Distributed Linear Quadratic Control and Filtering

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Distributed Linear Quadratic Control and Filtering:

a suboptimality approach

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To my parents

献给我亲爱的父母,
焦圣中、李敏

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Junjie Jiao
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Contents

1 Introduction

1.1 Background

Collective behavior of groups of individuals, for example animals, is ubiquitous in nature [80]. Examples are schooling of fish, flocking of birds, swarming of insects and pack hunting of wolves. These examples of collective animal behavior are believed to have many benefits. For instance, from a *protection-from-predator* perspective, one hypothesis states that fish schools or bird flocks may thwart predators [64]. This is because the many moving preys create a sensory overload on the visual channel of the predators and, consequently, it becomes difficult for the predators to pick out individual prey from the groups. Another hypothesis is that the collective behavior of animals (e.g. fish or birds) may save energy when swimming or flying together [6, 20]. These interesting phenomena have attracted much attention from researchers in many scientific disciplines, ranging from biological science [80] to physics [110], computer science [89] and control engineering [14, 34, 82].

In addition to the fact that we humans desire to understand this beautiful and fascinating collective behavior in nature, there also exist broad practical applications that are urgently calling for investigation in the area of control of interconnected systems. As examples, we mention satellite formation flying [79, 101], intelligent transportation systems [4, 5], distributed sensor networks [81] and power grids [17]. Due to the physical constraints on the interaction between these interconnected systems such as limited computational resources, local communication and local sensing capabilities, control problems in these application areas are very challenging.

Motivated by the above observations, in the past two decades, researchers in the systems and control community have put much effort into studying the problem of *distributed control of multi-agent systems*, see e.g. [11, 34, 54, 60, 85, 87, 94, 103, 111]. In this problem framework, a multi-agent system is a system that consists of a number of local systems, called the *agents*. Each agent exchanges information with the other agents according to a given communication topology. In this way, these agents together form a *network*. Consequently, the overall behavior of a network is determined not only by the behavior of the agents, but also by the communication between these agents, see e.g. [71, 85, 111]. Such communication topologies are represented by *graphs*. In general, a graph contains *nodes* being connected by *edges*, where each node represents an agent, and the edges represent the communication between these agents. It turns out that tools from algebraic graph theory are useful for tackling problems in the area of multi-agent systems [21, 63].

The essential idea of distributed control for multi-agent systems is that, while each agent makes use of only information obtained from local interactions according to the communication graph, the agents together will still achieve a common goal. Two typical examples are *consensus* and *synchronization*. Within the problem of consensus, the dynamics of the agents is often described by single or double integrators, and by proposing distributed control laws, the agents agree on a certain (possibly nonzero) *constant value* [29, 54, 85, 87, 111]. On the other hand, in the context of synchronization, the dynamics of the agents is typically characterized by a general higher dimensional linear or nonlinear system, and the proposed distributed protocols guarantee that the states or the outputs of the agents all converge to a common *time-varying trajectory* [28, 54, 94, 99, 103, 122]. If relative state information of the agents is available, it is often possible to design distributed protocols using *static state feedback* [10, 85, 87, 111]. However, if the models of the agents are described by higher dimensional dynamics, very often only relative output information is available instead of relative state information. In this case, the controlled multi-agent network may want to achieve synchronization by using *dynamic output feedback* based distributed protocols, see e.g. [54, 94, 103, 115].

In the literature on consensus and synchronization, based on whether all agents take equally important roles, multi-agent systems can be categorized into two types, namely, *leaderless* multi-agent systems and *leader-follower* multi-agent systems. In the leaderless case, all agents are equally important in the sense that they reach an agreement which depends on the dynamics of all agents, see e.g. [37, 54, 85, 87, 94, 103, 104]. On the other hand, in the leader-follower case there is (in most cases) one agent that takes the dominant position, called the *leader*, and the other agents are called the *followers*. The leaders are often taken as autonomous systems [35, 75, 122], or systems with unknown inputs [29, 56, 61]. Problems of consensus or synchronization for leader-follower systems are often referred to as *distributed tracking control problems*. The goal of a distributed tracking control problem is then to design distributed protocols for the followers such that their dynamics tracks that of the leader [29, 78].

For multi-agent systems, the models of the agents are not necessarily required to be identical. Depending on whether the agents have the same dynamics, multi-agent systems can also be classified into the following two subclasses, called *homogeneous* multi-agent systems and *heterogeneous* multi-agent systems. For homogeneous systems, the system models of the agents are identical, and here most existing work in the literature deals with *state* consensus or synchronization [54, 85, 103]. However, for heterogeneous systems, due to their very nature that the system models of the agents are allowed to be distinct, in particular their state space dimensions may even be different. For these systems it is therefore more natural and interesting to consider the collective *output* behavior [22, 46, 59, 99, 115].

While designing distributed control laws for a multi-agent system, one may not only want the controlled multi-agent network to achieve consensus or synchronization, but also would like the overall network to minimize a certain *optimality* criterion [7, 10, 23, 31, 41, 122]. Such problems are referred to as *distributed optimal control problems*. Within this framework, one of the important problems is the *distributed linear quadratic optimal control problem*. In the context of distributed linear quadratic optimal control, a global linear quadratic cost functional is introduced for a multi-agent system with *given initial states*. The objective is to design distributed control laws such that the given linear quadratic cost functional is minimized while the agents reach consensus or synchronization. Due to the particular form of distributed control laws, which capture the structure of the communication between the agents, the distributed linear quadratic optimal control problem is non-convex and very difficult to solve. It is also unclear whether in general a closed form solution exists.

As a consequence, the existing work in the literature on distributed linear quadratic optimal control either deals with *suboptimality versions* of this problem [7, 74, 76, 97, 98], or considers *special cases*, such as single integrator agent dynamics [10] and inverse optimality [75, 77, 123]. In Chapter 2 of this thesis, we investigate a *suboptimality* version of this problem. Given a leaderless multi-agent system and an associated global linear quadratic cost functional, we establish a design method for computing distributed control laws that guarantee the associated cost to be smaller than a given upper bound and achieve synchronization for the controlled network. In Chapter 3, we extend the results in Chapter 2 on distributed linear quadratic control for leaderless multi-agent systems to the case of distributed linear quadratic *tracking control* for leader-follower multi-agent systems. Both in the above distributed linear quadratic control problem and the distributed linear quadratic tracking problem, our computation of the proposed distributed control laws uses so-called *global information*, in the sense that, in order to compute the distributed control laws, knowledge of the entire network graph is required. To remove the dependence on this global information, in Chapter 4, for leaderless multi-agent systems with single integrator agent dynamics, we provide a *decentralized* method for computing distributed suboptimal control laws that do not involve global information.

Another important problem within the framework of distributed optimal control is the *distributed $\mathcal{H}_2$ optimal control* problem. In the context of distributed $\mathcal{H}_2$ optimal control, the dynamical model of a multi-agent system contains external disturbance inputs. An $\mathcal{H}_2$ cost functional is then introduced to quantify the influence of the disturbance inputs on the performance output of the overall network. The distributed $\mathcal{H}_2$ optimal control problem is the problem of finding a distributed protocol that minimizes the associated $\mathcal{H}_2$ cost while the controlled network achieves consensus or synchronization, see e.g. [36, 39, 53, 55, 112]. As before, due to the fact that the proposed distributed protocols have a particular structure imposed by the network graph, the problem of distributed $\mathcal{H}_2$ optimal control is a non-convex optimization problem. Again, it is unclear whether in general a closed form solution exists. Therefore, in Chapter 5 of this thesis, instead of considering the *actual* distributed $\mathcal{H}_2$ optimal control problem, we study a version of this problem that requires only *suboptimality*. Given a *homogeneous* multi-agent system and an associated $\mathcal{H}_2$ cost functional, we provide a design method for obtaining distributed protocols using *static relative state* information such that the associated $\mathcal{H}_2$ cost is smaller than an a priori given upper bound and the controlled network achieves state synchronization. In Chapter 6, we generalize the results in Chapter 5 on static relative state feedback to the general case of *dynamic relative output feedback*. The results in Chapters 5 and 6 on distributed $\mathcal{H}_2$ suboptimal control of *homogeneous* multi-agent systems are then further generalized in Chapter 7 to the case of *heterogeneous* multi-agent systems.

In parallel to the development of control design for consensus and synchronization of multi-agent systems, recent years have also witnessed an increasing interest in problems of *distributed state estimation* for spatially constrained large-scale systems. Applications can be found in power grids [32], industrial plants [107] and wireless sensor networks [86]. Due to physical constraints on the monitored systems, the measured output of a system is often not available to one single sensor. Consequently, standard estimation methods do not directly apply anymore. It might however be possible to monitor the state of a system by means of a *sensor network*. Such a sensor network consists a number of *local sensors*, where each of these has access to a certain portion of the measured output of the system. Each sensor then makes use of its obtained output portion to generate an estimate, and communicates this local estimate to the other local sensors according to a given communication graph. In this way, the states of all local sensors will reach synchronization to a common trajectory, which is then an estimate of the state of the measured system. Problems of monitoring the state of a spatially constrained system by a sensor network are often referred to as *distributed state estimation problems*.

The distributed state estimation problem has been studied mainly in two research directions, namely, *distributed observer design* [26, 27, 49, 65, 88, 113, 114] and *distributed filtering* [40, 47, 81, 83, 84, 105, 106]. In the distributed observer design problem, the system is noise/disturbance free and is monitored by a number of local sensors, called *local observers*. Each local observer makes use of its measured output portion of the monitored system and then communicates with the other local observers according to the given communication graph. In this way, the local observers together form a *distributed observer*. The aim is to design a distributed observer such that all local observers reconstruct the state of the system. On the other hand, in the context of distributed filtering, the dynamic model of the monitored system contains noise/disturbance inputs and its output is observed by a number of local sensors, which are referred to as *local filters*. Similarly, each local filter makes use of its measured output portion and then exchanges information with the other local filters according to the given communication graph. In this way, these local filters together form a *distributed filter*. The goal of the distributed filtering problem is to design a distributed filter such that the states of all local filters track that of the system and, in addition, this distributed filter is optimal with respect to a certain cost functional. A typical problem appearing in this research direction is the *distributed Kalman filtering problem*, see e.g. [81, 83, 84].

In the literature on distributed filtering, most of the existing work deals with *stochastic* versions of this problem. In Chapter 8 of this thesis, however, we consider two *deterministic* versions of the distributed optimal filtering problem for linear systems, more specifically, the distributed $\mathcal{H}_2$ and $\mathcal{H}_\infty$ optimal filtering problems. The distributed $\mathcal{H}_2$ and $\mathcal{H}_\infty$ optimal filtering problems are the problems of designing local filter gains such that the $\mathcal{H}_2$ or $\mathcal{H}_\infty$ norm of the transfer matrix from the disturbance input to the output estimation error is minimized while all local filters reconstruct the full system state asymptotically. Again, due to their *non-convex* nature, these problems are in general very challenging and it is not clear whether solutions exist. Therefore, instead in Chapter 8 we address *suboptimality* versions of these problems. In particular, we provide conceptual algorithms for obtaining $\mathcal{H}_2$ and $\mathcal{H}_\infty$ suboptimal distributed filters, respectively. The resulting distributed filters guarantee that the $\mathcal{H}_2$ or $\mathcal{H}_\infty$ norm of the transfer matrix from the disturbance input to the output estimation error is smaller than an a priori given upper bound, while all local filters reconstruct the full system state asymptotically.

1.2 Outline of this thesis

The organization of this thesis is as follows. Chapters 2 - 4 are concerned with the distributed linear quadratic optimal control problem. In Chapter 2, we study a *suboptimality* version of the distributed linear quadratic optimal control problem for *leaderless* homogeneous multi-agent systems. In Chapter 3, we extend the results in Chapter 2 on distributed linear quadratic suboptimal control for leaderless multi-agent systems to the case of distributed linear quadratic suboptimal tracking control for *leader-follower* multi-agent systems. The computation of the local control gains in Chapters 2 and 3 requires complete knowledge of the eigenvalues of the Laplacian matrix or of a given positive definite matrix associated with the communication graph interconnecting the agents, often called *global information*. In Chapter 4, we aim at removing this dependence on global information. For multi-agent systems with single integrator agent dynamics, we establish a *decentralized* computation method for computing suboptimal local control gains. Chapters 5 - 7 deal with the distributed $\mathcal{H}_2$ suboptimal control problem. In Chapter 5, we study this problem for *homogeneous* multi-agent systems by *static relative state feedback*, and the results are then generalized in Chapter 6 to the case of *dynamic relative output feedback*. In Chapter 7, we further generalize the results in Chapters 5 and 6, and investigate the distributed $\mathcal{H}_2$ suboptimal control problem for *heterogeneous* multi-agent systems. In Chapter 8, we study $\mathcal{H}_2$ and $\mathcal{H}_\infty$ suboptimal distributed filtering problems for linear systems. In Chapter 9, we formulate the conclusions of this thesis, and discuss directions for possible future research.

1.3 Publications during the PhD project

Journal papers

- J. Jiao, H. L. Trentelman and M. K. Camlibel, "A suboptimality approach to distributed linear quadratic optimal control", *IEEE Transactions on Automatic Control*, Volume: 65, Issue: 3, 2020. (Chapter 2)
- J. Jiao, H. L. Trentelman and M. K. Camlibel, "Distributed linear quadratic optimal control: compute locally and act globally", *IEEE Control Systems Letters*, Volume: 4, Issue: 1, 2020. (Chapter 4)
- J. Jiao, H. L. Trentelman and M. K. Camlibel, "A suboptimality approach to distributed $\mathcal{H}_2$ control by dynamic output feedback", Automatica, Volume 121, 109164, 2020. (Chapter 6)
- J. Jiao, H. L. Trentelman and M. K. Camlibel, "$\mathcal{H}_2$ suboptimal output synchronization of heterogeneous multi-agent systems", submitted for publication in *Systems and Control Letters*, 2020. (Chapter 7)
- J. Jiao, H. L. Trentelman and M. K. Camlibel, "$\mathcal{H}_2$ and $\mathcal{H}_\infty$ suboptimal distributed filter design for linear systems", submitted for publication in *IEEE Transactions on Automatic Control*, 2020. (Chapter 8)

Conference papers

- J. Jiao, H. L. Trentelman and M. K. Camlibel, "Distributed linear quadratic tracking control for leader-follower multi-agent systems: a suboptimality approach", *21st IFAC World Congress*, 2020. (Chapter 3)

- J. Jiao, H. L. Trentelman and M. K. Camlibel, "A suboptimality approach to distributed $\mathcal{H}_2$ optimal control", *7th IFAC Workshop on Distributed Estimation and Control in Networked Systems (NecSys18)*, 2018. (Chapter 5)

1.4 Notation

In this section, we will introduce some basic notation that will be used throughout this thesis.

We denote by $\mathbb{R}$ the field of real numbers and by $\mathbb{R}^n$ the space of n dimensional vectors over $\mathbb{R}$. For $x \in \mathbb{R}^n$, its Euclidean norm is defined by $\|x\| := \sqrt{x^\top x}$. For a given $r > 0$, we denote by $B(r) := \{x \in \mathbb{R}^n \mid \|x\| \leqslant r\}$ the closed ball of radius r. We write $\mathbf{1}_N$ for the n dimensional column vector with all its entries equal to 1. We denote by $\mathbb{R}^{n\times m}$ the space of real $n \times m$ matrices. For a given matrix A, we write $A^\top$ to denote its transpose and A^{-1} its inverse (if exists). For a symmetric matrix P, we denote $P > 0$ ($P \geqslant 0$) if it is positive (semi-)definite and $P < 0$ if its negative definite. We denote the identity matrix of dimension $n \times n$ by I_n. A matrix is called Hurwitz if all its eigenvalues have negative real parts. The trace of a square matrix A is denoted by $\mathrm{tr}(A)$. We denote by $\mathrm{diag}(d_1, d_2, \ldots, d_n)$ the $n \times n$ diagonal matrix with $d_1, d_2, \ldots, d_n$ on the diagonal. Given matrices $R_i \in \mathbb{R}^{m\times m}$, $i = 1, 2, \ldots, n$, we denote by $\mathrm{blockdiag}(R_i)$ the $nm \times nm$ block diagonal matrix with $R_1, R_2, \ldots, R_n$ on the diagonal and we denote by $\mathrm{col}(R_i)$ the $nm \times m$ column block matrix $\left(R_1^\top, R_2^\top, \ldots, R_n^\top\right)^\top$. The Kronecker product of two matrices A and B is denoted by $A \otimes B$. For a linear map $A : \mathcal{X} \to \mathcal{Y}$, the kernel and image of A are denoted by $\ker(A) := \{x \in \mathcal{X} \mid Ax = 0\}$ and $\mathrm{im}(A) := \{Ax \mid x \in \mathcal{X}\}$, respectively.

1.5 Graph theory

We will now review some basic concepts and elementary results on graph theory that will be used.

A directed weighted graph is denoted by $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ with node set $\mathcal{V} = \{1, 2, \ldots, N\}$ and edge set $\mathcal{E} = \{e_1, e_2, \ldots, e_M\}$ satisfying $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$, and where $\mathcal{A} = [a_{ij}]$ is the adjacency matrix with nonnegative elements a_{ij}, called the edge weights. If $(i, j) \in \mathcal{E}$ we have $a_{ji} > 0$. If $(i, j) \notin \mathcal{E}$ we have $a_{ji} = 0$. Given a graph $\mathcal{G}$, a directed path from node 1 to node p is a sequence of edges $(k, k + 1)$,

$k = 1, 2, \ldots, p-1$. A directed weighted graph is called *strongly connected* if for any pair of distinct nodes i and j, there exists a directed path from i to j. A graph is called undirected if $(i,j) \in \mathcal{E}$ implies $(j,i) \in \mathcal{E}$. It is called *simple* if $a_{ii} = 0$ for all i. A simple undirected graph is called *connected* if for each pair of nodes i and j there exists a directed path from i to j.

Given a graph $\mathcal{G}$, the degree matrix of $\mathcal{G}$ is the diagonal matrix, given by $\mathcal{D} = \text{diag}(\mathsf{d}_1, \mathsf{d}_2, \ldots, \mathsf{d}_N)$ with $\mathsf{d}_i = \sum_{j=1}^{N} a_{ij}$. The Laplacian matrix is defined as $L := \mathcal{D} - \mathcal{A}$. If $\mathcal{G}$ is a directed weighted graph, the associated Laplacian matrix L has a zero eigenvalue corresponding to the eigenvector $\mathbf{1}_N$. If moreover $\mathcal{G}$ is strongly connected, then the eigenvalue 0 has multiplicity one, and all the other eigenvalues lie in the open right half-plane. The Laplacian matrix of an undirected graph is symmetric and has only real nonnegative eigenvalues. A simple undirected weighted graph is connected if and only if its Laplacian matrix L has eigenvalue 0 with multiplicity one. In that case, there exists an orthogonal matrix U such that

$$U^\top L U = \Lambda = \text{diag}(0, \lambda_2, \ldots, \lambda_N) \tag{1.1}$$

with $0 < \lambda_2 \leqslant \cdots \leqslant \lambda_N$. Moreover, we can take $U = \begin{pmatrix} \frac{1}{\sqrt{N}}\mathbf{1}_N & U_2 \end{pmatrix}$ with $U_2 U_2^\top = I_N - \frac{1}{N}\mathbf{1}_N\mathbf{1}_N^\top$.

A simple undirected weighted graph obviously has an even number of edges M. Define $K := \frac{1}{2}M$. For such graph, an associated incidence matrix $R \in \mathbb{R}^{N \times K}$ is defined as a matrix $R = (r_1, r_2, \ldots, r_K)$ with columns $r_k \in \mathbb{R}^N$. Each column r_k corresponds to exactly one pair of edges $e_k = \{(i,j), (j,i)\}$, and the ith and jth entry of r_k are equal to 1 or -1, while they do not take the same value. The remaining entries of r_k are equal to 0. We also define the matrix

$$W = \text{diag}(\mathsf{w}_1, \mathsf{w}_2, \ldots, \mathsf{w}_K) \tag{1.2}$$

as the $K \times K$ diagonal matrix, where w_k is the weight on each of the edges in e_k for $k = 1, 2, \ldots, K$. The relation between the Laplacian matrix and the incidence matrix is captured by

$$L = RWR^\top.$$

For connected simple undirected graphs, we review the following result [29]:

Lemma 1.1. *Let $\mathcal{G}$ be a connected simple undirected graph with Laplacian matrix L. Let $g_1, g_2, \ldots, g_N$ be non-negative real numbers with at least one $g_i > 0$. Define $G = \text{diag}(g_1, g_2, \ldots, g_N)$. Then the matrix $L + G$ is positive definite.*

For strongly connected weighted directed graphs, we review the following result [8, 62]:

Lemma 1.2. *Let $\mathcal{G}$ be a strongly connected weighted directed graph with Laplacian matrix L. Then there exists a unique row vector $\theta = (\theta_1, \theta_2, \ldots, \theta_N)$, where $\theta_1, \theta_2, \ldots, \theta_N$ are all* *positive real numbers, such that* $\theta L = 0$ *and* $\theta \mathbf{1}_N = N$. *Define* $\Theta := \mathrm{diag}(\theta_1, \theta_2, \ldots, \theta_N)$, *then the matrix* $\mathcal{L} := \Theta L + L^\top \Theta$ *is a positive semi-definite matrix.*

Note that ΘL is the Laplacian matrix of the balanced directed graph obtained by adjusting the weights in the original graph. The matrix $\mathcal{L}$ is the Laplacian matrix of the undirected graph obtained by taking the union of the edges and their reversed edges in this balanced directed graph.

2 A suboptimality approach to distributed linear quadratic optimal control

This chapter is concerned with a suboptimality version of the distributed linear quadratic optimal control problem for multi-agent systems. Given a multi-agent system with identical agent dynamics and an associated global quadratic cost functional, our objective is to design distributed control laws that achieve synchronization and whose cost is smaller than an a priori given upper bound, for all initial states of the network that are bounded in norm by a given radius. A centralized design method is provided to compute such suboptimal controllers, involving the solution of a single Riccati inequality of dimension equal to the dimension of the agent dynamics, and the smallest nonzero and the largest eigenvalue of the Laplacian matrix. Furthermore, we relax the requirement of exact knowledge of the smallest nonzero and largest eigenvalue of the Laplacian matrix by using only lower and upper bounds on these eigenvalues. Finally, a simulation example is provided to illustrate our design method.

2.1 Introduction

In this chapter, we study the distributed linear quadratic optimal control problem for multi-agent networks. This problem deals with a number of identical agents represented by a finite dimensional linear input-state system, and an undirected graph representing the communication between these agents. Given is also a quadratic cost functional that penalizes the differences between the states of neighboring agents and the size of the local control inputs. The distributed linear quadratic control problem is the problem of finding a distributed diffusive control law that minimizes this cost functional, while achieving synchronization for the controlled network. This problem is non-convex and difficult to solve, and a closed form solution has not been provided in the literature up to now. It is also unknown under what conditions a distributed diffusive optimal control law exists in general [74]. Therefore, instead of addressing the problem formulated above, in the present chapter we will study a *suboptimality* version of this optimal control problem. In other words, our aim will be to design distributed diffusive suboptimal control laws that guarantee the controlled network to reach synchronization.

The distributed linear quadratic control problem has attracted extensive attention in the last decade, and has been studied from many different angles. For example, in [70, 109, 119] it was shown that if the quadratic cost functional involves the differences of states of neighboring agents, then, necessarily, the optimal control laws must be distributed and diffusive. However, these references do not address the problem of designing the optimal control laws. In [7], a design method was introduced for computing distributed suboptimal *stabilizing* controllers for decoupled linear systems. In this reference, the authors consider a global linear quadratic cost functional which contains terms that penalize the states and inputs of each agent and also the relative states between each agent and its neighboring agents. In [104, 122], methods were established for designing distributed *synchronizing* control laws for linear multi-agent systems, where the control laws are derived from the solution of an algebraic Riccati equation of dimension equal to the state space dimension of the agents. However, in these references, cost functionals were not taken explicitly into consideration.

