. The vector fields U (6.5) and Q (6.7) play not only an important role in the analysis of the fast-slow systems (6.3) and (6.4), being their desingularized slow flows on the critical manifold E_4 , respectively E_3 . In fact, U also appears by blowing up the complete desingularized slow flow Y (3.8) via

$$\Theta_{-a}: x = r_4x_4, y = r_4y_4, a = -r_4, \quad (6.8)$$

and restricting to the blow-up sphere $\{r_4 = 0\}$. Note that (6.8) is just the restriction of Φ_{-a} to the coordinates (x_4, y_4, r_4) . In other words, U naturally appears in the blow-up \bar{Y} , in the restricted chart κ_{-a} . Similarly Q arises in the blow-up of Y in the restricted chart κ_a .

We summarize this, in case of U , in the following remark.

Remark 6.3. Let $(\Theta_{-a})_*\bar{Y}^{-a} = Y$, where Y is the desingularized slow flow of X and Θ_{-a} as in (6.8). Then $U = \bar{Y}^{-a}|_{r_4=0}$. Denote the vector field (6.2) by \bar{X}^{-a} , such that $\bar{X}^{-a}|_{r_4=0}$ gives the fast-slow system (6.3). Then we have a commutative diagram:

$$\begin{array}{ccc}
 X & \xrightarrow{\text{slow flow desingularization}} & Y \\
 \downarrow \text{blow-up via } \Phi_{-a} & & \downarrow \text{blow-up via } \Theta_{-a} \\
 \bar{X}^{-a}|_{r_4=0} & \xrightarrow{\text{slow flow desingularization}} & \bar{Y}^{-a}|_{r_4=0} = U
 \end{array}$$

As should be clear from the preceding observations, one can employ U and Q to obtain the flow of the blown-up desingularized slow flow \bar{Y} on the blow-up sphere. That is, in the following we show how the flow on the blow-up sphere in figure 12 was obtained. We consider the case $B_0, C_0 > 0$. The folded saddle of U corresponding to ζ_1 organizes the dynamics on the critical manifold E_4 . Moreover, it organizes the dynamics of \bar{Y} on one half of the 2-d blow-up sphere, where \bar{Y} is the suitably rescaled blown-up desingularized slow flow (5.14). In other words, by the following Lemma we obtain the dynamics on the half-sphere $\bar{a} < 0$ in figure 12, if we take the time reversion in the saddle-type region into account.

Lemma 6.4. *Let $B_0, C_0 > 0$. Any trajectory of U has asymptotic behavior as indicated by $\zeta_{2,3,4,5}$ in figure 18. That is, neglecting the time reversion for U in the saddle-type region, the stable manifold of ζ_1 is asymptotic to ζ_2 for $y_4 \rightarrow -\infty$ and asymptotic to ζ_5 for $y_4 \rightarrow +\infty$. The unstable manifold of ζ_1 is asymptotic to ζ_4 for $y_4 \rightarrow +\infty$ and ζ_3 for $y_4 \rightarrow -\infty$. This determines the asymptotic behavior of any trajectory of U .*

Proof. Neglect the time reversion for U in the saddle-type region. The stable and unstable manifolds of ζ_1 are given by straight lines of the form

$$x_4 = \pm \sqrt{B_0/C_0} y_4 + \text{const.} \quad (6.9)$$

Employing the appropriate coordinate changes from Lemma A.3 to (6.9), one obtains the asymptotics in the assertion. Since U is a linear system, the saddle ζ_1 determines the forward and backward asymptotic behavior of all orbits. Hence, in each of the four sectors determined by the stable and unstable manifolds, any trajectory constitutes a heteroclinic connection with the corresponding behavior. \square

A similar analysis applies to Q . Combining the phase portraits of U and Q leads to the qualitative flow of the blown-up desingularized slow flow \bar{Y} on the 2-d blow-up sphere: The case $B_0 > C_0 > 0$ of \bar{Y} visualized in figure 12 is obtained by glueing the phase portraits from figures 19 and 18. That is, each desingularized slow flow in $\kappa_{\pm a}$ determines the flow on the half-spheres $\{\bar{a} > 0\}$ and $\{\bar{a} < 0\}$. The glueing takes place along the equator, on which heteroclinic orbits run. These appear in figure 11 along the x_1 -axis, similarly in $\kappa_{-x}, \kappa_x, \kappa_y$.

