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SECONDARY FREQUENCY CONTROL IN POWER SYSTEMS WITH ARBITRARY COMMUNICATION DELAYS*

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Abstract. In this paper, we consider a Kron-reduced microgrid that consists solely of generator units. The frequency control we consider consists of primary droop control at the generator units, as well as distributed averaging integral (DAI) control that enforces both synchronization among the generator frequencies and optimal power dispatch. To any generating unit, a DAI controller is attached that exchanges information with neighboring nodes. In practice, this is subject to time delays, which impedes the stability analysis. By performing scattering transformation of the input and output variables at each end of the communication link between two DAI controllers, and transmitting the scattered variables instead of the controller output, stability of the nominal frequency is guaranteed. The time delays are not restricted to any bounds.

Key words. scattering transformation, DAI control, swing equations, power systems

AMS subject classifications. 34D06, 37C75, 93A14, 93C23, 93D20

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1. Introduction.

Power systems are currently facing a transformation from an orderly grid with few generators that operate synchronously under centralized control to a much more heterogeneous grid with large-scale implementation of volatile renewable energy sources and components with complex dynamics. This causes the net load to change with a steeper gradient and by larger amounts. Moreover, prediction of these changes becomes more difficult. With the traditional control schemes, robust stability of the network is endangered, which illustrates the necessity to find more robust and flexible control strategies.

The model we consider is a microgrid, which is comprised of generating units at the buses, and modelled by the swing equations. The dynamics of these equations contribute to the frequency stability in two ways. In the first place, they encompass the dynamics of droop controllers for each of the generators, reducing large excursions of frequency deviations in a proportional fashion, which is categorized as primary frequency control. Second, they describe the active power flow between interconnected buses through transmission lines, which causes the rotor angles to synchronize in a distributed fashion for system states operating sufficiently close to a steady state.

The desired operation of the network at a state with constant voltage amplitudes and a synchronized nominal frequency among all generators relies heavily on the balance of power consumption and production in the network and on a net power inflow or outflow at each bus that does not exceed the capacity constraints of the transmission lines. As such, the system may not withstand the stressed conditions without further implementation of higher levels of frequency control.

In this paper, we focus on secondary frequency control, which is aimed at restoring the frequency to a nominal level [8], [20]. For this purpose, a distributed averaging

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integral (DAI) controller is implemented at every node in our power system that restores the frequency in an economically efficient way, while also power interchanges between adjacent control areas are restored. The DAI controllers annihilate the gap between the global net power consumption and production by means of integral control. Moreover, the distributed character of these controllers has the advantage that no central computation is needed. Delayed DAI controllers can be applied to a wide range of purposes; see, e.g., [2], [11], [21].

Passivity plays a key role in the control of networked systems and has been exploited in optimization problems such as resource allocation or steady-state optimality; see, e.g., [3]. In fact, DAI control can be seen as a consensus algorithm that relies on the passivity of the network elements. The passivity framework has been shown to facilitate Lyapunov-based stability analysis in port-Hamiltonian systems and in power systems; see, e.g., [3], [15], [19].

The DAI controllers communicate with each other through a communication network that differs from the physical network of transmission lines and together form a cyber-physical network. The communication links are often subject to time delays, message losses, and link failures, which may prevent secondary control from operating efficiently [7], [17]. Traditional power networks with a decentralized control architecture do not need to cope with this problem, since there is not always a communication layer involved in its secondary control. However, this problem is expected to show up in future power networks with a more distributed control architecture, such as microgrids [7], [18]. A complicating factor arises when delays are heterogeneous, as is studied in [14]. Secondary control strategies with communication delays are furthermore studied in, e.g., [1], [6].

Hatanaka et al. [11] showed that scattering transformation is a powerful tool to stabilize passive networked systems with constant communication delays. Moreover, the stability results are independent of the size of the delays. Scattering transformation in networked systems was introduced for bilateral teleoperators with time delays [5]. Matiakis, Hirche, and Buss [13] studied finite gain L_2 stability of networked control systems with arbitrary delays using scattering transformation. In our paper, we apply this technique to the output variables of our DAI controller and the input variables from the communication links in order to achieve local asymptotic stability using reasonable assumptions. As we do not measure delays, we do not aim at compensating for them.

The main contribution of this paper is the synthesis of the above mentioned power system comprising the dynamic DAI controller and the static scattering transformation, accompanied with sufficient conditions for delay-independent robust stability in the case of constant delays. This work should be regarded as complementary to the work of Schiffer, Dörfler, and Fridman [17], where robust stability is proven by means of delay-dependent linear matrix inequality (LMI) conditions derived from a LaSalle–Krasovskii type of argument for a special case of our system for which the scattering subsystem is absent. Instead, in their model, the DAI output is transmitted immediately, without a coordinate transformation in the input-output space. The feasibility of the resulting LMI depends on the chosen delay bounds. Although the authors consider fast-varying delays, no delay-independent conditions are given. The current work does not cover time-varying delays, but it provides novel insights by presenting a general stability theorem whose conditions are independent of the magnitude of the (constant) time delays.

In practice, the proposed controllers are implemented via packet-based control strategies, which are common in real-time digital communication setups [7]. In these setups, although the true delays of the information packets may vary in time, we can enforce a constant delay by applying buffering and a timestamp registration of the packets at each link, and the practical restrictions caused by assuming constant delays are limited. We may still freely choose the delay bounds and let them vary among the different edges.

We exploit passivity of the communication channel to construct a storage function of the scattering transformation and the delay blocks. This storage function is then combined with the storage function of the swing equations and the DAI controller to construct a Lyapunov functional that is used to show that trajectories will converge to a synchronous solution.

1.1. Notation.

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ denote the undirected graph in which $\mathcal{V}$ and $\mathcal{E}$ represent the node and edge sets, respectively. For each node $i \in \mathcal{V}$, $\mathcal{N}_i$ denotes the set of nodes that are connected to i through an edge in $\mathcal{E}$. $B = B(\mathcal{G}) \in \mathbb{R}^{|\mathcal{V}|\times|\mathcal{E}|}$ denotes the incidence matrix of $\mathcal{G}$ for which the kth column is associated with the edge $k = (i, j)$. Thus, $B_{ik} \in \{-1, 1\}$ if i is an endpoint of k; else $B_{ik} = 0$. Furthermore, $[x]$ is the diagonal matrix with diagonal entries $[x]_{ii} = x_i$, and $\mathrm{col}(x, y) = \begin{pmatrix} x^T & y^T \end{pmatrix}^T$ is the operator that stacks vectors. In equations where a time argument is not given explicitly, current time (t) is meant. Let $C([-\rho,\infty), \mathbb{R}^n)$ be the space of piecewise continuous functions $\mathbb{R}^n \to [-\rho, \infty)$. For a function $x \in C([-\rho,\infty), \mathbb{R}^n)$, the restriction $x_t \in C([-\rho, 0], \mathbb{R}^n)$ is defined as $x_t(\theta) = x(t + \theta)$ for $\theta \in [-\rho, 0]$ and $t \in [0,\infty)$. A delay differential equation (DDE) is a dynamical system of the form $\dot{x} = f(x_t)$, whereas a delay differential algebraic equation (DDAE) is a dynamical system of the form $E\dot{x} = f(x_t)$, where $E \in \mathbb{R}^{n\times n}$. Let $\phi \in C([-\rho, 0]\mathbb{R}^n)$. Then, a function $x \in C([-\rho,\infty), \mathbb{R}^n)$ satisfying the dynamics of the DDAE $E\dot{x} = f(x_t)$, and for which $x_0 = \phi$, is referred to as a solution through ϕ relative to $E\dot{x} = f(x_t)$. If this function is unique, we write $x = x(\phi, t)$, and $x_t = x_t(\phi, \theta)$ for its restrictions.

2. System formulation.

The nodes of the power system network consist of synchronous generators that rotate ideally with a frequency close to the nominal frequency ω^d. The generators are connected through transmission lines defined on the edges of $\mathcal{G}$. The transmission lines are assumed to be purely inductive and have a strictly positive susceptance. To every node, a DAI controller is attached for secondary frequency control. The generation unit at node i is modelled as

$$M_i\dot{\omega}_i = -D_i\omega_i - \sum_{j\in\mathcal{N}_i} f_{ij} + u_{DAI} + P_i^{net}.$$

Here, ω_i is the deviation of the rotational speed of the generator from the nominal frequency. $M_i > 0$ is the inertia coefficient, $D_i > 0$ the droop coefficient, f_{ij} is the active power flow through the transmission line that connects i with j and enforces consensus, and u_{DAI} is a control input that pursues restoration to the nominal frequencies. P^{net} is the net power from other sources injected into the node. We assume this quantity to be constant at the prevailing timescales of the dynamics.

For every link $k = (i, j)$, the phase angle difference of the generation units at i and j, denoted by η_k, admits the following dynamics:

$$\begin{aligned}\dot{\eta}_k &= B_{ik}\omega_i + B_{jk}\omega_j,\\ f_k &= \gamma_k \sin(\eta_k).\end{aligned}$$

The output f_k is the active power flow through link k, where $\gamma_k = V_iV_jb_k > 0$. In this expression, the constant voltages at the corresponding nodes are denoted by V_i and V_j, whereas b_k denotes the susceptance of link k.