The distributed linear quadratic optimal control problem was also addressed in [10] for multi-agent systems with *single integrator* agent dynamics. The authors obtained an expression for the optimal control law, with the optimal feedback gain given in terms of the initial conditions of *all* agents. In addition, in [98] a distributed optimal control problem was considered from the perspective of cooperative game theory. In that paper, the problem being studied was solved by transforming it into a maximization problem for linear matrix inequalities, taking into consideration the structure of the Laplacian matrix. For related work we also mention [15, 19, 73, 117], to name a few.

Also, in [76], a hierarchical control approach was introduced for linear leader-follower multi-agent systems. For the case that the weighting matrices in the cost functional are chosen to be of a special form, two suboptimal controller design methods were given in this reference. In addition, in [75], an inverse optimal control problem was addressed both for leader-follower and leaderless multi-agent systems. For a particular class of digraphs, the authors showed that distributed optimal controllers exist, and can be obtained if the weighting matrices are assumed to be of a special form, capturing the graph information. For other work related to distributed inverse optimal control, we refer to [77, 123].

In this chapter, our objective is to design distributed diffusive control laws that guarantee the controlled network to reach synchronization while the associated cost is smaller than an a priori given upper bound. The main contributions of this chapter are the following:

1. We present a design method for computing distributed diffusive suboptimal control laws, based on computing a positive definite solution of a single Riccati inequality of dimension equal to the dimension of the agent dynamics. In the computation of the local control gain, the smallest nonzero eigenvalue and the largest eigenvalue of the Laplacian matrix are involved.

2. For the case that exact information on the smallest nonzero eigenvalue and the largest eigenvalue of the Laplacian matrix is not available, we establish a design method using only lower and upper bounds on these Laplacian eigenvalues.

The remainder of this chapter is organized as follows. In Section 2.2, we formulate the distributed linear quadratic suboptimal control problem. Section 2.3 presents the analysis and design of linear quadratic suboptimal control for linear systems, collecting preliminary classical results for treating the actual distributed suboptimal control problem for multi-agent systems. Then, in Section 2.4, we study the distributed suboptimal control problem for linear multi-agent systems. To illustrate our results, a simulation example is provided in Section 2.5. Finally, in Section 2.6 we formulate some conclusions.

2.2 Problem formulation

In this chapter, we consider a multi-agent system consisting of N identical agents. It will be a standing assumption that the underlying graph is simple, undirected and connected. The corresponding Laplacian matrix is denoted by L. The dynamics of the identical agents is represented by the continuous-time linear time-invariant (LTI) system given by

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t), \quad x_i(0) = x_{i0}, \quad i = 1, 2, \dots, N, \tag{2.1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and $x_i \in \mathbb{R}^n, u_i \in \mathbb{R}^m$ are the state and input of the agent i, respectively, and x_{i0} is its initial state. Throughout this chapter, we assume that the pair (A, B) is stabilizable.

We consider the infinite horizon distributed linear quadratic optimal control problem for multi-agent system (2.1), where the global cost functional integrates the weighted quadratic difference of states between every agent and its neighbors, and also penalizes the inputs in a quadratic form. Thus, the cost functional considered in this chapter is given by

$$J(u_1, u_2, \dots, u_N) = \int_0^\infty \frac{1}{2}\sum_{i=1}^N \sum_{j\in\mathcal{N}_i} (x_i - x_j)^\top Q(x_i - x_j) + \sum_{i=1}^N u_i^\top R u_i \, dt, \tag{2.2}$$

where $Q \geqslant 0$ and $R > 0$ are given real weighting matrices.

We can rewrite multi-agent system (2.1) in compact form as

$$\dot{\mathbf{x}} = (I_N \otimes A)\mathbf{x} + (I_N \otimes B)\mathbf{u}, \quad \mathbf{x}(0) = \mathbf{x}_0 \tag{2.3}$$

with $\mathbf{x} = \left(x_1^\top, x_2^\top, \ldots, x_N^\top\right)^\top$, $\mathbf{u} = \left(u_1^\top, u_2^\top, \ldots, u_N^\top\right)^\top$, where $\mathbf{x} \in \mathbb{R}^{nN}$, $\mathbf{u} \in \mathbb{R}^{mN}$ contain the states and inputs of all agents, respectively. Note that, although the agents have identical dynamics, we allow the initial states of the individual agents to differ. These initial states are collected in the joint vector of initial states $\mathbf{x}_0 = \left(x_{10}^\top, x_{20}^\top, \ldots, x_{N0}^\top\right)^\top$. Moreover, we can also write the cost functional (2.2) in compact form as

$$J(\mathbf{u}) = \int_0^\infty \mathbf{x}^\top (L \otimes Q)\mathbf{x} + \mathbf{u}^\top (I_N \otimes R)\mathbf{u}\, dt. \tag{2.4}$$

The distributed linear quadratic optimal control problem is the problem of minimizing for all initial states $\mathbf{x}_0$ the cost functional (2.4) over all distributed diffusive control laws that achieve synchronization. By a distributed diffusive control law we mean a control law of the form

$$\mathbf{u} = (L \otimes K)\mathbf{x}, \tag{2.5}$$

where $K \in \mathbb{R}^{m \times n}$ is an identical feedback gain for all agents. The adjective diffusive refers to the fact that the input of each agent depends on the relative state variables with respect to its neighbors. The control law (2.5) is distributed in the sense that the local gains for all agents are identical.

By interconnecting the agents using this control law, we obtain the overall network dynamics

$$\dot{\mathbf{x}} = (I_N \otimes A + L \otimes BK)\mathbf{x}. \tag{2.6}$$

Foremost, we want the control law to achieve synchronization:

Definition 2.1. *We say the network reaches synchronization using control law (2.5) if for all $i, j = 1, 2, \ldots, N$ and for all initial states x_{i0} and x_{j0}, we have*

$$x_i(t) - x_j(t) \to 0 \text{ as } t \to \infty.$$

As a function of the to-be-designed local feedback gain K, the cost functional (2.4) can be rewritten as

$$J(K) = \int_0^\infty \mathbf{x}^\top \left(L \otimes Q + L^2 \otimes K^\top RK\right) \mathbf{x}\, dt. \tag{2.7}$$

In other words, the distributed linear quadratic optimal control problem is the problem of minimizing the cost functional (2.7) over all $K \in \mathbb{R}^{m\times n}$ such that the controlled network (2.6) reaches synchronization.

Due to the distributed nature of the control law (2.5) as imposed by the network topology, the distributed linear quadratic optimal control problem is a *non-convex* optimization problem. It is therefore difficult, if not impossible, to find a *closed form solution* for an optimal controller, or such optimal controller may not even exist. Therefore, as announced in the introduction, in this chapter we will study and resolve a version of this problem involving the design of distributed suboptimal control laws.

More specifically, let $B(r) = \{\mathbf{x} \in \mathbb{R}^{nN} \mid \|\mathbf{x}\| \leqslant r\}$ be the closed ball of radius r in the joint state space $\mathbb{R}^{nN}$ of the network (2.3). Then, for system (2.3) with initial states in such a closed ball of a given radius, we want to design a distributed diffusive controller such that synchronization is achieved and, for all initial states in the given ball, the associated cost is smaller than an a priori given upper bound. Thus, we will consider the following problem:

Problem 2.1. *Consider the multi-agent system* (2.3) *and associated cost functional given by (2.7). Let $r > 0$ be a given radius and let $\gamma > 0$ be an a priori given upper bound for the cost. The problem is to find a distributed diffusive controller of the form (2.5) such that the controlled network (2.6) reaches synchronization, and for all $\mathbf{x}_0 \in B(r)$ the associated cost (2.7) is smaller than the given upper bound, i.e., $J(K) < \gamma$.*

Remark 2.1. Note that we could also have formulated the alternative problem of finding a suboptimal controller for a *single, given, initial state* $\mathbf{x}_0$. In fact, this would be closer to the classical linear quadratic optimal control problem, which is usually formulated as the problem of minimizing the cost functional for a *given* initial state $\mathbf{x}_0$. In that context, however, the optimal controller is a state feedback that turns out to be optimal *for all initial states*. In order to capture in our problem formulation this property of being optimal for all initial states, we have formulated Problem 2.1 in terms of initial states contained in a ball of a given radius.

Before we address Problem 2.1, we will first briefly discuss the linear quadratic suboptimal control problem for a single linear system. This will be the subject of the next section.

2.3 Linear quadratic suboptimal control for linear systems

In this section, we consider a linear quadratic suboptimal control problem for single linear systems. The results presented in this section are standard and can be found scattered over the literature, see e.g. [33, 100, 102]. Exact references are however hard to give and therefore, in order to make this chapter self-contained, we will collect the required results here and provide their proofs.

We will first analyze the quadratic performance of a given autonomous system. Subsequently, we will discuss how to design suboptimal control laws for a linear system with inputs.

2.3.1 Quadratic performance analysis for autonomous linear systems

Consider the autonomous linear system

$$\dot{x}(t) = \bar{A}x(t), \quad x(0) = x_0, \tag{2.8}$$

where $\bar{A} \in \mathbb{R}^{n\times n}$ and $x \in \mathbb{R}^n$ is the state. We consider the quadratic performance of system (2.8), given by

$$J = \int_0^\infty x^\top \bar{Q}x \, dt, \tag{2.9}$$

where $\bar{Q} \geqslant 0$ is a given real weighting matrix. Note that the performance J is finite if system (2.8) is stable, i.e., $\bar{A}$ is Hurwitz.

We are interested in finding conditions such that the performance (2.9) of system (2.8) is smaller than a given upper bound. For this, we have the following lemma:

Lemma 2.2. *Consider system (2.8) with the corresponding quadratic performance (2.9). The performance* (2.9) *is finite if system (2.8) is stable, i.e., $\bar{A}$ is Hurwitz. In this case, it is given by*

$$J = x_0^\top Y x_0, \tag{2.10}$$

where Y is the unique positive semi-definite solution of

$$\bar{A}^\top Y + Y\bar{A} + \bar{Q} = 0. \tag{2.11}$$

Alternatively,

$$J = \inf\{x_0^\top P x_0 \mid P > 0 \text{ and } \bar{A}^\top P + P\bar{A} + \bar{Q} < 0\}. \tag{2.12}$$

Proof. The fact that the quadratic performance (2.9) is given by the quadratic expression (2.10) involving the Lyapunov equation (2.11) is well-known.

We will now prove (2.12). Let Y be the solution to Lyapunov equation (2.11) and let P be a positive definite solution to the Lyapunov inequality in (2.12). Define $X := P - Y$. Then we have

$$\bar{A}^\top (X + Y) + (X + Y)\bar{A} + \bar{Q} < 0.$$

So consequently,

$$\bar{A}^\top X + X\bar{A} < 0.$$

Since $\bar{A}$ is Hurwitz, it follows that $X > 0$. Thus, we have $P > Y$ and hence $J \leqslant x_0^\top P x_0$ for any positive definite solution P to the Lyapunov inequality.

Next we will show that for any $\epsilon > 0$ there exists a positive definite matrix P_ϵ satisfying the Lyapunov inequality such that $P_\epsilon < Y + \epsilon I$, and consequently $x_0^\top P_\epsilon x_0 \leqslant J + \epsilon \|x_0\|^2$. Indeed, for given ϵ, take P_ϵ equal to the unique positive definite solution of

$$\bar{A}^\top P + P\bar{A} + \bar{Q} + \epsilon I = 0.$$

Clearly then, $P_\epsilon = \int_0^\infty e^{\bar{A}^\top t}(\bar{Q} + \epsilon I)e^{\bar{A}t}\, dt$, so $P_\epsilon \downarrow Y$ as $\epsilon \downarrow 0$. This proves our claim. □

The following theorem now yields *necessary* and *sufficient* conditions such that, for a given upper bound $\gamma > 0$, the quadratic performance (2.9) satisfies $J < \gamma$.

Theorem 2.3. *Consider system (2.8) with the associated quadratic performance (2.9). For given $\gamma > 0$, we have that $\bar{A}$ is Hurwitz and $J < \gamma$ if and only if there exists a positive definite matrix P satisfying*

$$\bar{A}^\top P + P\bar{A} + \bar{Q} < 0, \tag{2.13}$$

$$x_0^\top P x_0 < \gamma. \tag{2.14}$$

Proof. (if) Since there exists a positive definite solution to the Lyapunov inequality (2.13), it follows that $\bar{A}$ is Hurwitz. Take a positive definite matrix P satisfying the inequalities (2.13) and (2.14). By Lemma 2.2, we then immediately have $J \leqslant x_0^\top P x_0 < \gamma$.

(only if) If $\bar{A}$ is Hurwitz and $J < \gamma$, then, again by Lemma 2.2, there exists a positive definite solution P to the Lyapunov inequality (2.13) such that $J \leqslant x_0^\top P x_0 < \gamma$. □

In the next subsection, we will discuss the suboptimal control problem for a linear system with inputs.

2.3.2 Linear quadratic suboptimal control for linear systems

In this section, we consider the linear time-invariant system given by

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \tag{2.15}$$

where $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}$, and $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ are the state and the input, respectively, and x_0 is a given initial state. Assume that the pair (A, B) is stabilizable.

The associated cost functional is given by

$$J(u) = \int_0^\infty x^\top Qx + u^\top Ru \, dt, \tag{2.16}$$

where $Q \geqslant 0$ and $R > 0$ are given weighting matrices that penalize the state and input, respectively.

Given $\gamma > 0$ and initial state x_0, we want to find a state feedback control law $u = Kx$ such that the closed system

$$\dot{x}(t) = (A + BK)x(t) \tag{2.17}$$

is stable and the corresponding cost

$$J(K) = \int_0^\infty x^\top (Q + K^\top RK)x \, dt \tag{2.18}$$

satisfies $J(K) < \gamma$.

The following theorem gives a sufficient condition for the existence of such control law.

Theorem 2.4. *Consider the system (2.15) with initial state x_0 and associated cost functional (2.16). Let $\gamma > 0$. Suppose that there exists a positive definite P satisfying*

$$A^\top P + PA - PBR^{-1}B^\top P + Q < 0, \tag{2.19}$$

$$x_0^\top Px_0 < \gamma. \tag{2.20}$$

Let $K := -R^{-1}B^\top P$. Then the controlled system (2.17) is stable and the control law $u = Kx$ is suboptimal, i.e., $J(K) < \gamma$.

Proof. Substituting $K := -R^{-1}B^\top P$ into (2.17) yields

$$\dot{x}(t) = (A - BR^{-1}B^\top P)x(t), \quad x(0) = x_0. \tag{2.21}$$

Since P satisfies (2.19), it should also satisfy

$$(A - BR^{-1}B^\top P)^\top P + P(A - BR^{-1}B^\top P) + Q + PBR^{-1}B^\top P < 0,$$

which implies that $A - BR^{-1}B^\top P$ is Hurwitz, i.e., the closed system (2.21) is stable. Consequently, the corresponding cost is finite and equal to

$$J(K) = \int_0^\infty x^\top (Q + PBR^{-1}B^\top P)x \, dt.$$

Since (2.20) holds, by taking $\bar{A} = A - BR^{-1}B^\top P$ and $\bar{Q} = Q + PBR^{-1}B^\top P$ in

Theorem 2.3, we immediately have $J(K) < \gamma$. $\square$

In the next section we will apply the above results to tackle the distributed linear quadratic suboptimal control problem as formulated in Problem 2.1.

2.4 Distributed linear quadratic suboptimal control for multi-agent systems

Again consider the multi-agent system with the dynamics of the identical agents represented by

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t), \quad x_i(0) = x_{i0}, \quad i = 1, 2, \dots, N, \tag{2.22}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and $x_i \in \mathbb{R}^n, u_i \in \mathbb{R}^m$ are the state and input of the i-th agent, respectively, and x_{i0} its initial state. We assume that the pair (A, B) is stabilizable.

Denoting $\mathbf{x} = \left(x_1^\top, x_2^\top, \dots, x_N^\top\right)^\top$, $\mathbf{u} = \left(u_1^\top, u_2^\top, \dots, u_N^\top\right)^\top$, we can rewrite the multi-agent system in compact form as

$$\dot{\mathbf{x}} = (I_N \otimes A)\mathbf{x} + (I_N \otimes B)\mathbf{u}, \quad \mathbf{x}(0) = \mathbf{x}_0. \tag{2.23}$$

The cost functional we consider was already introduced in (2.4). We repeat it here for convenience:

$$J(\mathbf{u}) = \int_0^\infty \mathbf{x}^\top (L \otimes Q)\mathbf{x} + \mathbf{u}^\top (I_N \otimes R)\mathbf{u}\, dt, \tag{2.24}$$

where $Q \geqslant 0$ and $R > 0$ are given real weighting matrices.

As formulated in Problem 2.1, given a desired upper bound $\gamma > 0$, for multi-agent system (2.23) with initial states contained in the closed ball $B(r)$ of given radius r we want to design a control law of the form

$$\mathbf{u} = (L \otimes K)\mathbf{x} \tag{2.25}$$

such that the controlled network

$$\dot{\mathbf{x}} = (I_N \otimes A + L \otimes BK)\mathbf{x} \tag{2.26}$$

reaches synchronization and, moreover, for all $\mathbf{x}_0 \in B(r)$ the associated cost

$$J(K) = \int_0^\infty \mathbf{x}^\top \left(L \otimes Q + L^2 \otimes K^\top RK\right) \mathbf{x}\, dt \tag{2.27}$$

is smaller than the given upper bound, i.e., $J(K) < \gamma$.

Let the matrix $U \in \mathbb{R}^{N \times N}$ be an orthogonal matrix that diagonalizes the Laplacian L. Define $\Lambda := U^\top L U = \mathrm{diag}(0, \lambda_2, \ldots, \lambda_N)$. To simplify the problem given above, by applying the state and input transformations $\bar{\mathbf{x}} = (U^\top \otimes I_n)\mathbf{x}$ and $\bar{\mathbf{u}} = (U^\top \otimes I_m)\mathbf{u}$ with $\bar{\mathbf{x}} = \left(\bar{x}_1^\top, \bar{x}_2^\top, \ldots, \bar{x}_N^\top\right)^\top$, $\bar{\mathbf{u}} = \left(\bar{u}_1^\top, \bar{u}_2^\top, \ldots, \bar{u}_N^\top\right)^\top$, system (2.23) becomes

$$\dot{\bar{\mathbf{x}}} = (I_N \otimes A)\bar{\mathbf{x}} + (I_N \otimes B)\bar{\mathbf{u}}, \quad \bar{\mathbf{x}}(0) = \bar{\mathbf{x}}_0 \tag{2.28}$$

with $\bar{\mathbf{x}}_0 = (U^\top \otimes I_n)\mathbf{x}_0$. Clearly, (2.25) is transformed to

$$\bar{\mathbf{u}} = (\Lambda \otimes K)\bar{\mathbf{x}}, \tag{2.29}$$

and the controlled network (2.26) transforms to

$$\dot{\bar{\mathbf{x}}} = (I_N \otimes A + \Lambda \otimes BK)\,\bar{\mathbf{x}}. \tag{2.30}$$

In terms of the transformed variables, the cost (2.27) is given by

$$J(K) = \int_0^\infty \sum_{i=1}^N \bar{x}_i^\top (\lambda_i Q + \lambda_i^2 K^\top R K)\bar{x}_i \, dt. \tag{2.31}$$

Note that the transformed states $\bar{x}_i$ and inputs $\bar{u}_i$, $i = 2, 3, \ldots, N$ appearing in system (2.30) and cost (2.31) are decoupled from each other, so that we can write system (2.30) and cost (2.31) as

$$\dot{\bar{x}}_1 = A\bar{x}_1, \tag{2.32}$$

$$\dot{\bar{x}}_i = (A + \lambda_i BK)\bar{x}_i, \quad i = 2, 3, \ldots, N, \tag{2.33}$$

and

$$J(K) = \sum_{i=2}^N J_i(K) \tag{2.34}$$

with

$$J_i(K) = \int_0^\infty \bar{x}_i^\top (\lambda_i Q + \lambda_i^2 K^\top R K)\bar{x}_i \, dt, \quad i = 2, 3, \ldots, N. \tag{2.35}$$

Note that $\lambda_1 = 0$, and that therefore (2.32) does not contribute to the cost $J(K)$.

We first record a well-known fact (see [54, 103]) that we will use later:

Lemma 2.5. *Consider the multi-agent system (2.23). Then the controlled network reaches synchronization with control law (2.25) if and only if, for $i = 2, 3, \ldots, N$, the systems (2.33) are stable.*

Thus we have transformed the problem of distributed suboptimal control for system (2.23) into the problem of finding a feedback gain $K \in \mathbb{R}^{m \times n}$ such that the systems (2.33) are stable and $J(K) < \gamma$. Moreover, since the pair (A, B) is stabilizable, there exists such a feedback gain K [103].

The following lemma gives a necessary and sufficient condition for a given feedback gain K to make all systems (2.33) stable and such that $J(K) < \gamma$ is satisfied for given initial states.

Lemma 2.6. *Let K be a feedback gain. Consider the systems (2.33) with given initial states $\bar{x}_{20}, \bar{x}_{30}, \ldots, \bar{x}_{N0}$ and associated cost functionals (2.34) and (2.35). Let $\gamma > 0$. Then all systems (2.33) are stable and $J(K) < \gamma$ if and only if there exist positive definite matrices P_i satisfying*

$$(A + \lambda_i BK)^\top P_i + P_i(A + \lambda_i BK) + \lambda_i Q + \lambda_i^2 K^\top RK < 0, \tag{2.36}$$

$$\sum_{i=2}^{N} \bar{x}_{i0}^\top P_i \bar{x}_{i0} < \gamma, \tag{2.37}$$

for $i = 2, 3, \ldots, N$, respectively.

Proof. (if) Since (2.37) holds, there exist sufficiently small $\epsilon_i > 0$, $i = 2, 3, \ldots, N$ such that $\sum_{i=2}^{N} \gamma_i < \gamma$ where $\gamma_i := \bar{x}_{i0}^\top P_i \bar{x}_{i0} + \epsilon_i$. Because there exists P_i such that (2.36) and $\bar{x}_{i0}^\top P_i \bar{x}_{i0} < \gamma_i$ holds for all $i = 2, 3, \ldots, N$, by taking $\bar{A} = A + \lambda_i BK$ and $\bar{Q} = \lambda_i Q + \lambda_i^2 K^\top RK$, it follows from Theorem 2.3 that all systems (2.33) are stable and $J_i(K) < \gamma_i$ for $i = 2, 3, \ldots, N$. Since $J(K) = \sum_{i=2}^{N} J_i(K)$, this implies that $J(K) < \sum_{i=2}^{N} \gamma_i < \gamma$.

(only if) Since $J(K) < \gamma$ and $J(K) = \sum_{i=2}^{N} J_i(K)$, there exist sufficiently small $\epsilon_i > 0$, $i = 2, 3, \ldots, N$ such that $\sum_{i=2}^{N} \gamma_i < \gamma$ where $\gamma_i := J_i(K) + \epsilon_i$. Because all systems (2.33) are stable and $J_i(K) < \gamma_i$ for $i = 2, 3, \ldots, N$, by taking $\bar{A} = A + \lambda_i BK$ and $\bar{Q} = \lambda_i Q + \lambda_i^2 K^\top RK$, it follows from Theorem 2.3 that there exist positive definite P_i such that (2.36) and $\bar{x}_{i0}^\top P_i \bar{x}_{i0} < \gamma_i$ hold for all $i = 2, 3, \ldots, N$. Since $\sum_{i=2}^{N} \gamma_i < \gamma$, this implies that $\sum_{i=2}^{N} \bar{x}_{i0}^\top P_i \bar{x}_{i0} < \sum_{i=2}^{N} \gamma_i < \gamma$. □

Lemma 2.6 establishes a necessary and sufficient condition for a given feedback gain K to stabilize all systems (2.33) and to satisfy, for given initial states of these systems, $J(K) < \gamma$. However, Lemma 2.6 does not yet provide a method to compute such K. In the following we present a method to find such K.

