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Robust output regulation for voltage control in DC networks with time-varying loads[☆]

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ABSTRACT

In this paper, we propose novel control schemes for regulating the voltage in Direct Current (DC) power networks, where the loads are the superposition of *time-varying* signals and *uncertain* constants. More precisely, the proposed control schemes are based on the robust output regulation methodology and, differently from the results in the literature, where the loads are assumed to be constant, we consider *time-varying* loads whose dynamics are described by a class of differential equations with parametric *uncertainty*. The proposed control schemes achieve voltage regulation and guarantee the local robust stability of the overall network in case of impedance (Z), current (I), and power (P) load types and the global robust stability in case of ZI loads. The simulation results illustrate excellent performance of the proposed control schemes in different scenarios, where real load data are considered.

1. Introduction

Power networks are categorized into Direct Current (DC) and Alternating Current (AC) networks. Recently, the design and operation of DC networks in presence of renewables and *new* types of load such as electric vehicles have attracted increasing research attention (Justo, Mwasilu, Lee, & Jung, 2013).

Voltage control is the main control purpose in DC networks ensuring the proper operation of the overall network, see for instance (Cucuzzella, Lazzari, Kawano, Kosaraju and Scherpen, 2019; Ferguson, Cucuzzella, & Scherpen, 2021; Iovine et al., 2018; Jeltsema & Scherpen, 2004; Kosaraju, Cucuzzella, Scherpen, & Pasumarthy, 2021; Machado, Arocas-Perez, He, Ortega, & Grino, 2018; Nahata, Soloperto, Tucci, Martinelli, & Ferrari-Trecate, 2020; Sadabadi, Shafiee, & Karimi, 2018; Strehle, Pfeifer, Malan, Krebs, & Hohmann, 2020). However, all these works and most of the results available in the literature ensure, to the best of our knowledge, voltage regulation and network stability in presence of *constant* load components only, while in practice loads are *time-varying* (see for instance Aguirre, Rodrigues, Lima, & Martinez, 2008; Chiang, Wang, Huang, Chen, & Huang, 1997; Choi et al., 2006; Verdejo, Awerkin, Saavedra, Kliemann, & Vargas, 2016; Vignesh, Chakrabarti, & Srivastava, 2014 and the references therein), making it more difficult to guarantee the stability of the power grid by using existing control strategies (Aguirre et al., 2008; Verdejo et al., 2016). Consequently, new control schemes providing robust stability guarantees of the power network in presence of *time-varying* loads and *uncertainty* need to be developed in order to make the overall power system more reliable and resilient (Sira-Ramirez & Rosales-Diaz, 2014; Wilson, Neely, Cook, & Glover, 2014). More precisely, we consider the parallel combination of impedance (Z), current (I), and power (P) load types. Additionally and differently from Cucuzzella, Lazzari et al. (2019), Ferguson et al. (2021), Iovine et al. (2018), Jeltsema and Scherpen (2004), Kosaraju et al. (2021), Machado et al. (2018), Nahata et al. (2020), Sadabadi et al. (2018), Strehle et al. (2020) and other related papers, for each type of load we consider the superposition of *time-varying* and *uncertain* constant components, where the *time-varying* components are expressed as the outputs of suitable dynamical exosystems. However, achieving voltage regulation in DC power networks (or, similarly, frequency control in AC power grids) in presence of *time-varying* loads is attracting growing research interest (see Ferguson et al., 2021; Silani, Cucuzzella, Scherpen, & Yazdanpanah, 2021b; Sira-Ramirez & Rosales-Diaz, 2014; Wilson et al., 2014 for DC grids and Silani, Cucuzzella, Scherpen, & Yazdanpanah, 2021a; Silani & Yazdanpanah, 2019; Trip, Burger, & De Persis, 2016 for AC grids). However, the solutions proposed for DC grids in Sira-Ramirez and Rosales-Diaz

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(2014) and Wilson et al. (2014) do not provide any closed-loop stability guarantee, while Ferguson et al. (2021) guarantee only (local) input-to-state stability, and the controller proposed in Silani et al. (2021b) considers only ZI loads and requires to solve Partial Differential Equations (PDEs). Differently, in this paper we propose robust control schemes based on the robust output regulation methodology (Huang, 2004), ensuring voltage regulation and robust stability also in presence of *time-varying* and *uncertain* constant components of ZIP loads.

The contributions of the paper can then be listed as follows: (i) the voltage control problem in DC networks is formulated as a robust output regulation problem; (ii) for the loads we consider the superposition of *time-varying* and *uncertain* constant impedance (Z), current (I) and power (P) components, where the *time-varying* components of loads are modeled as outputs of dynamical exosystems (see for instance Aguirre et al., 2008; Chiang et al., 1997; Choi et al., 2006; Sira-Ramirez & Rosales-Diaz, 2014; Trip et al., 2016; Vignesh et al., 2014), which is also conform with output regulation theory (Huang, 2004); (iii) we propose two control schemes achieving voltage regulation and ensuring *local* robust stability in presence of ZIP loads, where the local result is due to the use of linearization for control design; (iv) we also propose a control scheme achieving voltage regulation and ensuring *global* robust stability in presence of ZI loads.

1.1. Notation

The set of real and natural numbers are denoted by $\mathbb{R}$ and $\mathbb{N}$, respectively. The set of positive (nonnegative) real numbers is denoted by $\mathbb{R}_{>0}$ ($\mathbb{R}_{\geq 0}$). Let $\mathbf{0}$ be the vector of all zeros or the null matrix of suitable dimension(s) and let $\mathbf{1}_n \in \mathbb{R}^n$ be the vector containing all ones. The ith element of vector x is denoted by x_i. Given a vector $x \in \mathbb{R}^n$, $[x] \in \mathbb{R}^{n\times n}$ indicates the diagonal matrix whose diagonal entries are the components of x. Let $A \in \mathbb{R}^{n\times n}$ be a matrix. In case A is a positive definite (positive semi-definite) matrix, we write $A > \mathbf{0}$ ($A \geq \mathbf{0}$). Also, $\sigma(A)$ denotes the spectrum of matrix A. The $n \times n$ identity matrix is denoted by $\mathbb{I}_n$. Let $x \in \mathbb{R}^n, y \in \mathbb{R}^m$ be vectors and $\tilde{x} \in \mathbb{R}^{1\times n}, \tilde{y} \in \mathbb{R}^{1\times m}$ be row vectors, then we define $\mathrm{col}(x, y) := (x^\top \; y^\top)^\top \in \mathbb{R}^{n+m}$ and $\mathrm{row}(\tilde{x}, \tilde{y}) := (\tilde{x} \;\; \tilde{y}) \in \mathbb{R}^{1\times(n+m)}$. Consider the vector $x \in \mathbb{R}^n$ and functions $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}, h : \mathbb{R}^n \to \mathbb{R}^n$, then the Lie derivative of $h(x)$ along $g(x)$ is defined as $L_g h(x) := \frac{\partial h(x)}{\partial x} g(x)$ with $\frac{\partial h(x)}{\partial x} = \mathrm{col}\big(\frac{\partial h_1(x)}{\partial x}, \ldots, \frac{\partial h_n(x)}{\partial x}\big)$ and $\frac{\partial h_i(x)}{\partial x} = \big(\frac{\partial h_i(x)}{\partial x_1} \; \ldots \; \frac{\partial h_i(x)}{\partial x_n}\big)$ for $i = 1, \ldots, n$. Let $o^k(v)$ denote a generic function of v which is zero up to the kth order regardless of the dimension of its range space (see Huang, 2004, Definition 4.1). The bold symbols denote the solutions to regulator equations. A continuous function $\alpha : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ is said to be of class $\mathcal{K}$ if it is nondecreasing and $\alpha(0) = 0$, while it is said to be of class $\mathcal{K}_\infty$ if it also satisfies $\lim_{s\to\infty} \alpha(s) = \infty$.

2. Modeling and problem formulation

In this section, we introduce the considered DC power network model together with the dynamics of the load components. Then, the main control objective concerning the voltage regulation is introduced.

2.1. DC network model

The model of the considered DC network includes Distributed Generation Units (DGUs), loads and transmission lines Cucuzzella et al. (2019) and Tucci, Meng, Guerrero and Ferrari-Trecate (2018). Fig. 1 illustrates the structure of node i and Table 1 reports the description of the used symbols. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a connected and undirected graph that describes the topology of the considered DC network. The nodes and the edges are denoted by $\mathcal{V} = \{1, \ldots, n\}$ and $\mathcal{E} = \{1, \ldots, m\}$, respectively. Then, the topology of the network is represented by the corresponding incidence matrix $\mathcal{A} \in \mathbb{R}^{n\times m}$. Let the ends of each transmission line k be arbitrarily labeled with a '−' or a '+', then the entries of $\mathcal{A}$ are given by $\mathcal{A}_{ik} = +1$, if i is the positive end of k, $\mathcal{A}_{ik} = -1$, if i is the negative end of k, and $\mathcal{A}_{ik} = 0$, otherwise. Before presenting the dynamics of the overall network, we first introduce and discuss the models of each component of the network.

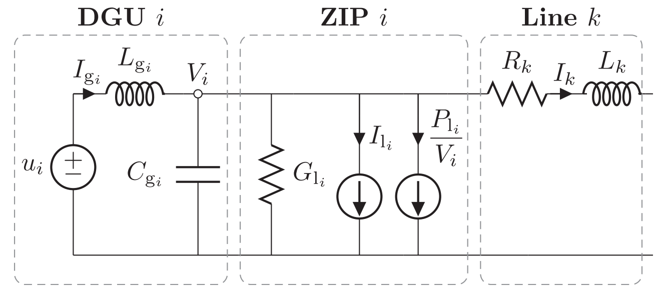

Fig. 1. Electrical scheme of DGU i, ZIP load i and transmission line k, with $i \in \mathcal{V}$ and $k \in \mathcal{E}$.

Table 1
Description of symbols.

G_{l_i}	Conductance load	u_i	Control input
I_{l_i}	Current load	L_k	Line inductance
I_{g_i}	Generated current	R_k	Line resistance
V_i	Load voltage	C_{g_i}	Shunt capacitor
I_k	Line current	L_{g_i}	Filter inductance

DGU model: The dynamic model of DGU $i \in \mathcal{V}$ is described by

$$\begin{aligned} L_{g_i}\dot{I}_{g_i} &= -V_i + u_i \\ C_{g_i}\dot{V}_i &= I_{g_i} - I_{l_i}(V_i) - \sum_{k\in\mathcal{E}_i} I_k, \end{aligned} \tag{1}$$

where $I_{g_i}, V_i, I_k, u_i : \mathbb{R}_{\geq 0} \to \mathbb{R}$, $L_{g_i}, C_{g_i} \in \mathbb{R}_{>0}$ and $\mathcal{E}_i$ is the set of the lines incident to node i. Moreover, $I_{l_i} : \mathbb{R} \to \mathbb{R}$ represents the current demand of load i possibly depending on the voltage V_i.

Load model: In this work, we consider a general *time-varying* load model including the parallel combination of the following types of load: (i) impedance (Z): $G^*_{l_i} + \hat{G}_{l_i}$, (ii) current (I): $I^*_{l_i} + \hat{I}_{l_i}$, (iii) power (P): $P^*_{l_i} + \hat{P}_{l_i}$, where $G^*_{l_i}, I^*_{l_i}, P^*_{l_i} \in \mathbb{R}_{>0}$ are *uncertain* constant components and $\hat{G}_{l_i}, \hat{I}_{l_i}, \hat{P}_{l_i} : \mathbb{R}_{\geq 0} \to \mathbb{R}$ are *time-varying* components, whose dynamics will be introduced in the next subsection. To refer to the load types above, the letters Z, I and P, respectively, are often used in the literature (see for instance De Persis, Weitenberg, & Dorfler, 2018). Thus, in presence of ZIP loads, $I_{l_i}(V_i)$ in (1) is given by

$$I_{l_i}(V_i) = (G^*_{l_i} + \hat{G}_{l_i})V_i + I^*_{l_i} + \hat{I}_{l_i} + V_i^{-1}(P^*_{l_i} + \hat{P}_{l_i}). \tag{2}$$

Line model: The dynamics of the current I_k exchanged between nodes i and j are described by

$$L_k\dot{I}_k = (V_i - V_j) - R_k I_k, \tag{3}$$

where $R_k, L_k \in \mathbb{R}_{>0}$.

Now, the dynamics of the overall network can be written compactly as

$$\begin{aligned} L_g\dot{I}_g &= -V + u \\ C_g\dot{V} &= I_g + \mathcal{A}I - [G_l]V - I_l - [V]^{-1}P_l \\ L\dot{I} &= -\mathcal{A}^\top V - RI, \end{aligned} \tag{4}$$

where $I_g, V, u : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^n$ are the generated current, load voltage and control input vector, respectively, $I : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^m$ is the line current vector, $L_g, C_g \in \mathbb{R}^{n\times n}_{>0}$ and $R, L \in \mathbb{R}^{m\times m}_{>0}$ are positive definite diagonal matrices, *e.g.*, $L_g = \text{diag}(L_{g_1}, \ldots, L_{g_n})$ (see Table 1 for description of the symbols). Moreover, $G_l = G_l^* + \hat{G}_l$, $I_l = I_l^* + \hat{I}_l$ and $P_l = P_l^* + \hat{P}_l$, where $G_l^*, I_l^*, P_l^* \in \mathbb{R}^n$ represent the *uncertain* constant components and $\hat{G}_l, \hat{I}_l, \hat{P}_l : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^n$ the *time-varying* ones.

We note that the considered graph $\mathcal{G}$ is undirected, *i.e.*, the edges are bidirectional, implying that the current through each edge can flow in both directions, and the information on the flow direction is enclosed in the incidence matrix $\mathcal{A}$. Finally, note also that for the sake of simplicity we consider load-connected topologies (see Fig. 1). This does not involve loss of generality in the case of linear loads (*e.g.*, ZI), where such topologies are obtained by a Kron reduction of the original network (Zhao & Dorfler, 2015).