The DAI controller attached to generating unit i is tasked with pushing the frequency to the nominal value in an economically efficient way. In accordance with [19] and [21], every DAI controller i has a state ξ_i and contributes to the total cost in a quadratic way. Cost minimization is represented as the following linearly constrained optimization problem, which is referred to as the *economic dispatch problem*:

$$\begin{aligned} \min_{\xi \in \mathbb{R}^n} \sum_{i \in \mathcal{V}} a_i \xi_i^2 \\ \text{s.t.} \sum_{i \in \mathcal{V}} \xi_i = \sum_{i \in \mathcal{V}} P_i^{net}. \end{aligned} \tag{2.1}$$

The a_i values are cost parameters. The constraint in this problem is a necessary condition for a steady state, which is elaborated in the proof of Lemma 3.1. The DAI controllers communicate with each other through a connected communication network that is represented as a directed graph $\mathcal{G}_c = (\mathcal{V}, \mathcal{E}_c)$ that differs from the physical network $\mathcal{G}$. The DAI controller has to steer the state ξ to the optimal solution ξ^* of (2.1). In [21] and [17], it is shown that this can be achieved with the dynamics

$$\kappa_i^{-1} \dot{\xi}_i = a_i \sum_{j \in \mathcal{N}_i} c_{ij}(a_j \xi_j - a_i \xi_i) + \omega_i. \tag{2.2}$$

Here, the first term is an edge damping term that forces identical marginal costs, i.e., $a_i \xi_i = a_j \xi_j$ for all $(i, j) \in \mathcal{E}$. The second term ω_i is the integrand that pursues frequency restoration. Furthermore, $\kappa_i > 0$ is a timescale parameter, while $c_{ij} = c_{ji} > 0$ is a weight factor corresponding to the link $(i, j) \in \mathcal{E}_c$. The closed-loop system can be written compactly as

$$\begin{aligned} \dot{\eta} &= B^T \omega, \\ M\dot{\omega} &= -D\omega - B\Gamma \sin(\eta) - \xi + P^{net}, \\ K^{-1}\dot{\xi} &= -AL(\mathcal{G}_c)A\xi + \omega, \end{aligned} \tag{2.3}$$

where $B = B(\mathcal{G})$ is the incidence matrix of the physical network, and the diagonal matrices $M, D, \Gamma, K, A \succ 0$ have diagonal entries denoted by M_i, D_i, γ_k, κ_i, and a_i, respectively. Throughout this paper, the variables without subscript are stacked vectors of the respective variables with subscript. Also, $L(\mathcal{G}_c) = B(\mathcal{G}_c)[c]B(\mathcal{G}_c)^T$ is the weighted Laplacian matrix with the edge weights given by c_{ij}. We refer to (2.3) as the nominal system, which is not subject to time delays.

2.1. Scattering transformation and time delay.

The DAI controller (2.2) exchanges information with neighboring controllers about their states. However, due to time delays, this information becomes inaccurate, and hence, we need to redefine the coupling between neighboring DAI controllers. First, we decouple the DAI controller by replacing ξ_j with the new input variable r_{ij}. The DAI controller at node i now reads as

$$\kappa_i^{-1} \dot{\xi}_i = a_i \sum_{j \in \mathcal{N}_i} c_{ij}(a_j r_{ij} - a_i \xi_i) + \omega_i. \tag{2.4}$$

From this point on, the system definition will differ from the DAI controlled power system defined in [17]. Also, we remind the reader that in equations where a time

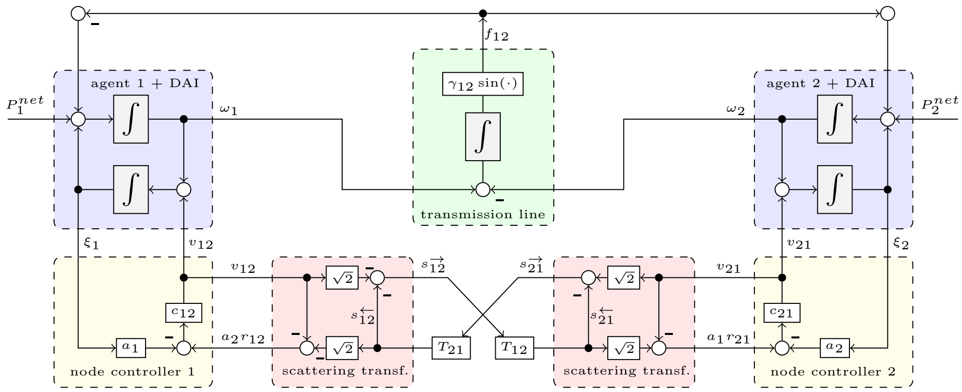

FIG. 1. *The subsystems at the link* $(1,2)$ *and their endpoints. Here, the node at the left is assumed to be the head of the edge.*

argument is not given explicitly, current time (t) is meant. Instead of exchanging the ξ_i variable with neighboring nodes directly, we place a static system at the ith end of the link $k=(i,j)$ that consists of two static subsystems: a scattering system, which performs a scattering transformation on its input and output signals, and a node controller that outputs an appropriate signal v_{ij} for the scattering system:

$$v_{ij} = c_{ij}(a_j r_{ij} - a_i \xi_i).$$

Here, $a_j r_{ij}$ is the output variable of the scattering system that is sent to the node controller. The node controllers are illustrated by the yellow areas in Figure 1 and are associated with both endpoints of the link. The scattering system performs a coordinate transformation of the input variable v_{ij} and output variable $a_j r_{ij}$, which results in scattering variables that are defined in an asymmetric fashion at both ends of the link; see the red areas of Figure 1. Let us assign a direction to every edge according to the incidence matrix, so that there is no ambiguity as to which end is referred to as the tail and which as the head. If node i is the head of the link $k=(i,j)\in\mathcal{E}_c$, we define the sending and receiving scattering variable as

$$\begin{aligned} s_{ij}^{\rightarrow} &= \frac{1}{\sqrt{2}}(-v_{ij} + a_j r_{ij}) \quad \text{and} \\ s_{ij}^{\leftarrow} &= \frac{1}{\sqrt{2}}(-v_{ij} - a_j r_{ij}), \end{aligned} \tag{2.5}$$

respectively. Otherwise, i is seen as the tail of k, in which case we define the scattering variables as

$$\begin{aligned} s_{ij}^{\rightarrow} &= \frac{1}{\sqrt{2}}(v_{ij} - a_j r_{ij}), \\ s_{ij}^{\leftarrow} &= \frac{1}{\sqrt{2}}(v_{ij} + a_j r_{ij}). \end{aligned} \tag{2.6}$$

The sending variable $s_{ij}^{\rightarrow}$ is transmitted to node j, but will be subjected to a time delay T_{ij} before it is received at the other end. We allow the time delay to be different for both directions:

$$\begin{aligned} s_{ji}^{\leftarrow}(t) &= s_{ij}^{\rightarrow}(t - T_{ij}), \\ s_{ij}^{\leftarrow}(t) &= s_{ji}^{\rightarrow}(t - T_{ji}). \end{aligned} \tag{2.7}$$

We can write the scattering system now in input-output form in the case when i is a head (left) or a tail (right):

$$a_j r_{ij} = -\sqrt{2} s_{ij}^{\leftarrow} - v_{ij}, \qquad a_j r_{ij} = \sqrt{2} s_{ij}^{\leftarrow} - v_{ij},$$
$$s_{ij}^{\rightarrow} = -s_{ij}^{\leftarrow} - \sqrt{2} v_{ij}, \qquad s_{ij}^{\rightarrow} = -s_{ij}^{\leftarrow} + \sqrt{2} v_{ij}.$$

The scattering system and the time delays are illustrated in the red area of Figure 1. Note that an algebraic loop is visible as v_{ij} and r_{ij} appear in each other's definitions. Although it is easily seen that this results in equations that have a unique solution, it is for simulation purposes useful to eliminate the algebraic loop by considering the input-state-output representation of the subsystem consisting of the node controllers, scattering subsystems, and time delay blocks associated with a communication link.