Lemma 2.7. *Consider the multi-agent system (2.23) with associated cost functional (2.27). Let $\mathbf{x}_0$ be a given initial state for the multi-agent system. Let $\gamma > 0$. Let c be any real number such that $0 < c < \frac{2}{\lambda_N}$. We distinguish two cases:*

(i) if

$$\frac{2}{\lambda_2 + \lambda_N} \leqslant c < \frac{2}{\lambda_N}, \tag{2.38}$$

then there exists $P > 0$ satisfying the Riccati inequality

$$A^\top P + PA + (c^2\lambda_N^2 - 2c\lambda_N)PBR^{-1}B^\top P + \lambda_N Q < 0. \tag{2.39}$$

(ii) if

$$0 < c < \frac{2}{\lambda_2 + \lambda_N}, \tag{2.40}$$

then there exists $P > 0$ satisfying

$$A^\top P + PA + (c^2\lambda_2^2 - 2c\lambda_2)PBR^{-1}B^\top P + \lambda_N Q < 0. \tag{2.41}$$

In both cases, if in addition P satisfies

$$\mathbf{x}_0^\top \left((I_N - \frac{1}{N}\mathbf{1}_N\mathbf{1}_N^\top) \otimes P \right) \mathbf{x}_0 < \gamma, \tag{2.42}$$

then the controlled network (2.26) with $K := -cR^{-1}B^\top P$ reaches synchronization and with the initial state $\mathbf{x}_0$ we have $J(K) < \gamma$.

Proof. We will only give the proof for case (i) above. Using the upper and lower bounds on c given by (2.38), it can be verified that $c^2\lambda_i^2 - 2c\lambda_i \leqslant c^2\lambda_N^2 - 2c\lambda_N < 0$ for $i = 2, 3, \ldots, N$. It is then easily seen that (2.39) has many positive definite solutions. Since also $\lambda_i \leqslant \lambda_N$, any such solution P is a solution to the $N-1$ Riccati inequalities

$$A^\top P + PA + (c^2\lambda_i^2 - 2c\lambda_i)PBR^{-1}B^\top P + \lambda_i Q < 0, \quad i = 2, 3, \ldots, N. \tag{2.43}$$

Equivalently, P also satisfies the Lyapunov inequalities

$$\begin{aligned}(A - c\lambda_i BR^{-1}B^\top P)^\top P + P(A - c\lambda_i BR^{-1}B^\top P)& \\ +\lambda_i Q + c^2\lambda_i^2 PBR^{-1}B^\top P < 0, \quad i = 2, 3, \ldots, N.&\end{aligned} \tag{2.44}$$

Next, recall that $\bar{\mathbf{x}} = (U^\top \otimes I_n)\mathbf{x}$ with $U = \begin{pmatrix} \frac{1}{\sqrt{N}}\mathbf{1}_N & U_2 \end{pmatrix}$. From this it is easily seen that $(\bar{x}_{20}^\top, \bar{x}_{30}^\top, \cdots, \bar{x}_{N0}^\top)^\top = (U_2^\top \otimes I_n)\mathbf{x}_0$. Also, $U_2U_2^\top = I_N - \frac{1}{N}\mathbf{1}_N\mathbf{1}_N^\top$. Since (2.42) holds, we have

$$\begin{aligned} \mathbf{x}_0^\top \left(U_2U_2^\top \otimes P\right)\mathbf{x}_0 < \gamma \Leftrightarrow& \\ ((U_2^\top \otimes I_n)\mathbf{x}_0)^\top \left(I_{N-1} \otimes P\right)((U_2^\top \otimes I_n)\mathbf{x}_0) < \gamma \quad \Leftrightarrow& \\ (\bar{x}_{20}^\top, \bar{x}_{30}^\top, \cdots, \bar{x}_{N0}^\top)\left(I_{N-1} \otimes P\right)(\bar{x}_{20}^\top, \bar{x}_{30}^\top, \cdots, \bar{x}_{N0}^\top)^\top < \gamma,& \end{aligned}$$

which is equivalent to

$$\sum_{i=2}^{N} \bar{x}_{i0}^\top P \bar{x}_{i0} < \gamma. \tag{2.45}$$

Taking $P_i = P$ for $i = 2, 3, \ldots, N$ and $K := -cR^{-1}B^\top P$ in inequalities (2.36) and (2.37) immediately gives us inequalities (2.44) and (2.45). Then it follows from Lemma 2.6 that all systems (2.33) are stable and $J(K) < \gamma$. Furthermore, it follows from Lemma 2.5 that the controlled network (2.26) reaches synchronization. □

We will now apply Lemma 2.7 to establish a solution to Problem 2.1. Indeed, the next main theorem gives a condition under which, for given radius r and upper bound γ, distributed diffusive suboptimal control laws exist, and explains how these can be computed.

Theorem 2.8. *Consider the multi-agent system (2.23) with associated cost functional (2.27). Let $r > 0$ be a given radius and let $\gamma > 0$ be an a priori given upper bound for the cost. Let c be any real number such that $0 < c < \frac{2}{\lambda_N}$. We distinguish two cases:*

(i) if

$$\frac{2}{\lambda_2 + \lambda_N} \leqslant c < \frac{2}{\lambda_N}, \tag{2.46}$$

then there exists $P > 0$ satisfying the Riccati inequality

$$A^\top P + PA + (c^2\lambda_N^2 - 2c\lambda_N)PBR^{-1}B^\top P + \lambda_N Q < 0. \tag{2.47}$$

(ii) if

$$0 < c < \frac{2}{\lambda_2 + \lambda_N}, \tag{2.48}$$

then there exists $P > 0$ satisfying

$$A^\top P + PA + (c^2\lambda_2^2 - 2c\lambda_2)PBR^{-1}B^\top P + \lambda_N Q < 0. \tag{2.49}$$

In both cases, if in addition P satisfies

$$P < \frac{\gamma}{r^2} I, \tag{2.50}$$

then the controlled network (2.26) with $K := -cR^{-1}B^\top P$ reaches synchronization and $J(K) < \gamma$ for all $\mathbf{x}_0 \in B(r)$.

Proof. Again, we only give the proof for case (i) above. Let $P > 0$ satisfy (2.47) and (2.50) holds. Our aim is to prove that (2.42) is satisfied for all $\mathbf{x}_0 \in B(r)$. First note that

$$\frac{1}{N}\mathbf{1}_N\mathbf{1}_N^\top \otimes P = \frac{1}{N}(\mathbf{1}_N \otimes P^{\frac{1}{2}})(\mathbf{1}_N \otimes P^{\frac{1}{2}})^\top,$$

which is therefore positive semi-definite. Now, for all $\mathbf{x}_0 \in B(r)$ we have

$$
\begin{aligned}
&\mathbf{x}_0^\top \left((I_N - \frac{1}{N}\mathbf{1}_N\mathbf{1}_N^\top) \otimes P \right) \mathbf{x}_0 \\
\leqslant &\mathbf{x}_0^\top \left(I_N \otimes P \right) \mathbf{x}_0 < \frac{\gamma}{r^2}\mathbf{x}_0^\top \mathbf{x}_0 \leqslant \gamma.
\end{aligned}
$$

By Lemma 2.7 then, the controlled network (2.26) with the given K reaches synchronization and $J(K) < \gamma$ for all $\mathbf{x}_0 \in B(r)$. □

Remark 2.9. Theorem 2.8 states that after choosing c satisfying the inequality (2.46) for case (i) and finding a positive definite P satisfying (2.47) and (2.50), the distributed control law with local gain $K = -cR^{-1}B^\top P$ is γ-suboptimal for all initial states of the network in the closed ball with radius r. By (2.50), the smaller the solution P of (2.47), the smaller the quotient $\frac{\gamma}{r^2}$ is allowed to be, leading to a smaller upper bound and a larger radius. The question then arises: how should we choose the parameter c in (2.46) so that the Riccati inequality (2.47) allows a positive definite solution that is as small as possible. In fact, one can find a positive definite solution $P(c,\epsilon)$ to (2.47) by solving the Riccati equation

$$
A^\top P + PA - PB\bar{R}(c)^{-1}B^\top P + \bar{Q}(\epsilon) = 0 \tag{2.51}
$$

with $\bar{R}(c) = \frac{1}{-c^2\lambda_N^2 + 2c\lambda_N}R$ and $\bar{Q}(\epsilon) = \lambda_N Q + \epsilon I_n$ where c is chosen as in (2.46) and $\epsilon > 0$. If c_1 and c_2 as in (2.46) satisfy $c_1 \leqslant c_2$, then we have $\bar{R}(c_1) \leqslant \bar{R}(c_2)$, so, clearly, $P(c_1,\epsilon) \leqslant P(c_2,\epsilon)$. Similarly, if $0 < \epsilon_1 \leqslant \epsilon_2$, we immediately have $\bar{Q}(\epsilon_1) \leqslant \bar{Q}(\epsilon_2)$. Again, it follows that $P(c,\epsilon_1) \leqslant P(c,\epsilon_2)$. Therefore, if we choose $\epsilon > 0$ very close to 0 and $c = \frac{2}{\lambda_2+\lambda_N}$, we find the 'best' solution to the Riccati inequality (2.47) in the sense explained above.

Likewise, if c satisfies (2.48) corresponding to case (ii), it can be shown that if we choose $\epsilon > 0$ very close to 0 and $c > 0$ very close to $\frac{2}{\lambda_2+\lambda_N}$, we find the 'best' solution to the Riccati inequality (2.49) in the sense explained above.

In Theorem 2.8, in order to compute a suitable feedback gain K, one needs to know λ_2 and λ_N, the smallest nonzero eigenvalue (the *algebraic connectivity*) and the largest eigenvalue of the Laplacian matrix, exactly. This requires so-called global information on the network graph which might not always be available. There exist algorithms to estimate λ_2 in a distributed way, yielding lower and upper bounds, see e.g. [2]. Moreover, also an upper bound for λ_N can be obtained in terms of the maximal node degree of the graph, see [1]. Then the question arises: can we still find a suboptimal controller reaching synchronization, using as information only a *lower bound* for λ_2 and an *upper bound* for λ_N? The answer to this question is affirmative, as shown in the following theorem.

Theorem 2.10. *Let a lower bound for λ_2 be given by $l_2 > 0$ and an upper bound for λ_N be given by L_N. Let $r > 0$ be a given radius and let $\gamma > 0$ be an a priori given upper bound for the cost. Choose c such that*

$$\frac{2}{l_2 + L_N} \leqslant c < \frac{2}{L_N}. \tag{2.52}$$

Then there exists $P > 0$ such that

$$A^\top P + PA + (c^2 L_N^2 - 2cL_N) PBR^{-1}B^\top P + L_N Q < 0. \tag{2.53}$$

If, in addition, P satisfies

$$P < \frac{\gamma}{r^2} I, \tag{2.54}$$

then the controlled network with local gain $K = -cR^{-1}B^\top P$ reaches synchronization and $J(K) < \gamma$ for all initial states $\mathbf{x}_0 \in B(r)$.

Furthermore, if we choose c such that

$$0 < c < \frac{2}{l_2 + L_N}, \tag{2.55}$$

then there exists $P > 0$ such that

$$A^\top P + PA + (c^2 l_2^2 - 2cl_2) PBR^{-1}B^\top P + L_N Q < 0. \tag{2.56}$$

If, in addition, P satisfies (2.54), *then the controlled network with $K := -cR^{-1}B^\top P$ reaches synchronization and $J(K) < \gamma$ for all $\mathbf{x}_0 \in B(r)$.*

Proof. A proof can be given along the lines of the proof of Theorem 2.8. □

Remark 2.11. Note that also in Theorem 2.10 the question arises how to choose $c > 0$ such that the Riccati inequalities (2.53) and (2.56) admit a positive definite solution that is as small as possible. Following the same ideas as in Remark 2.9, if we choose $\epsilon > 0$ very close to 0 and $c > 0$ equal to $\frac{2}{l_2 + L_N}$ in (2.53) (respectively very close to $\frac{2}{l_2 + L_N}$ in (2.56)), we find the 'best' solution to the Riccati inequalities (2.53) and (2.56).

Moreover, one may also ask the question: can we compare, with the same choice for c, solutions to (2.53) with solutions to (2.47), and also solutions to (2.56) with solutions to (2.49)? The answer is affirmative. Choose c that satisfies both conditions (2.46) and (2.52). One can then check that the computed positive definite solution to (2.53) is indeed 'larger' than that to (2.47) as explained in Remark 2.9. A similar remark holds for the positive definite solutions to (2.56) and corresponding solutions to (2.49) if c satisfies both (2.48) and (2.55). We conclude that if, instead of using the exact values λ_2 and λ_N, we use a lower bound, respectively upper bound for these eigenvalues, then the computed distributed control law is suboptimal for 'less' initial states of the agents.

Remark 2.12. As a final remark, we note that the theory developed in this chapter carries over unchanged to the case of undirected *weighted* graphs. In that case the expression for cost functional (2.2) should be changed to

$$J(u_1, u_2, \ldots, u_N) = \int_0^\infty \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N a_{ij}(x_i - x_j)^\top Q(x_i - x_j) + \sum_{i=1}^N u_i^\top R u_i \, dt,$$

in which $\mathcal{A} = [a_{ij}]$ is the *weighted* adjacency matrix. Denoting the corresponding weighted Laplacian matrix by L, also this cost functional can be represented in compact form by (2.4), and the subsequent development will remain the same.

2.5 Simulation example

In this section, we will use a simulation example borrowed from [76] to illustrate the proposed design method for distributed suboptimal controllers. Consider a group of 8 linear oscillators with identical dynamics

$$\dot{x}_i = Ax_i + Bu_i, \quad x_i(0) = x_{i0}, \quad i = 1, 2, \ldots, 8 \tag{2.57}$$

with

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Assume the underlying graph is the undirected line graph with Laplacian matrix

$$L = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}.$$

We consider the cost functional

$$J(\mathbf{u}) = \int_0^\infty \mathbf{x}^\top (L \otimes Q)\mathbf{x} + \mathbf{u}^\top (I_8 \otimes R)\mathbf{u} \, dt \tag{2.58}$$

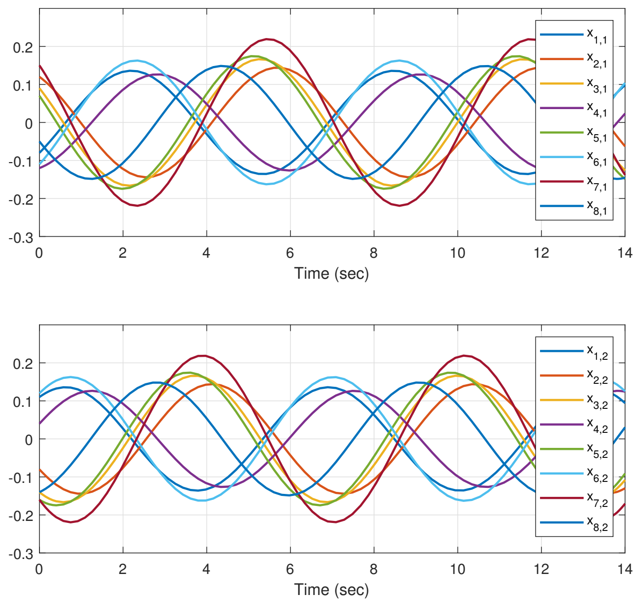

Figure 2.1: Plots of the state vector $\mathbf{x}^1 = (x_{1,1}, x_{2,1}, \ldots, x_{8,1})$ (upper plot) and $\mathbf{x}^2 = (x_{1,2}, x_{2,2}, \ldots, x_{8,2})$ (lower plot) of the 8 decoupled oscillators without control

where the matrices Q and R are chosen to be

$$Q = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \quad R = 1.$$

Let the desired upper bound for the cost functional (2.58) be given as $\gamma = 3$. Our goal is to design a control law $\mathbf{u} = (L \otimes K)\mathbf{x}$ such that the controlled network reaches synchronization and the associated cost is less than γ for all initial states $\mathbf{x}_0$ in a closed ball $B(r)$ with radius r. The radius r will be specified later on in this example.

In this example, we adopt the control design method given in case (i) of Theorem 2.8. The smallest nonzero and largest eigenvalue of L are $\lambda_2 = 0.0979$ and $\lambda_8 = 3.8478$. First, we compute a positive definite solution P to (2.47) by solving the Riccati equation

$$A^\top P + PA + (c^2\lambda_8^2 - 2c\lambda_8)PBR^{-1}B^\top P + \lambda_8 Q + \epsilon I_2 = 0$$

with $\epsilon > 0$ chosen small as mentioned in Remark 2.9. Here we choose $\epsilon = 0.001$. Moreover, we choose $c = \frac{2}{\lambda_2 + \lambda_8} = 0.5$, which is the 'best' choice as mentioned in

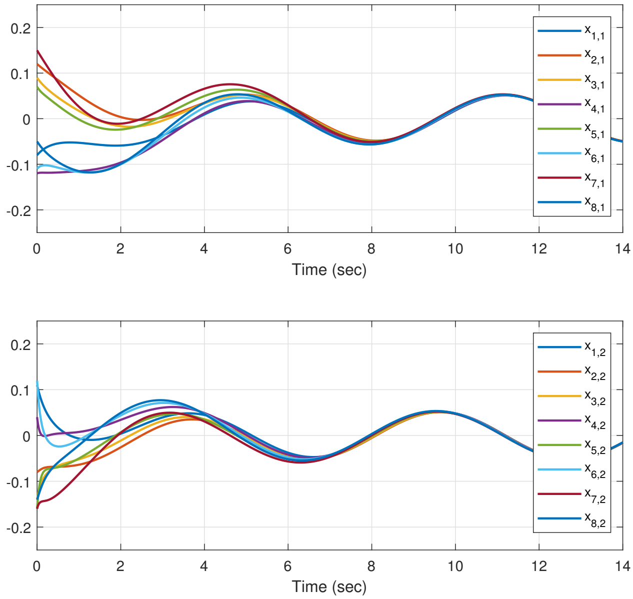

Figure 2.2: Plots of the state vector $\mathbf{x}^1 = (x_{1,1}, x_{2,1}, \ldots, x_{8,1})$ (upper plot) and $\mathbf{x}^2 = (x_{1,2}, x_{2,2}, \ldots, x_{8,2})$ (lower plot) of the controlled oscillator network

Remark 2.9. Then, by solving (2.5) in Matlab, we obtain

$$P = \begin{pmatrix} 12.1168 & 3.1303 \\ 3.1303 & 8.3081 \end{pmatrix}.$$

Correspondingly, the local feedback gain is then equal to

$$K = \begin{pmatrix} -1.5652 & -4.1541 \end{pmatrix}.$$

We now compute the radius r of a ball $B(r)$ of initial states for which the distributed control law $\mathbf{u} = (L \otimes K)\mathbf{x}$ is suboptimal, i.e. $J(K) < 3$. We compute that the largest eigenvalue of P is equal to 13.8765. Hence for every radius r such that $\frac{3}{r^2} > 13.8765$ the inequality (2.54) holds. Thus, the distributed controller with local gain K is suboptimal for all $\mathbf{x}_0$ with $\|\mathbf{x}_0\| \leqslant r$ with $r < 0.4650$.

As an example, the following initial states of the agents satisfy this norm bound: $x_{10}^\top = \begin{pmatrix} -0.08 & 0.11 \end{pmatrix}$, $x_{20}^\top = \begin{pmatrix} 0.12 & -0.08 \end{pmatrix}$, $x_{30}^\top = \begin{pmatrix} 0.09 & -0.14 \end{pmatrix}$, $x_{40}^\top = \begin{pmatrix} -0.12 & 0.04 \end{pmatrix}$, $x_{50}^\top = \begin{pmatrix} 0.07 & -0.16 \end{pmatrix}$, $x_{60}^\top = \begin{pmatrix} -0.11 & 0.12 \end{pmatrix}$, $x_{70}^\top = \begin{pmatrix} 0.15 & -0.16 \end{pmatrix}$, $x_{80}^\top = \begin{pmatrix} -0.05 & -0.14 \end{pmatrix}$. The plots of the eight decoupled oscillators without control are shown in Figure 2.1. Figure 2.2 shows that the controlled network of oscillators reaches synchronization.

2.6 Conclusions

In this chapter, we have studied a distributed linear quadratic suboptimal control problem for undirected linear multi-agent networks. We have considered a multi-agent system with identical linear agent dynamics and an associated global quadratic cost functional. For these, we have provided a design method to compute distributed diffusive control laws whose cost is bounded by a given upper bound for all initial states in a closed ball of a given radius, and such that the controlled network reaches synchronization. The computation of the local control gain involves finding solutions of a single Riccati inequality, whose dimension is equal to the dimension of the agent dynamics, and also involves the smallest nonzero and largest eigenvalue of the Laplacian matrix. As an extension, we have removed the requirement of having exact knowledge on the smallest nonzero and largest eigenvalue of the Laplacian matrix by, instead, using only lower and upper bounds for these eigenvalues.

3 Distributed linear quadratic tracking control: a suboptimality approach

In this chapter, we extend the results in Chapter 2 on distributed linear quadratic control for *leaderless* multi-agent systems to the case of distributed linear quadratic tracking control for *leader-follower* multi-agent systems. Given one autonomous leader and a number of homogeneous followers, we introduce an associated global quadratic cost functional. We assume that the leader shares its state information with at least one of the followers and the communication between the followers is represented by a connected simple undirected graph. Our objective is to design distributed control laws such that the controlled network reaches tracking consensus and, moreover, the associated cost is smaller than a given tolerance for all initial states bounded in norm by a given radius. We establish a centralized design method for computing such suboptimal control laws, involving the solution of a single Riccati inequality of dimension equal to the dimension of the local agent dynamics, and the smallest and the largest eigenvalue of a given positive definite matrix involving the underlying graph. The proposed design method is illustrated by a simulation example.

3.1 Introduction

Distributed control for multi-agent systems has drawn much attention in the past two decades due to its practical applications, e.g., formation control, intelligent transportation systems and power grids. In the literature, basically two types of multi-agent systems are considered, namely leaderless multi-agent systems and leader-follower multi-agent systems. In the leaderless case, the local agents reach agreement which depends on the dynamics of all agents [85, 103]. In the leader-follower case, the states or the outputs of the followers track that of the leader [29, 78]. One of the attractive directions in distributed control for multi-agent systems is to design distributed control laws that minimize certain global or local performances, while reaching an agreement for the controlled network.

In the past, quite some work has been devoted to distributed linear quadratic

(LQ) optimal control for leaderless multi-agent systems. In [104], an LQR based method was used to design distributed synchronizing control laws for a multi-agent system, without taking any performance into consideration. In [7], suboptimal distributed stabilizing control laws were established for a multi-agent system with general agent dynamics with respect to an associated global cost functional, while in [10], the optimal synchronizing control gain was computed for leaderless multi-agent systems with single integrator agent dynamics. In the meantime, the distributed LQ control problem was also considered in [98] by utilizing a game theoretic approach, in [75] by adopting an inverse optimal approach, and later in [37] by employing a suboptimality approach. For other papers related to this topic, see also [38].