2.2. Exosystems model

In this subsection, we introduce the dynamics of the *time-varying* components of the ZIP load, *i.e.*, $\hat{G}_l$, $\hat{I}_l$, $\hat{P}_l$ in (2), which are modeled as outputs of dynamical exosystems (see for instance Aguirre et al., 2008; Chiang et al., 1997; Choi et al., 2006; Sira-Ramirez & Rosales-Diaz, 2014; Trip et al., 2016; Vignesh et al., 2014). This is also conform with output regulation theory (Huang, 2004). Also, in Section 5 we use real data from the dataset (openei, 0000), which provides load profile data for residential buildings in the United States to model the exosystems. Let y (or y) denote G, I, P (or G, I, P) in case of Z, I, P loads, respectively. Then, the exosystem dynamics can be expressed as (Silani et al., 2021b, Equation (2), Silani et al., 2021a, Equation (8), Trip et al., 2016, Equations (25), (26))

$$\dot{d}_{y_i} = s_{y_i} d_{y_i}, \quad \hat{y}_{l_i} = \Gamma_{y_i} d_{y_i}, \tag{5}$$

where $d_{y_i} : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^{2n_d}$ is the state of the exosystem describing the *time-varying* components of y_{l_i}, which can be defined as $d_{y_i} := \text{col}\big(d^a_{y_i}, d^b_{y_i}\big)$, with $d^a_{y_i}, d^b_{y_i} : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^{n_d}$, $s_{y_i} \in \mathbb{R}^{2n_d \times 2n_d}$, and $\Gamma_{y_i} \in \mathbb{R}^{1\times 2n_d}$ is defined as $\Gamma_{y_i} := \big(\gamma_{y_i} \quad \mathbf{0}_{1\times n_d}\big)$, with $\gamma_{y_i} \in \mathbb{R}^{1\times n_d}$. Then, (5) can be written compactly as

$$\dot{d}_y = S_y d_y, \quad \hat{y}_l = \Gamma_y d_y, \tag{6}$$

where $d_y : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^{2nn_d}$ is defined as $d_y := \text{col}(d^a_{y_1}, \ldots, d^a_{y_n}, d^b_{y_1}, \ldots, d^b_{y_n})$, $y_l : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^n$, $S_y \in \mathbb{R}^{2nn_d \times 2nn_d}$ and $\Gamma_y := \big(\gamma_y \quad \mathbf{0}_{n\times nn_d}\big) \in \mathbb{R}^{n\times 2nn_d}$, with $\gamma_y := \text{diag}(\gamma_{y_1}, \ldots, \gamma_{y_n}) \in \mathbb{R}^{n\times nn_d}$. Now, following Huang (2004, Assumption 3.1), we introduce the following assumption.

Assumption 1 (*Stability of Exosystem*)**.** The exosystem (5) is Lyapunov stable,

$$S_y = \begin{pmatrix} \mathbf{0} & [\omega_y] \\ -[\omega_y] & \mathbf{0} \end{pmatrix}, s_{y_i} = \begin{pmatrix} \mathbf{0} & [\omega_{y_i}] \\ -[\omega_{y_i}] & \mathbf{0} \end{pmatrix}, \tag{7}$$

where $\omega_y := \text{col}\left(\omega_{y_1}, \ldots, \omega_{y_n}\right) \in \mathbb{R}^{nn_d}$, with $\omega_{y_i} \in \mathbb{R}^{n_d}$ has distinct nonzero elements and y denotes G, I or P in case of Z, I or P loads, respectively.

Note that the above assumption is required for establishing the necessary condition for the solvability of the local and global robust output regulation problem (Huang, 2004, Chapters 5, 6 and 7). Also, we notice that the exosystem model (6) is a general dynamical model, where the dimension of the state of (6) fulfilling Assumption 1 can be arbitrarily high. Indeed, choosing a higher state dimension leads to a better fitting for instance with the day-ahead load forecast. Then, the exosystem model (6) satisfying Assumption 1 is capable of reproducing a large class of signals. Thus, it is realistic to adopt exosystems and we use them because they are able to reproduce very accurately real load profiles.

2.3. Control objective

In this section, we introduce the main control objective of this paper, *i.e.*, voltage regulation. First, we notice that for a constant input $\bar{u}$, the steady-state solution $(\bar{I}_g, \bar{V}, \bar{I}, \bar{d}_G, \bar{d}_I, \bar{d}_P)$ to (4) and (6) satisfies

$$\begin{aligned} &\bar{V} = \bar{u}, \quad \mathcal{A}\bar{I} = I_l^* + \Gamma_I \bar{d}_I + \big([G_l^*] + [\Gamma_G \bar{d}_G]\big)\bar{V} \\ &+ [\bar{V}]^{-1}\left(P_l^* + \Gamma_P \bar{d}_P\right) - \bar{I}_g, \quad \bar{I} = -R^{-1}\mathcal{A}^\top \bar{V}, \\ &\mathbf{0} = S_G \bar{d}_G = S_I \bar{d}_I = S_P \bar{d}_P. \end{aligned} \tag{8}$$

The main control objective concerning the steady-state value of the voltage is defined as follows:

Objective 1 (*Voltage Regulation*)**.**

$$\lim_{t\rightarrow\infty} V(t) = V^*, \tag{9}$$

$V_i^* \in \mathbb{R}_{>0}$ being the voltage reference (*e.g.* nominal value) at node $i \in \mathcal{V}$.

3. Local robust output regulation

In this section, we formulate the voltage control problem as a standard output regulation problem (Huang, 2004) in order to design control schemes achieving Objective 1.

Let the network state $x : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^{m+2n}$ and the exosystems state $d : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^{6nn_d}$ be defined as $x := \text{col}(I_g, V, I)$ and $d := \text{col}(d_G, d_I, d_P)$, respectively, and $u : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^n$ be the control input. Moreover, let $w \in \mathbb{R}^{3n}$ be defined as $w := \text{col}(G_l^*, I_l^*, P_l^*)$, which represents the unknown constant components of the ZIP loads, *i.e.*, parametric uncertainty. Then, we can rewrite (4) and (6) as the following system:

$$\dot{x} = f(x, d, w) + g(x, d, w)u \tag{10a}$$

$$\dot{d} = Sd \tag{10b}$$

$$h(x, d) = V - V^*, \tag{10c}$$

where $h(x, d)$ is the output mapping representing the tracking error $e(t) := h(x, d)$, $S := \text{blockdiag}\left(S_G, S_I, S_P\right)$, $g(x, d, w) := \text{col}(L_g^{-1}, \mathbf{0}_{n\times n}, \mathbf{0}_{m\times n})$ and $f(x, d, w) := \text{col}\left(-L_g^{-1}V, C_g^{-1}(I_g + \mathcal{A}I - i_{ZIP}(V, d, w)), L^{-1}(-\mathcal{A}^\top V - RI)\right)$, with

$$\begin{aligned} i_{ZIP}(V, d, w) := I_l^* + \Gamma_I d_I + \big([G_l^*] + [\Gamma_G d_G]\big)V \\ + [V]^{-1}\left(P_l^* + \Gamma_P d_P\right). \end{aligned} \tag{11}$$

Now, we compute the relative degree of system (10), which will be used in the following subsections for analyzing the zero dynamics of system (10). Let

$$\begin{aligned} &f_a(x, d, w) := \text{col}(f(x, d, w), Sd) \\ &g_a(x, d, w) := \text{col}(g(x, d, w), \mathbf{0}), \end{aligned} \tag{12}$$

then, based on Huang (2004, Definition 2.47), the relative degree of the system (10) is given in the following lemma, whose proof is the same as that of Silani et al. (2021b, Lemma 1).

Lemma 1 (*Relative Degree of System (10)*)**.** *For each $i = 1, \ldots, n$, the ith output h_i of system (10) has relative degree $r_i = 2$ for all the trajectories (x, d).*

Now, in order to use in the following subsections some properties of the linearization technique around the equilibrium point $(\bar{x}, \bar{d} = \mathbf{0}_{6nn_d})$ with $u = \bar{u}$ and $w = \mathbf{0}_{3n}$, let $F_x(x, u, d, w) := f(x, d, w) + g(x, d, w)u$ and $A \in \mathbb{R}^{(m+2n)\times(m+2n)}$, $B \in \mathbb{R}^{(m+2n)\times n}$,

$C \in \mathbb{R}^{n\times(m+2n)}$ be defined as:

$$A := \frac{\partial F_x}{\partial x}(\bar{x}, \bar{u}, \mathbf{0}, \mathbf{0}) = \begin{pmatrix} \mathbf{0} & -L_g^{-1} & \mathbf{0} \\ C_g^{-1} & \mathbf{0} & C_g^{-1}\mathcal{A} \\ \mathbf{0} & -L^{-1}\mathcal{A}^\top & -L^{-1}R \end{pmatrix}$$
$$B := \frac{\partial F_x}{\partial u}(\bar{x}, \bar{u}, \mathbf{0}, \mathbf{0}) = \mathrm{col}\big(L_g^{-1}, \mathbf{0}, \mathbf{0}\big) \tag{13}$$
$$C := \frac{\partial h}{\partial x}(\bar{x}, \mathbf{0}) = \begin{pmatrix}\mathbf{0} & \mathbb{I}_n & \mathbf{0}\end{pmatrix},$$

where A, B, C describe the linearized form of (10a) with output (10c) around the equilibrium point. Note that, since the exosystem model (10b) is linear, we did not consider it in (13). Now, in the following lemma, we prove the stabilizability and detectability of the pairs (A, B) and (C, A), respectively.

Lemma 2 (*Stabilizability and Detectability of (13)*). *The pair (A, B) is stabilizable and the pair (C, A) is detectable, with A, B, C given in (13).*

Proof. To prove the stabilizability and detectability of (A, B) and (C, A), respectively, it is sufficient to show that A is Hurwitz, *i.e.*, it has all the eigenvalues with negative real part. Therefore, consider the linear system

$$\dot{x} = A(x - \bar{x}), \tag{14}$$

and the following Lyapunov function

$$E(x) = \frac{1}{2}(I_g - \bar{I}_g)^\top L_g(I_g - \bar{I}_g) + \frac{1}{2}(V - \bar{V})^\top C_g(V - \bar{V}) + \frac{1}{2}(I - \bar{I})^\top L(I - \bar{I}). \tag{15}$$

Note that in the following subsections, we will show that $\bar{V} = V^*$. Then, the derivative of the Lyapunov function (15) along the solutions to (14) satisfies

$$\dot{E}(x) = \frac{\partial E}{\partial x}A(x - \bar{x}) = -(I - \bar{I})^\top R(I - \bar{I}). \tag{16}$$

Then, we can conclude that A has no eigenvalue with positive real-part since $R > \mathbf{0}$. Thus, in order to verify that A has no eigenvalues with real part equal to zero, we compute the determinant of A using the Schur complement of A, *i.e.*,

$$\det(A) = \det\big(-L^{-1}R\big)\det\begin{pmatrix}\mathbf{0} & -L_g^{-1} \\ C_g^{-1} & -C_g^{-1}\mathcal{A}R^{-1}\mathcal{A}^\top\end{pmatrix} = (-1)^n \det\big(-L^{-1}R\big)\det(-C_g^{-1}L_g^{-1}) \neq 0, \tag{17}$$

where we use the property that interchanging any pair of rows of a matrix multiplies its determinant by -1. Consequently, we can conclude that all the eigenvalues of matrix A have negative real part. Then, the pair (A, B) is stabilizable and the pair (C, A) is detectable. ■

Before introducing the output regulation problem, we consider the following assumption in analogy with Huang (2004, Assumption 5.4), which guarantees that the Taylor series solution to the regulator equations exists (see Huang, 2004, Lemma 4.8 for more details). Indeed, it ensures that there exists an internal model of the k-fold exosystem (10b) (see Huang, 2004, Theorem 5.7 and Remark 5.8 (i)). Moreover, its physical meaning corresponds to the absence of resonance phenomena. Indeed, it does not allow that the linear combinations of the eigenvalues of S are equal to the eigenvalues of the linearized DC network.

Assumption 2 (*Condition on the Eigenvalues of S*). The matrix $\begin{pmatrix} A - \lambda\mathbb{I}_{2n+m} & B \\ C & \mathbf{0}\end{pmatrix}$, has full rank for $l \in \mathbb{N}$ and for all λ given by $\{\lambda = l_1\lambda_1^* + \cdots + l_{6nn_d}\lambda_{6nn_d}^*, l_1 + \cdots + l_{6nn_d} = l, l_1, \ldots, l_{6nn_d} = 0, 1, \ldots, l\}$, where $\lambda_1^*, \ldots, \lambda_{6nn_d}^*$ are the eigenvalues of the matrix S.

Now, given the parametric uncertainty w, we define the robust output regulation problem for system (10) as follows (see Huang, 2004, Section 5.1):

Problem 1 (*Local Robust Output Regulation*)**.** Design a state feedback controller

$$u(t) = k(x(t), \nu(t), d(t)) \\ \dot{\nu}(t) = \mathcal{Q}(\nu(t), e(t)), \tag{18}$$

such that the closed-loop system (10), (18), for every initial condition $\big(x(0), \nu(0), d(0)\big)$ sufficiently close to the equilibrium point $\big(\bar{x}, \bar{\nu}, \bar{d} = \mathbf{0}_{6nn_d}\big)$ and sufficiently small elements of the uncertainty vector w, has the following two properties:
Property 1 The trajectories $\mathrm{col}\big(x(t), \nu(t), d(t)\big)$ of the closed-loop system exist and are bounded for all $t \geq 0$,
Property 2 The trajectories $\mathrm{col}\big(x(t), \nu(t), d(t)\big)$ of the closed-loop system satisfy $\lim_{t\to\infty} e(t) = \mathbf{0}_n$, achieving Objective 1.[1]

If a controller (18) exists such that the closed-loop system (10), (18) satisfies Properties 1, 2, Problem 1 is *solvable*. Also, the kth-order robust output regulation problem is *solvable* if a controller exists such that the closed-loop system satisfies Properties 1, 2 with $\lim_{t\to\infty}\big(e(t) - o^k(d(t))\big) = \mathbf{0}_n$, *i.e.*, the steady-state tracking error is zero up to the kth order.

3.1. Linear robust controller

In this subsection, we propose a control scheme with a linear internal model achieving Objective 1 in presence of *time-varying* and *uncertain* constant ZIP loads.

First, we augment system (10) with a linear dynamic system incorporating an internal model of the k-fold exosystem (10b). Then, a controller stabilizing the linear approximation of the augmented system is proposed to solve the kth-order robust output regulation problem. Also, following Huang (2004, Theorem 3.8), the regulator equation associated with (10) can be expressed as

$$\frac{\partial \boldsymbol{x}(d, w)}{\partial d}Sd = f(\boldsymbol{x}(d, w), d, w) + g(\boldsymbol{x}(d, w), d, w)\boldsymbol{u}(d, w) \tag{19a}$$
$$\mathbf{0} = h(\boldsymbol{x}(d, w), d), \tag{19b}$$

whose solution we show to be polynomial in d, implying, according to Huang (2004, Theorem 5.12), that the proposed controller also solves Problem 1.