This subsystem has inputs $\xi_{t,i}, \xi_{t,j} \in C([-\rho, 0], \mathbb{R}^n)$, which contain the historic values of ξ_i and ξ_j between $t-\rho$ and the current time t. The outputs at the ith and jth end are $v_{ij}(t) = c_{ij}(a_j r_{ij} - a_i \xi_i)$ and $v_{ji}(t) = c_{ij}(a_i r_{ji} - a_j \xi_j)$, respectively, and will be obtained by focusing on the scattering subsystem variables $a_j r_{ij}(t)$ and $a_i r_{ji}(t)$. In what follows, a hat on a variable with subscript ij denotes evaluation with a delay T_{ij}. For example, $\hat{s}_{ij}^{\rightarrow}(t) := s_{ij}(t - T_{ij})$. If i is the head of a communication link, the output of the scattering subsystem writes as

$$\begin{aligned} a_j r_{ij} &= \frac{1}{\sqrt{2}}\left(s_{ij}^{\rightarrow} - s_{ij}^{\leftarrow}\right) = \frac{1}{\sqrt{2}}\left(s_{ij}^{\rightarrow} - \hat{s}_{ji}^{\rightarrow}\right) \\ &= \frac{1}{2}\left((1-c_{ij}) a_j r_{ij} + c_{ij} a_i \xi_i\right) \\ &\quad + \frac{1}{2}\left((1-c_{ij}) a_i \hat{r}_{ji} + c_{ij} a_j \hat{\xi}_j\right) \\ &= \frac{1}{1+c_{ij}}\left(c_{ij} a_i \xi_i + c_{ij} a_j \hat{\xi}_j + (1-c_{ij}) a_i \hat{r}_{ji}\right). \end{aligned} \tag{2.8}$$

The final equation holds for all $t \geq 0$ and is also valid if i is a tail. Hence, even though the sending and receiving variables at each end of the link are defined in an asymmetric fashion, the output terms $a_j r_{ij}$ and $a_i r_{ji}$ are symmetric at both ends with respect to the input-state-output representation of the subsystem. Using this symmetry and assuming that (2.8) is also valid in the initial condition,[1] we can write $a_i r_{ji}(t - T_{ji})$ in terms of $\xi_j(t - T_{ji})$, $\xi_i(t - T_{ij} - T_{ji})$, and $r_{ij}(t - T_{ij} - T_{ji})$ to obtain

$$a_j r_{ij}(t) = \underbrace{\frac{(1-c_{ij})^2}{(1+c_{ij})^2}}_{(\alpha_0)_{ij}} a_j r_{ij}(t - T_{ij} - T_{ji}) + g_{ij}(\xi_t), \tag{2.9}$$

where

$$\begin{aligned} g_{ij}(\xi_t) := &\underbrace{\frac{c_{ij}}{1+c_{ij}}}_{(\alpha_1)_{ij}} a_i \xi_i(t) + \underbrace{\frac{2c_{ij}}{(1+c_{ij})^2}}_{(\alpha_2)_{ij}} a_j \xi_j(t - T_{ji}) \\ &+ \underbrace{\frac{c_{ij}(1-c_{ij})}{(1+c_{ij})^2}}_{(\alpha_3)_{ij}} a_i \xi_i(t - T_{ij} - T_{ji}). \end{aligned} \tag{2.10}$$

[1]This assumption is made explicit in Assumption 2.1.

Equation (2.9) is a delay algebraic equation where g_{ij} is a linear function of ξ evaluated at the current time t and a finite number of times in the past within the interval $[t-\rho, t]$, with a time horizon $\rho = \max_{(i,j)\in\mathcal{E}}\{T_{ij}+T_{ji}\}$. Note, however, that there is no need for time measurements throughout the system as the delays are not part of the control architecture, but caused by the limitations of the communication network. DAI controller i (including the node controller) only has access to the delayed DAI controller state ξ_j of its neighbors and its own delayed value of ξ_i through the signal $a_j r_{ij}$. For example, the delay in the last term of (2.10) is a consequence of the propagation of ξ_i (which is present in the signal $a_i r_{ji}$) from i to j and back, thus accounting for the delay $T_{ij}+T_{ji}$.

In addition to the single hat notation for a delay T_{ij}, we introduce the double hat notation for denoting a delay $T_{ij}+T_{ji}$, e.g.,

$$\hat{\hat{r}}_{ij}(t) := r_{ij}(t - T_{ij} - T_{ji}).$$

We write the closed-loop system as the following DDAE:

$$\begin{aligned}
\dot{\eta} &= B^T\omega,\\
M\dot{\omega} &= -D\omega - B\Gamma\sin(\eta) - \xi + P^{net},\\
\kappa_i^{-1}\dot{\xi}_i &= a_i\sum_{j\in\mathcal{N}_i} c_{ij}(a_j r_{ij} - a_i\xi_i) + \omega_i \qquad \forall i\in\mathcal{V},\\
a_j r_{ij} &= (\alpha_0)_{ij} a_j \hat{\hat{r}}_{ij} + g_{ij}(\xi_t) \qquad \forall (i,j)\in\mathcal{E}.
\end{aligned} \tag{2.11}$$

Here, the variables without hats are evaluated at current time and the expression for g_{ij} can be found in (2.10). The latter two equations in (2.11) define the control architecture of the system, where ω is the input to be measured and ξ the actuation variable. We will refer to (2.11) as the *delayed system*. We pose the following assumption on the initial condition ϕ without any loss of generality.

Assumption 2.1. The algebraic equation (2.8) also applies to the initial condition $\phi \in C([-\rho,0],\mathbb{R}^{2n+3m})$, i.e., any solution of (2.11) through ϕ satisfies (2.8) for $t\in[-\rho,\infty)$.

As a consequence of this assumption, the conditions (2.8) and (2.9) are equivalent for all $t\in[-\rho,\infty)$. Note that the initial condition $\phi\in C([-\rho,0],\mathbb{R}^{2n+3m})$ of the above assumption is associated with the representation as given in (2.11). This initial condition is partly redundant for the purpose of simulation of the closed-loop system. In fact, only each delay block T_{ij} must have an initial condition consisting of a function in $[-T_{ij},0]$ that mimics past values of the corresponding outgoing scattering variable $s_{ij}^{\rightarrow}$.

Remark 2.2. For a better comparison between the delayed system (2.11) and the nominal system (2.3), we set the initial condition $r_{ij}(t)=\xi_j(t)$ for all $t\in[-\rho,0]$. For any $(i,j)\in\mathcal{E}$ and $t>0$, let $k(t)=\lceil\frac{t}{T_{ij}+T_{ji}}\rceil$ be the smallest nonnegative integer such that $t-k(t)(T_{ij}-T_{ji})\in[-\rho,0]$. We obtain that

$$\begin{aligned}
a_j r_{ij}(t) = (\alpha_0)^{k(t)} a_j r_{ij}(t-k(t)(T_{ij}+T_{ji})) + \sum_{l=0}^{k(t)-1}(\alpha_0)_{ij}^l \\
\big[(\alpha_1)_{ij} a_i \xi_i(t - l(T_{ij}+T_{ji})) \\
+(\alpha_2)_{ij} a_j \xi_j(t - lT_{ij} - (l+1)T_{ji}) \\
+(\alpha_3)_{ij} a_i \xi_i(t-(l+1)(T_{ij}+T_{ji}))\big].
\end{aligned} \tag{2.12}$$

The parameters add up to one. Indeed, note first that (2.9) is an affine combination as $\sum_{k=0}^{3}(\alpha_k)_{ij} = 1$. Thus,

$$(\alpha_0)^{k(t)} + \sum_{l=0}^{k(t)-1} (\alpha_0)_{ij}^l \underbrace{((\alpha_1)_{ij} + (\alpha_2)_{ij} + (\alpha_3)_{ij})}_{1-(\alpha_0)_{ij}} = 1. \tag{2.13}$$

Substituting (2.12) in (2.11), we notice that (2.11) can be obtained from (2.3) if we replace $a_j\xi_j$ by a weighted average of current and past values of $a_i\xi_i$ and $a_j\xi_j$ as specified in (2.12).

Remark 2.3. In the special case when $c_{ij} = 1$ for all $(i, j) \in \mathcal{E}$, (2.9) simplifies to $a_j r_{ij} = \frac{1}{2}(a_j\hat{\xi}_j + a_i\xi_i)$. Therefore, $v_{ij} = a_j r_{ij} - a_i\xi_i = \frac{1}{2}(a_j\hat{\xi}_j - a_i\xi_i)$. The DDAE (2.11) then also simplifies, as the latter two equations can be replaced by the single equation

$$\kappa_i^{-1}\dot{\xi}_i = a_i \sum_{j\in\mathcal{N}_i} \frac{1}{2}(a_j\hat{\xi}_j - a_i\xi_i) + \omega_i \quad \forall i \in \mathcal{V}.$$

Hence, by choosing unitary edge weights, the scattering subsystem becomes invisible, as the input v_{ij} of the DAI controller is merely a result of subtracting its own output $a_i\xi_i$ from the signal $a_j\hat{\xi}_j$ received from the other end. In fact, the system resembles the nominal system (2.3) with the most notable difference being that the signals ξ_j are subject to a delay in the communication channels. Thus, the DDAE (2.11) boils down to a regular DDE.

Assumption 2.4. Let $r \in \mathbb{R}^{2m}$ be the stacked vector of all r_{ij} (two per edge). The initial condition $\phi = \text{col}(\eta_0, \omega_0, \xi_0, r_0) \in C([-\rho, 0], \mathbb{R}^{2n+3m})$ of (2.11) is piecewise continuous and shows only discontinuities in r on a finite set $\mathcal{T}$ of time events in $[-\rho, 0]$.