Remark 6.5. *Restricted to the slow flow coordinates (x, y, a) , the charts $\kappa_{\pm a}$ in fact correspond to gnomonic projections, through which great circles on the blow-up sphere correspond to straight lines. Therefore the (un-)stable manifolds of $\zeta_{1,6}$ correspond to great circles, as depicted in figure 12.*

APPENDIX A.

A.1. Takens' constrained differential equations. Here we give Takens' definition of constrained differential equations and their solutions, taken from [29]. After this, we briefly compare Takens' notions to the singular limit of (2.1).

Definition A.1 ([29]). *Let $\pi: E \rightarrow B$ be a smooth fiber bundle. A constrained differential equation on E is a pair (V, X) with smooth potential $V: E \rightarrow \mathbb{R}$ and smooth vector field $X: E \rightarrow TB$ with $X(e) \in T_{\pi(e)}B$ such that*

- a) V restricted to any fiber is proper and bounded from below,
- b) the set $S_V = \{e \in E \mid V|_{\pi^{-1}(\pi(e))}$ has a critical point in $e\}$ is locally compact in the following sense: for each compact $K \subset B$ the set $S_V \cap \pi^{-1}(K)$ is compact.

Further, define

$$S_{V,\min} = \{e \in S_V \mid \text{the Hessian of } V|_{\pi^{-1}(\pi(e))} \text{ is positive (semi-)definite at } e\}.$$

Definition A.2 ([29]). *Let (V, X) be a constrained differential equation on E as above. A curve $\gamma: (\alpha, \beta) \rightarrow E$ is a solution of (V, X) if*

- i) $\gamma(t_0^+) := \lim_{t \downarrow t_0} \gamma(t)$ and $\gamma(t_0^-) := \lim_{t \uparrow t_0} \gamma(t)$ exist for all $t_0 \in (\alpha, \beta)$ and satisfy $\pi(\gamma(t_0^+)) = \pi(\gamma(t_0^-))$ and $\gamma(t_0^+), \gamma(t_0^-) \in S_{V,\min}$,
- ii) for all $t \in (\alpha, \beta)$, $X(\gamma(t^-))$ is the left derivative of $\pi \circ \gamma$ and $X(\gamma(t^+))$ is the right derivative of $\pi \circ \gamma$ at t ,
- iii) whenever $\gamma(t^-) \neq \gamma(t^+)$ for some $t \in (\alpha, \beta)$, there is a curve in $\pi^{-1}(\pi(\gamma(t^+)))$ from $\gamma(t^-)$ to $\gamma(t^+)$ along V decreases monotonically.

Let us briefly compare Takens' notion of a constrained differential equation to the singular limit of (2.1): The fibers of the bundle can be viewed as the fast subsystems. $S_{V,\min}$ can be viewed as the attracting region of the critical manifold with its surrounding singularities. Takens' solutions on $S_{V,\min}$ are essentially determined by the vector field X . This is similar to the slow flow of the slow subsystem of (2.1), which is well defined in regular points of the critical manifold since the critical manifold is a graph over the slow variables. Note that in the slow subsystem of (2.1) we allow dynamics on the full critical manifold, not only the attracting region and nearby singularities. When jumps occur in Takens' setting, they do so "infinitely fast" (producing discontinuities) along a fiber, such that the potential decreases along these jumps. In contrast to this, (2.1) has dynamics in the fast fibers, in fact gradient dynamics arising from a potential. The gradient dynamics force jumps along paths of steepest descent/ascent, depending on the used sign.