Now, let $G_1 \in \mathbb{R}^{6nn_d\times 6nn_d}$ and $G_2 \in \mathbb{R}^{6nn_d\times n}$ be defined as:

$$G_1 := \mathrm{blockdiag}\big(\varkappa_{11}, \ldots, \varkappa_{1n_d}, \ldots, \varkappa_{n1}, \ldots, \varkappa_{nn_d}\big) \tag{20}$$
$$G_2 := \mathrm{blockdiag}\big(\mathrm{col}(0, 1, \ldots, 0, 1), \ldots, \mathrm{col}(0, 1, \ldots, 0, 1)\big), \tag{21}$$

where

$$\varkappa_{ik} := \mathrm{blockdiag}\left(\begin{pmatrix} 0 & |\omega_{G_{ik}}| \\ -|\omega_{G_{ik}}| & 0\end{pmatrix}, \begin{pmatrix} 0 & |\omega_{I_{ik}}| \\ -|\omega_{I_{ik}}| & 0\end{pmatrix}, \begin{pmatrix} 0 & |\omega_{P_{ik}}| \\ -|\omega_{P_{ik}}| & 0\end{pmatrix}\right),$$

with $i = 1, \ldots, n$, $k = 1, \ldots, n_d$. Then, according to Huang (2004, Theorems 5.7, 5.12), the solvability of Problem 1 is established in the following theorem.

[1] Note that Property 2 implies $\bar{x} = \mathrm{col}(\bar{I}_g, V^*, \bar{I})$.

Theorem 1 (*Linear Robust Controller*). *Let Assumptions 1 and 2 hold.*

(i) The linear state feedback controller

$$u = K_1 x + K_2 \nu_a, \quad \dot{\nu}_a = G_1 \nu_a + G_2 e, \tag{22}$$

solves Problem 1, where $\nu_a : \mathbb{R}_{\geq 0} \to \mathbb{R}^{6nn_d}$ is the state of the controller, $K_1 \in \mathbb{R}^{n\times(2n+m)}$, $K_2 \in \mathbb{R}^{n\times 6nn_d}$ is chosen such that the matrix

$$\begin{pmatrix} A + BK_1 & BK_2 \\ G_2 C & G_1 \end{pmatrix} \tag{23}$$

is Hurwitz, with A, B, and C given in (13).

(ii) The linear output feedback controller

$$u = K \nu_b, \quad \dot{\nu}_b = \mathcal{G}_1 \nu_b + \mathcal{G}_2 e, \tag{24}$$

solves Problem 1, where $\nu_b : \mathbb{R}_{\geq 0} \to \mathbb{R}^{6nn_d+2n+m}$ is the state of the controller, $K := \begin{pmatrix} K_1 & K_2 \end{pmatrix} \in \mathbb{R}^{n\times(6nn_d+2n+m)}$, with K_1, K_2 given in part (i), and $\mathcal{G}_1 \in \mathbb{R}^{(6nn_d+2n+m)\times(6nn_d+2n+m)}$, $\mathcal{G}_2 \in \mathbb{R}^{(6nn_d+2n+m)\times n}$ are given by

$$\begin{aligned} \mathcal{G}_1 &:= \begin{pmatrix} A + BK_1 - \bar{L}C & BK_2 \\ \mathbf{0} & G_1 \end{pmatrix} \\ \mathcal{G}_2 &:= \text{col}\left(\bar{L}, \quad G_2\right), \end{aligned} \tag{25}$$

such that $A - \bar{L}C$ is Hurwitz, with $\bar{L} \in \mathbb{R}^{(2n+m)\times n}$.

Proof. The proof is provided in Appendix A due to lengthy calculations. ■

Remark 1 (*Controller Properties in Theorem 1*). Note that, even in presence of only ZI loads, the controllers (22) and (24) are incapable of ensuring global stability of the closed-loop system because of the linearity of the internal models in (22) and (24). Moreover, the controllers (22) and (24) do not guarantee the solvability of Problem 1 when the solutions to the regulator equation (19) are nonpolynomial in d, (*i.e.*, the solutions to the regulator equation (19) do not have polynomial expressions in d). Indeed, the solutions to the regulator equation (19) might be nonpolynomial in d if we consider for instance a nonlinear exosystem or an exosystem with a structure different from (6).

3.2. Stabilization technique for robust output regulation

In this subsection, we propose a control scheme solving Problem 1, but we use an internal model different from the ones in (22) and (24). Specifically, we follow the design framework introduced in Huang (2004, Chapter 6), which removes the assumption on the polynomial solutions to the regulator equation (19) and allows to incorporate global stabilization techniques which will be introduced in Section 4. This framework views the output regulation as a stabilization problem. However, for solving the "robust" output regulation problem (Problem 1), the solution to the regulator equation (19) cannot be used in the controller due to the parametric uncertainty w. To address this issue, we augment system (10) with a generalized internal model based on the steady-state generator for system (10), which is a dynamic system that can reproduce a partial or whole solution to the regulator equation (19). In the following lemma, according to Huang (2004, Definitions 6.1, 6.2), a steady-state generator for system (10) is derived.

Lemma 3 (*Steady-state Generator for System (10)*). *Let Assumption 2 hold. Then, system (10) has a steady-state generator $\{\theta, \alpha, \beta\}$ with output $g_0(x, u) = u$ with linear observability.*

Proof. We first form the pairwise coprime polynomials based on the solution to the regulator equation (19), whose expression is given in (A.6) in Appendix. Then, we obtain the companion matrix associated with the minimal zeroing polynomials of such pairwise coprime polynomials. Finally, following Huang (2004, Lemma 6.17), we obtain a steady-state generator $\{\theta, \alpha, \beta\}$ with output $g_0(x, u) = u$ with linear observability.

Following Huang (2004, Lemma 6.17), we form the pairwise coprime polynomials $\pi_1 : \mathbb{R}^{6nn_d+3n} \to \mathbb{R}^{nn_d}$, $\pi_2 : \mathbb{R}^{6nn_d+3n} \to \mathbb{R}^{nn_d}$, $\pi_3 : \mathbb{R}^{6nn_d+3n} \to \mathbb{R}^{nn_d}$ based on the solution to the regulator equation (19) for the output $g_0(x, u) = u$ as:

$$\begin{aligned} &\pi_1(d, w) = [\omega_G] d_G^b, \quad \pi_2(d, w) = \bar{C}_0 + [\omega_I] d_I^b \\ &\pi_3(d, w) = [\omega_P] d_P^b, \end{aligned} \tag{26}$$

with $\bar{C}_0 \in \mathbb{R}^{nn_d}$ satisfying $L_g \gamma_I \bar{C}_0 = V^*$. Then, the minimal zeroing polynomials $P_1 : \mathbb{R} \to \mathbb{R}^{nn_d\times nn_d}$, $P_2 : \mathbb{R} \to \mathbb{R}^{nn_d\times nn_d}$, $P_3 : \mathbb{R} \to \mathbb{R}^{nn_d\times nn_d}$ of (26) are given by

$$\begin{aligned} &P_1(\lambda) = \lambda^2 \mathbb{I}_{nn_d} + [\omega_G]^2, \quad P_2(\lambda) = \lambda^3 \mathbb{I}_{nn_d} + \lambda[\omega_I]^2 \\ &P_3(\lambda) = \lambda^2 \mathbb{I}_{nn_d} + [\omega_P]^2. \end{aligned} \tag{27}$$

Note that all the zeros of $P_j(\lambda)$, with $j = 1, 2, 3$, are simple and pure imaginary. Now, we define $\Lambda_{1ik} := \text{col}(\pi_{1ik}, \dot{\pi}_{1ik}) \in \mathbb{R}^2$, $\Lambda_{2ik} := \text{col}(\pi_{2ik}, \dot{\pi}_{2ik}, \ddot{\pi}_{2ik}) \in \mathbb{R}^3$, $\Lambda_{3ik} := \text{col}(\pi_{3ik}, \dot{\pi}_{3ik}) \in \mathbb{R}^2$, $\Lambda_{ji} := \text{col}(\Lambda_{ji1}, \ldots, \Lambda_{jin_d})$, $\Lambda_j := \text{col}(\Lambda_{j1}, \ldots, \Lambda_{jn})$, $\Lambda := \text{col}(\Lambda_1, \Lambda_2, \Lambda_3) \in \mathbb{R}^{7nn_d}$ with $i = 1, \ldots, n$, $j = 1, 2, 3$, $k = 1, \ldots, n_d$. Then, we have $g_0(\boldsymbol{x}, \boldsymbol{u}) = \Sigma(\Lambda)$, where $\Sigma : \mathbb{R}^{7nn_d} \to \mathbb{R}^n$ reproduces $\boldsymbol{u}(d, w)$ and is defined as

$$\Sigma(\Lambda) := L_g [V^*] \gamma_G \pi_1 + L_g \gamma_I \pi_2 + L_g [V^*]^{-1} \gamma_P \pi_3. \tag{28}$$

Note that (26) and (28) follow from the solution to the regulator equation (19), whose expression is given in (A.6) in Appendix. Now, the companion matrix associated with the minimal zeroing polynomials (27) can be expressed as $\Phi_j = \text{blockdiag}\left(\Phi_{j1}, \ldots, \Phi_{jn}\right)$, where $\Phi_{ji} = \text{blockdiag}\left(\Phi_{ji1}, \ldots, \Phi_{jin_d}\right)$,

$$\begin{aligned} &\Phi_{1ik} = \begin{pmatrix} 0 & 1 \\ -\omega_{G_{ik}}^2 & 0 \end{pmatrix}, \Phi_{2ik} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -\omega_{I_{ik}}^2 & 0 \end{pmatrix}, \\ &\Phi_{3ik} = \begin{pmatrix} 0 & 1 \\ -\omega_{P_{ik}}^2 & 0 \end{pmatrix}, \end{aligned} \tag{29}$$

with $i = 1, \ldots, n$, $j = 1, 2, 3$, $k = 1, \ldots, n_d$. Then, we have $\Phi = \text{blockdiag}(\Phi_1, \Phi_2, \Phi_3) \in \mathbb{R}^{7nn_d\times 7nn_d}$. Now, the Jacobian of $\Sigma(\Lambda)$ at the origin can be given by $\Psi = \text{row}\left(\Psi_1, \Psi_2, \Psi_3\right) \in \mathbb{R}^{n\times 7nn_d}$, where $\Psi_j = \text{blockdiag}\left(\Psi_{j1}, \ldots, \Psi_{jn}\right)$ with $\Psi_{ji} = \text{row}\left(\Psi_{ji1}, \ldots, \Psi_{jin_d}\right)$, $\Psi_{1ik} = \text{row}\left(L_{g_i} V_i^* \gamma_{G_{ik}}, 0\right) \in \mathbb{R}^2$, $\Psi_{2ik} = \text{row}\left(L_{g_i} \gamma_{I_{ik}}, 0, 0\right) \in \mathbb{R}^3$, $\Psi_{3ik} = \text{row}\left(L_{g_i} V_i^{*-1} \gamma_{P_{ik}}, 0\right) \in \mathbb{R}^2$, $i = 1, \ldots, n$, $j = 1, 2, 3$, $k = 1, \ldots, n_d$. Then, the steady-state generator $\theta : \mathbb{R}^{6nn_d+3n} \to \mathbb{R}^{7nn_d}$, $\alpha : \mathbb{R}^{7nn_d} \to \mathbb{R}^{7nn_d}$, $\beta : \mathbb{R}^{7nn_d} \to \mathbb{R}^n$ with output $g_0(x, u) = u$ is given by

$$\begin{aligned} &\theta(d, w) = T\Lambda, \quad \alpha(\theta) = T\Phi T^{-1}\theta, \\ &\beta(\theta) = \Sigma(T^{-1}\theta) = \Psi T^{-1}\theta, \end{aligned} \tag{30}$$

where $T \in \mathbb{R}^{7nn_d\times 7nn_d}$ is any nonsingular matrix. Since the pair $\left(\Psi_{jik}, \Phi_{jik}\right)$ has a companion form, the pair $\left(\Psi, \Phi\right)$ is observable; therefore, the steady-state generator $\{\theta, \alpha, \beta\}$ is linearly observable. Also, it can be inferred from (30) that $\dot{\theta} = \alpha(\theta)$. Moreover, we note that the solution to the regulator equation (19) relates system (10) to the steady-state generator (30), while $\beta(\theta)$ in (30) relates (28) to (30). ■

In the following, by virtue of Lemma 3, we use the steady-state generator (30) in order to find an internal model for system (10). Now, we define

$$\begin{aligned} &M := \text{blockdiag}(M_1, M_2, M_3) \in \mathbb{R}^{7nn_d\times 7nn_d} \\ &N := \text{col}(N_1, N_2, N_3) \in \mathbb{R}^{7nn_d\times n}, \end{aligned} \tag{31}$$

where $M_j := \text{blockdiag}(M_{j1}, \dots, M_{jn})$, $N_j := \text{blockdiag}(N_{j1}, \dots, N_{jn})$ with $M_{ji} := \text{blockdiag}(M_{ji1}, \dots, M_{jin_d})$, $N_{ji} := \text{col}(N_{ji1}, \dots, N_{jin_d})$,

$$M_{1ik} := \begin{pmatrix} 0 & 1 \\ -a_{1ik} & -b_{1ik} \end{pmatrix}, N_{1ik} := \text{col}(0, 1),$$
$$M_{2ik} := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_{2ik} & -b_{2ik} & -c_{2ik} \end{pmatrix}, N_{2ik} := \text{col}(0, 0, 1),$$
$$M_{3ik} := \begin{pmatrix} 0 & 1 \\ -a_{3ik} & -b_{3ik} \end{pmatrix}, N_{3ik} := \text{col}(0, 1),$$

$a_{jik}, b_{jik}, c_{jik} \in \mathbb{R}_{>0}$, $i = 1 \dots, n$, $j = 1, 2, 3$, $k = 1, \dots, n_d$. Then, T in (30) is defined as $T := \text{blockdiag}(T_1, T_2, T_3)$, $T_j = \text{blockdiag}(T_{j1}, \dots, T_{jn})$, $T_{ji} = \text{blockdiag}(T_{ji1}, \dots, T_{jin_d})$, $T_{1ik}, T_{3ik} \in \mathbb{R}^{2\times 2}$, $T_{2ik} \in \mathbb{R}^{3\times 3}$, $i = 1, \dots, n$, $j = 1, 2, 3$, $k = 1, \dots, n_d$. Then, T can be obtained by solving the following Sylvester equation

$$T_{ji}\Phi_{ji} - M_{ji}T_{ji} = N_{ji}\Psi_{ji}. \tag{32}$$

Note that there exists a unique nonsingular matrix T_{ji} satisfying the Sylvester equation (32) since the pair (M_{ji}, N_{ji}) is controllable, (Ψ_{ji}, Φ_{ji}) is observable and the spectra of M_{ji} and Φ_{ji} are disjoint. Note that the controllability of the pair (M_{ji}, N_{ji}) and the observability of the pair (Ψ_{ji}, Φ_{ji}) follow from the controllable canonical form of the pair (M_{jik}, N_{jik}) and the companion form of the pair (Ψ_{jik}, Φ_{jik}), respectively.