LEMMA 2.5. *For any initial condition $\phi \in C([-\rho, 0], \mathbb{R}^{2n+3m})$, the delayed system* (2.11) *admits a unique and piecewise continuous solution through ϕ for $t \in [-\rho, \infty)$.*

Proof. Let $t_1 > 0$ be the first event in time for which $x(\phi, t)$ depends explicitly on a time event in $\mathcal{T}^0 := \mathcal{T} \cup \{0\}$, i.e., $t_1 - T_{ij} \in \mathcal{T}^0$, or $t_1 - T_{ij} - T_{ji} \in \mathcal{T}^0$ for some $i \in \mathcal{V}$, $j \in \mathcal{N}_i$. Then (2.11) restricted to the interval $(0, t_1)$ boils down to an ordinary differential equation of the form $\dot{x} = f(t, x(t))$, which is globally Lipschitz continuous in x and continuous in t as the $a_j r_{ij}$ terms are piecewise continuous on this interval. By the global version of the Picard–Lindelöf theorem, there is a unique solution $x(t)$ for $-\rho \le t \le t_1$. For the solution beyond t_1, we proceed as follows: for $k \ge 1$, define recursively the sets $\mathcal{T}^k = \{t > 0 \mid t - T_{ij} \in \mathcal{T}^{k-1},\ i \in \mathcal{V},\ j \in \mathcal{N}_i\}$ as sets of time events for which discontinuities may appear. For any given $T > 0$, we have that $\mathcal{T}(T) := \cup_{k\ge 0}\mathcal{T}^k \cap [0, T]$ is a finite set and thus, for the interval $(t_2, t_3]$, where $0 < t_2, t_3 < T$ are two consecutive time events in $\mathcal{T}(T)$, we can compute a solution through $x(t)\mid_{-\rho\le t\le t_2}$ relative to (2.11) in a similar way as for the interval $(0, t_1]$. By repeating this procedure, a piecewise continuous function on $(-\rho, \infty)$ is obtained in a unique way. ∎

3. Conditions on synchronization. We will now study the synchronous conditions of the delayed system (2.11). Given an initial condition $\phi \in C([-\rho, 0], \mathbb{R}^{2n+3m})$, we refer to a synchronous solution as a solution $x(\phi, t)$ for which $\omega(t) \equiv \mathbb{1}\omega^s$ for some constant $\omega^s \in \mathbb{R}$. First, we assume that $x = \text{col}(\eta, \omega, \xi, r)$ is contained in the invariant set

$$\Omega_0 := \{x = \text{col}(\eta, \omega, \xi, r) \in \mathbb{R}^{2n+3m} \mid \eta \in \text{im}(B^T)\}.$$

In what follows, we will focus mainly on the subset of Ω_0, where the phase angle differences are less than $\frac{\pi}{2}$ from 0:

$$\Omega := \Big\{x \in \mathbb{R}^{2n+3m} \mid \eta \in \text{im}(B^T), |\eta| < \frac{\pi}{2}\Big\}.$$

Now, let the union of ω-limit sets of synchronous solutions of (2.11) in the set Ω be denoted by $\bar{\Omega}$. Similarly, $\bar{\Omega}_{nom}$ is the union of ω-limit sets of synchronous solutions that satisfy $\eta \in \text{im}(B^T)$ and $|\eta| < \frac{\pi}{2}$ relative to the nominal system (2.3). It is well known [17] that points in $\bar{\Omega}_{nom}$ are equilibrium points of (2.3). For the convenience of the reader, we restate the lemma with a proof.

LEMMA 3.1. *Let* $\bar{x} = col(\bar{\eta}, \bar{\omega}, \bar{\xi}) \in \bar{\Omega}_{nom}$. *Then* $\bar{x}$ *is an equilibrium point of* (2.3) *with* $\bar{\omega} = 0$, $\bar{\xi} = A^{-1}\mathbb{1}\xi^s$, *and* $\xi^s = \frac{\mathbb{1}^T P^{net}}{\mathbb{1}^T A^{-1}\mathbb{1}}$.

Proof. Let $\bar{\omega} = \mathbb{1}\omega^s$ for some $\omega^s \in \mathbb{R}$. Then $\dot{\bar{\eta}} = B^T\mathbb{1}\omega^s = 0$. Therefore, the terms in the right-hand side of the dynamics for $0 = M\dot{\bar{\omega}} = -D\bar{\omega} - B\Gamma\sin(\bar{\eta}) - \bar{\xi} + P^{net}$ are constant over time; hence $\dot{\bar{\xi}} = 0$. Summing up the rows gives $\mathbb{1}^T D\mathbb{1}\omega^s = \mathbb{1}^T(-\bar{\xi} + P^{net})$. The active power flow term vanishes since $\mathbb{1}^T B = 0$. Setting $\dot{\bar{\xi}} = 0$ yields $AL(\mathcal{G}_c)A\bar{\xi} = -\mathbb{1}\omega^s$. By positive definiteness of A and $\text{im}(L(\mathcal{G}_c)) \perp \text{im}(\mathbb{1})$, we have $\text{im}(AL(\mathcal{G}_c)) \cap \text{im}(\mathbb{1}) = \{0\}$, implying that $A\bar{\xi} \in \ker(AL(\mathcal{G}_c)) = \text{span}\{\mathbb{1}\}$ and $\omega^s = 0$. This implies, in turn, that $\bar{\xi} = A^{-1}\mathbb{1}\xi^s$ for some $\xi^s \in \mathbb{R}$ and from $\dot{\bar{\omega}} = 0$, $\mathbb{1}^T\bar{\xi} = \mathbb{1}^T P^{net}$, which yields the given value for ξ^s. ∎

Furthermore, the equilibrium value of $\bar{\xi}$ coincides with the solution to the economic dispatch problem (2.1); see also [19] and [21]. It is shown in, e.g., [16] that the equilibrium of the nominal system, if it exists, is unique, i.e.,

$$|\bar{\Omega}_{nom}| \leq 1. \tag{3.1}$$

Unfortunately, the general lower bound for the cardinality of $\bar{\Omega}_{nom}$ is zero. Therefore, we need the following assumption.

Assumption 3.2. $|\bar{\Omega}_{nom}| = 1$, i.e., the system parameters allow for the existence of a nominal equilibrium in the set defined by $\eta \in \text{im}(B^T)$ and $|\eta| < \frac{\pi}{2}$.

Remark 3.3. In practice, this means that the active power flow capacities should not be exceeded, which is achieved by choosing the a_i parameters carefully. Indeed, a necessary condition for a steady state is that $|P^{net} - \bar{\xi}| \leq |B|\Gamma\mathbb{1}$, where $|B|\Gamma\mathbb{1}$ must be interpreted as a weighted degree vector. For each node $i \in \mathcal{V}$, this boils down to

$$\left|\frac{P_i^{net}}{\sum_{j\in\mathcal{V}} P_j^{net}} - \frac{a_i^{-1}}{\sum_{j\in\mathcal{V}} a_j^{-1}}\right| \leq \frac{\sum_{\substack{k=(i,j)\\ j\in\mathcal{N}_i}} \gamma_k}{|\sum_{j\in\mathcal{V}} P_j^{net}|}.$$

Assumption 3.2 is automatically obeyed if we choose a constant ratio $\frac{a_i^{-1}}{P_i^{net}}$ for all $i \in \mathcal{V}$, in which case $\bar{\xi} = P^{net}$, and no active power flow is needed in the steady state. Choosing a_i^{-1} inversely proportional to some reference power $\bar{P}_i$ for all $i \in \mathcal{V}$ leads to a condition which is known as *fair proportional power sharing* [21].

Assumption 3.2 guarantees the existence of a unique equilibrium point of the nominal system. Moreover, it also implies that the delayed system (2.11) has a unique equilibrium in the subset of synchronous solutions.

LEMMA 3.4. $|\bar{\Omega}| = 1$, *i.e., every synchronous solution of the delayed system* (2.11) *converges to a unique equilibrium point* $\bar{x} = col(\bar{\eta}, \bar{\omega}, \bar{\xi}, \bar{r})$. *This equilibrium point satisfies* $\bar{\omega} = 0$, $\bar{\xi} = A^{-1}\mathbb{1}\xi^s$ *with* $\xi^s = \frac{\mathbb{1}^T P^{net}}{\mathbb{1}^T A^{-1}\mathbb{1}}$ *and* $a_j\bar{r}_{ij} = \xi^s$ *for all* $i \in \mathcal{V}$ *and* $j \in \mathcal{N}_i$.