On the other hand, distributed LQ tracking control for leader-follower multi-agent systems has also attracted much attention. In [122], distributed synchronizing control laws were established using an LQR based approach without optimizing any performance. Later on, in [13], distributed suboptimal control laws were proposed for achieving guaranteed cost. In [76], a hierarchical LQR based method was presented to design suboptimal synchronizing control laws for leader-follower systems, and an inverse optimal approach was introduced in [75], see also [123].

In the present chapter we extend the results from [37] on distributed LQ control for leaderless multi-agent systems to the case of distributed LQ tracking control for leader-follower multi-agent systems. Given a leader-follower system with one autonomous leader and a number of followers, we introduce an associated global quadratic cost functional. We assume that the leader shares its state information with at least one of the followers, and the communication between the followers is represented by a connected simple undirected graph. Our aim is then to design distributed diffusive control laws such that the controlled network reaches tracking synchronization, i.e., the states of the followers track the state of the leader asymptotically and the associated cost is smaller than an a priori given upper bound.

The outline of this chapter is as follows. Section 3.2 provides some preliminaries on quadratic performance analysis for autonomous linear systems. In Section 3.3, we formulate the distributed linear quadratic suboptimal tracking control problem for leader-follower multi-agent systems. We then address this distributed suboptimal tracking control problem in Section 3.4. A simulation example is presented in Section 3.5 to illustrate our design method. Finally, Section 3.6 concludes this chapter.

3.2 Quadratic performance analysis for autonomous linear systems

In this subsection, we will analyze the quadratic performance of a linear autonomous system. Consider the autonomous system

$$\dot{x}(t) = \bar{A}x(t), \quad x(0) = x_0 \tag{3.1}$$

where $\bar{A} \in \mathbb{R}^{n \times n}$ and $x \in \mathbb{R}^n$ is the state. We consider the quadratic performance of system (3.1), given by

$$J = \int_0^\infty x^\top(t)\bar{Q}x(t)\, dt \tag{3.2}$$

where $\bar{Q} \geqslant 0$ is a given real weighting matrix. Note that the performance J is finite if system (3.1) is asymptotically stable, i.e., $\bar{A}$ is Hurwitz.

The following well-known result ([37, 100]) provides a *necessary* and *sufficient* condition such that, for a given tolerance $\gamma > 0$, the performance (3.2) satisfies $J < \gamma$.

Theorem 3.1. *Consider system (3.1) with associated performance (3.2). For given $\gamma > 0$, we have that $\bar{A}$ is Hurwitz and $J < \gamma$ if and only if there exists $P > 0$ satisfying*

$$\bar{A}^\top P + P\bar{A} + \bar{Q} < 0, \tag{3.3}$$

$$x_0^\top P x_0 < \gamma. \tag{3.4}$$

In the next section, we will formulate the problem that we will address in this chapter.

3.3 Problem formulation

In this chapter, we consider a leader-follower multi-agent system, consisting of one leader and N followers. The dynamics of the leader is represented by the linear time-invariant autonomous system

$$\dot{x}_r(t) = Ax_r(t), \quad x_r(0) = x_{r0}. \tag{3.5}$$

where $A \in \mathbb{R}^{n \times n}$, $x_r \in \mathbb{R}^n$ is the state of the leader and x_{r0} is its initial state. The dynamics of the followers are identical and represented by the linear time-invariant systems

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t), \quad x_i(0) = x_{i0}, \quad i = 1, 2, \dots, N \tag{3.6}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and $x_i \in \mathbb{R}^n, u_i \in \mathbb{R}^m$ are the state and input of follower i, respectively, and x_{i0} is its initial state. Throughout this chapter, we assume that the pair (A, B) is stabilizable. Moreover, we make the following two standard assumptions regarding the communication between the leader and the followers:

Assumption 3.1. *We assume that at least one follower receives the state information of the leader.*

Assumption 3.2. *We also assume that the underlying graph $\mathcal{G}$ of the communication between the followers is a connected simple undirected graph.*

We consider the infinite horizon distributed linear quadratic tracking control problem for the leader-follower system (3.5) and (3.6), where the global cost functional integrates the weighted quadratic difference of states between every follower and its neighbors and the weighted quadratic difference of states between the leader and the followers communicating with the leader, and where the cost functional also penalizes the inputs in a quadratic form.

Note that, as mentioned in Assumption 3.1, at least one follower receives the state information of the leader. Thus, the leader-follower system (3.5) and (3.6) can be interconnected by a distributed diffusive control law of the form

$$u_i(t) = K\sum_{j=1}^{N} a_{ij}(x_i(t) - x_j(t)) + Kg_i(x_i(t) - x_r(t)) \tag{3.7}$$

where a_{ij} is the ij-th entry of the adjacency matrix $\mathcal{A}$ of the underlying graph $\mathcal{G}$, $K \in \mathbb{R}^{m\times n}$ is an identical feedback gain for all followers and we have $g_i > 0$ for at least one $i = 1, 2, \ldots, N$. Accordingly, the cost functional considered in this chapter is given by

$$\begin{aligned} J(u_1, u_2, \ldots, u_N) = \int_0^\infty & \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}(x_i - x_j)^\top Q(x_i - x_j) \\ & + \sum_{i=1}^{N} g_i(x_i - x_r)^\top Q(x_i - x_r) + \sum_{i=1}^{N} u_i^\top Ru_i \, dt \end{aligned} \tag{3.8}$$

where $Q \geqslant 0$ and $R > 0$ are given real weighting matrices of suitable dimensions.

The distributed linear quadratic tracking problem is then the problem of minimizing the cost functional (3.8) for all initial states x_{r0} and x_{i0}, $i = 1, 2, \ldots, N$ over all distributed diffusive control laws (3.7) such that the states of all followers track the state of the leader asymptotically. In that case we say the network reaches *tracking synchronization*:

Definition 3.1. *We say the control law* (3.7) *achieves tracking synchronization for the leader-follower system* (3.5) *and* (3.6) *if for all* $i = 1, 2, \ldots, N$ *and for all initial states* x_{r0} *and* x_{i0}*, we have*

$$x_i(t) - x_r(t) \to 0 \text{ as } t \to \infty.$$

Due to the distributed nature of the control law (3.7) as imposed by the network topology, the distributed linear quadratic tracking problem is a non-convex optimization problem [74]. It is therefore difficult, if not impossible, to find a closed form solution for an optimal controller, or such optimal controller may not even exist. Therefore, in this chapter we will design distributed control laws which solve a *suboptimality* version of this problem.

To proceed, for the ith follower we introduce the following error state

$$e_i = x_i - x_r,$$

for $i = 1, 2, \ldots, N$. Subsequently, the dynamics of e_i is given by

$$\dot{e}_i = Ae_i + Bu_i, \quad i = 1, 2, \ldots, N. \tag{3.9}$$

Denoting $\mathbf{x} = \left(x_1^\top, \ldots, x_N^\top\right)^\top$, $\mathbf{u} = \left(u_1^\top, \ldots, u_N^\top\right)^\top$, and $\mathbf{e} = \left(e_1^\top, \ldots, e_N^\top\right)^\top$, we can then rewrite the error system (3.9) in compact form as

$$\dot{\mathbf{e}} = (I_N \otimes A)\mathbf{e} + (I_N \otimes B)\mathbf{u}, \quad \mathbf{e}(0) = \mathbf{e}_0. \tag{3.10}$$

Note that

$$\mathbf{e} = \mathbf{x} - \mathbf{1}_N \otimes x_r.$$

Correspondingly, by using the fact $(L \otimes K)(\mathbf{1} \otimes x_r) = 0$, the control law (3.7) can be given by

$$\mathbf{u}(t) = (\Gamma \otimes K)\mathbf{e} \tag{3.11}$$

where $\Gamma = L + G$ and $G = \text{diag}(g_1, g_2, \ldots, g_N)$. Similarly, the cost functional (3.8) can be written in terms of e and u as

$$J(\mathbf{u}) = \int_0^\infty \mathbf{e}^\top(\Gamma \otimes Q)\mathbf{e} + \mathbf{u}^\top(I_N \otimes R)\mathbf{u}\, dt. \tag{3.12}$$

Now, by substituting the control law (3.11) into the error dynamics (3.10), we obtain the closed-loop error system

$$\dot{\mathbf{e}} = (I_N \otimes A + \Gamma \otimes BK)\mathbf{e}, \quad \mathbf{e}(0) = \mathbf{e}_0. \tag{3.13}$$

and the associated cost is now given by

$$J(K) = \int_0^\infty \mathbf{e}^\top \Big(\Gamma \otimes Q + \Gamma^2 \otimes K^\top R K \Big) \mathbf{e}\, dt \tag{3.14}$$

Note that the controlled leader-follower system (3.5) and (3.6) reaches tracking synchronization, i.e., the states of all followers track the state of the leader asymptotically, if and only if the error dynamics (3.13) is stable.

Let

$$B(r) = \{ \mathbf{e}_0 \in \mathbb{R}^{nN} \mid \|\mathbf{e}_0\| \leqslant r \} \tag{3.15}$$

be the closed ball of radius r in the state space $\mathbb{R}^{nN}$ of the error system (3.13). Then, for the leader-follower system (3.5) and (3.6) with initial states such that the error initial state is contained in a closed ball of a given radius, we want to design a distributed diffusive controller such that tracking synchronization is achieved and, for all initial states satisfying (3.15), the associated cost is smaller than an a priori given upper bound. Thus, the problem that we will address is the following:

Problem 3.1. *Consider the leader-follower multi-agent system* (3.5) *and* (3.6) *and the associated cost functional* (3.8). *Let $r > 0$ be a given radius and let $\gamma > 0$ be an a priori given upper bound for the cost. The problem is to find a distributed diffusive control law of the form* (3.7) *such that the controlled leader-follower system reaches tracking synchronization and, for all initial conditions $\boldsymbol{x}_0$ and x_{r0} such that $\boldsymbol{e}_0 = \boldsymbol{x}_0 - \mathbf{1}_N \otimes x_{r0}$ satisfies* (3.15), *the associated cost* (3.8) *is smaller than the given upper bound, i.e., $J(K) < \gamma$.*

3.4 Distributed suboptimal tracking control for leader-follower multi-agent systems

In this section, we will address Problem 3.1 and provide a suitable control design method. As mentioned before, the distributed control law (3.7) achieves tracking synchronization and suboptimal performance for the leader-follower system (3.5) and (3.6) with respect to the given tolerance on the cost functional (3.8) if and only if the error dynamics (3.13) is stable and $J(K) < \gamma$.

Now, let $U \in \mathbb{R}^{N \times N}$ be an orthogonal matrix that diagonalizes $\Gamma = L + G$. Define

$$U^\top \Gamma U := \Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N).$$

It follows from Lemma 1.1 that $\lambda_i > 0$ for all $i = 1, 2, \ldots, N$. To simplify the problem formulated in the previous section, by applying the state transformation

$\bar{\mathbf{e}} = (U^\top \otimes I_n)\mathbf{e}$, system (3.13) becomes

$$\dot{\bar{\mathbf{e}}} = (I_N \otimes A + \Lambda \otimes BK)\,\bar{\mathbf{e}}, \quad \bar{\mathbf{e}}(0) = \bar{\mathbf{e}}_0 \tag{3.16}$$

where $\bar{\mathbf{e}} = \left(\bar{e}_1^\top, \ldots, \bar{e}_N^\top\right)^\top$. In terms of the transformed variable, the cost (3.14) is then given by

$$J(K) = \int_0^\infty \sum_{i=1}^{N} \bar{e}_i^\top (\lambda_i Q + \lambda_i^2 K^\top RK)\bar{e}_i \, dt. \tag{3.17}$$

Note that the transformed states $\bar{e}_i$, $i = 1, 2, \ldots, N$ appearing in system (3.16) and cost (3.17) are decoupled from each other. Then we can write system (3.16) as

$$\dot{\bar{e}}_i = (A + \lambda_i BK)\bar{e}_i, \quad i = 1, 2, \ldots, N. \tag{3.18}$$

Also, the cost (3.17) equals

$$J(K) = \sum_{i=1}^{N} J_i(K) \tag{3.19}$$

with

$$J_i(K) = \int_0^\infty \bar{e}_i^\top (\lambda_i Q + \lambda_i^2 K^\top RK)\bar{e}_i \, dt, \quad i = 1, 2, \ldots, N. \tag{3.20}$$

Clearly, the controlled leader-follower system (3.5) and (3.6) reaches tracking synchronization with control law (3.7) if and only if, for $i = 1, 2, \ldots, N$, the systems (3.18) are stable. In addition, the control law (3.7) is suboptimal if $J(K) < \gamma$.

So far, we have transformed the problem of distributed suboptimal control for the leader-follower system (3.5) and (3.6) into the problem of finding one single static feedback gain $K \in \mathbb{R}^{m\times n}$ such that the systems (3.18) are stable for $i = 1, 2, \ldots, N$ and $J(K) < \gamma$. Since the pair (A, B) is stabilizable, there exists such a feedback gain K [54, 122].

The following lemma then provides a necessary and sufficient condition for a given feedback gain K to stabilize all systems (3.18) and for given initial states guarantee that $J(K) < \gamma$.

Lemma 3.2. *Let K be a feedback gain. Consider the systems* (3.18) *with given initial states $\bar{e}_{10}, \bar{e}_{20}, \ldots, \bar{e}_{N0}$ and associated cost functionals* (3.19) *and* (3.20). *Let $\gamma > 0$. Then all systems (3.18) are stable and $J(K) < \gamma$ if and only if there exist $P_i > 0$ satisfying*

$$(A + \lambda_i BK)^\top P_i + P_i(A + \lambda_i BK) + \lambda_i Q + \lambda_i^2 K^\top RK < 0 \tag{3.21}$$

and

$$\sum_{i=1}^{N} \bar{e}_{i0}^\top P_i \bar{e}_{i0} < \gamma, \tag{3.22}$$

for $i = 1, 2, \dots, N$, respectively.

Proof. (if) Since (3.22) holds, there exist $\gamma_i := \bar{e}_{i0}^\top P_i \bar{e}_{i0} + \epsilon_i$ with sufficiently small $\epsilon_i > 0$, $i = 1, 2, \dots, N$ such that $\sum_{i=1}^N \gamma_i < \gamma$. Because there exists $P_i > 0$ such that (3.21) and $\bar{e}_{i0}^\top P_i \bar{e}_{i0} < \gamma_i$ holds for all $i = 1, 2, \dots, N$, by taking $\bar{A} = A + \lambda_i BK$ and $\bar{Q} = \lambda_i Q + \lambda_i^2 K^\top RK$, it follows from Theorem 3.1 that all systems (3.18) are stable and $J_i(K) < \gamma_i$ for $i = 1, 2, \dots, N$. Since $J(K) = \sum_{i=1}^N J_i(K)$, this implies that $J(K) < \sum_{i=1}^N \gamma_i < \gamma$.

(only if) Since $J(K) < \gamma$ and $J(K) = \sum_{i=1}^N J_i(K)$, there exist $\gamma_i := J_i(K) + \epsilon_i$ with sufficiently small $\epsilon_i > 0$, $i = 1, 2, \dots, N$ such that $\sum_{i=1}^N \gamma_i < \gamma$. Because all systems (3.18) are stable and $J_i(K) < \gamma_i$ for $i = 1, 2, \dots, N$, by taking $\bar{A} = A + \lambda_i BK$ and $\bar{Q} = \lambda_i Q + \lambda_i^2 K^\top RK$, it again follows from Theorem 3.1 that there exist $P_i > 0$ such that (3.21) and $\bar{x}_{i0}^\top P_i \bar{x}_{i0} < \gamma_i$ hold for all $i = 1, 2, \dots, N$. Since $\sum_{i=1}^N \gamma_i < \gamma$, this implies that $\sum_{i=1}^N \bar{x}_{i0}^\top P_i \bar{x}_{i0} < \sum_{i=1}^N \gamma_i < \gamma$. □

Lemma 3.2 establishes a necessary and sufficient condition for a given feedback gain K to stabilize all systems (3.18) and to satisfy, for given initial states of these systems, $J(K) < \gamma$. However, Lemma 3.2 does not yet provide a design method for computing such K. Therefore, in the following we will provide a method to find such K.

Lemma 3.3. *Consider the leader-follower system* (3.5) *and* (3.6) *with associated cost functional* (3.8). *Let x_{r0} be the given initial state of the leader and x_{i0}, $i = 1, 2 \dots, N$ be the given initial states of the followers, respectively. Let $\gamma > 0$ be a given tolerance. Let c be any real number such that $0 < c < \frac{2}{\lambda_N}$. We distinguish two cases:*

(i) if

$$\frac{2}{\lambda_1 + \lambda_N} \leqslant c < \frac{2}{\lambda_N}, \tag{3.23}$$

then there exists $P > 0$ satisfying

$$A^\top P + PA + (c^2 \lambda_N^2 - 2c\lambda_N) PBR^{-1}B^\top P + \lambda_N Q < 0. \tag{3.24}$$

(ii) if

$$0 < c < \frac{2}{\lambda_1 + \lambda_N}, \tag{3.25}$$

then there exists $P > 0$ satisfying

$$A^\top P + PA + (c^2 \lambda_1^2 - 2c\lambda_1) PBR^{-1}B^\top P + \lambda_N Q < 0. \tag{3.26}$$

In both cases, if in addition P satisfies

$$\sum_{i=1}^{N}(x_{i0} - x_{r0})^\top P(x_{i0} - x_{r0}) < \gamma, \tag{3.27}$$

then the distributed control law (3.7) *with $K := -cR^{-1}B^\top P$ achieves tracking synchronization for the controlled leader-follower system* (3.5) *and* (3.6)*, and with the initial states x_{r0} and x_{i0} we have $J(K) < \gamma$.*

Proof. We will only give the proof for case (i) above. Using the upper and lower bounds on c given by (3.23), it can be verified that $c^2\lambda_i^2 - 2c\lambda_i \leqslant c^2\lambda_N^2 - 2c\lambda_N < 0$ for $i = 1, 2, \ldots, N$. It is then easily seen that (3.24) has many positive definite solutions. Since also $\lambda_i \leqslant \lambda_N$, any such solution P is a solution to the $N-1$ Riccati inequalities

$$A^\top P + PA + (c^2\lambda_i^2 - 2c\lambda_i)PBR^{-1}B^\top P + \lambda_i Q < 0, \quad i = 1, 2, \ldots, N. \tag{3.28}$$

Equivalently, P also satisfies the Lyapunov inequalities

$$\begin{aligned}(A - c\lambda_i BR^{-1}B^\top P)^\top P + P(A - c\lambda_i BR^{-1}B^\top P)\\ + \lambda_i Q + c^2\lambda_i^2 PBR^{-1}B^\top P < 0, \quad i = 1, 2, \ldots, N.\end{aligned} \tag{3.29}$$

Next, by substituting $\bar{\mathbf{e}} = (U^\top \otimes I_n)\mathbf{e}$ into (3.22) we have $\sum_{i=1}^{N} e_{i0}^\top P e_{i0} < \gamma$, which is equal to (3.27).

Next, taking $P_i = P$ for $i = 1, 2, \ldots, N$ and $K := -cR^{-1}B^\top P$ in (3.21) and (3.22) immediately gives us (3.29) and (3.27). Then it follows from Lemma 3.2 that all systems (3.18) are stable and $J(K) < \gamma$. Subsequently, the controlled leader-follower system reaches tracking synchronization and $J(K) < \gamma$. $\square$

We will now apply Lemma 3.3 to establish a solution to Problem 3.1. The next theorem provides a condition under which, for given radius r and upper bound γ, distributed diffusive suboptimal control laws exist, and explains how these can be computed.

Theorem 3.4. *Consider the leader-follower system* (3.5) *and* (3.6) *with associated cost functional* (3.8)*. Let $r > 0$ be a given radius and let $\gamma > 0$ be an a priori given upper bound for the cost. Let c be any real number such that $0 < c < \frac{2}{\lambda_N}$. We distinguish two cases:*

(i) if

$$\frac{2}{\lambda_1 + \lambda_N} \leqslant c < \frac{2}{\lambda_N}, \tag{3.30}$$

then there exists $P > 0$ satisfying

$$A^\top P + PA + (c^2\lambda_N^2 - 2c\lambda_N)PBR^{-1}B^\top P + \lambda_N Q < 0. \tag{3.31}$$

(ii) if

$$0 < c < \frac{2}{\lambda_1 + \lambda_N}, \tag{3.32}$$

then there exists $P > 0$ satisfying

$$A^\top P + PA + (c^2\lambda_1^2 - 2c\lambda_1)PBR^{-1}B^\top P + \lambda_N Q < 0. \tag{3.33}$$

In both cases, if in addition P satisfies

$$P < \frac{\gamma}{r^2}I, \tag{3.34}$$

then the distributed control law (3.7) *with $K := -cR^{-1}B^\top P$ achieves tracking synchronization for the controlled leader-follower system* (3.5) *and* (3.6) *and $J(K) < \gamma$ for all initial states x_{r0} and $\mathbf{x}_0$ satisfying*

$$\mathbf{x}_0 - \mathbf{1}_N \otimes x_{r0} \in B(r). \tag{3.35}$$

Proof. Again, we only give proof for case (i). Let $P > 0$ satisfy (3.31) and (3.34). Next, we will show that if the initial states x_{r0} and $\mathbf{x}_0$ satisfy $\mathbf{x}_0 - \mathbf{1}_N \otimes x_{r0} \in B(r)$, then (3.27) holds. Indeed, if $\|\mathbf{x}_0 - \mathbf{1}_N \otimes x_{r0}\| \leqslant r$, then

$$\begin{aligned}
&\sum_{i=1}^{N}(x_{i0} - x_{r0})^\top P(x_{i0} - x_{r0}) \\
=&(\mathbf{x}_0 - \mathbf{1}_N \otimes x_{r0})^\top (I \otimes P)(\mathbf{x}_0 - \mathbf{1}_N \otimes x_{r0}) \\
<&\frac{\gamma}{r^2}\|\mathbf{x}_0 - \mathbf{1}_N \otimes x_{r0}\|^2 \leqslant \gamma.
\end{aligned}$$

It then follows from Lemma 3.3 that the controlled leader-follower system (3.5) and (3.6) reaches tracking synchronization with the given K and $J(K) < \gamma$ for all initial states x_{r0} and $\mathbf{x}_0$ satisfying (3.35). □

Remark 3.5. Theorem 3.4 states that after choosing c satisfying the inequality (3.30) for case (i) and finding $P > 0$ satisfying (3.31) and (3.34), the distributed control law with local gain $K = -cR^{-1}B^\top P$ is γ-suboptimal for all initial states of the leader-follower system satisfying the condition (3.35). According to (3.34), the smaller the solution P of (3.31), the smaller the quotient $\frac{\gamma}{r^2}$ is allowed to be, leading to a smaller upper bound and a larger radius. The question then arises: how should we choose the parameter c in (3.30) so that the Riccati inequality (3.31) allows a positive definite solution that is as small as possible. In fact, one can find a positive definite solution $P(c, \epsilon)$ to (3.31) by solving the Riccati equation

$$A^\top P + PA - PB\bar{R}(c)^{-1}B^\top P + \bar{Q}(\epsilon) = 0 \tag{3.36}$$

with $\bar{R}(c) = \frac{1}{-c^2\lambda_N^2 + 2c\lambda_N} R$ and $\bar{Q}(\epsilon) = \lambda_N Q + \epsilon I_n$ where c is chosen as in (3.30) and $\epsilon > 0$. If c_1 and c_2 as in (3.30) satisfy $c_1 \leqslant c_2$, then we have $\bar{R}(c_1) \leqslant \bar{R}(c_2)$, so, clearly, $P(c_1, \epsilon) \leqslant P(c_2, \epsilon)$. Similarly, if $0 < \epsilon_1 \leqslant \epsilon_2$, we immediately have $\bar{Q}(\epsilon_1) \leqslant \bar{Q}(\epsilon_2)$. Again, it follows that $P(c, \epsilon_1) \leqslant P(c, \epsilon_2)$. Therefore, if we choose $\epsilon > 0$ very close to 0 and $c = \frac{2}{\lambda_1 + \lambda_N}$, we find the 'best' solution to the Riccati inequality (3.31) in the sense explained above.