Now, according to Huang (2004, Proposition 6.21), in the following proposition we introduce an internal model for system (10) different from the ones in (22) and (24).

Proposition 1 (*Internal Model for System* (10)). *Let Assumption 2 hold and consider the steady-state generator (30) with output $g_0(x, u) = u$. Then, the internal model for system (10) is given by*

$$\dot{\xi} = \mu(\xi, u) := M\xi + Nu, \tag{33}$$

where $\xi : \mathbb{R}_{\geq 0} \to \mathbb{R}^{7nn_d}$ is the state of the internal model, $\mu : \mathbb{R}^{7nn_d} \times \mathbb{R}^n \to \mathbb{R}^{7nn_d}$, and M, N are given in (31).

Proof. Since $\Sigma(\Lambda)$ given in (28) is a linear function, we can write $\beta_i(\xi_{ji}) = \Psi_{ji}T_{ji}^{-1}\xi_{ji}$, with $i = 1 \dots, n, j = 1, 2, 3$. Thus, we can write $\mu_{ji}(\xi_{ji}, u) = M_{ji}\xi_{ji} + N_{ji}\big(g_{0i}(x, u) - \beta_i(\xi_{ji}) + \Psi_{ji}T_{ji}^{-1}\xi_{ji}\big)$. Then, by virtue of the Sylvester equation (32) and in analogy with (Huang, 2004, Proposition 6.21), we can write

$$\begin{aligned} \mu_{ji}(\theta_{ji}, \boldsymbol{u}) &= M_{ji}\theta_{ji}(d, w) + N_{ji}\Psi_{ji}T_{ji}^{-1}\theta_{ji}(d, w) \\ &= T_{ji}\Phi_{ji}T_{ji}^{-1}\theta_{ji}(d, w) \\ &= \alpha_{ji}\big(\theta_{ji}(d, w)\big), \end{aligned} \tag{34}$$

where $\boldsymbol{u}$ is given in (A.6) in the Appendix and $i = 1, \dots, n, j = 1, 2, 3$. Note that in (34) we apply $g_{0i}(\boldsymbol{x}, \boldsymbol{u}) = \beta_i(\theta_{ji}(d, w))$, where $\beta_i(\theta_{ji}(d, w))$ is given in (30) and $M_{ji} + N_{ji}\Psi_{ji}T_{ji}^{-1} = T_{ji}\Phi_{ji}T_{ji}^{-1}$, which is obtained from the Sylvester equation (32). Now, we rewrite (34) compactly as $\mu(\theta, \boldsymbol{u}) = \alpha\big(\theta(d, w)\big)$. Consequently, by Huang (2004, Definition 6.6), (33) is an internal model for system (10) with output $g_0(x, u) = u$. Also, we note that the Sylvester equation (32) relates the steady-state generator (30) to the internal model (33). ∎

In the following theorem, we use the internal model (33) to design a robust controller solving Problem 1.

Theorem 2 (*Linear Robust Controller with Internal Model* (33)). *Let Assumptions 1 and 2 hold. Consider system (10), the steady-state generator (30) and the internal model (33). Then, the output feedback controller*

$$\begin{aligned} u &= K\delta + \Psi T^{-1}\xi \\ \dot{\delta} &= \begin{pmatrix} A & B_{\text{‰}} \\ \mathbf{0} & M + N\Psi T^{-1} \end{pmatrix}\delta + \begin{pmatrix} B \\ N \end{pmatrix}K\delta \\ &\quad - \bar{L}\begin{pmatrix} C & \mathbf{0} \end{pmatrix}\delta + \bar{L}e \\ \dot{\xi} &= M\xi + Nu, \end{aligned} \tag{35}$$

solves Problem 1, where $\delta : \mathbb{R}_{>0} \to \mathbb{R}^{7nn_d+2n+m}$ is the state of the controller, A, B, C are given in (13), $B_{\text{‰}} := \text{col}\left(L_g^{-1}\Psi T^{-1}, \mathbf{0}_{n\times 7nn_d}, \mathbf{0}_{m\times 7nn_d}\right) \in \mathbb{R}^{(m+2n)\times 7nn_d}$, and $K \in \mathbb{R}^{n\times(7nn_d+2n+m)}$, $\bar{L} \in \mathbb{R}^{(7nn_d+2n+m)\times n}$ are chosen such that

$$\begin{pmatrix} A & B_\xi \\ \mathbf{0} & M + N\Psi T^{-1} \end{pmatrix} + \begin{pmatrix} B \\ N \end{pmatrix}K, \tag{36}$$
$$\begin{pmatrix} A & B_\xi \\ \mathbf{0} & M + N\Psi T^{-1} \end{pmatrix} - \bar{L}\begin{pmatrix} C & \mathbf{0} \end{pmatrix}, \tag{37}$$

are Hurwitz.

Proof. According to Lemma 3, system (10) has the linearly observable steady-state generator (30) with output $g_0(x, u) = u$. Also, following Proposition 1, (33) is an internal model for system (10). Then, consider the following transformation based on the steady-state generator (30) and internal model (33) as:

$$\begin{aligned} \hat{x}_a &:= I_g - \boldsymbol{I}_g(d, w) \\ &= I_g - \mathcal{A}R^{-1}\mathcal{A}^\top V^* - I_1^* - \Gamma_1 d_1 - ([G_1^*] \\ &\quad + [\Gamma_G d_G])V^* - [V^*]^{-1}(P_1^* + \Gamma_P d_P) \\ \hat{x}_b &:= V - \boldsymbol{V}(d, w) = V - V^* \\ \hat{x}_c &:= I - \boldsymbol{I}(d, w) = I + R^{-1}\mathcal{A}^\top V^* \\ \hat{\xi} &:= \xi - \theta(d, w) \\ \hat{u} &:= u - \beta(\xi) = u - \Psi T^{-1}\xi, \end{aligned} \tag{38}$$

where $\boldsymbol{I}_g(d, w), \boldsymbol{V}(d, w), \boldsymbol{I}(d, w)$ are given in (A.6) in Appendix. Then, the augmented system composed of the DC network dynamics (10a) and internal model (33) in the new coordinates is given by

$$\begin{aligned} \dot{\hat{x}}_a &= L_g^{-1}\big(-\hat{x}_b - V^* + \hat{u} + \beta(\hat{\xi} + \theta)\big) \\ &\quad - \Gamma_1 S_1 d_1 - [V^*]\Gamma_G S_G d_G - [V^*]^{-1}\Gamma_P S_P d_P \\ \dot{\hat{x}}_b &= C_g^{-1}\left(\hat{x}_a + \mathcal{A}\hat{x}_c - ([G_1^*] + [\Gamma_G d_G])\hat{x}_b\right. \\ &\quad \left. - [\hat{x}_b]^{-1}(P_1^* + \Gamma_P d_P)\right) \\ \dot{\hat{x}}_c &= L^{-1}\big(-\mathcal{A}^\top\hat{x}_b - R\hat{x}_c\big) \\ \dot{\hat{\xi}} &= (M + N\Psi T^{-1})\hat{\xi} + N\hat{u}. \end{aligned} \tag{39}$$

According to Huang (2004, Corollary 6.9), a controller that locally stabilizes the equilibrium point of system (39) with $d = \mathbf{0}_{6nn_d}$ and $w = \mathbf{0}_{3n}$ solves Problem 1. Thus, we linearize system (39) around its equilibrium point and obtain

$$\dot{\hat{x}} = A\hat{x} + B_{\text{‰}}\hat{\xi} + B\hat{u}, \quad \dot{\hat{\xi}} = (M + N\Psi T^{-1})\hat{\xi} + N\hat{u}, \tag{40}$$

where $\hat{x} = \text{col}(\hat{x}_a, \hat{x}_b, \hat{x}_c)$ and A, B and $B_{\text{‰}}$ are given in (13) and (35), respectively. In analogy with the proof of Huang (2004, Theorem 6.23), by virtue of Assumption 2 and Lemma 2, it can be shown that (40) is stabilizable and detectable. Thus, the controller $\hat{u} = K\delta$, with δ as in (35), stabilizes system (40). Consequently, according to Huang (2004, Corollary 6.9), controller (35) solves Problem 1. ∎

Remark 2 (*Local and Global Robust Stability*). The solvability of Problem 1 guarantees the boundedness of the trajectories of the closed-loop system (10), (18) and the asymptotic regulation of the error when the initial conditions are sufficiently close to the equilibrium point and the elements of the uncertainty vector w sufficiently small. Even in the case of ZI loads, the linear controllers (22), (24) and (35) can ensure only local stability. In the next subsection, we design a controller that guarantees the global boundedness of the trajectories of the closed-loop system in presence of ZI loads and the asymptotic regulation of the error for any initial condition of the closed-loop system and uncertain parameter w, *i.e.*, global robust output regulation.

4. Global robust output regulation

In this section, we first convert the system into a system in lower triangular form. To do so, we use a suitable coordinate transformation and dynamic extension. Then, we use the control design approach introduced in Section 3.2, *i.e.*, we convert the robust output regulation problem into a robust stabilization problem for a suitably augmented system. However, to this end, we consider the DC network model with ZI loads only, *i.e.*, system (10) with

$$\begin{aligned} f(x,d,w) &:= \begin{pmatrix} -L_g^{-1}V \\ C_g^{-1}\left(I_g + \mathcal{A}I - i_{ZI}(V,d,w)\right) \\ L^{-1}\left(-\mathcal{A}^\top V - RI\right)\end{pmatrix} \\ i_{ZI}(V,d,w) &:= I_l^* + \Gamma_I d_I + ([G_l^*] + [\Gamma_G d_G])V \\ S &:= \operatorname{blockdiag}(S_G, S_I) \\ d &:= \operatorname{col}(d_G, d_I), \quad w := \operatorname{col}(G_l^*, I_l^*). \end{aligned} \tag{41}$$

Also, we notice that for a constant input $\bar{u}$, the steady-state solution $\bar{x}$ to (10) with (41) satisfies

$$\mathbf{0} = f(\bar{x}, \bar{d}, w) + g(\bar{x}, \bar{d}, w)\bar{u}, \quad \mathbf{0} = S\bar{d}, \tag{42}$$

where $\bar{d} = \mathbf{0}_{4nn_d}$. Now, according to Huang (2004, Section 7.1), we define the global robust output regulation problem for system (10) with (41) as follows:

Problem 2 (*Global Robust Output Regulation*)**.** Design a state feedback controller

$$\begin{aligned} u(t) &= k(x(t), v(t), e(t)) \\ \dot{v}(t) &= \mathcal{Q}(x(t), v(t), e(t)), \end{aligned} \tag{43}$$

such that the closed-loop system (10) with (41), (43), for any compact set $\mathcal{D} \in \mathbb{R}^{4nn_d}$ with a known bound and any compact set $\mathcal{W} \in \mathbb{R}^{2n}$ with a known bound, has the following two properties:
Property 3 For all $d(0) \in \mathcal{D}$ and $w \in \mathcal{W}$, the trajectories $\operatorname{col}\big(x(t), v(t), d(t)\big)$ of the closed-loop system starting from any initial state $x(0)$ exist and are bounded for all $t \geq 0$,
Property 4 The trajectories $\operatorname{col}\big(x(t), v(t), d(t)\big)$ of the closed-loop system satisfy $\lim_{t\to\infty} e(t) = \mathbf{0}_n$, achieving Objective 1.[2]

If a controller exists such that the closed-loop system satisfies Properties 3, 4, Problem 2 is *solvable*.

Now, we define the known bounds for the compact sets $\mathcal{D}$ and $\mathcal{W}$ as

$$\begin{aligned} &d_{y_i}^{\min} \leq d_{y_i} \leq d_{y_i}^{\max} \\ &G_{l_i}^{\min} \leq G_{l_i}^* \leq G_{l_i}^{\max}, \quad I_{l_i}^{\min} \leq I_{l_i}^* \leq I_{l_i}^{\max}, \end{aligned} \tag{44}$$

where y denotes G or I in case of Z or I loads, respectively. Now, let the state $\hat{x} : \mathbb{R}_{\geq 0} \to \mathbb{R}^{n+m}$ and $v_x : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ be defined as $\hat{x} := \operatorname{col}(I_g, I)$ and $v_x := V$, respectively. Then, the system (10) with (41) can be rewritten as

$$\begin{aligned} \dot{\hat{x}} &= \bar{F}\hat{x} + \bar{G}v_x + \bar{g}u \\ \dot{v}_x &= \bar{H}\hat{x} + \bar{K}(d,w)v_x + \bar{D}(d,w) \\ \dot{d} &= Sd \\ e &= v_x - V^*, \end{aligned} \tag{45}$$

where $\bar{F} := \operatorname{diag}(\mathbf{0}_{n\times n}, -L^{-1}R) \in \mathbb{R}^{(n+m)\times(n+m)}$, $\bar{G} := \operatorname{col}(-L_g^{-1}, -L^{-1}\mathcal{A}^\top) \in \mathbb{R}^{(n+m)\times n}$, $\bar{g} := \operatorname{col}(L_g^{-1}, \mathbf{0}_{m\times n}) \in \mathbb{R}^{(n+m)\times n}$, $\bar{H} := \begin{pmatrix} C_g^{-1} & C_g^{-1}\mathcal{A}\end{pmatrix} \in \mathbb{R}^{n\times(n+m)}$, $\bar{K}(d,w) := -C_g^{-1}([G_l^*] + [\Gamma_G d_G]) \in \mathbb{R}^{n\times n}$, and $\bar{D}(d,w) := -C_g^{-1}(I_l^* + \Gamma_I d_I) \in \mathbb{R}^n$.