Proof. Consider a synchronous solution $\omega = \mathbb{1}\omega^s$ with $\omega^s \in \mathbb{R}$. Then, similar to the proof of Lemma 3.1, we verify that η and ξ are constant. At steady state, also the values r_{ij} turn out to be constant. Indeed, the algebraic equation (2.9) of $a_j r_{ij}$ with the steady-state condition $a_i\xi_i = a_i\hat{\hat{\xi}}_i$ boils down to

$$a_j r_{ij} = (\alpha_0)_{ij} a_j \hat{\hat{r}}_{ij} + (\alpha_{13})_{ij} a_i \xi_i + (\alpha_2)_{ij} a_j \xi_j \tag{3.2}$$

with $(\alpha_{13})_{ij} = (\alpha_1)_{ij} + (\alpha_3)_{ij} = \frac{2c_{ij}}{(1+c_{ij})^2}$. Note that (3.2) is a stable linear difference equation with gain $|(\alpha_0)_{ij}| = |\frac{(1-c_{ij})^2}{(1+c_{ij})^2}| < 1$ and constant input. Hence, we obtain $\bar{r}_{ij} := \lim_{t\to\infty} r_{ij}(t) = \lim_{t\to\infty} \hat{\hat{r}}_{ij}(t)$ and we have

$$\begin{aligned} a_j\bar{r}_{ij} =& \frac{(\alpha_{13})_{ij}}{1-(\alpha_0)_{ij}} a_i\xi_i + \frac{(\alpha_2)_{ij}}{1-(\alpha_0)_{ij}} a_j\xi_j \\ =& \frac{1}{2}\left(a_i\xi_i + a_j\xi_j\right). \end{aligned} \tag{3.3}$$

Substituting into $v_{ij} = c_{ij}(a_j\bar{r}_{ij} - a_i\xi_i)$ yields

$$v_{ij} = \frac{1}{2} c_{ij}(a_j\xi_j - a_i\xi_i).$$

The dynamics of ξ then become

$$0 = \dot{\xi}_i = \frac{1}{2} a_i \sum_{j\in\mathcal{N}_i} c_{ij}(a_j\xi_j - a_i\xi_i) + \omega^s.$$

In matrix-vector notation, we write this as $\dot{\xi} = -\frac{1}{2}AL(\mathcal{G}_c)A\xi + \mathbb{1}\omega^s = 0$. The rest of the proof follows along the same lines as the proof of Lemma 3.1, where $\omega^s = 0$ and $\tilde{\xi}^s = \frac{\mathbb{1}^T P^{net}}{\mathbb{1}^T A^{-1}\mathbb{1}} = \xi^s$. In addition, (3.3) boils down to $a_j\bar{r}_{ij} = \frac{1}{2}(\xi^s + \xi^s) = \xi^s$. ∎

Remark 3.5. The proof of Lemma 3.1 reveals that all synchronous solutions of the nominal system (2.3) not only converge to $\bar{\Omega}_{nom}$, but are also fully contained in $\bar{\Omega}_{nom}$ for $t \in [-\rho, \infty)$, and hence are constant functions. This is not the case for the delayed system (2.11), which allows for time-varying transient behavior of synchronous solutions under some conditions. As shown in the proof of Lemma 3.4, ξ is constant in synchronous solutions, and thus, it must assume the steady-state value of $\bar{\xi} = A^{-1}\mathbb{1}\xi^s$. The steady-state values of η, ω, and ξ are maintained if and only if $v_i = \sum_{j\in\mathcal{N}_i} c_{ij}(a_j r_{ij} - \xi^s) = 0$, where we can express $a_j r_{ij}$ in terms of its past values and ξ^s similar to (2.12):

$$a_j r_{ij}(t) =(\alpha_0)^l_{ij} a_j r_{ij}(t - lT_{ij} - lT_{ji}) + (1 - (\alpha_0)^l_{ij})\xi^s,$$

where l is any nonnegative integer.

Make a partition of $\mathcal{N}_i$ in clusters $\mathcal{N}_i^k$ of constant variables, i.e., with the property that $T_{ij}+T_{ji} = T^k$, $c_{ij} = c^k$ (and consequently, $(\alpha_0)_{ij} = \alpha_0^k$) for all $j \in \mathcal{N}_i^k$. For any cluster $\mathcal{N}_i^k$, choose any set of initial conditions $r_{ij} \in \mathcal{C}([-\rho, 0], \mathbb{R})$, $j \in \mathcal{N}_i^k$, such that

$$\sum_{j\in\mathcal{N}_i^k} c^k(a_j r_{ij}(t) - \xi^s) = 0 \text{ for all } t \in [-\rho, 0]. \tag{3.4}$$

Then

$$\begin{aligned} v_i(t) &= \sum_k \sum_{j\in\mathcal{N}_i^k} c^k(a_j r_{ij}(t) - \xi^s) \\ &= \sum_k c^k \alpha_0^k \sum_{j\in\mathcal{N}_i^k} (a_j r_{ij}(t - T^k) - \xi^s) \\ &\ \vdots \\ &= \sum_k c^k (\alpha_0^k)^{l_k} \sum_{j\in\mathcal{N}_i^k} (a_j r_{ij}(t - l_k T^k) - \xi^s), \end{aligned}$$

where each $l_k \in \mathbb{N}$ is such that $t - l_k T^k \in [-\rho, 0]$ Note that the inner summation is zero by the initial condition we have set. Consequently, a lower bound for the dimension of the feasible subspace of the variables $(r_{ij_1}, \dots, r_{ij_{|\mathcal{N}_i|}}) \in \mathbb{R}^{|\mathcal{N}_i|}$ in the initial condition for a synchronous solution in Ω is $|\mathcal{N}_i|$ minus the number of clusters $\mathcal{N}_i^k$.

4. Stability. In [21], it is shown that the nominal system will converge to the unique equilibrium point $(\bar{\eta}, \bar{\omega}, \bar{\xi}) \in \bar{\Omega}_{nom}$ through Lyapunov analysis. To show that the equilibrium point $\bar{x}$ of the delayed system (2.11) is stable, we use an extension of LaSalle's invariance principle. The procedure to describe this stability result in this section is self-contained and is adopted from section 5.3 of Hale and Verduyn Lunel [10]. First, we use the definition of Lyapunov functionals for DDAEs and adapt it for our delayed system.

DEFINITION 4.1 (see [10, Def. 3.1, p. 143]). *A continuous function $H : G \to \mathbb{R}$, with*

$$G := C([-\rho, 0], \Omega), \tag{4.1}$$

is a Lyapunov functional relative to the DDAE (2.11) *if H is continuous on $\bar{G}$ (the closure of G), $H \geq 0$, and $\dot{H} \leq 0$ on G, where*

$$\dot{H}(x_t) = \lim \sup_{h\to 0+} \frac{1}{h} \left(H(x_{t+h}) - H(x_t)\right)$$

and $x_t(\theta) = x(\phi, t + \theta)\, |_{\theta\in[-\rho,0]} \in C([-\rho, 0], \Omega)$, the restriction of the unique solution $x(\phi, t)$ of (2.11).

We propose $H(x_t) = S(x_t(0)) + \sum_{(i,j)\in\mathcal{E}} V_{ij}(x_t)$ as a candidate Lyapunov functional. Here, S is the Lyapunov function which is commonly used for stability analysis in the case without time delays, given as

$$\begin{aligned} S = &\, U(\eta) - U(\bar{\eta}) - \nabla_\eta U(\bar{\eta})^T(\eta - \bar{\eta}) \\ &+ \frac{1}{2}\omega^T M \omega + \frac{1}{2}(\xi - \bar{\xi})^T K^{-1}(\xi - \bar{\xi}), \end{aligned} \tag{4.2}$$

where $U(\eta) = -\mathbb{1}^T \Gamma \cos(\eta)$. The extra terms V_{ij} are defined as follows:

$$\begin{aligned} V_{ij} =& \frac{1}{2} \int_0^t \left(\|\Delta s_{ij}^{\rightarrow}\|^2 - \|\Delta s_{ij}^{\leftarrow}\|^2 + \|\Delta s_{ji}^{\rightarrow}\|^2 - \|\Delta s_{ji}^{\leftarrow}\|^2 \right) d\tau \\ &+ \underbrace{\frac{1}{2} \int_{-T_{ij}}^{0} \|\Delta s_{ij}^{\rightarrow}\|^2 d\tau}_{V_{ij}^*} + \underbrace{\frac{1}{2} \int_{-T_{ji}}^{0} \|\Delta s_{ji}^{\rightarrow}\|^2 d\tau}_{V_{ji}^*} . \end{aligned} \tag{4.3}$$

Here, Δ denotes the difference w.r.t. the equilibrium $\bar{x}$ of Lemma 3.4, which can be determined using (2.5) and (2.6), together with the equilibrium conditions $a_j \bar{r}_{ij} = a_i \bar{\xi}_i$ and $\bar{v}_{ij} = c_{ij}(a_j \bar{r}_{ij} - a_i \bar{\xi}_i) = 0$. This yields the following explicit expression for if i is a head (left) or a tail (right):

$$\begin{aligned} \Delta s_{ij}^{\rightarrow} =& s_{ij}^{\rightarrow} - \frac{1}{\sqrt{2}} a_i \bar{\xi}_i, \qquad & \Delta s_{ij}^{\rightarrow} =& s_{ij}^{\rightarrow} + \frac{1}{\sqrt{2}} a_i \bar{\xi}_i, \\ \Delta s_{ij}^{\leftarrow} =& s_{ij}^{\leftarrow} + \frac{1}{\sqrt{2}} a_i \bar{\xi}_i, \qquad & \Delta s_{ij}^{\leftarrow} =& s_{ij}^{\leftarrow} - \frac{1}{\sqrt{2}} a_i \bar{\xi}_i. \end{aligned}$$

LEMMA 4.2. *H is a Lyapunov functional on G relative to the delayed system* (2.11).