A.2. Coordinate changes between the charts κ_\bullet . Let $\kappa_{\bullet \rightarrow \circ}$ denote the coordinate change from κ_\bullet to κ_\circ , defined on the overlap of the involved charts. That is, $\kappa_{\bullet \rightarrow \circ} = \kappa_\circ \circ \kappa_\bullet^{-1}$. The overlap is apparent from our notation: For example, to switch from κ_{-y} to κ_ε , we need $\varepsilon_1 > 0$ in κ_{-y} . The formula for $\kappa_{-y \rightarrow \varepsilon}$ can then be computed by relating Φ_{-y} and Φ_ε . The chart κ_{ex} plays an exceptional role and intuitively corresponds to the "direction $x + y$ ". For example, to apply $\kappa_{\varepsilon \rightarrow \text{ex}}$ we need points in the overlap with $x_2 + y_2 > 0$ in κ_ε . For convenience of the reader, the following Lemma states the relevant coordinate changes, most of them used extensively and without reference throughout the analysis in section 5.

Lemma A.3. *The relevant coordinate changes $\kappa_{\bullet \rightarrow \circ}$ are given by the following maps:*

$$\kappa_{-y \rightarrow \varepsilon}: \begin{cases} x_2 &= x_1 \varepsilon_1^{-1/3} \\ y_2 &= -\varepsilon_1^{-1/3} \\ a_2 &= a_1 \varepsilon_1^{-1/3} \\ b_2 &= b_1 \varepsilon_1^{-2/3} \\ c_2 &= c_1 \varepsilon_1^{-2/3} \\ r_2 &= r_1 \varepsilon_1^{1/3}, \end{cases} \quad \kappa_{-y \rightarrow -x}: \begin{cases} r_0 &= -r_1 x_1 \\ y_0 &= x_1^{-1} \\ a_0 &= a_1 (-x_1)^{-1} \\ b_0 &= b_1 (-x_1)^{-2} \\ c_0 &= c_1 (-x_1)^{-2} \\ \varepsilon_0 &= \varepsilon_1 (-x_1)^{-3}, \end{cases}$$

$$\begin{array}{l}
\kappa_{-y \rightarrow a} : \begin{cases} x_3 = x_1 a_1^{-1} \\ y_3 = -a_1^{-1} \\ r_3 = r_1 a_1 \\ b_3 = b_1 a_1^{-2} \\ c_3 = c_1 a_1^{-2} \\ \varepsilon_3 = \varepsilon_1 a_1^{-3} \end{cases} \\
\kappa_{\varepsilon \rightarrow a} : \begin{cases} x_3 = x_2 a_2^{-1} \\ y_3 = y_2 a_2^{-1} \\ r_3 = r_2 a_2 \\ b_3 = b_2 a_2^{-2} \\ c_3 = c_2 a_2^{-2} \\ \varepsilon_3 = a_2^{-3} \end{cases} \\
\kappa_{\varepsilon \rightarrow -y} : \begin{cases} x_1 = x_2 (-y_2)^{-1} \\ r_1 = -r_2 y_2 \\ a_1 = a_2 (-y_2)^{-1} \\ b_1 = b_2 (-y_2)^{-2} \\ c_1 = c_2 (-y_2)^{-2} \\ \varepsilon_1 = (-y_2)^{-3}, \end{cases} \\
\kappa_{a \rightarrow \text{ex}} : \begin{cases} x_5 = (x_3 - y_3)(x_3 + y_3)^{-1} \\ r_5 = r_3(x_3 + y_3)/2 \\ a_5 = 2(x_3 + y_3)^{-1} \\ b_5 = 4b_3(x_3 + y_3)^{-2} \\ c_5 = 4c_3(x_3 + y_3)^{-2} \\ \varepsilon_5 = 8\varepsilon_3(x_3 + y_3)^{-3} \end{cases} \\
\kappa_{-y \rightarrow -a} : \begin{cases} x_4 = x_1(-a_1)^{-1} \\ y_4 = a_1^{-1} \\ r_4 = -r_1 a_1 \\ b_4 = b_1(-a_1)^{-2} \\ c_4 = c_1(-a_1)^{-2} \\ \varepsilon_4 = \varepsilon_1(-a_1)^{-3} \end{cases} \\
\kappa_{\varepsilon \rightarrow -a} : \begin{cases} x_4 = x_2(-a_2)^{-1} \\ y_4 = y_2(-a_2)^{-1} \\ r_4 = -r_2 a_2 \\ b_4 = b_2(-a_2)^{-2} \\ c_4 = c_2(-a_2)^{-2} \\ \varepsilon_4 = (-a_2)^{-3} \end{cases} \\
\kappa_{\varepsilon \rightarrow \text{ex}} : \begin{cases} x_5 = (x_2 - y_2)(x_2 + y_2)^{-1} \\ r_5 = r_2(x_2 + y_2)/2 \\ a_5 = 2a_2(x_2 + y_2)^{-1} \\ b_5 = 4b_2(x_2 + y_2)^{-2} \\ c_5 = 4c_2(x_2 + y_2)^{-2} \\ \varepsilon_5 = 8(x_2 + y_2)^{-3} \end{cases} \\
\kappa_{-a \rightarrow \text{ex}} : \begin{cases} x_5 = (x_4 - y_4)(x_4 + y_4)^{-1} \\ r_5 = r_4(x_4 + y_4)/2 \\ a_5 = 2(x_4 + y_4)^{-1} \\ b_5 = 4b_4(x_4 + y_4)^{-2} \\ c_5 = 4c_4(x_4 + y_4)^{-2} \\ \varepsilon_5 = 8\varepsilon_4(x_4 + y_4)^{-3} \end{cases}
\end{array}$$