We first convert system (45) into a system in lower triangular form. For this, we recall that the relative degree of system (10) with (41) is (as in Lemma 1) equal to 2. Then, according to Huang (2004, Section 7.3), we use the following dynamic extension:

$$\dot{\zeta} = -\zeta + u, \tag{46}$$

where $\zeta : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ is the state of the dynamic extension. Subsequently, we apply the following coordinate transformation to system (45):

$$\begin{aligned} z &:= \hat{x} - \bar{g}\zeta - (\bar{F} + \mathbb{I}_{n+m})\bar{g}\big(\bar{H}\bar{g}\big)^{-1}v_x \\ &= \hat{x} - \operatorname{col}\big(L_g^{-1}, \mathbf{0}\big)\zeta - \operatorname{col}\big(C_g, \mathbf{0}\big)v_x. \end{aligned} \tag{47}$$

Combining (45) with (46), we can write

$$\dot{z} = Fz + G(d,w)v_x + D(d,w) \tag{48a}$$

$$\dot{v}_x = \bar{H}z + K(d,w)v_x + b\zeta + \bar{D}(d,w) \tag{48b}$$

$$\dot{\zeta} = -\zeta + u \tag{48c}$$

$$\dot{d} = Sd \tag{48d}$$

$$e = v_x - V^*, \tag{48e}$$

$$\begin{aligned} &F := \begin{pmatrix} -\mathbb{I}_n & -\mathcal{A} \\ \mathbf{0} & -L^{-1}R\end{pmatrix}, \quad D(d,w) := \begin{pmatrix} I_l^* + \Gamma_I d_I \\ \mathbf{0}\end{pmatrix}, \\ &K(d,w) := \mathbb{I}_n - C_g^{-1}\big([G_l^*] + [\Gamma_G d_G]\big), b := C_g^{-1}L_g^{-1}, \\ &G(d,w) := \begin{pmatrix} -C_g - L_g^{-1} + [G_l^*] + [\Gamma_G d_G] \\ -L^{-1}\mathcal{A}^\top \end{pmatrix}. \end{aligned}$$

System (48) will be used later to obtain the lower triangular form for system (45). First, similarly to the approach discussed in Section 3.2, we find a steady-state generator and an internal model for system (48).

Lemma 4 (*Steady-State Generator for System* (48))**.** *Let Assumption 2 hold. Then, system* (48) *has a steady-state generator* $\{\bar{\theta}, \bar{\alpha}, \bar{\beta}\}$ *with output* $\bar{g}_0(z, v_x, \zeta, u) = \operatorname{col}(\zeta, u)$ *with linear observability.*

Proof. We first compute the solution to the regulator equations associated with system (48) and form the pairwise coprime polynomials. According to Huang (2004, Section 7.3), recalling that for each $i = 1, \ldots, n$, the ith output h_i of system (10) with (41) has relative degree equal to 2 (similar to Lemma 1), the solution to the regulator equations associated with system (48) is given by $\operatorname{col}(\boldsymbol{z}(d,w), \boldsymbol{v}_x(d,w), \boldsymbol{\zeta}(d,w))$, with $\boldsymbol{z} : \mathbb{R}^{4nn_d+2n} \to \mathbb{R}^{n+m}$, $\boldsymbol{v}_x : \mathbb{R}^{4nn_d+2n} \to \mathbb{R}^n$, $\boldsymbol{\zeta} : \mathbb{R}^{4nn_d+2n} \to \mathbb{R}^n$, and $\boldsymbol{u} : \mathbb{R}^{4nn_d+2n} \to \mathbb{R}^n$, satisfying $\forall d \in \mathbb{R}^{4nn_d}$, $\forall w \in \mathbb{R}^{2n}$

$$\begin{aligned} \frac{\partial \boldsymbol{z}(d,w)}{\partial d}Sd &= F\boldsymbol{z}(d,w) + G(d,w)V^* + D(d,w) \\ \boldsymbol{v}_x(d,w) &= V^* \\ \boldsymbol{\zeta}(d,w) &= b^{-1}(-\bar{H}\boldsymbol{z}(d,w) - K(d,w)V^* - \bar{D}(d,w)) \\ \boldsymbol{u}(d,w) &= \dot{\boldsymbol{\zeta}}(d,w) + \boldsymbol{\zeta}(d,w). \end{aligned} \tag{49}$$

Let $\boldsymbol{z}(d,w)$ in (49) be partitioned as $\boldsymbol{z}(d,w) = \operatorname{col}(\boldsymbol{z}_a(d,w), \boldsymbol{z}_b(d,w))$ with $\boldsymbol{z}_a : \mathbb{R}^{4nn_d+2n} \to \mathbb{R}^n$ and $\boldsymbol{z}_b : \mathbb{R}^{4nn_d+2n} \to \mathbb{R}^m$,

[2] Note that Property 4 implies $\bar{x} = \operatorname{col}(\bar{I}_g, V^*, \bar{I})$.

then the solution to (49) can be expressed as

$$
\begin{aligned}
\boldsymbol{z}_{\mathrm{a}}(d,w) =& -[V^*]\gamma_{\mathrm{G}}\big([\omega_{\mathrm{G}}]^2-\mathbb{I}_{nn_{\mathrm{d}}}\big)^{-1}d_{\mathrm{G}}^{\mathrm{a}} \\
&- [V^*]\gamma_{\mathrm{G}}\big([\omega_{\mathrm{G}}]^2-\mathbb{I}_{nn_{\mathrm{d}}}\big)^{-1}[\omega_{\mathrm{G}}]d_{\mathrm{G}}^{\mathrm{b}} \\
&+ \gamma_{\mathrm{l}}\big([\omega_{\mathrm{l}}]^2+\mathbb{I}_{nn_{\mathrm{d}}}\big)^{-1}d_{\mathrm{l}}^{\mathrm{a}} \\
&- \gamma_{\mathrm{l}}\big([\omega_{\mathrm{l}}]^2+\mathbb{I}_{nn_{\mathrm{d}}}\big)^{-1}[\omega_{\mathrm{l}}]d_{\mathrm{l}}^{\mathrm{b}} - L_{\mathrm{g}}^{-1}V^* \\
&+ [G_{\mathrm{l}}^*]V^* - C_{\mathrm{g}}V^* + I_{\mathrm{l}}^* + \mathcal{A}R^{-1}\mathcal{A}^\top V^* \\
\boldsymbol{z}_{\mathrm{b}}(d,w) =& -R^{-1}\mathcal{A}^\top V^* \\
\boldsymbol{v}_{\mathrm{x}}(d,w) =& V^* \\
\boldsymbol{\zeta}(d,w) =& -L_{\mathrm{g}}\Big(\boldsymbol{z}_{\mathrm{a}}(d,w)+\mathcal{A}\boldsymbol{z}_{\mathrm{b}}(d,w)+C_{\mathrm{g}}V^* \\
&- ([G_{\mathrm{l}}^*]+[\Gamma_{\mathrm{G}}d_{\mathrm{G}}])V^* + I_{\mathrm{l}}^* + \Gamma_{\mathrm{l}}d_{\mathrm{l}}\Big) \\
\boldsymbol{u}(d,w) =& L_{\mathrm{g}}\Big([V^*]\gamma_{\mathrm{G}}\Big(2([\omega_{\mathrm{G}}]^2-\mathbb{I}_{nn_{\mathrm{d}}})^{-1}[\omega_{\mathrm{G}}] \\
&+ [\omega_{\mathrm{G}}]\Big)d_{\mathrm{G}}^{\mathrm{b}} - 2\gamma_{\mathrm{l}}d_{\mathrm{l}}^{\mathrm{a}} - \gamma_{\mathrm{l}}[\omega_{\mathrm{l}}]d_{\mathrm{l}}^{\mathrm{b}} \\
&+ L_{\mathrm{g}}^{-1}V^* - 2I_{\mathrm{l}}^*\Big).
\end{aligned}
\tag{50}
$$

Then, in analogy with Huang (2004, Lemma 7.19), we form the following pairwise coprime polynomials $\bar{\pi}_1 : \mathbb{R}^{4nn_{\mathrm{d}}+2n} \to \mathbb{R}^{nn_{\mathrm{d}}}$, $\bar{\pi}_2 : \mathbb{R}^{4nn_{\mathrm{d}}+2n} \to \mathbb{R}^{nn_{\mathrm{d}}}$ for output $\bar{g}_0(z, v_{\mathrm{x}}, \zeta, u) = \mathrm{col}(\zeta, u)$ as:

$$
\begin{aligned}
\bar{\pi}_1(d,w) =& \Big(\big([\omega_{\mathrm{G}}]^2-\mathbb{I}_{nn_{\mathrm{d}}}\big)^{-1}+\mathbb{I}_{nn_{\mathrm{d}}}\Big)d_{\mathrm{G}}^{\mathrm{a}} \\
&+ \big([\omega_{\mathrm{G}}]^2-\mathbb{I}_{nn_{\mathrm{d}}}\big)^{-1}[\omega_{\mathrm{G}}]d_{\mathrm{G}}^{\mathrm{b}} \\
\bar{\pi}_2(d,w) =& \big([\omega_{\mathrm{l}}]^2+\mathbb{I}_{nn_{\mathrm{d}}}\big)^{-1}[\omega_{\mathrm{l}}]d_{\mathrm{l}}^{\mathrm{b}} \\
&- \Big(\big([\omega_{\mathrm{l}}]^2+\mathbb{I}_{nn_{\mathrm{d}}}\big)^{-1}+\mathbb{I}_{nn_{\mathrm{d}}}\Big)d_{\mathrm{l}}^{\mathrm{a}} + \bar{C}_1,
\end{aligned}
\tag{51}
$$

with $\bar{C}_1 \in \mathbb{R}^{nn_{\mathrm{d}}}$ satisfying $L_{\mathrm{g}}\gamma_{\mathrm{l}}\bar{C}_1 = V^* - 2L_{\mathrm{g}}I_{\mathrm{l}}^*$. Then, the minimal zeroing polynomials $\bar{P}_1 : \mathbb{R} \to \mathbb{R}^{nn_{\mathrm{d}}\times nn_{\mathrm{d}}}$, $\bar{P}_2 : \mathbb{R} \to \mathbb{R}^{nn_{\mathrm{d}}\times nn_{\mathrm{d}}}$ of (51) are given by

$$
\bar{P}_1(\lambda) = \lambda^2\mathbb{I}_{nn_{\mathrm{d}}} + [\omega_{\mathrm{G}}]^2, \quad \bar{P}_2(\lambda) = \lambda^3\mathbb{I}_{nn_{\mathrm{d}}} + \lambda[\omega_{\mathrm{l}}]^2. \tag{52}
$$

Note that all the zeros of $\bar{P}_j(\lambda)$, with $j = 1, 2$ are simple and pure imaginary. Now, we define $\bar{\Lambda}_{1ik} := \mathrm{col}(\bar{\pi}_{1ik}, \dot{\bar{\pi}}_{1ik})$, $\bar{\Lambda}_{2ik} := \mathrm{col}(\bar{\pi}_{2ik}, \dot{\bar{\pi}}_{2ik}, \ddot{\bar{\pi}}_{2ik})$, $\bar{\Lambda}_{ji} := \mathrm{col}(\bar{\Lambda}_{ji1}, \ldots, \bar{\Lambda}_{jin_{\mathrm{d}}})$, $\bar{\Lambda}_j := \mathrm{col}(\bar{\Lambda}_{j1}, \ldots, \bar{\Lambda}_{jn})$, $\bar{\Lambda} := \mathrm{col}(\bar{\Lambda}_1, \bar{\Lambda}_2)$ with $i = 1, \ldots, n$, $j = 1, 2$, $k = 1, \ldots, n_{\mathrm{d}}$. Then, $\bar{\Sigma}_1 : \mathbb{R}^{5nn_{\mathrm{d}}} \to \mathbb{R}^n$ is defined as $\bar{\Sigma}_1(\bar{\Lambda}) := L_{\mathrm{g}}[V^*]\gamma_{\mathrm{G}}\bar{\pi}_1 + L_{\mathrm{g}}\gamma_{\mathrm{l}}\bar{\pi}_2$, which reproduces $\boldsymbol{\zeta}(d, w)$ given in (50). Thus, the companion matrix associated with the minimal zeroing polynomials (52) can be expressed as $\bar{\Phi}_j = \mathrm{blockdiag}(\bar{\Phi}_{j1}, \ldots, \bar{\Phi}_{jn})$, where $\bar{\Phi}_{ji} = \mathrm{blockdiag}(\bar{\Phi}_{ji1}, \ldots, \bar{\Phi}_{jin_{\mathrm{d}}})$,

$$
\bar{\Phi}_{1ik} = \begin{pmatrix} 0 & 1 \\ -\omega_{\mathrm{G}_{ik}}^2 & 0 \end{pmatrix}, \bar{\Phi}_{2ik} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -\omega_{\mathrm{l}_{ik}}^2 & 0 \end{pmatrix}, \tag{53}
$$

with $i = 1, \ldots, n$, $j = 1, 2$, $k = 1, \ldots, n_{\mathrm{d}}$. Then, we have

$$
\bar{\Phi} = \mathrm{blockdiag}(\bar{\Phi}_1, \bar{\Phi}_2) \in \mathbb{R}^{5nn_{\mathrm{d}}\times 5nn_{\mathrm{d}}}. \tag{54}
$$

Now, the Jacobian of $\bar{\Sigma}_1(\bar{\Lambda})$ at the origin can be given by

$$
\bar{\Psi} = \mathrm{row}(\bar{\Psi}_1, \bar{\Psi}_2) \in \mathbb{R}^{n\times 5nn_{\mathrm{d}}}, \tag{55}
$$

where $\bar{\Psi}_j = \mathrm{blockdiag}(\bar{\Psi}_{j1}, \ldots, \bar{\Psi}_{jn})$ with $\bar{\Psi}_{ji} = \mathrm{row}(\bar{\Psi}_{ji1}, \ldots, \bar{\Psi}_{jin_{\mathrm{d}}})$, $\bar{\Psi}_{1ik} = \mathrm{row}(L_{\mathrm{g}_i}V_i^*\gamma_{\mathrm{G}_{ik}}, 0)$, $\bar{\Psi}_{2ik} = \mathrm{row}(L_{\mathrm{g}_i}\gamma_{\mathrm{l}_{ik}}, 0, 0)$, $i = 1, \ldots, n$, $j = 1, 2$, $k = 1, \ldots, n_{\mathrm{d}}$. Then, the steady-state generator $\bar{\theta} : \mathbb{R}^{4nn_{\mathrm{d}}+2n} \to \mathbb{R}^{5nn_{\mathrm{d}}}$, $\bar{\alpha} : \mathbb{R}^{5nn_{\mathrm{d}}} \to \mathbb{R}^{5nn_{\mathrm{d}}}$, $\bar{\beta} : \mathbb{R}^{5nn_{\mathrm{d}}} \to \mathbb{R}^n$ with output $\bar{g}_0(z, v_{\mathrm{x}}, \zeta, u) = \mathrm{col}(\zeta, u)$ is given by

$$
\begin{aligned}
&\bar{\theta}(d,w) = \bar{T}\bar{\Lambda}, \quad \bar{\alpha}(\bar{\theta}) = \bar{T}\bar{\Phi}\bar{T}^{-1}\bar{\theta}, \\
&\quad \bar{\beta}(\bar{\theta}) = \mathrm{col}(\bar{\beta}_1(\bar{\theta}), \bar{\beta}_2(\bar{\theta})),
\end{aligned}
\tag{56}
$$

where $\bar{\beta}_1(\bar{\theta}) = \bar{\Sigma}_1(\bar{T}^{-1}\bar{\theta}) = \bar{\Psi}\bar{T}^{-1}\bar{\theta}$, $\bar{\beta}_2(\bar{\theta}) = \dot{\bar{\beta}}_1(\bar{\theta}) + \bar{\beta}_1(\bar{\theta}) = \bar{\Psi}\bar{\Phi}\bar{T}^{-1}\bar{\theta} + \bar{\Psi}\bar{T}^{-1}\bar{\theta}$ and $\bar{T} \in \mathbb{R}^{5nn_{\mathrm{d}}\times 5nn_{\mathrm{d}}}$ is any nonsingular matrix. Since the pair $(\bar{\Psi}_{jik}, \bar{\Phi}_{jik})$ has a companion form, the pair $(\bar{\Psi}, \bar{\Phi})$ is observable; therefore, the steady-state generator $\{\bar{\theta}, \bar{\alpha}, \bar{\beta}\}$ is linearly observable. Also, it can be inferred from (56) that $\dot{\bar{\theta}} = \bar{\alpha}(\bar{\theta})$. ■