Proof. By construction, V is lower bounded by zero. Indeed, if we substitute the time delay coupling equations (2.7) into (4.3), we obtain

$$V_{ij} = \frac{1}{2} \int_{t-T_{ij}}^{t} \|\Delta s_{ij}^{\rightarrow}(\tau)\|^2 d\tau + \frac{1}{2} \int_{t-T_{ji}}^{t} \|\Delta s_{ji}^{\rightarrow}(\tau)\|^2 d\tau \geq 0. \tag{4.4}$$

Also, S is lower bounded by zero as $U(\eta) - U(\bar{\eta}) - \nabla_\eta U(\bar{\eta})^T (\eta - \bar{\eta})$ defines a Bregman divergence [4], which is nonnegative whenever $|\eta| < \frac{\pi}{2}$.

Now we show that $\dot{H} \leq 0$ everywhere. S admits the following partial gradients:

$$\begin{aligned} \nabla_\eta S =& \nabla_\eta U(\eta) - \nabla_\eta U(\bar{\eta}) =: f(\eta), \\ \nabla_\omega S =& M\omega, \\ \nabla_\xi S =& K^{-1}(\xi - \bar{\xi}) =: K^{-1} \Delta\xi. \end{aligned}$$

Let us write the dynamics of ξ in matrix-vector notation as $\dot{\xi} = Av + \omega$, where A is as before (see (2.3)), and $v_i = \sum_{j \in \mathcal{N}_i} v_{ij}$. The time derivative of S is given as (note that $P^{net} = \bar{\xi} + B\nabla U(\bar{\eta})$ and $\bar{v} = 0$, hence $\Delta v = v$):

$$\begin{aligned} \dot{S} =& \nabla_\eta S^T \dot{\eta} + \nabla_\omega S^T \dot{\omega} + \nabla_\xi S^T \dot{\xi} \\ =& f(\eta)^T B^T \omega + \omega^T(-D\omega - Bf(\eta) - \Delta\xi) \\ &+ \Delta\xi^T (Av + \omega) \\ =& -\omega^T D\omega + \Delta\xi^T A \Delta v \\ =& -\omega^T D\omega + \sum_{i \in \mathcal{V}} a_i \Delta\xi_i \sum_{j \in \mathcal{N}_i} \Delta v_{ij} \\ =& -\omega^T D\omega + \sum_{(i,j) \in \mathcal{E}} \left(a_i \Delta\xi_i \Delta v_{ij} + a_j \Delta\xi_j \Delta v_{ji} \right). \end{aligned}$$

By (2.5) and (2.6), it follows readily that

$$\|\Delta s_{ij}^{\rightarrow}\|^2 - \|\Delta s_{ij}^{\leftarrow}\|^2 = -2a_j \Delta r_{ij} \Delta v_{ij}.$$

Substituting in (4.3) gives

$$V_{ij} = \int_0^t \left(-a_j \Delta r_{ij} \Delta v_{ij} - a_i \Delta r_{ji} \Delta v_{ji}\right) d\tau + V_{ij}^* + V_{ji}^*,$$

and therefore,

$$\dot{V} = \sum_{(i,j)\in\mathcal{E}} \dot{V}_{ij} = \sum_{(i,j)\in\mathcal{E}} \left(-a_j \Delta r_{ij} \Delta v_{ij} - a_i \Delta r_{ji} \Delta v_{ji}\right).$$

The derivative of the Hamiltonian $H = S + \sum_{(i,j)\in\mathcal{E}} V_{ij}$ then becomes

$$\begin{aligned}\dot{H} = &- \omega^T D\omega - \sum_{(i,j)\in\mathcal{E}} ((a_j \Delta r_{ij} - a_i \Delta\xi_i)\Delta v_{ij} \\ &+ (a_i \Delta r_{ji} - a_j \Delta\xi_j)\Delta v_{ji}) \\ = &- \omega^T D\omega - \sum_{(i,j)\in\mathcal{E}} \left(\frac{1}{c_{ij}} \Delta v_{ij}^2 + \frac{1}{c_{ij}} \Delta v_{ji}^2\right).\end{aligned}$$

As a consequence, $\dot{H} \leq 0$. ◻

Note that in contrast to H, its derivative $\dot{H}$ does not depend on past states, but only on current variables. Analogous to the classical version of LaSalle's invariance principle, we aim at finding $\mathcal{M}$, the largest invariant subset of

$$\mathcal{S} := \{\phi = \mathrm{col}(\eta_0, \omega_0, \xi_0, r_0) \in G \mid \dot{H}(\phi) = 0\},$$

where the set G is defined in (4.1), and the invariance is with respect to the dynamics (2.11).

Since the equilibrium point $\bar{x} = \mathrm{col}(\bar{\eta}, \bar{\omega}, \bar{\xi}, \bar{r})$ of Lemma 3.4 is contained in $\mathcal{S}$ and $\dot{H}$ does not depend on past states, this gives

$$\mathcal{S} = \{\phi \in G \mid \omega(0) = 0,\ a_j \Delta r_{ij}(0) = a_i \Delta\xi_i(0)\ \forall i \in \mathcal{V},\ j \in \mathcal{N}_i\}.$$

Under the dynamics (2.11), we have shown in the proof of Lemma 3.4 that the synchronous solution $\bar{\omega} = 0$ implies that ξ and r converge to the equilibrium values of $\bar{\xi}$ and $\bar{r}$ as given in Lemma 3.4 as $t \to \infty$. Thus,

$$\mathcal{M} = \{\phi \in G \mid \omega \equiv 0, \Delta\xi(0) \equiv 0, \Delta r \equiv 0\},$$

which means that $\mathcal{M}$ is a singleton that coincides with the constant function $[0, \rho] \to \bar{x}$. We will work towards our main result that any solution in G converges to $\mathcal{M}$ under the dynamics (2.11). This requires that $|\eta| < \frac{\pi}{2}$ for all $t \geq -\rho$, which will lead to local asymptotic stability of $\bar{x}$.

As we have proven in Lemma 2.5, the solution $x(\phi, t)$ through ϕ is unique and piecewise continuous. With this notation, we write

$$\mathcal{M} = \{\phi \in G \mid x(\phi, t) \equiv \bar{x}\}. \tag{4.5}$$

Before we state the stability result, we need the following lemma, which shows that solutions of the delayed system (2.11) through an initial condition ϕ contained in Ω are bounded, provided that the value of the Lyapunov function H is sufficiently small on the initial condition ϕ, and the initial condition $\phi = (\omega_0, \eta_0, \xi_0, r)$ is bounded.

Since ω_0, η_0, and ξ_0 are bounded by the first condition, the latter condition boils down to boundedness of the r component of ϕ. We pose the latter as an assumption, which, together with Assumption 2.1 and Assumption 2.4, forms the conditions on the initial condition ϕ.

Assumption 4.3. For every initial condition $\phi = \mathrm{col}(\eta_0, \omega_0, \xi_0, r_0) \in G$, with $G = C([-\rho, 0], \Omega)$ as before, the control variable r_0 is bounded on its entire domain, i.e., there exists $M \in \mathbb{R}$ such that $|r_0(t)| \leq M$ for all $t \in [-\rho, 0]$.

Hence, the initial condition of the control variables cannot be arbitrary, but must satisfy a suitable bound. In the nondelay case, this corresponds to initializing the controller to values in a bounded subset instead of the whole control space.

LEMMA 4.4. *Consider the Lyapunov functional* $H(x_t) = S(x_t(0)) + \sum_{(i,j)\in\mathcal{E}} V_{ij}(x_t)$, *with* S *and* V_{ij} *as in* (4.2) *and* (4.3). *Define* $\mathcal{U}_l := \{\phi \in G \mid H(\phi) < l\}$. *Then*

1. *there exists* $L > 0$ *such that* $\mathcal{U}_l$ *is positive invariant with respect to the dynamics* (2.11) *for all* $l \leq L$;
2. *for all* $\phi \in \mathcal{U}_l$ *with* $l \leq L$, *there exists* $K = K(l, \phi)$ *such that* $\|x_t(\phi, 0)\| < K$ *for all* $t \geq 0$.