A.3. Another universal unfolding for the hyperbolic umbilic. Consider

$$\tilde{V} = \frac{1}{3}\tilde{y}^3 + \tilde{y}\tilde{x}^2 + \tilde{a}(\tilde{y}^2 - \tilde{x}^2) + \tilde{b}\tilde{x} + \tilde{c}\tilde{y} \quad (\text{A.1})$$

from (5.46). Here we briefly show that \tilde{V} is a universal unfolding (among others) for the hyperbolic umbilic catastrophe, compare to table 1.

We briefly recall: Let \mathcal{E}_n denote the local ring of germs at $0 \in \mathbb{R}^n$. Let \mathfrak{m}_n denote the ideal of germs vanishing at 0. \mathfrak{m}_n is the maximal ideal in \mathcal{E}_n , which means that there does not exist a proper ideal of \mathcal{E}_n strictly containing \mathfrak{m}_n . Let $\Delta(f)$ denote the Jacobi ideal of $f \in \mathfrak{m}_n^2$, that is, the ideal generated by the partial derivatives of f over \mathcal{E}_n . Let the codimension of f be defined by the dimension of $\mathfrak{m}_n/\Delta(f)$ as an \mathbb{R} -vector space. The following theorem can be found in [22], for example.

Theorem A.4. *Let $f \in \mathfrak{m}_n^2$ be a germ with finite codimension and let b_1, \dots, b_d be representatives of a basis for $\mathfrak{m}_n/\Delta(f)$. Then $V(z, \alpha) = f(z) + \sum_{i=1}^d b_i(z)\alpha_i$ is a universal unfolding of f .*

Consider the germ \tilde{V} unfolds, that is, let $f = \frac{1}{3}y^3 + yx^2$. The hyperbolic umbilic germ $x^3 + y^3$ is right-equivalent to $2\tilde{x}^3 + 6\tilde{x}\tilde{y}^2$ by putting $x = \tilde{x} + \tilde{y}$ and $y = \tilde{x} - \tilde{y}$. By scaling the variables, the germ $2\tilde{x}^3 + 6\tilde{x}\tilde{y}^2$ is right equivalent to f . Hence f is a valid germ to represent the hyperbolic umbilic catastrophe. Now consider $\mathfrak{m}_2/\Delta(f)$. The Jacobi-ideal $\Delta(f)$ is generated by $x^2 + y^2$ and $2xy$ over \mathcal{E}_2 . Surely x and y are not contained in $\Delta(f)$. But $x^k = x^{k-2}(x^2 + y^2) - \frac{1}{2}x^{k-1}y(2xy)$ for $k \geq 3$, and similarly y^k for $k \geq 3$ is contained in $\Delta(f)$. Certain quadratic germs are still missing in $\Delta(f)$. We can choose $y^2 - x^2$ together with x and y to form a basis for the \mathbb{R} -vector space $\mathfrak{m}_2/\Delta(f)$. This gives the universal unfolding \tilde{V} .

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