In the following, we use the steady-state generator (56) in order to find an internal model for system (48). Thus, we define

$$
\begin{aligned}
\bar{M} &:= \mathrm{blockdiag}(\bar{M}_1, \bar{M}_2) \in \mathbb{R}^{5nn_{\mathrm{d}}\times 5nn_{\mathrm{d}}} \\
\bar{N} &:= \mathrm{col}(\bar{N}_1, \bar{N}_2) \in \mathbb{R}^{5nn_{\mathrm{d}}\times nn_{\mathrm{d}}},
\end{aligned}
\tag{57}
$$

where $\bar{M}_j := \mathrm{blockdiag}(\bar{M}_{j1}, \ldots, \bar{M}_{jn})$, $\bar{N}_j := \mathrm{blockdiag}(\bar{N}_{j1}, \ldots, \bar{N}_{jn})$ with $\bar{M}_{ji} := \mathrm{blockdiag}(\bar{M}_{ji1}, \ldots, \bar{M}_{jin_{\mathrm{d}}})$, $\bar{N}_{ji} := \mathrm{col}(\bar{N}_{ji1}, \ldots, \bar{N}_{jin_{\mathrm{d}}})$,

$$
\bar{M}_{1ik} := \begin{pmatrix} 0 & 1 \\ -\bar{a}_{1ik} & -\bar{b}_{1ik} \end{pmatrix}, \bar{N}_{1ik} := \mathrm{col}(0, 1),
$$

$$
\bar{M}_{2ik} := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\bar{a}_{2ik} & -\bar{b}_{2ik} & -\bar{c}_{2ik} \end{pmatrix}, \bar{N}_{2ik} := \mathrm{col}(0, 0, 1),
$$

$\bar{a}_{jik}, \bar{b}_{jik}, \bar{c}_{jik} \in \mathbb{R}_{>0}$, $i = 1, \ldots, n$, $j = 1, 2$, $k = 1, \ldots, n_{\mathrm{d}}$. Then, $\bar{T}$ in (56) is defined as $\bar{T} := \mathrm{blockdiag}(\bar{T}_1, \bar{T}_2)$, where $\bar{T}_j = \mathrm{blockdiag}(\bar{T}_{j1}, \ldots, \bar{T}_{jn})$ with $\bar{T}_{ji} = \mathrm{blockdiag}(\bar{T}_{ji1}, \ldots, \bar{T}_{jin_{\mathrm{d}}})$, $\bar{T}_{1ik}, \in \mathbb{R}^{2\times 2}$, $\bar{T}_{2ik} \in \mathbb{R}^{3\times 3}$, $i = 1, \ldots, n$, $j = 1, 2$, $k = 1, \ldots, n_{\mathrm{d}}$. Then, $\bar{T}$ can be obtained by solving the following Sylvester equation

$$
\bar{T}_{ji}\bar{\Phi}_{ji} - \bar{M}_{ji}\bar{T}_{ji} = \bar{N}_{ji}\bar{\Psi}_{ji}. \tag{58}
$$

Note that there exists a unique, nonsingular matrix $\bar{T}_{ji}$ satisfying the Sylvester equation (58) since the pair $(\bar{M}_{ji}, \bar{N}_{ji})$ is controllable, $(\bar{\Psi}_{ji}, \bar{\Phi}_{ji})$ is observable and the spectra of $\bar{M}_{ji}$ and $\bar{\Phi}_{ji}$ are disjoint. Note that the controllability of the pair $(\bar{M}_{ji}, \bar{N}_{ji})$ and the observability of the pair $(\bar{\Psi}_{ji}, \bar{\Phi}_{ji})$ follow from the controllable canonical form of the pair $(\bar{M}_{jik}, \bar{N}_{jik})$ and the companion form of the pair $(\bar{\Psi}_{jik}, \bar{\Phi}_{jik})$, respectively.

Now, in the following proposition, we introduce an internal model for system (48) based on Huang (2004, Proposition 6.21).

Proposition 2 (*Internal model for system* (48)). *Let Assumption 2 hold and consider the steady-state generator* (56) *with output* $\bar{g}_0(z, v_{\mathrm{x}}, \zeta, u) = \mathrm{col}(\zeta, u)$. *Then, the internal model for system* (48) *is given by*

$$
\dot{\eta} = \bar{\mu}(\eta, \zeta) := \bar{M}\eta + \bar{N}\zeta, \tag{59}
$$

where $\eta : \mathbb{R}_{\geq 0} \to \mathbb{R}^{5nn_{\mathrm{d}}}$ *is the state of the internal model,* $\bar{\mu} : \mathbb{R}^{5nn_{\mathrm{d}}} \times \mathbb{R}^n \to \mathbb{R}^{5nn_{\mathrm{d}}}$*, and* $\bar{M}$*,* $\bar{N}$ *are given in* (57).

Proof. The proof is similar to the one of Proposition 1. It is sufficient to replace ξ and u in (33) by η and ζ, respectively. ■

Before introducing the global robust controller with internal model (59), we define the following quantities, which are used to obtain a gain function for the lower triangular form of system (10) with (41) and represent the parameters of the proposed global robust controller:

$$
\begin{aligned}
\rho :=& \Big(\min_{i=1,\ldots,n}\Big(\frac{1}{C_{\mathrm{g}_i}L_{\mathrm{g}_i}}, 1\Big)\Big)^{-1}\Big(16\frac{\lambda_{\max}(\bar{P})}{\lambda_{\min}(\bar{P})}\upsilon\varrho^2\vartheta^2\|\bar{P}\|^2 \\
&\Big(\left\|\begin{matrix} C_{\mathrm{g}}^{-1} & C_{\mathrm{g}}^{-1}\mathcal{A} \\ \mathbf{0} & \mathbf{0} \end{matrix}\right\| + \left\|\begin{matrix} C_{\mathrm{g}}^{-1}L_{\mathrm{g}}^{-1}\bar{\Psi}\bar{T}^{-1} \\ \mathbf{0} \end{matrix}\right\| + 1\Big)^2 + 3 \\
&+ 2\big(\|\mathbb{I}_n\| + \|C_{\mathrm{g}}^{-1}\mathrm{G}\| + \|C_{\mathrm{g}}^{-1}L_{\mathrm{g}}^{-1}\bar{\Psi}\bar{T}^{-1}\bar{N}L_{\mathrm{g}}C_{\mathrm{g}}\|\big) \\
&+ 2\left\|\begin{matrix} C_{\mathrm{g}}^{-1}L_{\mathrm{g}}^{-1} \\ -(\mathbb{I}_n + \bar{\Psi}\bar{T}^{-1}\bar{N}) \end{matrix}\right\| + \Big(\max_{i=1,\ldots,n}\Big(\frac{1}{C_{\mathrm{g}_i}L_{\mathrm{g}_i}}, 1\Big)\Big)^2\Big),
\end{aligned}
\tag{60}
$$

$$\vartheta := \left\| \begin{matrix} \|C_g\| + \|L_g^{-1}\| + \|\mathsf{G}\| \\ \|L^{-1}\mathcal{A}\| \\ \|\bar{M}\bar{N}L_gC_g\| + \|\bar{N}L_gC_g\| + \|\bar{N}L_g\mathsf{G}\| \end{matrix} \right\|, \tag{61}$$

where $\mathsf{G} := [G_l^{\max}] + [\Gamma_G d_G^{\max}]$, $\upsilon > 2$, $\varrho > 1$ are constant parameters and $\bar{P}$ is a symmetric positive-definite matrix of appropriate dimensions satisfying

$$\bar{P}\varpi + \varpi^\top \bar{P} = -\mathbb{I}_{5nn_d+m+n}, \tag{62}$$

where

$$\varpi := \begin{pmatrix} -\mathbb{I}_n & \mathcal{A} & \mathbf{0} \\ \mathbf{0} & -L^{-1}R & \mathbf{0} \\ -\bar{N}L_g & -\bar{N}L_g\mathcal{A} & \bar{M} \end{pmatrix} \tag{63}$$

is Hurwitz. Moreover, $\lambda_{\min}(\bar{P})$, $\lambda_{\max}(\bar{P})$ are the minimum and maximum eigenvalues of $\bar{P}$, respectively.

In the following theorem, we use the internal model (59) to design a robust controller solving Problem 2.

Theorem 3 (*Global Robust Controller*). *Let Assumptions 1 and 2 hold. Consider system (10) with (41), dynamic extension (46), steady-state generator (56) and internal model (59). Then, the feedback controller*

$$\begin{aligned} u &= -\rho\zeta - \rho^2 e + (\rho+1)\bar{\Psi}\bar{T}^{-1}\eta + \bar{\Psi}\bar{\Phi}\bar{T}^{-1}\eta \\ \dot{\eta} &= \bar{M}\eta + \bar{N}\zeta \\ \dot{\zeta} &= -\zeta + u, \end{aligned} \tag{64}$$

solves Problem 2, where ρ is given in (60).

Proof. The proof is provided in Appendix B due to lengthy calculations. ■

Remark 3 (*Controller Properties*). Note that the structures of the controllers (22), (24), (35) and (64) we propose in this section are more complex than other controllers proposed in the literature (see *e.g.*, Cucuzzella, Lazzari et al., 2019; Ferguson et al., 2021; Iovine et al., 2018; Jeltsema & Scherpen, 2004; Kosaraju et al., 2021; Machado et al., 2018; Nahata et al., 2020; Sadabadi et al., 2018; Strehle et al., 2020). More precisely, the proposed controllers require some information about the network parameters (*e.g.*, L_g, C_g, L and R). However, in addition to the *uncertain* constant components of loads, the uncertainties of the network parameters can also be considered in the uncertainty vector w, then the robustness of the proposed controllers can be guaranteed with respect to the uncertainties of the load components and network parameters as well. Note that the higher complexity is associated with the more challenging control objective we achieve. Indeed, differently from Cucuzzella, Lazzari et al. (2019), Ferguson et al. (2021), Iovine et al. (2018), Jeltsema and Scherpen (2004), Kosaraju et al. (2021), Machado et al. (2018), Nahata et al. (2020), Sadabadi et al. (2018) and Strehle et al. (2020), the proposed controllers achieve voltage regulation in DC networks including *time-varying* and *uncertain* constant loads. Indeed, existing controllers in the literature cannot guarantee voltage regulation and stability in presence of load components that continuously vary with time. Finally, notice that we do not consider P loads in the global output regulation problem because finding a gain function satisfying (B.7) in the Appendix is very challenging and left as a future work. Also, we do not consider the resistance of the buck converter in (4) because it is very small in practice and acts as an uncertain damping, thus not affecting the stability.

Remark 4 (*Comparison with (Silani et al., 2021b)*). Note that the controller proposed in Silani et al. (2021b) is designed for ZI loads only and does not guarantee stability in presence of P loads. Also, the controller in Silani et al. (2021b) strongly depends on the solution to a Partial Differential Equation (PDE). On the other hand, the controllers (22), (24) and (35) achieve voltage regulation, guaranteeing local robust stability in presence of ZIP loads while the controller (64) additionally ensures global robust stability in presence of ZI loads. Furthermore, the controllers (22), (24), (35) and (64) guarantee robust stability with respect to the uncertain constants $G_{l_i}^*$, $I_{l_i}^*$ and $P_{l_i}^*$ while the controller presented in Silani et al. (2021b) only ensures local stability (not robust stability). Also, in this paper we provide the analytical solutions (A.6) and (50) to the regulator Eqs. (19) and (49), respectively, while the analytical solution for the PDE in Silani et al. (2021b) is not provided.

5. Simulation results

In this section, the performance of the control schemes proposed in Theorems 1–3 is evaluated. We consider a DC network composed of 4 nodes, whose electric parameters are equal to those reported in Cucuzzella, Trip et al. (2019, Tables II, III) and are identical or very similar to those used in Nahata et al. (2020), Strehle et al. (2020) and Tucci, Meng et al. (2018) for simulations and in Cucuzzella, Lazzari et al. (2019) for experimental validation. In the following, we consider a mismatch between the actual load profile and the one generated by the corresponding exosystem, showing that the controlled system is Robust Input-to-State Stable (RISS) with respect to such a mismatch and uncertain loads.