Likewise, if c satisfies (3.32) corresponding to case (ii), it can be shown that if we choose $\epsilon > 0$ very close to 0 and $c > 0$ very close to $\frac{2}{\lambda_1 + \lambda_N}$, we find the 'best' solution to the Riccati inequality (3.33) in the sense explained above.

3.5 Simulation example

In this section, we will use a numerical example borrowed from [76] to illustrate the design method for the distributed suboptimal control laws given in Theorem 3.4.

Consider a leader-follower multi-agent system, consisting of one leader and five followers. The dynamics of the leader is given by

$$\dot{x}_r(t) = Ax_r(t), \quad x_r(0) = x_{r0},$$

and the dynamics of the followers are identical and represented by

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t), \quad x_i(0) = x_{i0}, \quad i = 1, 2, \ldots, 5$$

where

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The pair (A, B) is stabilizable. Assume the underlying graph representing the communication between the leader and the followers is given as in Figure 3.1. The graph representing the communication between the followers is then the

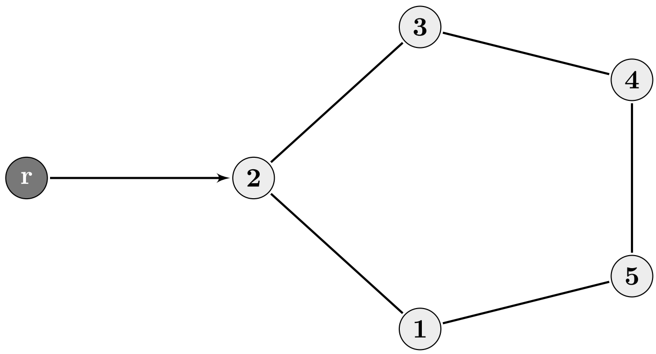

Figure 3.1: The underlying graph of the communication between the leader and the followers.

undirected cycle graph with the Laplacian matrix

$$L = \begin{pmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{pmatrix}.$$

Since the leader shares its state information only with follower 2, it follows from Lemma 1.1 that the associated diagonal matrix $G = \mathrm{diag}(g_1, g_2, \ldots, g_5) = \mathrm{diag}(0, 1, 0, 0, 0)$. Furthermore, we consider the cost functional

$$J(u_1, u_2, \ldots, u_5) = \int_0^\infty \frac{1}{2} \sum_{i=1}^5 \sum_{j=1}^5 a_{ij}(x_i - x_j)^\top Q(x_i - x_j) \\ + \sum_{i=1}^5 g_i(x_i - x_r)^\top Q(x_i - x_r) + \sum_{i=1}^5 u_i^\top R u_i \, dt$$

with

$$Q = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \quad R = 1.$$

Let the desired tolerance for the cost functional be $\gamma = 20$. Our aim is then to design a control law of the form

$$u_i(t) = K \sum_{j=1}^5 a_{ij}(x_i(t) - x_j(t)) + K g_i(x_i(t) - x_r(t)) \tag{3.37}$$

such that the controlled leader-follower system reaches tracking synchronization

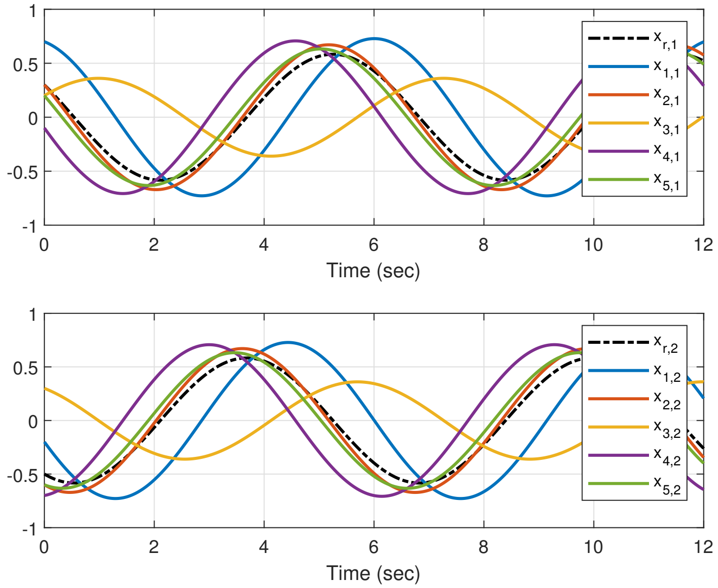

Figure 3.2: Plots of the states x_{r1} and $\mathbf{x}^1 = (x_{1,1}, x_{2,1}, \ldots, x_{5,1})$ (upper plot) and x_{r2} and $\mathbf{x}^2 = (x_{1,2}, x_{2,2}, \ldots, x_{5,2})$ (lower plot) of the six decoupled local agents without control

and the associated cost satisfies $J(K) < 20$ for all initial states $\mathbf{x}_0$ and x_{r0} satisfying the condition $\|\mathbf{x}_0 - \mathbf{1}_5 \otimes x_{r0}\| \leqslant r$ with radius r to be specified later.

In this simulation example, we will use the design method of case (i) in Theorem 3.4. For $\Gamma = L + G$ the smallest and largest eigenvalues are $\lambda_1 = 0.1392$ and $\lambda_5 = 4.1149$, respectively. We first compute a solution $P > 0$ to (3.31) by solving

$$A^\top P + PA + (c^2\lambda_5^2 - 2c\lambda_5)PBR^{-1}B^\top P + \lambda_5 Q + \epsilon I_2 = 0 \tag{3.38}$$

with ϵ sufficiently small as mentioned in Remark 3.5. Here we choose $\epsilon = 0.01$. In addition, we choose $c = \frac{2}{\lambda_1+\lambda_5} = 0.4701$, which is the 'best' choice as mentioned in Remark 3.5. Then by solving (3.38) using Matlab, we compute

$$P = \begin{pmatrix} 13.2553 & 3.3886 \\ 3.3886 & 9.2760 \end{pmatrix}.$$

Correspondingly, the control gain is equal to $K = \begin{pmatrix} 1.5931 & 4.3610 \end{pmatrix}$. We now compute the radius r of a ball $B(r)$ of initial states for which the distributed control law (3.37) is suboptimal, i.e. $J(K) < 20$. We compute that the largest eigenvalue of P is equal to 15.1952. Hence for every radius r such that $\frac{20}{r^2} > 15.1952$ the

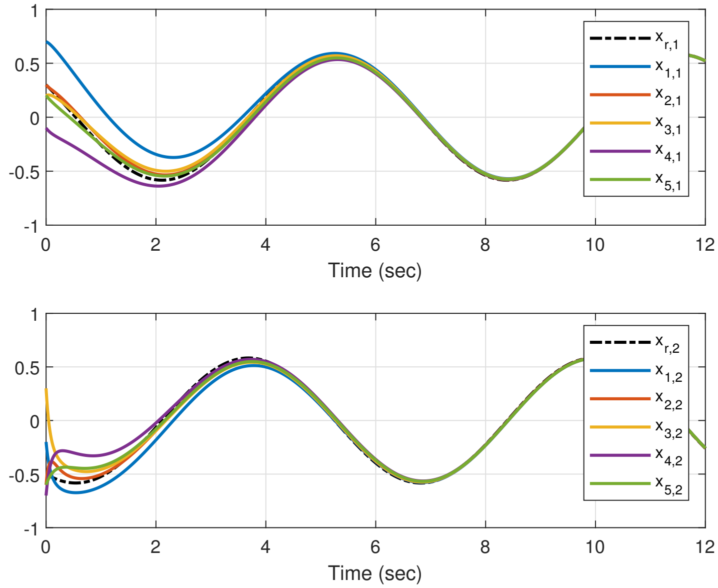

Figure 3.3: Plots of the states x_{r1} and $\mathbf{x}^1 = (x_{1,1}, x_{2,1}, \ldots, x_{5,1})$ (upper plot) and x_{r2} and $\mathbf{x}^2 = (x_{1,2}, x_{2,2}, \ldots, x_{5,2})$ (lower plot) of the controlled leader-follower system

inequality (3.34) holds. Thus, the distributed controller with local gain K is suboptimal for all x_{r0} and $\mathbf{x}_0$ satisfying $\|\mathbf{x}_0 - \mathbf{1}_5 \otimes x_{r0}\| \leqslant r$ with $r < 1.1473$.

As an example, the following initial states of the agents satisfy this norm bound: $x_{r0}^\top = \begin{pmatrix}0.3 & -0.5\end{pmatrix}$, $x_{10}^\top = \begin{pmatrix}0.7 & -0.2\end{pmatrix}$, $x_{20}^\top = \begin{pmatrix}0.3 & -0.6\end{pmatrix}$, $x_{30}^\top = \begin{pmatrix}0.2 & 0.3\end{pmatrix}$, $x_{40}^\top = \begin{pmatrix}-0.1 & -0.7\end{pmatrix}$, $x_{50}^\top = \begin{pmatrix}0.2 & -0.6\end{pmatrix}$. The plots of the state of the six local agents without control are shown in Figure 3.2. Figure 3.3 shows that the controlled leader-follower system reaches tracking synchronization.

3.6 Conclusions

In this chapter, we have studied the distributed linear quadratic tracking control problem for leader-follower multi-agent systems. We have considered leader-follower systems consisting of one autonomous leader and N followers, together with an associated global cost functional. We assume that the leader shares its state information with at least one of the followers and the underlying graph connecting the followers is a connected simple undirected graph. For this type of leader-follower systems, we have provided a design method to compute distributed suboptimal control laws such that the controlled network reaches tracking synchronization and the associated cost is smaller than a given tolerance for all initial states bounded in norm by a given radius. The computation of the local gain involves the solution of a single Riccati inequality, whose dimension is equal to the dimension of the agent dynamics, and also involves the largest and smallest eigenvalue of a positive definite matrix capturing the underlying graph structure.

4 Distributed linear quadratic control: compute locally and act globally

In this chapter we consider the distributed linear quadratic control problem for networks of agents with single integrator dynamics. We first establish a general formulation of the distributed LQ problem and show that the optimal control gain depends on global information on the network. Thus, the optimal protocol can only be computed in a centralized fashion. In order to overcome this drawback, we propose the design of protocols that are computed in a decentralized way. We will write the global cost functional as a sum of *local* cost functionals, each associated with one of the agents. In order to achieve 'good' performance of the controlled network, each agent then computes its own local gain, using sampled information of its neighboring agents. This decentralized computation will only lead to suboptimal global network behavior. However, we will show that the resulting network will reach consensus.

4.1 Introduction

The distributed linear quadratic (LQ) optimal control problem is the problem of interconnecting a finite number of identical agents according to a given network graph to achieve consensus optimally. Each agent receives input only from its neighbors, in the form of a linear feedback of the relative states amplified by a certain constant gain. Such control law is called a *distributed diffusive control law*. The problem of minimizing a given quadratic cost functional over all distributed diffusive control laws that achieve consensus is then called the distributed LQ problem corresponding to this cost functional.

In the case that the agent dynamics is given by a general state space system, this optimal control problem is non-convex and difficult to solve, and it is unclear whether a solution exists in general, see [37]. In contrast, for the case of single integrator dynamics it is fairly easy to find an explicit expression for the distributed diffusive optimal control law, see, for example, [10]. Although a solution to the problem is available, it turns out however that *global information* on the network is needed to compute this optimal control law. More specifically, the distributed diffusive optimal control law can be computed only by a (virtual) supervisor that knows the network graph and the initial states of all the agents. Thus, although the resulting optimal control law *operates* in a distributed fashion, its actual computation can only be performed in a *centralized* way.

Formulating the distributed LQ problem as a problem of minimizing a *global* cost functional is therefore not practical. Indeed, the centralized computation requires that the local optimal gains needs to be re-designed by the supervisor in case that changes in the network occur. For example, by adding or removing agents from the network, its graph will change, and new initial states will occur while existing ones will disappear.

In this chapter we will address this drawback and present a *decentralized* design method to compute a distributed controller: each agent will compute its own local control law. For this computation, the agent will not need knowledge of the network graph or the initial states of all other agents. This will then enable 'plug-and-play' operations on the network, since each agent will be able to automatically recompute its local gain whenever a new agent is added or removed.

In order to achieve this decentralized computation scheme we will write the original global cost functional as the sum of *local* LQ tracking cost functionals, each associated with one of the agents. The agents can not solve these optimal tracking problems explicitly because the reference signals depend on the future dynamics of the neighbors. However, using sampling, suboptimal local gains are obtained. This decentralized computation will not necessarily result in optimality of the global network behavior. We will however show that the resulting network will reach consensus.

The distributed LQ control problem has attracted much attention in the past, see e.g. [7, 10, 73, 74]. In [7], a distributed suboptimal controller for a global cost functional was developed to stabilize a network with general agent dynamics. A similar cost functional was also considered in [16] for designing distributed controllers with guaranteed performance. The distributed LQ control problem with general agent dynamics was also dealt with in [75] and [123] by adopting an inverse optimal control approach. In [98] a game theoretic approach was considered to obtain a suboptimal solution. Also, [37] considers a suboptimal version of this problem. In [76], a suboptimal consensus controller design was developed by employing a hierarchical LQ control approach for an appropriately chosen global performance index, and a similar idea for constructing a particular cost functional was employed in [77] to design a reduced order distributed controller. In [90] a distributed optimal control method was adopted to decouple a class of linear multi-agent systems with state coupled nonlinear uncertainties.

The common feature of all work referred to above is that the computation of the control gains needs global information on the network. This disadvantage can be avoided by adopting adaptive control methods [52] or by using reinforcement learning [108], [68]. In [70] and [109], it was shown that diffusive couplings are necessary for minimization of cost functionals of a particular form, involving the weighted squared synchronization error.

Below we list the contributions of this chapter.

1. We show that for agents with single integrator dynamics, in any distributed LQ cost functional, the state weighting matrix must be equal to a weighted square of the Laplacian of the network graph.

2. We give a solution to this general distributed LQ problem, and show that computation of the optimal protocol requires exact knowledge of the Laplacian as well as the initial state of the entire network.

3. We represent the global cost functional as a sum of local LQ tracking cost functionals, one for each agent. Using sampling, suboptimal local gains are obtained. Computation of these gains is completely decentralized.

4. We show that these gains lead to a protocol that achieves consensus of the network.

The outline of this chapter is as follows. In Section 4.2, we derive a general formulation of the distributed LQ problem. In Section 4.3, we show that computation of the optimal control laws requires complete knowledge of the network graph and the initial state of the entire network. In Section 4.4, we propose a decentralized method to compute suboptimal (local) control laws. In order to do this, we need to apply ideas from linear quadratic tracking, and these are reviewed in Section 4.5. Then, in Section 4.6, we compute these local control laws, and show that the network reaches consensus if all agents apply their own local gain. To illustrate the designed control protocol, a simulation example is provided in Section 4.7. Finally, in Section 4.8, we give some concluding remarks.

4.2 The general form of a distributed linear quadratic cost functional

In this section we will show that in any distributed LQ cost functional, the state weighting matrix must be a weighted square of the Laplacian of the network graph. We will also give two important examples of distributed LQ cost functionals.

We consider a network of agents described by scalar single integrator dynamics

$$\dot{x}_i(t) = u_i(t), \quad x_i(0) = x_{i0}, \quad i = 1, 2, \ldots, N, \tag{4.1}$$

with $x_{i0} \in \mathbb{R}$ the initial state of agent i. By collecting the states and inputs of the individual agents into the vectors $\mathbf{x} = (x_1, x_2, \dots, x_N)^\top$ and $\mathbf{u} = (u_1, u_2, \dots, u_N)^\top$, (4.1) can be written as

$$\dot{\mathbf{x}}(t) = \mathbf{u}(t), \quad \mathbf{x}(0) = \mathbf{x}_0. \tag{4.2}$$

A general class of LQ cost functionals are those of the form

$$J(\mathbf{x}_0, \mathbf{u}) = \int_0^\infty \mathbf{x}^\top(t) Q \mathbf{x}(t) + \mathbf{u}^\top(t) R \mathbf{u}(t) dt, \tag{4.3}$$

where $Q \in \mathbb{R}^{N \times N}$, $R \in \mathbb{R}^{N \times N}$ and $Q \geqslant 0$ and $R > 0$.

In the context of distributed LQ control we only allow distributed diffusive control laws that achieve consensus, i.e. the controlled trajectories converge to $\mathrm{im}(\mathbf{1}_N)$, the span of the vector of ones. Thus the class of control laws over which we want to minimize (4.3) consists of those of the form $\mathbf{u} = -gL\mathbf{x}$, with $L \in \mathbb{R}^{N \times N}$ the Laplacian of the network graph and where $g > 0$, see e.g. [85].

We will now show that for a cost functional (4.3) to make sense in this context, the weighting matrix Q must be of the form $Q = LWL$ for some positive semi-definite matrix W.

Lemma 4.1. *$J(\mathbf{x}_0, \mathbf{u}) < \infty$ for all $\mathbf{x}_0 \in \mathbb{R}^N$ and control laws of the form $\mathbf{u} = -gL\mathbf{x}$ with $g > 0$ only if there exists a positive semi-definite $W \in \mathbb{R}^{N \times N}$ such that $Q = LWL$.*

Proof. Write $Q = C^T C$ for some C. Now, let $\bar{\mathbf{x}}(t)$ denote any nonzero state trajectory generated by the control law $\mathbf{u} = -gL\mathbf{x}$ with $g > 0$ and let $\bar{\mathbf{u}}(t) = -gL\bar{\mathbf{x}}(t)$. It is well known that this control law achieves consensus (see [85]) so we have $\bar{\mathbf{x}}(t) \to c\mathbf{1}_N$ for some nonzero constant c. Now assume that the control law $\mathbf{u} = -gL\mathbf{x}$ gives finite cost, i.e. $J(\mathbf{x}_0, \bar{\mathbf{u}}) < \infty$. This implies

$$\int_0^\infty \bar{\mathbf{x}}^\top(t) C^\top C \bar{\mathbf{x}}(t) dt < \infty$$

and hence $C\bar{\mathbf{x}}(t) \to 0$. Thus we obtain $\mathbf{1}_N \in \ker(C)$, equivalently, $\ker(L) \subset \ker(C)$. We thus conclude that there exists a matrix V such that $C = VL$ so the state weighting matrix Q must be of the form $Q = LV^\top VL$ for some matrix V. This proves our claim. □

We have thus shown that, for a general LQ cost functional to make sense in the context of distributed diffusive control for multi-agent systems, it must necessarily be of the form

$$J(\mathbf{x}_0, \mathbf{u}) = \int_0^\infty \mathbf{x}^\top(t) LWL \mathbf{x}(t) + \mathbf{u}^\top(t) R \mathbf{u}(t) dt, \tag{4.4}$$

for some $W \geqslant 0$ and $R > 0$. The corresponding distributed LQ problem is to minimize, for the system (4.2) with initial state $\mathbf{x}_0$, the cost functional (4.4) over all control laws of the form $\mathbf{u} = -gL\mathbf{x}$ with $g > 0$.

As an illustration, we will now provide two important special cases of LQ cost functionals. The first one was studied before in [37] and [10]:

$$J(\mathbf{x}_0, \mathbf{u}) = \sum_{i=1}^{N} \int_0^\infty \sum_{j \in \mathcal{N}_i} q(x_i(t) - x_j(t))^2 + ru_i^2(t)dt, \tag{4.5}$$

where q and r are positive real numbers. Clearly, (4.5) is equal to

$$J(\mathbf{x}_0, \mathbf{u}) = \int_0^\infty \mathbf{x}^\top(t) 2qL\mathbf{x}(t) + r\mathbf{u}^\top(t)\mathbf{u}(t)dt.$$

Note that $2qL = L(2qL^\dagger)L$ with $L^\dagger$ the Moore-Penrose inverse of L (which is indeed positive semi-definite). Thus this cost functional is of the form (4.4) with $W = 2qL^\dagger$ and $R = rI$.

As a second example, consider

$$J(\mathbf{x}_0, \mathbf{u}) = \sum_{i=1}^{N} \int_0^\infty q\left(x_i(t) - a_i(t)\right)^2 + ru_i^2(t)dt, \tag{4.6}$$

with

$$a_i(t) := \frac{1}{d_i + 1}\big(x_i(t) + \sum_{j \in \mathcal{N}_i} x_j(t)\big). \tag{4.7}$$

Here, q and r are positive weights, d_i denotes the node degree of agent i and $\mathcal{N}_i$ its set of neighbors. The idea of the cost functional (4.6) is to minimize the sum of the deviations between the state $x_i(t)$ and the average $a_i(t)$ of the states of its neighbors (including itself) and the control energy. In order to put this in the form (4.4), define

$$G := (D + I_N)^{-1}(A + I_N) \in \mathbb{R}^{N \times N}, \tag{4.8}$$

where $D \in \mathbb{R}^{N \times N}$ is the degree matrix and $A \in \mathbb{R}^{N \times N}$ the adjacency matrix. Then clearly

$$\mathbf{a}(t) = G\mathbf{x}(t),$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_N)^\top$ and $\mathbf{a} = (a_1, a_2, \ldots, a_N)^\top$. It is then easily seen that

$$J(\mathbf{x}_0, \mathbf{u}) = \int_0^\infty q\mathbf{x}^\top(t)(I_N - G)^\top(I_N - G)\mathbf{x}(t) + r\mathbf{u}^\top(t)\mathbf{u}(t)dt.$$

Since $(I_N - G)^\top(I_N - G) = L(D + I_N)^{-2}L$, we conclude that (4.6) is a special case of (4.4) with $W = q(D + I_N)^{-2}$ and $R = rI_N$.

4.3 Centralized computation of the optimal control gain

In this section we will briefly give a solution to the general distributed LQ problem with cost functional (4.4) as introduced in Section 4.2, thus generalizing the result from [10] to general distributed LQ cost functionals. We will show that, indeed, computation of the optimal protocol requires global information on the network graph and the initial state of the entire network.