Proof. We start with the first statement. By Lemma 4.2, we have $\dot{H}(x_t) \leq 0$ on $G \supseteq \mathcal{U}_l$, and we need only prove the existence of $L > 0$ such that $x(\phi, t) \in \Omega$, i.e., $|\eta(t)| < \frac{\pi}{2}$ for all $t > 0$ and $\phi \in \mathcal{U}_l$ with $l \leq L$. Note that the first three terms in (4.2) define a Bregman distance between $U(\eta)$ and the value of the first order Taylor expansion of U around $\bar{\eta}$ evaluated at η. We write this Bregman distance as $d(\eta) := \sum_{i\in\mathcal{E}} d_i(\eta_i)$, where $d_i(\eta_i) = -\Gamma_{ii}(\cos(\eta_i) - \cos(\bar{\eta}_i)) - \Gamma_{ii}\sin(\bar{\eta}_i)(\eta_i - \bar{\eta}_i)$. Note that d restricted to the closed set $\Theta := \{\eta \mid |\eta_i| \leq \frac{\pi}{2}\}$ is convex, attains its minimum at $\eta = \bar{\eta}$ with $d(\bar{\eta}) = 0$, and is a maximum in $\hat{\eta}$, where

$$\hat{\eta}_i = \begin{cases} -\frac{\pi}{2}, & \bar{\eta}_i \geq 0, \\ \frac{\pi}{2}, & \bar{\eta}_i < 0. \end{cases}$$

Now, select an index $i_0 \in 1, \dots, n$ for which $d_{i_0}(-\hat{\eta}_{i_0})$ is minimum and set $L = d(\hat{\eta}^*)$ with

$$\hat{\eta}_i^* = \begin{cases} -\hat{\eta}_i, & i = i_0, \\ \bar{\eta}_i & \text{otherwise.} \end{cases} \tag{4.6}$$

Then it holds that $L > 0$ since $d_{i_0}(-\hat{\eta}_i) > 0$ and $d_i(\bar{\eta}_i) = 0$ for all $i \neq i_0$. By construction, $\hat{\eta}^*$ satisfies the property that $\min_{\eta\in\partial\Theta} d(\eta) = d(\hat{\eta}^*) = L$, where $\partial\Theta$ denotes the boundary of Θ. Now, for any solution $x(\phi, t) = \mathrm{col}(\eta, \omega, \xi, r)$ through $\phi \in \mathcal{U}_l$ with $l \leq L$, we have $d(\eta_t(0)) \leq H(x_t) < l \leq L$ and $|\eta_i(0)| < \frac{\pi}{2}$ for all $i \in \mathcal{V}$. We conclude that $\eta(t)$ remains in the interior of Θ for all $t \geq 0$.

For the second statement, consider any initial condition $\phi \in \mathcal{U}_l$ and the corresponding solution $x(\phi, t) = \mathrm{col}(\omega, \eta, \xi, r)$. From the Lyapunov characteristics $H(\phi) < l$ and $\dot{H}(x_t) \leq 0$ and nonnegativity of (1) the Bregman distance $d(\eta)$, (2) the quadratic terms of ω and $\Delta\xi$ in $S(x_t(0))$, and (3) $V(x_t)$, both $\|\omega\|$ and $\|\xi\|$ are bounded by some numbers l_ω and l_ξ, respectively, on $[-\rho, \infty)$. These bounds are only dependent on l. Since the function $g_{ij}(\xi_t)$ of (2.9) is a linear combination of $\xi_i(t)$, $\xi_j(t - T_{ji})$, and $\xi_i(t - T_{ij} - T_{ji})$ with $(\alpha_1)_{ij} + (\alpha_2)_{ij} + (\alpha_3)_{ij} = 1 - (\alpha_0)_{ij}$, we have $a_j|r_{ij}| \leq (\alpha_0)_{ij} a_j |\hat{\hat{r}}_{ij}| + (1 - (\alpha_0)_{ij}) l_\xi$. Hence, for any $t > 0$, and $k(t) \in \mathbb{N}$ such that

$t - k(T_{ij} + T_{ji}) \in [-\rho, 0]$, we write this in terms of the initial condition as

$$\begin{aligned} a_j|r_{ij}(t)| &\leq (\alpha_0)_{ij}^k a_j |r_{ij}(t - k(T_{ij} + T_{ji}))| \\ &\quad + \left(1 - (\alpha_0)_{ij}^k\right) l_\xi \\ &< \sup_{\substack{i \in \mathcal{V}, j \in \mathcal{N}_i \\ \tau \in [-\rho, 0]}} a_j |r_{ij}(\tau)| + l_\xi. \end{aligned}$$

Hence, by Assumption 4.3, r is bounded on the entire domain $[-\rho, \infty)$. Also, $|\eta| < \pi/2$ by definition of $\mathcal{U}_l$. Thus, we have shown that every variable stacked in $x(t) = x_t(0)$ with $t \geq 0$ is bounded. Depending on the chosen norm, we can set up a constant $K = K(l, \phi)$ such that $x_t \in \mathcal{U}_l$ satisfies $\|x_t(0)\| < K$. □

We are ready to state our main result on the local asymptotic stability of (2.11); see [10, Thm. 3.1, p. 143]. We recall from Lemma 3.4 that synchronous solutions of the delayed system (2.11) converge to a unique equilibrium point. This result can be extended for nonsynchronous solutions in a neighborhood of the equilibrium point.

THEOREM 4.5. *The equilibrium point* $\bar{x} = col(\bar{\eta}, \bar{\omega}, \bar{\xi}, \bar{r})$ *of the delayed system* (2.11) *to which every synchronous solution converges, i.e.,* $\bar{\omega} = 0$, $\bar{\xi} = A^{-1}\mathbb{1}\xi^s$ *with* $\xi^s = \frac{\mathbb{1}^T P^{net}}{\mathbb{1}^T A^{-1} \mathbb{1}}$ *and* $a_j \bar{r}_{ij} = \xi^s$ *for all* $i \in \mathcal{V}$ *and* $j \in \mathcal{N}_i$, *is locally asymptotically stable under the dynamics of* (2.11).

Proof. Let $x(\phi, t)$ be the unique solution through an initial condition $\phi \in \mathcal{U}_l$ relative to (2.11) with $l < L = d(\hat{\eta}^*)$ as defined in (4.6), and consider the function $\gamma_\phi : [0, \infty) \to C([-\rho, 0], \mathbb{R}^n) : t \to x_t(\phi)$. By Lemma 4.4, the range $\{x_t(\phi)\}_{t \geq 0}$ of γ_ϕ belongs to a compact set and has a nonempty ω-limit set denoted by $\omega(\gamma_\phi)$. Since $H(\gamma_\phi(t))$ is nonincreasing and bounded from below by zero, we have that for any $\psi \in \omega(\gamma_\phi)$ it holds that $H(\gamma_\psi(t)) \equiv c$ for some $c \geq 0$. Thus, $\omega(\gamma_\phi) \subseteq \mathcal{S}$, and since the ω-limit set is invariant, it must be contained in $\mathcal{M}$. Thus, since γ_ϕ is bounded, we have that $\gamma_\phi(t) \to \omega(\gamma_\phi) \subseteq \mathcal{M}$ as $t \to \infty$, and from (4.5), $x(\phi, t)$ converges to $\bar{x}$. □

5. Simulation.

We perform a case study on the two-area test system described in [12, Ex. 12.6, p. 813] and [17]. In this network, two areas, each consisting of two generators, are connected to each other by a weak tie, i.e., with smaller susceptance, as is illustrated in Figure 2.

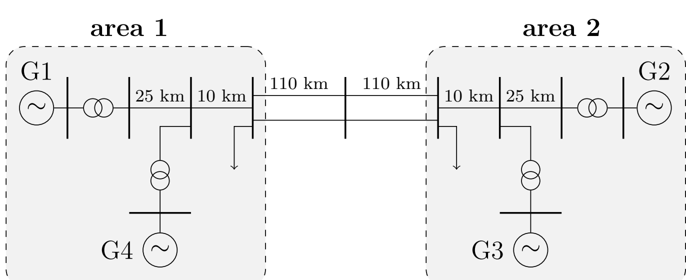

FIG. 2. *The power network consisting of two areas and four generators from* [12] *and* [17].

We model this configuration as a ring graph consisting of the generators 1, 2, 3, and 4. The parameters are expressed in a per-unit (pu) system, which is commonly used for the analysis of power systems. We set the base voltage equal to the rated voltage of 20 kV in both areas and the base power to 100 MAV. The choice of the system parameter values is based on the ones used in [12, Ex. 12.6, p. 813], except for the droop coefficients, which we set to $D_i = 2.5$ for all $i \in \mathcal{V}$.

We consider two operating conditions, starting with the machine loads in P_0^{net}. After 10 seconds, we let the system operate under more stressed conditions by increasing the machine loads of the generators in area 2, changing P^{net} instantly to P_1^{net}, where

$$P_0^{net} = \begin{pmatrix} 7.49 \\ 2.92 \\ 6.25 \\ 5.26 \end{pmatrix} \text{pu}, \qquad P_1^{net} = \begin{pmatrix} 6.65 \\ 0.43 \\ 1.53 \\ 5.56 \end{pmatrix} \text{pu}.$$

We set the rated machine power for generators 1 and 2 to 9 pu, and for generators 3 and 4 to 7 pu. We choose as DAI cost parameters a_i the reciprocals of the rated machine power, which conforms to a fair power sharing principle. We set the DAI timescale parameters $\kappa_i = 250$, and communication edge weights $c_{ij} = 0.5$ for all $i \in \mathcal{V}$ and $j \in \mathcal{N}_i$. Furthermore, the transmission line capacity parameters $\gamma_{12} = \gamma_{34} = 3.41$ account for the weak ties between the areas, whereas $\gamma_{23} = \gamma_{41} = 39.6$ hold for the stronger ties that lie within each of the two areas.