Let $\Xi_1 := 1.43\sin(0.08t-0.12)+0.45\sin(1.37t-3.5)+1$, and $\Xi_2 := 12.41\sin(0.477t-1.1)+11.98\sin(0.495t+1.97)+0.5$. Then, consider the following load variations: $\Delta I_{l_i} = \Xi_1$ A, $\Delta G_{l_i} = 0.005\,\Xi_1\ \Omega^{-1}$, $\Delta P_{l_i} = 0.1\,\Xi_1$ W, for $i = 1, 2, 3$ and $\Delta I_{l_4} = \Xi_2$ A, $\Delta G_{l_4} = 0.005\,\Xi_2\ \Omega^{-1}$, $\Delta P_{l_4} = 0.1\,\Xi_2$ W. Thus, the exosystem (6) can be expressed as

$$\begin{aligned} \dot{d}_{y_i} &= \begin{pmatrix} \mathbf{0}_{2\times 2} & [\omega_{y_i}] \\ -[\omega_{y_i}] & \mathbf{0}_{2\times 2} \end{pmatrix} d_{y_i} \\ \hat{y}_{l_i} &= \Gamma_{y_i} d_{y_i}, \end{aligned} \tag{65}$$

where $d_{y_i} : \mathbb{R}_{\geq 0} \to \mathbb{R}^4$ is the state of the exosystem and $\omega_{y_i} \in \mathbb{R}^2$ is defined as $\omega_{y_i} := \mathrm{col}(\omega_{y_i}^{\cdot}, \omega_{y_i}^{\cdot})$ with $\omega_{y_i}^{\cdot}, \omega_{y_i}^{\cdot}$ equal to 0.08 and 1.37 rad/s for Nodes 1, 2 and 3, and 0.477 and 0.495 rad/s for Node 4, respectively. Moreover, the elements of the matrix $\Gamma_{y_i} \in \mathbb{R}^{1\times 4}$, with y (or y) denoting G, I, P (or G, I, P) in case of Z, I, P loads, respectively, can be obtained from the amplitude of the sinusoidal terms in ΔI_l, ΔG_l, and ΔP_l. Consider the case in which the initial conditions of the voltages are not sufficiently close to the desired value, *i.e.*, $V(0) = \mathrm{col}(340, 340, 340, 340)$. We can notice from Fig. 2 that only by applying the controller proposed in Theorem 3 the voltages converge to their desired values (*i.e.*, 380 V). Differently, by applying the controllers proposed in Theorems 1 and 2, the voltages oscillate and do not converge to their desired values. Now, let the system initially be at the steady-state with $I_l(0) = \mathrm{col}\big(30, 15, 30, 26\big)$ A, $G_l(0) = \mathrm{col}\big(0.07, 0.05, 0.06, 0.08\big)\ \Omega^{-1}$ and $P_l(0) = \mathrm{col}\ \big(\ 8, 4, 5,\ 12\ \big)$ W. Then, at the time instant $t = 1$ s the loads vary according to the real data[3] in openei (0000) while the controller uses the information of the exosystems, which differ from the real data. We can observe from Fig. 3 that by applying the controllers proposed in Theorems 1 and 2, the voltage at each node is kept very close to the corresponding desired value, showing that the controlled system is RISS with respect to the mismatch between the actual

[3] Specifically, we use real load profile data of four different consumers in the State of New York on March 21st, 2019.

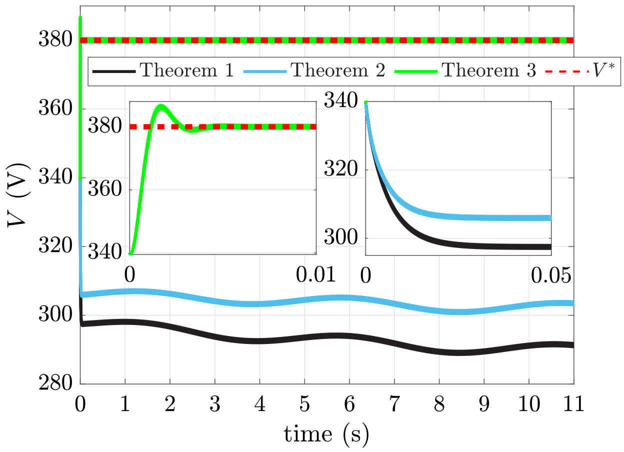

Fig. 2. Controllers in Theorems 1–3: time evolution of the voltages in presence of ZI loads together with the corresponding desired values (dashed lines).

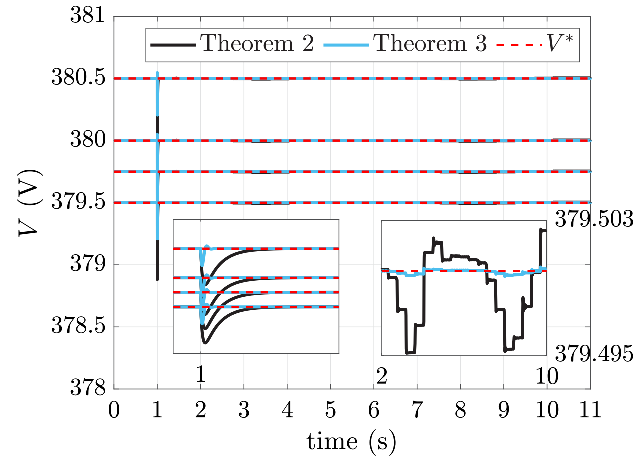

Fig. 4. Controllers in Theorems 2 and 3: time evolution of the voltages in presence of ZI loads together with the corresponding desired values (dashed lines).

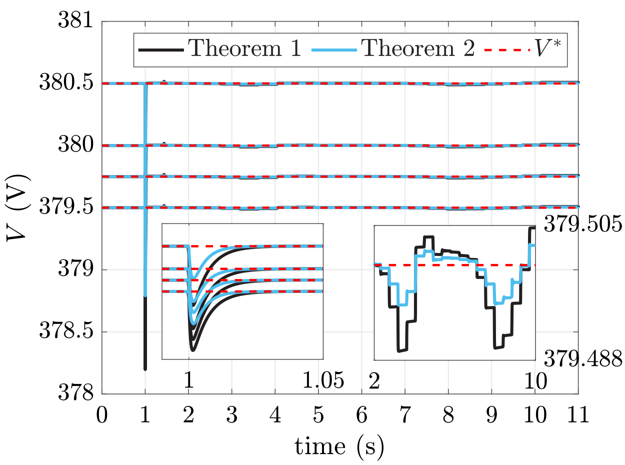

Fig. 3. Controllers in Theorems 1 and 2: time evolution of the voltages in presence of ZIP loads together with the corresponding desired values (dashed lines).

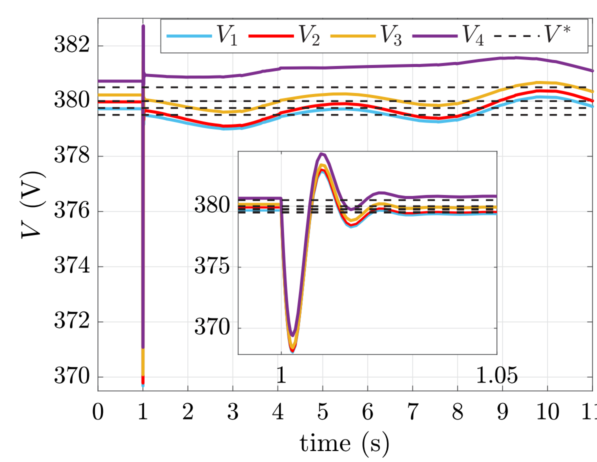

Fig. 5. Controller (Tucci, Riverso et al., 2018): time evolution of the voltages at each node together with the corresponding desired values (dashed lines).

load profile and the one generated by the exosystem, achieving in practice voltage regulation (Objective 1). We also observe that the controller proposed in Theorem 2 performs better than the one proposed in Theorem 1 in terms of voltage undershoots and deviations. Also, Fig. 4 shows that in presence ZI loads and by applying the controllers proposed in Theorems 2 and 3, the voltage at each node converges to the desired value, achieving voltage regulation (see Objective 1) and the controlled system is RISS with respect to the actual load profile. We also observe that the controller proposed in Theorem 3 performs better than the one proposed in Theorem 2 in terms of voltage undershoots, deviations and time response. Finally, for the sake of comparison, we apply (to the same network under the same conditions) the controller in Tucci, Riverso and Ferrari-Trecate (2018), which is designed to deal with constant loads only. Fig. 5 depicts that in presence of *time-varying* ZIP loads the controller in Tucci, Riverso et al. (2018) is not able to achieve voltage regulation.

Remark 5 (*Region of Attraction*)**.** Note that for the controllers proposed in Theorems 1 and 2 when at the initial time instant, the initial conditions are outside of the region of attraction, then, only during the initialization phase, a conventional controller can be used to steer the system trajectories within such region. However, if the region of attraction is not exactly known, the conventional controller can use a sufficiently small region around the equilibrium point instead of the region of attraction.

6. Conclusions and future work

In this paper, we have considered *time-varying* dynamics with constant uncertainties for the load components of a DC network. Then, we have proposed control schemes achieving voltage regulation and guaranteeing robust stability of the overall network. Finally, we have proposed a control scheme achieving voltage regulation and guaranteeing global robust stability. Future research directions include the use of (global) robust output regulation theory to tackle the problem of average voltage regulation and current sharing in DC networks with *time-varying* loads. Additionally, we would like to extend the global robust stability also to the case of P loads via finding a suitable gain function for the lower triangular form of the system.

Appendix

The proofs of Theorems 1 and 3 are presented in the following.

Appendix A. Proof of Theorem 1

Proof. We first form the regulator equation associated with (10) and show that its solution is a degree k polynomial in d. Then, based on this solution we obtain the minimal polynomial of the k-fold exosystem (10b) to show that the pair (G_1, G_2) given in (20) and (21) incorporates an internal model of the k-fold exosystem (10b). Finally, following Huang (2004, Theorems 5.7, 5.12), we show that the controllers (22) and (24) solve Problem 1.

In analogy with Huang (2004, Theorem 3.26), we first compute $H_e(x,d,w) = \begin{pmatrix} h(x,d) \\ L_{f_a}h(x,d) \end{pmatrix}$. Then, we notice that the solution to $H_e(x,d,w) = \mathbf{0}_{2n}$ for system (10) can be expressed as follows:

$$V = V^*, I_g = -\mathcal{A}I + i_{ZIP}(V^*, d, w), \tag{A.1}$$

with i_{ZIP} given by (11). Thus, the partition $x^a := \mathrm{col}(I_g, V)$, $x^b := I$ and a smooth function $\tau(x^b,d,w) := \mathrm{col}(-\mathcal{A}I + I_I^* + \Gamma_I d_I + ([G_I^*] + [\Gamma_G d_G])V^* + [V^*]^{-1}(P_I^* + \Gamma_P d_P), V^*)$ exist such that $H_e(x,d,w)|_{x^a=\tau(x^b,d,w)} = \mathbf{0}_{2n}$. Recalling the relative degree of system (10) (see Lemma 1), the equivalent control input $u_e(x,d,w)$ can be computed by posing the second-time derivative of the output mapping (10c) equal to zero, *i.e.*,

$$L_{f_a}^2 h(x,d) + L_{g_a}L_{f_a}h(x,d)u_e(x,d,w) = \mathbf{0}, \tag{A.2}$$

Now, let $u_e^*(x,d,w) := u_e(x,d,w)|_{x^a=\tau(x^b,d,w)}$. By replacing V and I_g in the solution to (A.2) with the right-hand side of (A.1), we obtain

$$\begin{aligned} u_e^*(x,d,w) = {} & V^* + L_g\mathcal{A}L^{-1}\mathcal{A}^\top V^* + L_g\mathcal{A}L^{-1}RI \\ & + L_g[V^*]\Gamma_G S_G d_G + L_g\Gamma_I S_I d_I \\ & + L_g[V^*]^{-1}\Gamma_P S_P d_P. \end{aligned} \tag{A.3}$$

According to Lemma 1, the zero dynamics of (10) can be expressed as

$$L\dot{I} = -\mathcal{A}^\top V^* - RI, \quad \dot{d} = Sd. \tag{A.4}$$

Then, $\boldsymbol{I}(d,w)$ in (19) can be obtained by solving

$$\frac{\partial \boldsymbol{I}(d,w)}{\partial d}Sd = -L^{-1}\big(\mathcal{A}^\top V^* + R\boldsymbol{I}(d)\big). \tag{A.5}$$

Since the exosystem model (6) is linear, we can find the analytical expression for the solutions to (A.5). Thus, by virtue of Assumption 1, Lemma 2 and following Huang (2004, Theorem 3.26), the solutions to (19) can be given by

$$\begin{aligned} \boldsymbol{x}(d,w) &= \begin{pmatrix} \mathcal{A}R^{-1}\mathcal{A}^\top V^* + i_{ZIP}(V^*,d,w) \\ V^* \\ -R^{-1}\mathcal{A}^\top V^* \end{pmatrix} \\ \boldsymbol{u}(d,w) &= u_e^*(\boldsymbol{x}(d,w),d,w) \\ &= V^* + L_g\,\big(\,[V^*]\Gamma_G S_G d_G + \Gamma_I S_I d_I \\ &\quad + [V^*]^{-1}\Gamma_P S_P d_P\,\big), \end{aligned} \tag{A.6}$$

with i_{ZIP} given by (11). Then, we can infer from (A.6) that the solutions $\boldsymbol{x}(d,w)$ and $\boldsymbol{u}(d,w)$ to the regulator equation (19) are linear in d, *i.e.*, degree $k=1$ polynomials in d. Consequently, according to Huang (2004, Theorem 5.16), the minimal polynomial of the k-fold exosystem (10b) with $k=1$ can be expressed as:

$$\alpha^k(\lambda) = \prod_{i=1}^{n}\prod_{h=1}^{n_d}(\lambda^2 + \omega_{G_{ih}}^2)(\lambda^2 + \omega_{I_{ih}}^2)(\lambda^2 + \omega_{P_{ih}}^2). \tag{A.7}$$

Thus, it follows that the pair (G_1, G_2) given by (20) and (21) is the minimal internal model of the k-fold exosystem (10b) with $k=1$. Moreover, by virtue of Assumption 2, it can be inferred from (A.7) that G_1 satisfies (Huang, 2004, Property 1.5), *i.e.*, for all $\lambda \in \sigma(G_1)$ the matrix $\begin{pmatrix} A - \lambda\mathbb{I}_{2n+m} & B \\ C & \mathbf{0} \end{pmatrix}$ has full rank. Then, by virtue of Assumptions 1, 2 and Lemma 2, we have

(i) In analogy with Huang (2004, Theorem 5.7 (i)), the controller (22) solves the first order ($k=1$) robust output regulation in the sense as described in Huang (2004, Section 5.1). Furthermore, since the solutions $\boldsymbol{x}(d,w)$ and $\boldsymbol{u}(d,w)$ to the regulator equation (19) are degree $k=1$ polynomials in d, following Huang (2004, Theorem 5.12 (i)), the controller (22) solves Problem 1.

(ii) In analogy with Huang (2004, Theorem 5.7 (ii)), the controller (24) solves the first order ($k=1$) robust output regulation in the sense as described in Huang (2004, Section 5.1). Furthermore, since the solution $\boldsymbol{u}(d,w)$ to the regulator equation (19) is a degree $k=1$ polynomial in d, following Huang (2004, Theorem 5.12 (ii)), the controller (24) solves Problem 1. ■

Appendix B. Proof of Theorem 3

Proof. We first apply a coordinate transformation to system (48), (59) to find a lower triangular form for system (10) with (41). Then, following Huang (2004, Remark 7.23), we investigate the Robust Input-to-State Stability (RISS) (Huang, 2004, Definition 2.22) of the internal model in the new coordinate to show that the controller (64) satisfies the condition given in Huang (2004, Theorem 7.21 and Remark 7.22). Finally, we show that ρ given in (60) fulfills the conditions given in Huang (2004, Remark 7.10).