Consider the cost functional (4.4) together with the dynamics (4.2) with given initial state $\mathbf{x}_0$. Since the admissible control laws are given by $\mathbf{u} = -gL\mathbf{x}$, the associated state trajectory is $\mathbf{x}(t) = e^{-gLt}\mathbf{x}_0$ and $\mathbf{u}(t) = -gL\mathbf{x}(t)$. Substituting this into the cost functional yields

$$J(g) := \mathbf{x}_0^\top (\int_0^\infty e^{-gLt} \left(LWL + g^2 LRL\right) e^{-gLt} dt)\mathbf{x}_0 \tag{4.9}$$

Clearly, we need to minimize $J(g)$ over $g > 0$. Substituting $gt = \tau$, we find

$$J(g) := \mathbf{x}_0^\top \int_0^\infty e^{-\tau L} \left(\frac{1}{g} LWL + gLRL\right) e^{-\tau L} d\tau\, \mathbf{x}_0.$$

Define

$$X_0 := \int_0^\infty e^{-\tau L} LWL e^{-\tau L} d\tau$$

and

$$Y_0 := \int_0^\infty e^{-\tau L} LRL e^{-\tau L} d\tau.$$

It turns out that both integrals indeed exist, and can be computed as particular solutions of the Lyapunov equations

$$-LX - XL + LWL = 0, \tag{4.10a}$$

$$-LY - YL + LRL = 0. \tag{4.10b}$$

Indeed, although L is not Hurwitz, these equations do have positive semi-definite solutions X and Y and, in fact, X_0 is the unique positive semi-definite solution X to (4.10a) with the property that $\mathrm{im}(\mathbf{1}_N) \subset \ker(X)$. Likewise Y_0 is the unique positive semi-definite solution Y of (4.10b) with the property that $\mathrm{im}(\mathbf{1}_N) \subset \ker(Y)$ (see Proposition 1 in [44]). It follows from (4.10b) that, in fact, $\ker(Y_0) = \mathrm{im}(\mathbf{1}_N)$. Thus we see that

$$J(g) = \frac{1}{g}\mathbf{x}_0^\top X_0 \mathbf{x}_0 + g\mathbf{x}_0^\top Y_0 \mathbf{x}_0.$$

In order to minimize $J(g)$ we distinguish three cases. (i) If $\mathbf{x}_0 \in \ker(Y_0) = \mathrm{im}(\mathbf{1}_N)$ then we must have $\mathbf{x}_0 \in \ker(X_0)$ as well, so $J(g) = 0$ for all θ and every $g > 0$ is optimal. (ii) If $\mathbf{x}_0^\top Y_0 \mathbf{x}_0 > 0$ and $\mathbf{x}_0^\top X_0 \mathbf{x}_0 = 0$ then no optimal $g > 0$ exists. (iii) If $\mathbf{x}_0^\top Y_0 \mathbf{x}_0 > 0$ and $\mathbf{x}_0^\top X_0 \mathbf{x}_0 > 0$ then an optimal $g > 0$ exists and can be shown to be equal to

$$g^* = \left(\frac{\mathbf{x}_0^\top X_0 \mathbf{x}_0}{\mathbf{x}_0^\top Y_0 \mathbf{x}_0} \right)^{\frac{1}{2}} .$$

It is clear that the computation of the optimal gain g requires exact knowledge of the network graph in the form of the Laplacian L. Also, the optimal gain clearly depends on the global initial state of the network.

4.4 Towards decentralized computation

In this section we will propose a new approach to compute 'good' local gains that can be computed in a decentralized way. Instead of doing this for the general LQ cost functional (4.4), we will zoom in on the particular case given by (4.6) - (4.7).

In order to decentralize the computation, instead of minimizing the global cost functional (4.6) for the multi-agent system (4.2), we write it as a sum of *local* cost functionals, one for each agent in the network.

More specifically, the associated local cost functional for agent i is given by

$$J_i(u_i) = \int_0^\infty q\left(x_i(t) - a_i(t)\right)^2 + r u_i^2(t)\, dt, \tag{4.11}$$

where $a_i(t)$ is defined in (4.7), for $i = 1, 2, \ldots, N$. This local cost functional penalizes the squared difference between the state of the ith agent and the average of the states of its neighboring agents (including itself), and the local control energy. By minimizing (4.11), agent i would make the difference between its own state and the average of the states of its neighbors (including itself) small. Note, however, that it is impossible for agent i to minimize this local cost functional since the trajectory $a_i(t)$ for $t \in [0, \infty)$ associated with the neighboring agents is *not known*, so also not available to the ith agent. Thus, because direct minimization of (4.11) is impossible, as an alternative we will replace each of these local optimal control problems by *a sequence of linear quadratic tracking problems* that do turn out to be tractable.

More specifically, we choose a sampling period $T > 0$, and introduce the following sampling procedure. For each nonnegative integer k, at time $t = kT$ the ith agent receives the sampled state value $x_j(kT)$ of its neighboring agents and takes the average of these, which is given by

$$a_i(kT) = \frac{1}{d_i + 1}\big(x_i(kT) + \sum_{j \in \mathcal{N}_i} x_j(kT)\big). \tag{4.12}$$

Then, the ith agent minimizes the cost functional

$$J_{i,k}(u) = \int_0^\infty e^{-2\alpha t}\left(q\left(x_i(t) - a_i(kT)\right)^2 + r u_i^2(t)\right) dt. \tag{4.13}$$

In fact, this is a discounted linear quadratic tracking problem with constant reference signal $a_i(kT)$ and discount factor $\alpha > 0$. By solving this linear quadratic tracking problem, agent i obtains an optimal control law over an infinite time interval. However, agent i applies this control law only on the time interval $[kT, (k+1)T)$.

Then, at time $t = (k+1)T$ the above procedure is repeated, i.e. agent i receives the updated average $a_i((k+1)T)$, and subsequently solves the discounted tracking problem with cost functional $J_{i,k+1}(u)$ which involves the constant updated reference signal $a_i((k+1)T)$. By performing this control design procedure sequentially at each sampling time kT, we then obtain a single control law for agent i over the entire interval $[0, \infty)$.

Based on this control design procedure for the individual agents, we will obtain a distributed control protocol for the entire multi-agent system, simply by letting all agents compute their own control law. In the sequel we will analyze this protocol and show that it achieves consensus for the network:

Definition 4.1. *A distributed control protocol is said to achieve consensus for the network if $x_i(t) - x_j(t) \to 0$ as $t \to \infty$ for all initial states of agents i and j, for all $i, j = 1, 2, \ldots, N$.*

In order to obtain an explicit expression for the control protocol proposed above, we will study the linear quadratic tracking problem for a single linear system. This will be done in the next section.

4.5 The discounted linear quadratic tracking problem

In this section, we will deal with the discounted linear quadratic tracking problem for a given linear system. The linear quadratic tracking problem has been studied before, see e.g. [67]. Here, however, we will solve it by transforming it into a standard linear quadratic control problem.

Consider the linear time-invariant system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \tag{4.14}$$

with $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$, and where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ denote the state and the input, respectively. We assume that the pair (A, B) is stabilizable. Given is also a constant reference signal $r_{\text{ref}}(t) = r$ with $r \in \mathbb{R}^n$. Next, we introduce a discounted quadratic cost functional given by

$$J(u) = \int_0^\infty e^{-2\alpha t}[(x(t) - r)^\top Q\,(x(t) - r) + u^\top(t)Ru(t)]dt \tag{4.15}$$

where $Q \in \mathbb{R}^{n\times n}$, $R \in \mathbb{R}^{m\times m}$ and $Q > 0$ and $R > 0$ are given weight matrices and $\alpha > 0$ is a discount factor [67]. The linear quadratic tracking problem is to determine for every initial state x_0 a piecewise continuous input function $u(t)$ that minimizes the cost functional (4.15).

To solve this problem, we introduce the variables

$$z(t) = e^{-\alpha t}x(t), \quad z_r(t) = e^{-\alpha t}r, \quad v(t) = e^{-\alpha t}u(t), \tag{4.16}$$

and denote $\xi(t) = (z^\top(t), z_r^\top(t))^\top$. Then we obtain an auxiliary system in terms of ξ and v, given by

$$\dot{\xi}(t) = A_e\xi(t) + B_e v(t), \quad \xi_0 = (x_0^\top, r^\top)^\top,$$

where $\xi_0 \in \mathbb{R}^{2n}$ is the initial state and

$$A_e = \begin{pmatrix} A - \alpha I_n & 0 \\ 0 & -\alpha I_n \end{pmatrix}, \quad B_e = \begin{pmatrix} B \\ 0 \end{pmatrix}.$$

In terms of the new variables ξ and v, the cost functional (4.15) can be written as

$$J(v) = \int_0^\infty \xi^\top(t)Q_e\xi(t) + v^\top(t)Rv(t)\,dt,$$

where

$$Q_e = \begin{pmatrix} Q & -Q \\ -Q & Q \end{pmatrix} \in \mathbb{R}^{2n\times 2n}.$$

The problem is now to find, for every initial state ξ_0, a piecewise continuous input function $v(t)$ that minimizes this cost functional. This is a so-called a *free endpoint* standard LQ control problem, see [102, pp. 218]. Since the pair (A, B) is stabilizable, the pair (A_e, B_e) is also stabilizable and hence the input function $v(t)$ that minimizes the cost functional $J(v)$ is generated by the feedback law

$$v(t) = -R^{-1}B_e^\top P_e^- \xi(t), \tag{4.17}$$

where $P_e^- \in \mathbb{R}^{2n\times 2n}$ is the smallest positive semi-definite solution of the Riccati equation

$$A_e^\top P_e^- + P_e^- A_e - P_e^- B_e R^{-1} B_e^\top P_e^- + Q_e = 0. \tag{4.18}$$

Now, partition

$$P_e^- := \begin{pmatrix} P_1 & P_{12} \\ P_{12}^\top & P_2 \end{pmatrix},$$

where all blocks have dimension $n \times n$. Recalling (4.16) and (4.17), we then immediately find an expression for the input function $u(t)$ that minimizes the cost functional (4.15) for the system (4.14) and reference signal $r_{\mathrm{ref}}(t) = r$.

Theorem 4.2. *The input function $u(t)$ that minimizes the cost functional* (4.15) *is generated by the control law*

$$u(t) = K_1 x(t) + K_2 r, \tag{4.19}$$

where $K_1 = -R^{-1}B^\top P_1$ and $K_2 = -R^{-1}B^\top P_{12}$.

The proof follows immediately from the above considerations. See also [67].

Remark 4.3. Let $e(t) := x(t) - r$ denote the tracking error. Because $Q > 0$, the control law (4.19) only guarantees that $\bar{e}(t) := e^{-\alpha t} e(t)$ tends to zero as t goes to infinity. Thus, the feedback law that minimizes the LQ tracking cost functional (4.15) only guarantees the actual tracking error $e(t)$ to be exponentially bounded with growth rate $\alpha > 0$. Note that $\alpha > 0$ can be taken arbitrarily small.

It will be shown however that, for the multi-agent system case, the control design method established in this section will, nevertheless, lead to a protocol that achieves consensus.

4.6 Consensus analysis

In this section, we will show that, by adopting the control design method for the multi-agent system (4.2) as proposed in Section 4.4, the resulting distributed control protocol achieves consensus for the entire network.

As already explained in Section 4.4, we choose a sampling period $T > 0$ and introduce a sampling procedure. For each nonnegative integer k, at time $t = kT$ the ith agent receives the sampled state value of its neighboring agents (including itself) and minimizes the cost functional (4.13), which is a discounted linear quadratic tracking problem with constant reference signal $r_{\mathrm{ref}}(t) = a_i(kT)$ and discount factor $\alpha > 0$.

According to the theory on the discounted LQ tracking problem described in Section 4.5, the local optimal control law for agent i at time $t = kT$ over the whole time horizon $[0, \infty)$ is therefore of the form

$$u_{i,k}(t) = g_{i,k}x_i(t) + g'_{i,k}a_i(kT), \tag{4.20}$$

in which the control gains $g_{i,k}$ and $g'_{i,k}$ can be computed explicitly by solving the Riccati equation (4.18) associated with the LQ tracking problem for agent i.

Lemma 4.4. *Consider, at time $t = kT$, the ith agent of the multi-agent system* (4.1) *with associated local cost functional* (4.13)*. Denote*

$$\bar{A} = \begin{pmatrix} -\alpha & 0 \\ 0 & -\alpha \end{pmatrix}, \quad \bar{B} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \bar{Q} = \begin{pmatrix} q & -q \\ -q & q \end{pmatrix}.$$

Let $\bar{P} := \begin{pmatrix} p_1 & p_{12} \\ p_{12} & p_2 \end{pmatrix}$ be the smallest positive semi-definite solution of the Riccati equation

$$\bar{A}^\top \bar{P} + \bar{P}\bar{A} - r^{-1}\bar{P}\bar{B}\bar{B}^\top \bar{P} + \bar{Q} = 0. \tag{4.21}$$

Then the local control law (4.20) *with $g_{i,k} := -r^{-1}p_1$ and $g'_{i,k} := -r^{-1}p_{12}$ minimizes the cost* (4.13) *for agent i.*

Proof. This follows immediately from Theorem 4.2. □

Next, agent i applies the control law (4.20) only on the time interval $[kT, (k+1)T)$. Then, at time $t = (k+1)T$ the above procedure is repeated.

Since, for all $i = 1, 2, \ldots, N$ and $k = 0, 1, \ldots$, the matrices $\bar{A}$, $\bar{B}$ and $\bar{Q}$ are independent of i and k, the same holds for the gains $g_{i,k}$ and $g'_{i,k}$. In the sequel, we will therefore drop the subscripts in the control gains $g_{i,k}$ and $g'_{i,k}$ and denote them by g and g', respectively. Moreover, using (4.21), we compute $g = r^{-1}(\alpha - \sqrt{\alpha^2 + rq})$ and $g' = -g$.

By performing this procedure sequentially at each sampling time kT, we then obtain a single control law for agent i over the entire interval $[0, \infty)$ as

$$u_{i,k}(t) = gx_i(t) - ga_i(kT), \quad t \in [kT, (k+1)T), \tag{4.22}$$

where $g = r^{-1}(\alpha - \sqrt{\alpha^2 + rq}) < 0$.

Recall that $\mathbf{a}(t) = G\mathbf{x}(t)$, with G given by (4.8), and that $\mathbf{a}(kT) = G\mathbf{x}(kT)$. Therefore, the local control laws for the individual agents lead to a distributed control protocol

$$\mathbf{u}_k(t) = g\mathbf{x}(t) - gG\mathbf{x}(kT), \quad t \in [kT, (k+1)T). \tag{4.23}$$

Now, by applying the protocol (4.23) to the multi-agent system (4.1), we find that the controlled network is represented by

$$\dot{\mathbf{x}}(t) = g\mathbf{x}(t) - gG\mathbf{x}(kT), \quad t \in [kT, (k+1)T). \tag{4.24}$$

In the remainder of this section, we will analyze this representation, and show that consensus is achieved, i.e. for each initial state $\mathbf{x}(0) = \mathbf{x}_0$ we have $x_i(t) - x_j(t) \to 0$ as t tends to infinity.

In order to do this, note that the solution of (4.24) with initial state $\mathbf{x}(0) = \mathbf{x}_0$ is given by

$$\mathbf{x}(t) = e^{g(t-kT)}\mathbf{x}(kT) - \int_{kT}^{t} e^{g(t-\tau)} gG\,\mathbf{x}(kT)\,d\tau, \tag{4.25}$$

for $t \in [kT, (k+1)T)$, $k = 0, 1, 2, \ldots$. Obviously, for each initial state $\mathbf{x}_0$, the corresponding solution $\mathbf{x}(t)$ is continuous. From (4.25) we see that the sequence of network states $\mathbf{x}(kT)$ evaluated at the discrete time instances kT, $k = 0, 1, \ldots$ satisfies the difference equation

$$\mathbf{x}((k+1)T) = \Gamma\mathbf{x}(kT), \tag{4.26}$$

$\Gamma = e^{gT}I_N - (e^{gT} - 1)G \in \mathbb{R}^{N \times N}$.

Clearly, the network reaches consensus if and only if for each $\mathbf{x}_0$, $x_i(kT) - x_j(kT) \to 0$ as t tends to infinity.

We proceed with analyzing the eigenvalues of G.

Lemma 4.5. *The matrix G has an eigenvalue 1 with algebraic multiplicity equal to one and associated eigenvector $\mathbf{1}_N$. The remaining eigenvalues of G are all real and have absolute value strictly less than 1.*

Proof. Since $L = D - A$, we have $G = I_N - (D + I_N)^{-1}L$. Hence we have $\tilde{D}^{\frac{1}{2}}G\tilde{D}^{-\frac{1}{2}} = I_N - \tilde{D}^{-\frac{1}{2}}L\tilde{D}^{-\frac{1}{2}}$ where $\tilde{D} = D + I_N$. Note that the right hand side is symmetric and hence has only real eigenvalues. Thus, by matrix similarity, G also has only real eigenvalues.

Next, we show that G has a simple eigenvalue 1 with associated eigenvector $\mathbf{1}_N$. First note that

$$G\mathbf{1}_N = (I_N - (D + I_N)^{-1}L)\mathbf{1}_N = \mathbf{1}_N. \tag{4.27}$$

Hence, indeed, 1 is an eigenvalue of G with eigenvector $\mathbf{1}_N$. Since G is similar to a symmetric matrix, it is diagonalizable, so the algebraic multiplicity of its eigenvalue 1 must be equal to its geometric multiplicity. Suppose now that 1 is not a simple eigenvalue. Then there must exist a second eigenvector, say v, which is linearly independent of $\mathbf{1}_N$. This implies $Gv = v$. Then $Lv = 0$, so v must be a multiple of $\mathbf{1}_N$. This is a contradiction. We conclude that the eigenvalue 1 is indeed simple.

Finally, it follows from Gershgorin's Theorem [30] that every eigenvalue λ of G satisfies $-1 < \lambda \leqslant 1$. $\square$

Before we give the main result of this chapter, we first review the following proposition.

Proposition 4.6. *Consider the discrete-time system*

$$x(k+1) = Ax(k), \quad x(0) = x(0), \quad y(k) = Cx(k)$$

with $A \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{p\times n}$, where $x(k) \in \mathbb{R}^n$ is the state, x_0 is the initial state and $y(k) \in \mathbb{R}^p$ is the output. Then, $y(k) \to 0$ as $k \to \infty$ for all initial states x_0 if and only if $X_+(A) \subset \ker(C)$. Here, $X_+(A)$ is the unstable subspace, i.e., the sum of the generalized eigenspaces of A associated with its eigenvalues in $\{\lambda \in \mathbb{C} \mid |\lambda| \geqslant 1\}$.

Proof. A proof can be given by generalizing the results [102, pp. 99] to the discrete time case. $\square$

We are now ready to present the main result of this chapter.

Theorem 4.7. *Consider the multi-agent system* (4.1). *Let $T > 0$ be a sampling period, $\alpha > 0$ a discount factor, and let $q, r > 0$ be given weights. Let $\bar{P}$ be the smallest positive semi-definite solution of the Riccati equation* (4.21) *and partition $\bar{P} := \begin{pmatrix} p_1 & p_{12} \\ p_{12} & p_2 \end{pmatrix}$. Then the distributed control protocol* (4.23) *with $g = -r^{-1}p_1$ and $g' = -r^{-1}p_{12}$ achieves consensus for the controlled network* (4.24).

Proof. The network reaches consensus if and only if $L\mathbf{x}(kT) \to 0$ as $k \to \infty$. Since $\ker(L) = \mathrm{im}(\mathbf{1}_N)$, it then follows from Proposition 4.6 that consensus is achieved if and only if $X_+(\Gamma) \subset \ker(L)$, equivalently, the sum of the generalized eigenspaces of Γ corresponding to the eigenvalues λ with $\mid \lambda \mid \geqslant 1$ is equal to $\mathrm{im}(\mathbf{1}_N)$.

Indeed, we will show that all eigenvalues λ of Γ are real and satisfy $-1 < \lambda \leqslant 1$, and $\lambda = 1$ is a simple eigenvalue with associated eigenvector $\mathbf{1}_N$.

Recall that $\Gamma = e^{gT} I_N - (e^{gT} - 1)G$. Hence, μ is an eigenvalue of Γ if and only if $\mu = e^{gT} - \lambda(e^{gT} - 1)$ where λ is an eigenvalue of G. It was shown in Lemma 4.5 that all eigenvalues λ of G are real and satisfy $-1 < \lambda \leqslant 1$ and, moreover, $\lambda = 1$ is a simple eigenvalue. Using the fact that $g < 0$ we thus obtain that the eigenvalues μ of Γ satisfy $-1 < \mu \leqslant 1$ and $\mu = 1$ is a simple eigenvalue of Γ.

Finally, we will show $\mu = 1$ has eigenvector $\mathbf{1}_N$. Indeed, this follows from $\Gamma\mathbf{1}_N = (e^{gT} I_N - (e^{gT} - 1)G)\mathbf{1}_N = \mathbf{1}_N$. This completes the proof. $\square$

Remark 4.8. By analyzing the eigenvalues μ of Γ satisfying $-1 < \mu < 1$, it can be seen that, for given α, the convergence rate of the difference equation (4.26) increases with increasing sampling period T. The total time it takes to reach a disagreement smaller than a given tolerance is then the product of the number of iterations in (4.26) and this sampling period. It might therefore be more advantageous to use a smaller sampling period with a larger number of required iterations, but yet leading to a smaller total time. In other words, the choice of sampling period is a trade-off between the total time required to obtain an acceptable disagreement, and the number of iterations in (4.26).

4.7 Simulation example

In this section, we provide a simple simulation example to illustrate the proposed control design method.

Consider a network of six agents with single integrator dynamics

$$\dot{x}_i(t) = u_i(t), \quad i = 1, 2, \dots, 6,$$

where the initial states are $x_{10} = 1$, $x_{20} = 2$, $x_{30} = -1$, $x_{40} = -2$, $x_{50} = 1$ and $x_{60} = 3$. We assume that the communication among these agents is represented by an undirected cycle graph with six nodes. The associated Laplacian matrix is given by

$$L = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}. \tag{4.28}$$

First, we take the sampling period to be equal $T = 10$. On the time interval $t \in [kT, (k+1)T)$, $k = 0, 1, \dots$, we consider the local cost functional (4.13) for agent i. We choose the weights to be $q = 2$, $r = 1$ and the discount factor $\alpha = 0.01$. We adopt the control design proposed in Theorem 4.7 and compute the smallest positive semi-definite of the Riccati equation

$$A^\top P + PA - r^{-1} PBB^\top P + Q = 0$$

with

$$A = \begin{pmatrix} -0.01 & 0 \\ 0 & -0.01 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}.$$

This Riccati equation has a unique positive semi-definite solution which is given by

$$P = \begin{pmatrix} 1.4042 & -1.4042 \\ -1.4042 & 1.4042 \end{pmatrix}.$$

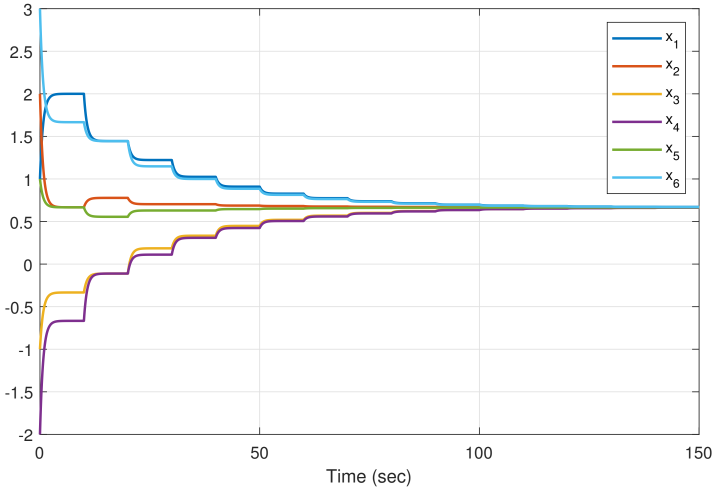

Figure 4.1: Plot of the states of the controlled network with $T = 10$

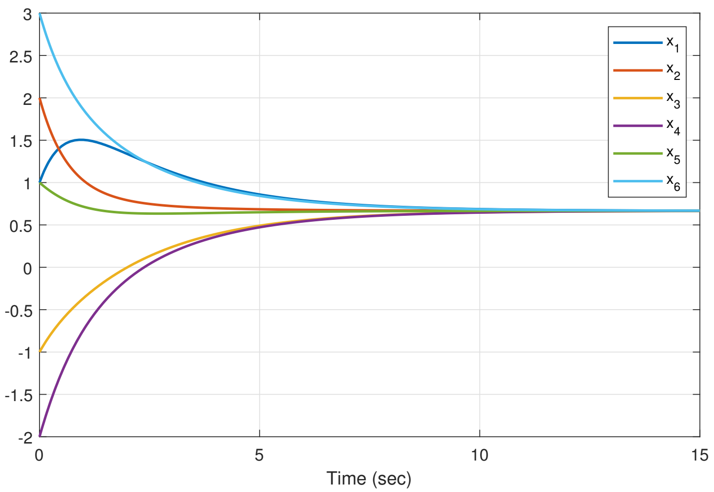

Figure 4.2: Plot of the states of the controlled network with $T = 0.1$

Thus we find the control gains $g = -1.4042$ and $g' = 1.4042$. Subsequently, the local control law for agent i is given by $u_{i,k}(t) = -1.4042x_i(t) + 1.4042a_i(kT)$ and the dynamics of the controlled agents is given by

$$\dot{x}_i(t) = -1.4042x_i(t) + 1.4042a_i(kT)$$

for $t \in [kT, (k+1)T)$ and $i = 1, 2, 3$ and $k = 0, 1, \ldots$. In Figure 4.1 we have plotted the controlled trajectories of the individual agents. It can be seen that the protocol resulting from the local control laws indeed achieves consensus. Note that the states of these agents achieves consensus at around $T = 150$. The results of a second simulation, this time with sampling period $T = 0.1$, are plotted in Figure 4.2. By comparing Figure 4.1 and 4.2, it can be seen that the network reaches consensus faster by taking a smaller sampling period.