The simulation is carried out with Python/Numpy. As an initial condition, we choose the equilibrium point corresponding to the steady-state operating condition with the initial P_0^{net} as described in Lemma 3.1. That is, the frequency deviation variables $\bar{\omega}_i$ are all zero, whereas the scattering subsystem output variables $a_j \bar{r}_{ij} = \xi^s = 0.685$ pu and the DAI controller state $\bar{\xi}_i = a_i^{-1} \xi^s$. The delays are symmetric and heterogeneous: 0.8 and 1.3 seconds for the shorter ties within each area, and 3.4 and 4.7 seconds for the longer ties connecting the two areas.

Figure 3 confirms the stability of this system, both before and after the change in machine loads. We observe a frequency drop at the generators, especially for those in area 2. The frequency stabilizes shortly afterwards due to droop control, while for the next ∼30 seconds, secondary control takes effect to restore the frequency. The rotor angle differences corresponding to the ties that connect the two areas reveal that the network is more stressed in the new steady state. This can be seen by a further shift of the values η_{12} and η_{34} towards the boundaries of the stability region $[-\frac{1}{2}\pi, \frac{1}{2}\pi]$. We observe that ξ is restored to values proportional to the reciprocals of the cost parameters a_i^{-1} in such a way that in the dynamics of the generators, the power delivered by this actuation variable compensates for the net machine load, together with the active power flow. The controller variables $a_j r_{ij}$ converge to the new constant $\xi^s = 0.443$ pu.

6. Conclusion. We focused on the Kron-reduced microgrid consisting of generator units that is modelled by the swing equations and controlled by distributed averaging integral (DAI) controllers. These controllers are placed at every node to enforce synchronization among the generator frequencies in an economically efficient way. Each DAI controller sends a signal to its neighboring nodes that is subject to a scattering transformation, a delay, and another scattering transformation before it is received at the other end of the communication channel. The closed-loop system can be modelled by a delay differential-algebraic system of equations (DDAE) which has a state in the space of piecewise continuous functions from $-\rho$ to 0, where ρ denotes an appropriate time horizon. Assuming piecewise continuity, a finite amount of discontinuities, and partial boundedness in the initial state, the system possesses a

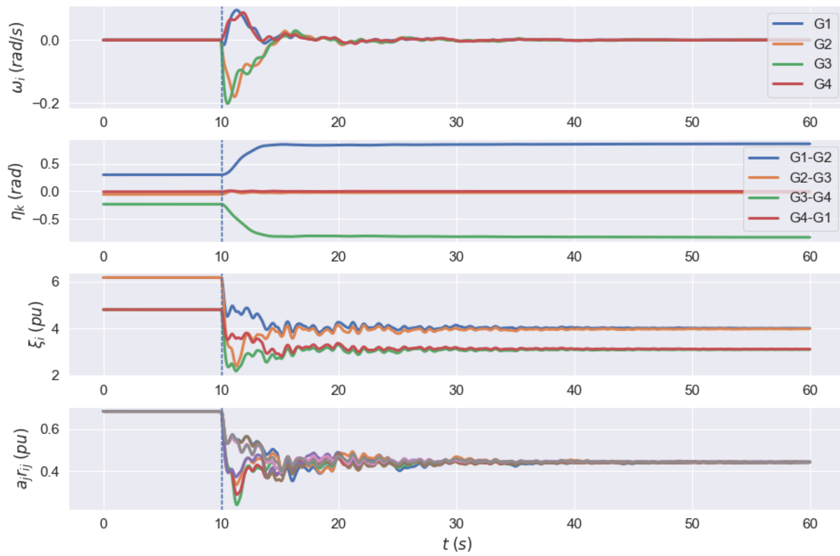

FIG. 3. *The upper plot shows that the frequency deviation ω_i of all four generators in the two-area system reverts to zero after a change in the network load P^{net} for the delayed differential algebraic system described in* (2.11). *The other state variables are converging to the equilibrium point corresponding to the new situation too, as is illustrated by the plots of the rotor angle differences η_k, DAI controller state ξ_i, and scattering subsystem output variables $a_j r_{ij}$.*

locally asymptotically stable equilibrium point that exhibits synchronization of rotational speed and coincides with the solution of the optimal power dispatch problem. Furthermore, the asymptotic stability is independent of the value of the delays.

Future work will include finding a port-Hamiltonian representation of the delayed system, and more generally, stating a general result on PI control of port-Hamiltonian systems over a network when feedback measurement is delayed. Also, higher-dimensional models of the system components can be considered, such as third order swing equations.

REFERENCES

[1] C. Ahumada, R. Cárdenas, D. Sáez, and J. M. Guerrero, *Secondary control strategies for frequency restoration in islanded microgrids with consideration of communication delays*, IEEE Trans. Smart Grid, 7 (2016), pp. 1430–1441.

[2] M. Andreasson, H. Sandberg, D. V. Dimarogonas, and K. H. Johansson, *Distributed integral action: Stability analysis and frequency control of power systems*, in Proceedings of the IEEE 51st Annual Conference on Decision and Control, IEEE, 2012, pp. 2077–2083.

[3] M. Bürger, D. Zalazo, and F. Allgöwer, *Duality and network theory in passivity-based cooperative control*, Automatica J. IFAC, 50 (2014), pp. 2051–2061.

[4] C. De Persis and N. Monshizadeh, *Bregman storage functions for microgrid control*, IEEE Trans. Autom. Control, 63 (2018), pp. 53–68.

[5] N. Chopra, M. W. Spong, and R. Lozano, *Synchronization of bilateral teleoperators with time delay*, Automatica J. IFAC, 44 (2008), pp. 2142–2148.

[6] S. Ci, J. Qian, D. Wu, and A. Keyhani, *Impact of wireless communication delay on load sharing among distributed generation systems through smart microgrids*, IEEE Trans. Wireless Commun., 19 (2012), pp. 24–29.

[7] E. A. A. Coelho, D. Wu, J. M. Guerrero, J. C. Vasquez, T. Dragičević, Č. Stefanović,

and P. Popovski, *Small-signal analysis of the microgrid secondary control considering a communication time delay*, IEEE Trans. Ind. Electron., 63 (2016), pp. 2657–6269.

[8] F. Dörfler, J. W. Simpson-Porco, and F. Bullo, *Breaking the hierarchy: Distributed control and economic optimality in microgrids*, IEEE Trans. Control Netw. Syst., 3 (2016), pp. 241–253.

[9] E. Fridman, *Introduction to Time-delay Systems: Analysis and Control*, Birkhäuser, Basel, 2014.

[10] J. Hale and S. Verduyn Lunel, *Introduction to Functional Differential Equations*, Appl. Math. Sci. 99, Springer-Verlag, New York, 1993.

[11] T. Hatanaka, N. Chopra, T. Ishizaki, and N. Li, *Passivity-based distributed optimization with communication delays using PI consensus algorithm*, IEEE Trans. Automat. Control, 63 (2018), pp. 4421–4428.

[12] P. Kundur, *Power System Stability and Control*, McGraw-Hill, New York, 1994.

[13] T. Matiakis, S. Hirche, and M. Buss, *Independent-of-delay stability of nonlinear networked control systems by scattering transformation*, in Proceedings of the 2006 IEEE American Control Conference, IEEE, 2006, pp. 2801–2806.

[14] U. Münz, A. Papachristodoulou, and F. Allgöwer, *Delay robustness in non-identical multi-agent systems*, IEEE Trans. Automat. Control, 57 (2012), pp. 1597–1603.

[15] R. Ortega, A. van der Schaft , B. Maschke, and G. Escobar, *Interconnection and damping assignment passivity-based control of port-controlled Hamiltonian systems*, Automatica J. IFAC, 38 (2002), pp. 585–596.

[16] J. Schiffer and F. Dörfler, *On stability of a distributed averaging PI frequency and active power controlled differential-algebraic power system model*, in Proceedings of the 2016 IEEE European Control Conference, IEEE, 2016, pp. 1487–1492.

[17] J. Schiffer, F. Dörfler, and E. Fridman, *Robustness of distributed averaging control in power systems: Time delays & dynamics communication topology*, Automatica J. IFAC, 80 (2017), pp. 261–271.

[18] G. Strbac, N. Hatziargyriou, J. P. Lopes, C. Moreira, A. Dimeas, and D. Papadaskalopoulos, *Microgrids: Enhancing the resilience of the European megagrid*, IEEE Power Energy Mag., 13 (2015), pp. 35–43.

[19] S. Trip, M. Bürger, and C. De Persis, *An internal model approach to (optimal) frequency regulation in power grids with time-varying voltages*, Automatica J. IFAC, 64 (2016), pp. 240–253.

[20] E. Weitenberg, and C. De Persis, *Robustness to noise of distributed averaging integral controllers in power networks*, Systems Control Lett., 119 (2018), pp. 1–7.

[21] E. Weitenberg, Y. Jiang, C. Zhao, E. Mallada, C. De Persis, and F. Dörfler, *Robust decentralized secondary frequency control in power systems: Merits and tradeoffs*, IEEE Trans. Automat. Control, 64 (2019), pp. 3967–3982.