Now, we apply the following coordinate transformation to system (48), (59) in order to find a lower triangular form for system (10) with (41)

$$\begin{aligned} \hat{z} &:= z - \boldsymbol{z}(d,w), e := v_x - V^*, \hat{\zeta} := \zeta - \bar{\beta}_1(\eta) \\ &= \zeta - \bar{\Psi}\bar{T}^{-1}\eta, \hat{\eta} := \eta - \bar{\theta}(d,w), \hat{u} := u - \beta_2(\eta) \\ &= u - \bar{\Psi}\bar{\Phi}\bar{T}^{-1}\eta - \bar{\Psi}\bar{T}^{-1}\eta, \end{aligned} \tag{B.1}$$

where $\boldsymbol{z}(d,w)$ is given in (50), $\bar{\Phi}, \bar{\Psi}$ are given in (54), (55), respectively, and $\bar{T}$ is the solution to the Sylvester equation (58). Then, the augmented system (48a), (48b), (48c), (59) in the new coordinates is given by

$$\begin{aligned} \dot{\hat{z}} &= F\hat{z} + G(d,w)e \\ \dot{e} &= \bar{H}\hat{z} + K(d,w)e + b\bar{\Psi}\bar{T}^{-1}\hat{\eta} + b\hat{\zeta} \\ \dot{\hat{\zeta}} &= -\left(\mathbb{I}_n + \bar{\Psi}\bar{T}^{-1}\bar{N}\right)\hat{\zeta} + \hat{u} \\ \dot{\hat{\eta}} &= \left(\bar{M} + \bar{N}\bar{\Psi}\bar{T}^{-1}\right)\hat{\eta} + \bar{N}\hat{\zeta}, \end{aligned} \tag{B.2}$$

with F, $\bar{H}$, b, $G(d,w)$, $K(d,w)$ as in (48) and $\bar{M}$, $\bar{N}$ in (57). Now, we consider the coordinate transformation $\tilde{\eta} := \hat{\eta} - \bar{N}b^{-1}e$. Then, we have

$$\dot{\tilde{\eta}} = \bar{M}\tilde{\eta} + \bar{M}\bar{N}b^{-1}e - \bar{N}b^{-1}K(d,w)e - \bar{N}b^{-1}\bar{H}\hat{z}. \tag{B.3}$$

Now, let define $z^{tr} := \mathrm{col}(z_a^{tr}, z_b^{tr}) := \mathrm{col}(\hat{z}, \tilde{\eta})$, $x^{tr} := \mathrm{col}(x_a^{tr}, x_b^{tr}) := \mathrm{col}(e, \hat{\zeta})$. Then, the lower triangular form of the system (10) with (41) can be expressed as

$$\dot{z}_a^{tr} = Fz_a^{tr} + Gx_a^{tr} \tag{B.4a}$$

$$\dot{z}_b^{tr} = \bar{M}z_b^{tr} + \bar{M}\bar{N}b^{-1}x_a^{tr} - \bar{N}b^{-1}Kx_a^{tr} - \bar{N}b^{-1}\bar{H}z_a^{tr} \tag{B.4b}$$

$$\dot{x}_a^{tr} = \bar{H}z_a^{tr} + Kx_a^{tr} + b\bar{\Psi}\bar{T}^{-1}(z_b^{tr} + \bar{N}b^{-1}x_a^{tr}) + bx_b^{tr} \tag{B.4c}$$

$$\dot{x}_b^{tr} = -\left(\mathbb{I}_n + \bar{\Psi}\bar{T}^{-1}\bar{N}\right)x_b^{tr} + \hat{u}. \tag{B.4d}$$

Now, let (B.3) view $\tilde{\eta}$ as state and $\mathrm{col}(\hat{z}, e)$ as input. Then, in analogy with Huang (2004, Remark 7.23), we investigate the

RISS (Huang, 2004, Definition 2.22) of (B.3) with respect to the uncertainty w. Suppose that there exists a constant value $0 < r_0 < 1$, satisfying $0 \le (1-r_0)\tilde{\eta}^\top\tilde{\eta}$. Moreover, since $\bar{M}$ is Hurwitz, there exists a symmetric positive-definite matrix $P \in \mathbb{R}^{5nn_d \times 5nn_d}$ satisfying $P\bar{M} + \bar{M}^\top P = -\mathbb{I}_{5nn_d}$. Let $S_1(\tilde{\eta}) = \frac{2}{r_0}\tilde{\eta}^\top P\tilde{\eta}$, then $\frac{2}{r_0}\lambda_{\min}(P)\|\tilde{\eta}\|^2 \le S_1(\tilde{\eta}) \le \frac{2}{r_0}\lambda_{\max}(P)\|\tilde{\eta}\|^2$. The derivative of $S_1(\tilde{\eta})$ satisfies

$$\begin{aligned}\dot{S}_1(\tilde{\eta}) =& \frac{2}{r_0}\Big(2\tilde{\eta}^\top P\bar{M}\tilde{\eta} + 2\tilde{\eta}^\top P\left(\bar{M}\bar{N}b^{-1}e\right.\\ &\left.- \bar{N}b^{-1}K(d,w)e - \bar{N}b^{-1}\bar{H}\hat{z}\right)\Big)\\ \le& -\|\tilde{\eta}\|^2 + \left\|\frac{2}{r_0}P\left(\bar{M}\bar{N}b^{-1}e\right.\right.\\ &\left.\left.- \bar{N}b^{-1}K(d,w)e - \bar{N}b^{-1}\bar{H}\hat{z}\right)\right\|^2,\end{aligned} \tag{B.5}$$

along the solutions to (B.3). Then, we have

$$\begin{aligned}&\left\|\frac{2}{r_0}P\big(\bar{M}\bar{N}b^{-1}e - \bar{N}b^{-1}K(d,w)e - \bar{N}b^{-1}\bar{H}\hat{z}\big)\right\|\\ &\le a\left\|\mathrm{col}(\hat{z}, e)\right\|,\end{aligned} \tag{B.6}$$

where $a := 1 + \big\|\frac{2}{r_0}P\|\ \mathrm{row}\big(\|\bar{N}L_g\|, \|\bar{N}L_g\mathcal{A}\|, \|\bar{M}\bar{N}L_g C_g\| + \|\bar{N}L_gC_g\| + \|\bar{N}L_g([G_I^{\max}] + [\Gamma_G d_G^{\max}])\|\big)\big\|$. Let define the class $\mathcal{K}_\infty$ function $\bar{a}(s) := sa$, then we have $\dot{S}_1(\tilde{\eta}) \le -\|\tilde{\eta}\|^2 + \bar{a}^2\big(\|\mathrm{col}(\hat{z},e)\|\big)$. Therefore, for any $0 < \varepsilon < 1$ such that $\frac{\bar{a}(\|\mathrm{col}(\hat{z},e)\|)}{1-\varepsilon} \le \|\tilde{\eta}\|$, we have $\dot{S}_1(\tilde{\eta}) \le -\varepsilon\|\tilde{\eta}\|^2$. Thus, following Huang (2004, Theorem 2.16), let $\underline{a}_1(s) := \frac{2}{r_0}\lambda_{\min}(P)s^2$, $\bar{a}_1(s) := \frac{2}{r_0}\lambda_{\max}(P)s^2$, $\chi_1(s) := \frac{1}{1-\varepsilon}\bar{a}(s)$, then (B.3) is RISS with gain function $\kappa_1(s)$ satisfying

$$\underline{a}_1^{-1}\big(\bar{a}_1(\chi_1(s))\big) = \frac{\bar{a}(s)}{1-\varepsilon}\sqrt{\lambda_{\max}(P)/\lambda_{\min}(P)} \le \kappa_1(s). \tag{B.7}$$

Moreover, we know that the eigenvalues of matrix F given in (48) have negative real-parts, matrix b given in (48) is positive definite and $\bar{\Psi}$, $\bar{\Phi}$ given in (55), (54), respectively, are observable. Consequently, according to Huang (2004, Theorem 7.21 and Remark 7.22), Problem 2 is solvable by the following controller

$$u = \iota_2(\hat{x}_b^{\mathrm{tr}}) + \bar{\beta}_2(\eta), \quad \hat{x}_b^{\mathrm{tr}} = \zeta - \bar{\beta}_1(\eta) - \iota_1(e), \tag{B.8}$$

where the functions $\iota_1, \iota_2 : \mathbb{R}^n \to \mathbb{R}^n$ are defined as

$$\iota_1(e) := -\rho e, \quad \iota_2(\hat{x}_b^{\mathrm{tr}}) := -\rho\hat{x}_b^{\mathrm{tr}}, \tag{B.9}$$

with ρ given in (60). Now, we will show that the control parameter ρ given in (60) satisfies the conditions given in Huang (2004, Remark 7.10). To do this, we should first obtain the gain function $\kappa_2(s)$ introduced in Huang (2004, Theorem 2.16) for system (B.4a), (B.4b). Since ϖ given in (63) is Hurwitz, the symmetric positive-definite matrix $\bar{P}$ exists such that (62) is satisfied. Let $S_2(z^{\mathrm{tr}}) = z^{\mathrm{tr}\top}\bar{P}z^{\mathrm{tr}}$, then $\lambda_{\min}(\bar{P})\|z^{\mathrm{tr}}\|^2 \le S_2(z^{\mathrm{tr}}) \le \lambda_{\max}(\bar{P})\|z^{\mathrm{tr}}\|^2$ and the derivative of $S_2(z^{\mathrm{tr}})$ satisfies

$$\begin{aligned}\dot{S}_2(z^{\mathrm{tr}}) =& z^{\mathrm{tr}\top}\big(\bar{P}\varpi + \varpi^\top\bar{P}\big)z^{\mathrm{tr}}\\ &+ z^{\mathrm{tr}\top}\bar{P}\begin{pmatrix} G & \mathbf{0}\\ \bar{M}\bar{N}b^{-1} - \bar{N}b^{-1}K & \mathbf{0}\end{pmatrix}x^{\mathrm{tr}}\\ &+ x^{\mathrm{tr}\top}\begin{pmatrix} G & \mathbf{0}\\ \bar{M}\bar{N}b^{-1} - \bar{N}b^{-1}K & \mathbf{0}\end{pmatrix}^\top\bar{P}z^{\mathrm{tr}}\\ \le& -\|z^{\mathrm{tr}}\|^2 + 2\vartheta\|\bar{P}\|\|z^{\mathrm{tr}}\|\|x^{\mathrm{tr}}\|,\end{aligned} \tag{B.10}$$

along the solution to (B.4a), (B.4b), where ϑ is given in (61). Therefore, for any $0 < \bar{\varepsilon} < 1$ such that $\frac{2\vartheta\|\bar{P}\|}{1-\bar{\varepsilon}}\|x^{\mathrm{tr}}\| \le \|z^{\mathrm{tr}}\|$, we have $\dot{S}_2(z^{\mathrm{tr}}) \le -\bar{\varepsilon}\|z^{\mathrm{tr}}\|^2$. Thus, following Huang (2004, Theorem 2.16), let $\underline{a}_2(s) := \lambda_{\min}(P)s^2$, $\bar{a}_2(s) := \lambda_{\max}(P)s^2$, $\chi_2(s) := \frac{2\vartheta\|\bar{P}\|}{1-\bar{\varepsilon}}s$, then the gain function $\kappa_2(s)$ for system (B.4a), (B.4b) is given by $\kappa_2(s) := 2\sqrt{\lambda_{\max}(\bar{P})/\lambda_{\min}(\bar{P})}\vartheta\varrho\|\bar{P}\|s$ satisfying

$$\kappa_2(s) \ge \underline{a}_2^{-1}\big(\bar{a}_2(\chi_2(s))\big) = \sqrt{\lambda_{\max}(\bar{P})/\lambda_{\min}(\bar{P})}\frac{2\vartheta\|\bar{P}\|}{1-\bar{\varepsilon}}s, \tag{B.11}$$

with $\varrho > 1$. Now, let define the right hand side of (B.4c) and (B.4d) for $\hat{u} = 0$ as $\phi(z^{\mathrm{tr}}, x^{\mathrm{tr}}, d, w)$. Then, we have

$$\|\phi(z^{\mathrm{tr}}, x^{\mathrm{tr}}, d, w)\| \le \|x^{\mathrm{tr}}\|\phi_0 + \|z^{\mathrm{tr}}\|\phi_1, \tag{B.12}$$

where

$$\begin{aligned}\phi_0 :=& \|\mathbb{I}_n\| + \left\|C_g^{-1}\mathrm{G}\right\| + \left\|C_g^{-1}L_g^{-1}\bar{\Psi}\bar{T}^{-1}\bar{N}L_gC_g\right\|\\ &+ \left\|\begin{matrix} C_g^{-1}L_g^{-1}\\ -\big(\mathbb{I}_n + \bar{\Psi}\bar{T}^{-1}\bar{N}\big)\end{matrix}\right\| + 1\\ \phi_1 :=& \left\|\begin{matrix} C_g^{-1} & C_g^{-1}\mathcal{A}\\ \mathbf{0} & \mathbf{0}\end{matrix}\right\| + \left\|\begin{matrix} C_g^{-1}L_g^{-1}\bar{\Psi}\bar{T}^{-1}\\ \mathbf{0}\end{matrix}\right\| + 1.\end{aligned} \tag{B.13}$$

Now, following Huang (2004, Remark 7.10), ρ given in (60) satisfies

$$\begin{aligned}\rho \ge& \Big(2\min_{i=1,\dots,n}\Big(\frac{1}{C_{g_i}L_{g_i}}, 1\Big)\Big)^{-1}\Big(16\frac{\lambda_{\max}(\bar{P})}{\lambda_{\min}(\bar{P})}\upsilon\varrho^2\vartheta^2\|\bar{P}\|^2\phi_1^2\\ &+ 2\phi_0 + 1 + \Big(\max_{i=1,\dots,n}\Big(\frac{1}{C_{g_i}L_{g_i}}, 1\Big)\Big)^2\Big),\end{aligned} \tag{B.14}$$

with $\upsilon > 2$. Consequently, following Huang (2004, Theorem 7.21 and Remark 7.22), the controller (64) solving Problem 2 is obtained by replacing (B.9) in (B.8). ■

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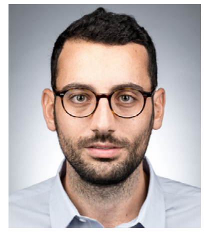

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