4.8 Conclusions

We have studied the distributed linear quadratic control problem for a network of agents with single integrator dynamics. We have shown that the computation of control gains that minimize global cost functionals needs global information, in particular knowledge of the initial states of all agents and the Laplacian matrix. We have also shown that this drawback can be overcome by transforming the global cost functional into discounted local cost functionals and assigning each of these to an associated agent. In such a way, each agent computes its own control gain, using sampled information of its neighboring agents. Finally, we have shown that the resulting control protocol achieves consensus for the network.

5 A suboptimality approach to distributed $\mathcal{H}_2$ optimal control by state feedback

This chapter deals with the distributed $\mathcal{H}_2$ optimal control problem for linear multi-agent systems by static relative state feedback. In particular, we consider a *suboptimality* version of this problem. Given a linear multi-agent system with identical agent dynamics, an associated $\mathcal{H}_2$ cost functional, and a desired upper bound for the cost, our aim is to design a distributed diffusive static protocol such that the protocol achieves state synchronization while the associated cost is smaller than the given upper bound. To that end, we first analyze the $\mathcal{H}_2$ performance of linear systems and then apply the results to linear multi-agent systems. Based on these results, we provide a design method for computing such a distributed suboptimal protocol. The expression for the local control gain involves a solution to a single Riccati inequality of dimension equal to the dimension of the individual agent dynamics, and the smallest nonzero and the largest eigenvalue of the Laplacian matrix.

5.1 Introduction

The design of distributed protocols for multi-agent systems has received extensive attention in the past decade [85]. This increase in attention is partly due to the broad range of applications of multi-agent systems, e.g., formation control [79], intelligent transportation systems [5], and smart grids [17]. One of the challenging problems in the context of multi-agent systems is to develop distributed diffusive protocols that minimize given cost criteria, while the agents of the network reach a common goal, e.g., state synchronization. The difficulties of designing such distributed diffusive optimal protocols are due to the structural constraints on the communication among these agents, that is, each agent can only receive information from certain other agents. Therefore, in general, distributed optimal control problems are non-convex and difficult to solve.

To overcome this problem, much effort has been devoted to the design of distributed suboptimal protocols for multi-agent systems. In [7], the authors established a design method to compute distributed suboptimal stabilizing controllers with respect to a global linear quadratic cost functional, which contains terms that penalize the states and inputs of each agent and also the relative states between each agent and its neighboring agents. Later on, an inverse optimal control problem was addressed in [75]. In that paper, the authors showed that there exists a global optimal synchronizing controller if the weighting matrices of the linear quadratic cost functional are chosen to be of a certain special form. For other papers related to distributed linear quadratic optimal control, see also [37, 74, 77].

On the other hand, there has been some work on the design of structured stabilizing controllers for large-scale systems. In [91], the aim was to design decentralized optimal controllers, subject to some constraints on the controller structure, to minimize the closed-loop norm of a feedback system. The authors showed that if the constraints on the controller structure have the property of quadratic invariance, the solution of such problems can be computed efficiently via convex programming. In more recent work, [19] studied the distributed optimal problem for linear discrete-time systems. The authors showed that the problem can be relaxed to a semidefinite program, and a globally optimal distributed controller can be obtained if the semidefinite program relaxation has a rank one solution. In [18], the authors derived a condition under which, given a centralized optimal controller, there exists a distributed suboptimal controller whose state and input trajectories are close to those of the closed-loop system by using this centralized controller.

All the existing work mentioned above deals with finding distributed suboptimal protocols whose performance is very close to being optimal, or to find distributed optimal protocols for certain special cost functionals. In this chapter, however, we want to find a distributed diffusive suboptimal static protocol for linear multi-agent networks such that the associated cost functional is smaller than an, a priori given, desired tolerance (upper bound). We consider a group of identical agents whose dynamics is represented by a finite dimensional linear system. In addition, a connected, simple, undirected weighted graph is given, representing the communication between these agents. Furthermore, we introduce an $\mathcal{H}_2$ cost functional. Our aim is to design a distributed diffusive static protocol that achieves state synchronization and guarantees the associated cost to be smaller than a given upper bound.

The outline of this chapter is as follows. In Section 5.2, we formulate the distributed $\mathcal{H}_2$ suboptimal control problem for linear multi-agent systems. We then present the analysis and design of $\mathcal{H}_2$ suboptimal control laws for general linear systems in Section 5.3, providing necessary results for treating the distributed $\mathcal{H}_2$ suboptimal control problem. In Section 5.4, we deal with the distributed $\mathcal{H}_2$ suboptimal control problem. Finally, Section 5.5 concludes this chapter.

5.2 Problem formulation

In this chapter, we consider a multi-agent system consisting of N agents with identical dynamics. The interconnection topology among the agents is assumed to be represented by a connected, simple undirected weighted graph with associated Laplacian matrix L. The dynamics of agent i is represented by the following continuous-time linear-time-invariant (LTI) system

$$\begin{aligned} \dot{x}_i(t) &= Ax_i(t) + Bu_i(t) + Ed_i(t), \\ z_i(t) &= Cx_i(t) + Du_i(t), \end{aligned} \qquad i = 1, 2, \ldots, N, \tag{5.1}$$

where $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$, $z_i \in \mathbb{R}^p$ and $d_i \in \mathbb{R}^q$ are the state, the coupling input, the output and the external disturbance input of the ith agent, respectively. The matrices A, B, C, D and E have suitable dimensions. We assume that the pair (A, B) is stabilizable. In this chapter, we consider the case that the agents (5.1) are interconnected by means of a distributed diffusive static protocol of the form

$$u_i = K \sum_{j=1}^{N} a_{ij}(x_j - x_i), \quad i = 1, 2, \ldots, N, \tag{5.2}$$

where $K \in \mathbb{R}^{m \times n}$ is a feedback gain to be designed. The coefficients a_{ij} are the entries of the adjacency matrix $\mathcal{A}$ of the underlying graph.

Denote the aggregate vectors as

$$\begin{aligned} \mathbf{x} &= (x_1^\top, x_2^\top, \ldots, x_N^\top)^\top \in \mathbb{R}^{nN}, \quad \mathbf{u} = (u_1^\top, u_2^\top, \ldots, u_N^\top)^\top \in \mathbb{R}^{mN}, \\ \mathbf{z} &= (z_1^\top, z_2^\top, \ldots, z_N^\top)^\top \in \mathbb{R}^{pN}, \quad \mathbf{d} = (d_1^\top, d_2^\top, \ldots, d_N^\top)^\top \in \mathbb{R}^{qN}. \end{aligned}$$

We can then write system (5.1) in compact form as

$$\begin{aligned} \dot{\mathbf{x}} &= (I_N \otimes A)\mathbf{x} + (I_N \otimes B)\mathbf{u} + (I_N \otimes E)\mathbf{d}, \\ \mathbf{z} &= (I_N \otimes C)\mathbf{x} + (I_N \otimes D)\mathbf{u}, \end{aligned} \tag{5.3}$$

the protocol (5.2) is now of the form

$$\mathbf{u} = (L \otimes K)\mathbf{x}. \tag{5.4}$$

Foremost, we want our protocol to achieve state synchronization for the overall network. This is defined as follows.

Definition 5.1. *The protocol* (5.4) *is said to achieve state synchronization if, whenever the disturbance input is equal to zero, i.e.* $\mathbf{d} = 0$*, then for all* $i, j = 1, 2, \ldots, N$ *we have* $x_i(t) - x_j(t) \to 0$ *as* $t \to \infty$.

The distributed $\mathcal{H}_2$ optimal control problem is to minimize a given $\mathcal{H}_2$ cost functional for multi-agent system (5.3) over all protocols (5.4) that achieve state synchronization. Note that in the context of distributed control for multi-agent systems, we are interested in the differences of the state and output values of the agents in the controlled network. Observe also that the differences of the state and output values of communicating agents are captured by the incidence matrix R of the underlying graph. Therefore, we define a new output variable as

$$\boldsymbol{\zeta} = (W^{\frac{1}{2}} R^\top \otimes I_p)\mathbf{z}$$

with $\boldsymbol{\zeta} = (\zeta_1^\top, \zeta_2^\top, \ldots, \zeta_M^\top)^\top \in \mathbb{R}^{pM}$, where W is the weight matrix given by (1.2). Thus, the output $\boldsymbol{\zeta}$ reflects the weighted disagreement between the outputs of the agents in accordance with the weights of the edges connecting these agents. Subsequently, we have the following input/state/output model

$$\begin{aligned} \dot{\mathbf{x}} &= (I_N \otimes A)\mathbf{x} + (I_N \otimes B)\mathbf{u} + (I_N \otimes E)\mathbf{d}, \\ \boldsymbol{\zeta} &= (W^{\frac{1}{2}} R^\top \otimes C)\mathbf{x} + (W^{\frac{1}{2}} R^\top \otimes D)\mathbf{u}. \end{aligned} \tag{5.5}$$

Next, by substituting protocol (5.4) into equations (5.5), we obtain the following equations for the controlled network

$$\begin{aligned} \dot{\mathbf{x}} &= (I_N \otimes A + L \otimes BK)\mathbf{x} + (I_N \otimes E)\mathbf{d}, \\ \boldsymbol{\zeta} &= (W^{\frac{1}{2}} R^\top \otimes C + W^{\frac{1}{2}} R^\top L \otimes DK)\mathbf{x}. \end{aligned}$$

Denote $\tilde{A} := I_N \otimes A + L \otimes BK$, $\tilde{E} := I_N \otimes E$, $\tilde{C} := W^{\frac{1}{2}} R^\top \otimes C + W^{\frac{1}{2}} R^\top L \otimes DK$. The impulse response from the disturbance $\mathbf{d}$ to the output $\boldsymbol{\zeta}$ is then given by

$$T_K(t) = \tilde{C} e^{\tilde{A}t} \tilde{E}.$$

Subsequently, we define the associated $\mathcal{H}_2$ cost functional as

$$J(K) := \int_0^\infty \operatorname{tr}\left[T_K^\top(t) T_K(t)\right] dt. \tag{5.6}$$

Since the protocol (5.4) has a special form which contains the Kronecker product of the to be designed feedback gain K with the given Laplacian matrix L, the distributed $\mathcal{H}_2$ optimal control problem is non-convex, and therefore difficult to solve in general. Therefore, instead of trying to solve the distributed $\mathcal{H}_2$ optimal control problem itself, we will address a suboptimality version of this problem, i.e., we want to design a state synchronizing, distributed diffusive, static protocol such that the associated cost is smaller than an a priori given upper bound. More concretely, the problem we want to address is the following:

Problem 5.1. *Consider multi-agent system (5.3), with interconnection topology among the agents represented by a connected, simple undirected weighted graph with associated Laplacian matrix L, together with cost functional $J(K)$ given by (5.6). Let $\gamma > 0$ be a given tolerance. Our aim is to design a matrix $K \in \mathbb{R}^{n\times m}$ such that the distributed diffusive static protocol $\mathbf{u} = (L \otimes K)\mathbf{x}$ achieves state synchronization and $J(K) < \gamma$.*

Before we address Problem 5.1, we will first briefly discuss the $\mathcal{H}_2$ suboptimal control problem for general linear systems, in that way collecting the required preliminary results to treat the actual distributed $\mathcal{H}_2$ suboptimal control problem for multi-agent systems. This will be the subject of the next section.

5.3 $\mathcal{H}_2$ suboptimal control for linear systems by static state feedback

In this section, we consider the $\mathcal{H}_2$ suboptimal control problem for linear systems. We will first analyze the $\mathcal{H}_2$ performance of a given system with disturbance inputs. Subsequently, we will discuss how to design suboptimal protocols for a linear system with control inputs and disturbance inputs.

5.3.1 $\mathcal{H}_2$ performance analysis for linear systems with disturbance inputs

In this subsection, we will analyze the $\mathcal{H}_2$ performance for linear systems with disturbance inputs. More specifically, we consider the following linear system

$$\begin{aligned} \dot{x}(t) &= \bar{A}x(t) + \bar{E}d(t), \\ z(t) &= \bar{C}x(t), \end{aligned} \tag{5.7}$$

where $x \in \mathbb{R}^n$ represents the state, $d \in \mathbb{R}^q$ the disturbance input and $z \in \mathbb{R}^p$ the output. The matrices $\bar{A}$, $\bar{C}$ and $\bar{E}$ have suitable dimensions. The impulse response matrix of system (5.7) from the disturbance d to the output z is $T(t) = \bar{C}e^{\bar{A}t}\bar{E}$. The associated $\mathcal{H}_2$ performance is given by

$$J = \int_0^\infty \operatorname{tr}\left[T^\top(t)T(t)\right] dt, \tag{5.8}$$

which measures the performance of system (5.7) as the square of the $\mathcal{L}_2$-norm of its impulse response matrix. Note that performance (5.8) is finite if the system is internally stable, i.e., $\bar{A}$ is Hurwitz. Our aim is to find conditions such that the performance (5.8) is smaller than a given upper bound. For this, we have the following lemma. See also [124] or [93].

Lemma 5.1. *Consider system (5.7) with associated performance (5.8). The performance is finite if $\bar{A}$ is Hurwitz. In that case, we have*

$$J = \text{tr}\left(\bar{E}^\top Y \bar{E}\right), \tag{5.9}$$

where Y is the unique positive semidefinite solution of

$$\bar{A}^\top Y + Y\bar{A} + \bar{C}^\top \bar{C} = 0. \tag{5.10}$$

Alternatively,

$$J = \inf\{\text{tr}\left(\bar{E}^\top P \bar{E}\right) \mid P > 0 \text{ and } \bar{A}^\top P + P\bar{A} + \bar{C}^\top \bar{C} < 0\}. \tag{5.11}$$

For a proof of Lemma 5.1, see the proof of Theorem 4.6.2 in [100].

The following lemma now establishes a *necessary* and *sufficient* condition [33], such that the system (5.7) is stable and, for a given upper bound $\gamma > 0$, the performance (5.8) satisfies $J < \gamma$.

Lemma 5.2. *Consider system (5.7) with associated performance (5.8). Given $\gamma > 0$. Then $\bar{A}$ is Hurwitz and $J < \gamma$ if and only if there exists a positive definite matrix P satisfying*

$$\bar{A}^\top P + P\bar{A} + \bar{C}^\top \bar{C} < 0, \tag{5.12}$$

$$\text{tr}\left(\bar{E}^\top P \bar{E}\right) < \gamma. \tag{5.13}$$

Proof. (if) Let $P > 0$ satisfy (5.12). Then $\bar{A}^\top P + P\bar{A} < 0$. Note also that $P > 0$, which implies that $\bar{A}$ is Hurwitz. If $P > 0$ also satisfies (5.13), then it follows from Lemma 5.1 that $J \leqslant \text{tr}\left(\bar{E}^\top P \bar{E}\right) < \gamma$.

(only if) If $\bar{A}$ is Hurwitz and $J < \gamma$, it follows again from Lemma 5.1 that there exists a positive definite solution P to (5.12) and (5.13) such that $J \leqslant \text{tr}\left(\bar{E}^\top P \bar{E}\right) < \gamma$. □

5.3.2 $\mathcal{H}_2$ suboptimal control for linear systems with control and disturbance inputs

In this subsection, we will discuss the $\mathcal{H}_2$ suboptimal control problem for linear systems with control inputs and disturbance inputs. More specifically, we consider the linear system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + Ed(t), \\ z(t) &= Cx(t) + Du(t), \end{aligned} \tag{5.14}$$

where $x \in \mathbb{R}^n$ represents the state, $u \in \mathbb{R}^m$ the control input, $z \in \mathbb{R}^p$ the output, and $d \in \mathbb{R}^q$ the external disturbance input. The matrices A, B, C, D and E have suitable dimensions. We assume that the pair (A, B) is stabilizable. Using the static state feedback

$$u = Kx \tag{5.15}$$

yields the closed-loop system

$$\begin{aligned} \dot{x} &= (A + BK)x + Ed, \\ z &= (C + DK)x. \end{aligned} \tag{5.16}$$

We measure the performance of system (5.16) by considering the square of the $\mathcal{L}_2$-norm of its impulse response matrix. Therefore, we define the associated $\mathcal{H}_2$ cost functional as

$$J(K) = \int_0^\infty \text{tr}\left[T_K^\top(t) T_K(t)\right] dt, \tag{5.17}$$

where $T_K(t) = (C + DK)e^{(A+BK)t}E$ is the closed-loop impulse response matrix of system (5.16) from the disturbance input d to the output z. Let $\gamma > 0$ be a given upper bound for the cost $J(K)$. We are interested in finding a static state feedback of the form (5.15) such that $A + BK$ is Hurwitz and the associated cost is smaller than the given upper bound γ, i.e. $J(K) < \gamma$.

The following lemma yields a sufficient condition for the existence of such a static state feedback and how to compute one.

Lemma 5.3. *Consider system (5.14) with associated cost functional (5.17). Let $\gamma > 0$. Assume that the pair (A, B) is stabilizable. Assume that $D^\top C = 0$ and $D^\top D = I_m$. Suppose that there exists a positive definite matrix P satisfying*

$$A^\top P + PA - PBB^\top P + C^\top C < 0, \tag{5.18}$$

$$\text{tr}\left(E^\top PE\right) < \gamma. \tag{5.19}$$

Let $K = -B^\top P$. Then $A + BK$ is Hurwitz and $J(K) < \gamma$.

Proof. Substituting $K = -B^\top P$ into system (5.16) gives us

$$\begin{aligned} \dot{x} &= (A - BB^\top P)x + Ed, \\ z &= (C - DB^\top P)x. \end{aligned}$$

Since $D^\top C = 0$ and $D^\top D = I_m$, inequality (5.18) is equivalent to

$$(A - BB^\top P)^\top P + P(A - BB^\top P) + (C - DB^\top P)^\top (C - DB^\top P) < 0. \tag{5.20}$$

Since $P > 0$ is a solution of (5.18), it also satisfies (5.20), which implies that $A - BB^\top P$ is Hurwitz. Since (5.19) also holds, by taking $\bar{A} = A - BB^\top P$, $\bar{C} = C - DB^\top P$ and $\bar{E} = E$, it immediately follows from Theorem 5.2 that $J(K) < \gamma$. $\square$

5.4 Distributed $\mathcal{H}_2$ suboptimal protocols by static state feedback

In the previous section, we have dealt with the $\mathcal{H}_2$ suboptimal control problem for linear systems, collecting the necessary results for treating the distributed $\mathcal{H}_2$ suboptimal control problem. In the present section, we will deal with the distributed $\mathcal{H}_2$ suboptimal control problem for multi-agent networks with identical linear agent dynamics.

As has already been shown in Section 5.2, the dynamics of the multi-agent network we consider is given by

$$\begin{aligned} \dot{\mathbf{x}} &= (I_N \otimes A + L \otimes BK)\mathbf{x} + (I_N \otimes E)\mathbf{d}, \\ \boldsymbol{\zeta} &= (W^{\frac{1}{2}} R^\top \otimes C + W^{\frac{1}{2}} R^\top L \otimes DK)\mathbf{x}. \end{aligned} \tag{5.21}$$

For convenience, we also repeat here the associated $\mathcal{H}_2$ cost functional

$$J(K) = \int_0^\infty \mathrm{tr}\left[T_K^\top(t) T_K(t)\right] dt, \tag{5.22}$$

where $T_K(t) = \tilde{C} e^{\tilde{A}t} \tilde{E}$ is the impulse response matrix from the disturbance input $\mathbf{d}$ to the output $\boldsymbol{\zeta}$ with $\tilde{A} := I_N \otimes A + L \otimes BK$, $\tilde{E} := I_N \otimes E$ and $\tilde{C} := W^{\frac{1}{2}} R^\top \otimes C + W^{\frac{1}{2}} R^\top L \otimes DK$.

The distributed $\mathcal{H}_2$ suboptimal control problem is to find a distributed diffusive static protocol (5.4) with gain matrix K that achieves state synchronization and such that the associated cost (5.22) is smaller than a given upper bound $\gamma > 0$, i.e. $J(K) < \gamma$. We further assume that $D^\top C = 0$ and $D^\top D = I_m$, i.e. we assume that the distributed $\mathcal{H}_2$ suboptimal control problem is in *standard form*.

Next, we will first apply the state transformation

$$\bar{\mathbf{x}} = (U^\top \otimes I_n)\mathbf{x}$$

where the orthogonal matrix U is defined in (1.1). After this state transformation, the equations of the controlled network become

$$\begin{aligned} \dot{\bar{\mathbf{x}}} &= (I_N \otimes A + \Lambda \otimes BK)\bar{\mathbf{x}} + (U^\top \otimes E)\mathbf{d}, \\ \boldsymbol{\zeta} &= (W^{\frac{1}{2}} R^\top U \otimes C + W^{\frac{1}{2}} R^\top L U \otimes DK)\bar{\mathbf{x}}, \end{aligned}$$

and our cost functional is equal to

$$J(K) = \int_0^\infty \mathrm{tr}\left[T_K^\top(t) T_K(t)\right] dt, \tag{5.23}$$

where

$$T_K(t) = C_o e^{A_o t} E_o \tag{5.24}$$

is the impulse response matrix from the disturbance input $\mathbf{d}$ to the output $\boldsymbol{\zeta}$ with $A_o := I_N \otimes A + \Lambda \otimes BK$, $C_o := W^{\frac{1}{2}} R^\top U \otimes C + W^{\frac{1}{2}} R^\top LU \otimes DK$ and $E_o := U^\top \otimes E$. Note that, by applying the state transformation, only the system model has changed while the impulse response and the associated cost remain the same.

In order to proceed, we introduce the following $N-1$ auxiliary linear systems

$$\begin{aligned} \dot{\xi}_i &= A\xi_i + \lambda_i B v_i + E\delta_i, \\ \eta_i &= \sqrt{\lambda_i} C \xi_i + \lambda_i \sqrt{\lambda_i} D v_i, \end{aligned} \qquad i = 2, 3, \ldots, N, \tag{5.25}$$

where $\lambda_i, i = 2, 3, \ldots, N$ are the nonzero eigenvalues of the graph Laplacian L. Using in all systems (5.25) the identical static state feedback

$$v_i = K\xi_i, \quad i = 2, 3, \ldots, N \tag{5.26}$$

yields the closed-loop systems

$$\begin{aligned} \dot{\xi}_i &= (A + \lambda_i BK)\xi_i + E\delta_i, \\ \eta_i &= (\sqrt{\lambda_i} C + \lambda_i \sqrt{\lambda_i} DK)\xi_i, \end{aligned} \qquad i = 2, 3, \ldots, N. \tag{5.27}$$

We further introduce the associated $\mathcal{H}_2$ cost functionals

$$J_i(K) = \int_0^\infty \operatorname{tr}\left[T_{i,K}^\top(t) T_{i,K}(t)\right] dt, \quad i = 2, 3, \ldots, N, \tag{5.28}$$

where

$$T_{i,K} = (\sqrt{\lambda_i} C + \lambda_i \sqrt{\lambda_i} DK) e^{(A + \lambda_i BK)t} E, \tag{5.29}$$

for $i = 2, 3, \ldots, N$, are the closed-loop impulse response matrices from the disturbance δ_i to the output η_i, respectively.

It turns out that our original cost functional can be expressed as the sum of the cost functionals associated with the auxiliary systems (5.25). In fact, the following theorem holds.

Theorem 5.4. *Consider the network (5.21) with associated cost (5.22) and the systems (5.27) with associated costs (5.28) for $i = 2, 3, \ldots, N$, respectively. Then the protocol (5.4) achieves state synchronization for the network (5.21) if and only if the static state feedback*

(5.26) internally stabilizes all systems (5.25). Moreover,

$$J(K) = \sum_{i=2}^{N} J_i(K). \tag{5.30}$$

Proof. It is a standard result that the protocol (5.4) achieves state synchronization for the network (5.21) if and only if the static state feedback (5.26) internally stabilizes all systems (5.25). See e.g. [54] or [103].

We now prove (5.30). Let K be such that synchronization is achieved. Then we have

$$J(K) = \int_0^{\infty} \operatorname{tr}\left(\bar{E}_e^\top e^{\bar{A}_e^\top t}\bar{C}_e^\top \bar{C}_e e^{\bar{A}_e t}\bar{E}_e\right) dt$$

with

$$\bar{A}_e = I_N \otimes A + \Lambda \otimes BK, \quad \bar{C}_e = W^{\frac{1}{2}}R^\top U \otimes C + W^{\frac{1}{2}}R^\top LU \otimes DK, \quad \bar{E}_e = U^\top \otimes E.$$

Since $U^\top LU = \Lambda$, $L = RWR^\top$ and $D^\top C = 0$, we have $\bar{C}_e^\top \bar{C}_e = \tilde{C}_e^\top \tilde{C}_e$ with

$$\tilde{C}_e := \Lambda^{\frac{1}{2}} \otimes C + \Lambda^{\frac{3}{2}} \otimes DK.$$

We also have $\bar{E}_e \bar{E}_e^\top = \tilde{E}_e \tilde{E}_e^\top$ with $\tilde{E}_e := I_N \otimes E$. Thus we find that

$$\operatorname{tr}\left(\bar{E}_e^\top e^{\bar{A}_e^\top t}\bar{C}_e^\top \bar{C}_e e^{\bar{A}_e t}\bar{E}_e\right) = \operatorname{tr}\left(\tilde{E}_e^\top e^{\bar{A}_e^\top t}\tilde{C}_e^\top \tilde{C}_e e^{\bar{A}_e t}\tilde{E}_e\right). \tag{5.31}$$

We now analyze the matrix function $\tilde{C}_e e^{\bar{A}_e t}\tilde{E}_e$ appearing in (5.31). It is straightforward to show that

$$\tilde{C}_e e^{\bar{A}_e t}\tilde{E}_e = \operatorname{blockdiag}\left(0, C_2 e^{A_2 t}E, \ldots, C_N e^{A_N t}E\right),$$

where $A_i = A + \lambda_i BK$ and $C_i = \sqrt{\lambda_i}C + \lambda_i\sqrt{\lambda_i}DK$. Thus we find that

$$J(K) = \int_0^{\infty} \sum_{i=2}^{N} \operatorname{tr}\left[T_{i,K}^\top(t) T_{i,K}(t)\right] dt.$$

The claim (5.30) then follows immediately. □

Based on Theorem 5.4, we have transformed the problem of distributed $\mathcal{H}_2$ suboptimal control for the network (5.21) into $\mathcal{H}_2$ suboptimal control problems for $N-1$ linear systems (5.27) with the same feedback gain K. Next, we want to establish conditions under which all $N-1$ systems (5.27) are internally stable and the state feedback (5.26) is suboptimal.

The following lemma yields a *necessary* and *sufficient* condition for a given gain matrix $K \in \mathbb{R}^{m\times n}$ such that all systems (5.27) are internally stable and $\sum_{i=2}^{N} J_i(K) < \gamma$.

Lemma 5.5. *Consider the closed-loop systems (5.27) with the associated cost functionals* (5.28)*. Let $\gamma > 0$ be a given tolerance. The static state feedback* (5.26) *internally stabilizes all systems and $\sum_{i=2}^{N} J_i(K) < \gamma$ if and only if there exist positive definite matrices $P_i, i = 2, 3, \ldots, N$ satisfying*

$$(A+\lambda_i BK)^\top P_i + P_i(A+\lambda_i BK) \\ +(\sqrt{\lambda_i}C + \lambda_i\sqrt{\lambda_i}DK)^\top(\sqrt{\lambda_i}C + \lambda_i\sqrt{\lambda_i}DK) < 0, \tag{5.32}$$

$$\sum_{i=2}^{N} \operatorname{tr}\left(E^\top P_i E\right) < \gamma. \tag{5.33}$$

Proof. (if) Since (5.33) holds, there exist sufficiently small $\epsilon_i > 0, i = 2, \ldots, N$ such that $\sum_{i=2}^{N}\gamma_i < \gamma$ where $\gamma_i := \operatorname{tr}\left(E^\top P_i E\right) + \epsilon_i$. Because there exists P_i such that (5.32) and $\operatorname{tr}\left(E^\top P_i E\right) < \gamma_i$ hold for all $i = 2, \ldots, N$, by taking $\bar{A} = A + \lambda_i BK$ and $\bar{C} = \sqrt{\lambda_i}C + \lambda_i\sqrt{\lambda_i}DK$, it follows from Theorem 5.2 that all systems (5.27) are internally stable and $J_i(K) < \gamma_i$ for $i = 2, \ldots, N$. Therefore, $\sum_{i=2}^{N} J_i(K) < \gamma$.

(only if) Since $\sum_{i=2}^{N} J_i(K) < \gamma$, there exist sufficiently small $\epsilon_i > 0, i = 2, \ldots, N$ such that $\sum_{i=2}^{N}\gamma_i < \gamma$ where $\gamma_i := J_i(K) + \epsilon_i$. Because all systems (5.27) are internally stable and $J_i(K) < \gamma_i$ for $i = 2, \ldots, N$, by taking $\bar{A} = A + \lambda_i BK$ and $\bar{C} = \sqrt{\lambda_i}C + \lambda_i\sqrt{\lambda_i}DK$, it follows from Theorem 5.2 that there exist positive semi-definite matrices P_i such that (5.32) and $\operatorname{tr}\left(E^\top P_i E\right) < \gamma_i$ hold for all $i = 2, \ldots, N$. Since $\sum_{i=2}^{N}\gamma_i < \gamma$, this implies that $\sum_{i=2}^{N}\operatorname{tr}\left(E^\top P_i E\right) < \gamma$. $\square$

Lemma 5.5 establishes a necessary and sufficient condition for a given gain matrix K to internally stabilize all closed-loop systems (5.27) and to achieve $\sum_{i=2}^{N} J_i(K) < \gamma$. However, it does yet not provide a method to compute such gain matrix K. To this end, in the following theorem, we will provide a design method for computing such a gain matrix K and, correspondingly, a distributed suboptimal protocol for multi-agent system (5.1) together with cost functional (5.22).

Theorem 5.6. *Consider multi-agent system (5.1) with the associated cost functional (5.22). Let $\gamma > 0$ be a given tolerance. Let c be any real number such that $0 < c < \frac{2}{\lambda_N^2}$. We distinguish two cases:*

(i) if

$$0 < c \leqslant \frac{2}{\lambda_2^2 + \lambda_2\lambda_N + \lambda_N^2}, \tag{5.34}$$

then there exists a positive definite matrix P satisfying

$$A^\top P + PA + (c^2\lambda_2^3 - 2c\lambda_2)PBB^\top P + \lambda_N C^\top C < 0. \tag{5.35}$$

(ii) if

$$\frac{2}{\lambda_2^2 + \lambda_2\lambda_N + \lambda_N^2} < c < \frac{2}{\lambda_N^2}, \tag{5.36}$$

then there exists a positive definite matrix P satisfying

$$A^\top P + PA + (c^2\lambda_N^3 - 2c\lambda_N)PBB^\top P + \lambda_N C^\top C < 0. \tag{5.37}$$

In both cases, if in addition P satisfies

$$\text{tr}\left(E^\top PE\right) < \frac{\gamma}{N-1}. \tag{5.38}$$

Then protocol (5.4) with $K := -cB^\top P$ achieves synchronization, and it is suboptimal, i.e. $J(K) < \gamma$.

Proof. We will only provide the proof for case (i) above. Using the upper and lower bound on c given by (5.34), it can be verified that $c^2\lambda_i^3 - 2c\lambda_i \leqslant c^2\lambda_2^3 - 2c\lambda_2 < 0$ for $i = 2, 3, \ldots, N$. Since also $\lambda_i \leqslant \lambda_N$, one can see that the positive definite solution P of (5.35) also satisfies the $N-1$ Riccati inequalities

$$A^\top P + PA + (c^2\lambda_i^3 - 2c\lambda_i)PBB^\top P + \lambda_i C^\top C < 0 \tag{5.39}$$

for $i = 2, \ldots, N$. Equivalently, P also satisfies the Lyapunov inequalities

$$(A - c\lambda_i BB^\top P)^\top P + P(A - c\lambda_i BB^\top P) + c^2\lambda_i^3 PBB^\top P + \lambda_i C^\top C < 0,$$

for $i = 2, \ldots, N$. Taking $P_i = P$ for $i = 2, 3, \ldots, N$ and $K := -cB^\top P$ in inequalities (5.32) and (5.33) immediately gives us inequalities (5.39) and

$$\sum_{i=2}^{N} \text{tr}\left(E^\top PE\right) < \gamma.$$

Then it follows from Lemma 5.5 that all systems (5.27) are internally stable and $\sum_{i=2}^{N} J_i(K) < \gamma$. Furthermore, it follows from Theorem 5.4 that the protocol (5.4) achieves state synchronization for the network (5.21) and $J(K) < \gamma$. □

Remark 5.7. Theorem 5.6 states that by choosing c satisfying (5.34) for case (i) and $P > 0$ satisfying (5.35), the distributed static protocol with local gain $K = -cB^\top P$ is suboptimal if P also satisfies (5.38). Then the question arises: how should we choose c and P such that the local gain of the suboptimal protocol is 'best' in the sense that we have tr $\left(E^\top PE\right)$ and, consequently, $J(K)$ as small as possible? Since smaller P leads to smaller tr $\left(E^\top PE\right)$ and, consequently, smaller $J(K)$, we should therefore try to find P as small as possible. In fact, one can find a positive definite solution $P = P(c, \epsilon)$ to (5.35) by solving

$$A^\top P + PA - PBR(c)^{-1}B^\top P + Q(\epsilon) = 0$$

with $R(c) = \frac{1}{-c^2\lambda_2^3+2c\lambda_2}I_n$ and $Q(\epsilon) = \lambda_N C^\top C + \epsilon I_n$ where c is chosen as in (5.34) and $\epsilon > 0$ arbitrary. If c_1 and c_2 as in (5.34) satisfy $c_1 \leqslant c_2$, then we have $R(c_1) \leqslant R(c_2)$, so, clearly, $P(c_1, \epsilon) \leqslant P(c_2, \epsilon)$. Similarly, if $0 < \epsilon_1 \leqslant \epsilon_2$, we immediately have $Q(\epsilon_1) \leqslant Q(\epsilon_2)$. Again, it follows that $P(c, \epsilon_1) \leqslant P(c, \epsilon_2)$. Therefore, if we choose $\epsilon > 0$ very close to 0 and $c = \frac{2}{\lambda_2^2+\lambda_2\lambda_N+\lambda_N^2}$, we find the 'best' solution to the Riccati inequality (5.37) in the sense explained above.

Likewise, if we choose c satisfying (5.36) corresponding to case (ii), it can be shown that if we choose $\epsilon > 0$ very close to 0 and $c > 0$ very close to $\frac{2}{\lambda_2^2+\lambda_2\lambda_N+\lambda_N^2}$, we find the 'best' solution to the Riccati inequality (5.37).

5.5 Conclusions

In this chapter, we have studied a distributed $\mathcal{H}_2$ suboptimal control problem for linear multi-agent systems with connected, simple undirected weighted graph. Given a multi-agent system with identical agent dynamics, an associated global $\mathcal{H}_2$ cost functional and also a desired upper bound for the cost, we have provided a design method for computing a distributed suboptimal protocol such that the protocol achieves state synchronization and the associated cost is smaller than the given upper bound. The expression for the local control gain is provided in terms of solutions of a single Riccati inequality, whose dimension is equal to the dimension of the individual agent dynamics, and also involves the largest and the smallest nonzero eigenvalue of the Laplacian matrix.

6 Distributed $\mathcal{H}_2$ suboptimal control by dynamic output feedback

This chapter deals with distributed $\mathcal{H}_2$ suboptimal control by dynamic relative output feedback for homogeneous linear multi-agent systems. Given a linear multi-agent system, together with an associated $\mathcal{H}_2$ cost functional, the objective is to design dynamic output feedback protocols that guarantee the associated cost to be smaller than an a priori given upper bound while synchronizing the controlled network. A design method is provided to compute such protocols. The computation of the two local gains in these protocols involves two Riccati inequalities, each of dimension equal to the dimension of the state space of the agents. The largest and smallest nonzero eigenvalue of the Laplacian matrix of the network graph are also used in the computation of one of the two local gains. A simulation example is provided to illustrate the performance of the proposed protocols.

6.1 Introduction

The design of distributed protocols for networked multi-agent systems has been one of the most active research topics in the field of systems and control over the last two decades, see e.g. [11, 85]. This is partly due to the broad range of applications of multi-agent systems, e.g. smart grids [17], formation control [79, 118], and intelligent transportation systems [4]. One of the challenging problems in the context of linear multi-agent systems is the problem of developing distributed protocols to minimize given quadratic cost criteria while the agents reach a common goal, e.g., synchronization. Due to the structural constraints that are imposed on the control laws by the communication topology, such optimal control problems are difficult to solve. These structural constraints make distributed optimal control problems non-convex, and it is unclear under what conditions solutions exist in general.

In the existing literature, many efforts have been devoted to addressing distributed linear quadratic optimal control problems. In [7], distributed suboptimal stabilizing controllers were computed to stabilize multi-agent networks with identical agent dynamics subject to a global linear quadratic cost functional. For a network of agents with single integrator dynamics, an explicit expression for the optimal gain was given in [10], see also [38]. In [75] and [123], a distributed linear quadratic control problem was dealt with using an inverse optimality approach. This approach was further employed in [77] to design reduced order controllers. Recently, also in [37], the distributed LQ suboptimal control problem was considered. In parallel to the above, much work has been put into the problem of distributed $\mathcal{H}_2$ optimal control. Given a particular global $\mathcal{H}_2$ cost functional, [55] and [53] proposed distributed suboptimal stabilizing protocols involving static state feedback for multi-agent systems with undirected graphs. Later on, in [112] these results were generalized to directed graphs. For a given $\mathcal{H}_2$ cost criterion that penalizes the weighted differences between the outputs of the communicating agents, in [36] a distributed suboptimal synchronizing protocol based on static relative state feedback was established.

In the past, also the design of structured controllers for large-scale systems has attracted much attention. In [91], the notion of quadratic invariance was adopted to develop decentralized controllers that minimize the performance of the feedback system with constraints on the controller structure. In [57], the so called alternating direction method of multipliers was adopted to design sparse feedback gains that minimize an $\mathcal{H}_2$ performance. In [18], conditions were provided under which, for a given optimal centralized controller, a distributed suboptimal controller exists so that the resulting closed loop state and input trajectories are close in a certain sense.

The distributed $\mathcal{H}_2$ optimal control problem for multi-agent systems by dynamic relative output feedback is to find an optimal distributed dynamic protocol that achieves synchronization for the controlled network and that minimizes the $\mathcal{H}_2$ cost functional. This problem, however, is a non-convex optimization problem, and therefore it is unclear whether such optimal protocol exists, or whether a closed form solution can be given. Therefore, in this chapter, we look at an alternative version of this problem that requires only *suboptimality*. More precisely, we extend our preliminary results in Chapter 5 on static relative state feedback to the general case of dynamic protocols using relative measurement outputs. The main contributions of this chapter are the following.

1. We solve the open problem of finding, for a single continuous-time linear system, a separation principle based $\mathcal{H}_2$ suboptimal dynamic output feedback controller. This result extends the recent result in [25] on the separation principle in $\mathcal{H}_2$ suboptimal control for discrete-time systems.

2. Based on the above result, we provide a method for computing distributed $\mathcal{H}_2$ suboptimal dynamic output feedback protocols for linear multi-agent

systems.

The outline of this chapter is as follows. In Section 6.2, we formulate the distributed $\mathcal{H}_2$ suboptimal control problem by dynamic relative output feedback for linear multi-agent systems. In order to solve this problem, in Section 6.3, we first study $\mathcal{H}_2$ suboptimal control by dynamic output feedback for a single linear system. In Section 6.4 we then treat the problem introduced in Section 6.2. To illustrate our method, a simulation example is provided in Section 6.5. Finally, Section 6.6 concludes this chapter.

6.2 Problem formulation

In this chapter, we consider a homogeneous multi-agent system consisting of N identical agents, where the underlying network graph is a connected, simple undirected weighted graph with associated adjacency matrix $\mathcal{A}$ and Laplacian matrix L. The dynamics of the ith agent is represented by a finite-dimensional linear time-invariant system

$$
\begin{aligned}
\dot{x}_i &= Ax_i + Bu_i + Ed_i, \\
y_i &= C_1x_i + D_1d_i, \qquad\qquad i = 1, 2, \ldots, N, \\
z_i &= C_2x_i + D_2u_i,
\end{aligned} \tag{6.1}
$$

where $x_i \in \mathbb{R}^n$ is the state, $u_i \in \mathbb{R}^m$ is the coupling input, $d_i \in \mathbb{R}^q$ is an unknown external disturbance, $y_i \in \mathbb{R}^r$ is the measured output and $z_i \in \mathbb{R}^p$ is the output to be controlled. The matrices A, B, C_1, D_1, C_2, D_2 and E are of compatible dimensions. Throughout this chapter we assume that the pair (A, B) is stabilizable and the pair (C_1, A) is detectable. The agents (6.1) are to be interconnected by means of a dynamic output feedback protocol. Following [103] and [121], we consider observer based dynamic protocols of the form

$$
\begin{aligned}
\dot{w}_i &= Aw_i + B\sum_{j=1}^{N} a_{ij}(u_i - u_j) + G\left(\sum_{j=1}^{N} a_{ij}(y_i - y_j) - C_1w_i\right), \\
u_i &= Fw_i, \quad i = 1, 2, \ldots, N,
\end{aligned} \tag{6.2}
$$

where $G \in \mathbb{R}^{n\times r}$ and $F \in \mathbb{R}^{m\times n}$ are local gains to be designed. The coefficients a_{ij} are the entries of the adjacency matrix $\mathcal{A}$ of the underlying network graph. We briefly explain the structure of this protocol. Each local controller of the protocol (6.2) observes the weighted sum of the relative input signals $\sum_{j=1}^{N} a_{ij}(u_i - u_j)$ and the weighted sum of the disagreements between the measured output signals $\sum_{j=1}^{N} a_{ij}(y_i - y_j)$. The first equation in (6.2) in fact represents an asymptotic observer for the weighted sum of the relative states of agent i, and the state of this observer is an estimate of this value. Note that, for the error $e_i := w_i - \sum_{j=1}^{N} a_{ij}(x_i - x_j)$, the error dynamics is $\dot{e}_i = (A - GC_1)e_i + \sum_{j=1}^{N} a_{ij}(GD_1 - E)(d_i - d_j)$. If the disturbance inputs to all the agents are zero, i.e. $d_i = 0$ for $i = 1, 2, \ldots, N$, then the error dynamics is asymptotically stable if and only if $A - GC_1$ is Hurwitz. An estimate of the weighted sum of the relative states of each agent is then fed back to this agent using a static gain.

Denote by $\mathbf{x} = (x_1^\top, x_2^\top, \ldots, x_N^\top)^\top$ the aggregate state vector and likewise define $\mathbf{u}$, $\mathbf{y}$, $\mathbf{z}$, $\mathbf{d}$ and $\mathbf{w}$. The multi-agent system (6.1) can then be written in compact form as

$$\begin{aligned} \dot{\mathbf{x}} &= (I_N \otimes A)\mathbf{x} + (I_N \otimes B)\mathbf{u} + (I_N \otimes E)\mathbf{d}, \\ \mathbf{y} &= (I_N \otimes C_1)\mathbf{x} + (I_N \otimes D_1)\mathbf{d}, \\ \mathbf{z} &= (I_N \otimes C_2)\mathbf{x} + (I_N \otimes D_2)\mathbf{u}, \end{aligned} \tag{6.3}$$

and the dynamic protocol (6.2) is represented by

$$\begin{aligned} \dot{\mathbf{w}} &= (I_N \otimes (A - GC_1) + L \otimes BF)\, \mathbf{w} + (L \otimes G)\mathbf{y}, \\ \mathbf{u} &= (I_N \otimes F)\mathbf{w}. \end{aligned} \tag{6.4}$$

By interconnecting the network (6.3) using the dynamic protocol (6.4), we obtain the controlled network

$$\begin{aligned} \begin{pmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{w}} \end{pmatrix} &= \begin{pmatrix} I_N \otimes A & I_N \otimes BF \\ L \otimes GC_1 & I_N \otimes (A - GC_1) + L \otimes BF \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{w} \end{pmatrix} + \begin{pmatrix} I_N \otimes E \\ L \otimes GD_1 \end{pmatrix} \mathbf{d}, \\ \mathbf{z} &= \begin{pmatrix} I_N \otimes C_2 & I_N \otimes D_2F \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{w} \end{pmatrix}. \end{aligned} \tag{6.5}$$

Foremost, we want the dynamic protocol (6.4) to achieve synchronization for the network.

Definition 6.1. *The protocol* (6.4) *is said to synchronize the network if, whenever the external disturbances of all agents are equal to zero, i.e.* $\mathbf{d} = 0$, *we have* $x_i(t) - x_j(t) \to 0$ *and* $w_i(t) - w_j(t) \to 0$ *as* $t \to \infty$, *for all* $i, j = 1, 2, \ldots, N$.

The distributed $\mathcal{H}_2$ optimal control problem by dynamic output feedback is to minimize a given global $\mathcal{H}_2$ cost functional over all dynamic protocols of the form (6.4) that achieve synchronization for the controlled network. In the context of distributed control for multi-agent systems, we are interested in the differences of the state and output values of the agents in the controlled network, see e.g. [45] or [69]. Note that these differences are captured by the incidence matrix R of the underlying graph. Therefore, we introduce a new output variable as

$$\boldsymbol{\zeta} = (W^{\frac{1}{2}} R^\top \otimes I_p)\mathbf{z} \tag{6.6}$$

with $\boldsymbol{\zeta} = (\zeta_1^\top, \zeta_2^\top, \ldots, \zeta_M^\top)^\top \in \mathbb{R}^{pM}$, where W is the weight matrix of the underlying graph, as defined in (1.2). Thus, the output $\boldsymbol{\zeta}$ is the vector of weighted disagreements between the outputs of the agents, in which the weights are given by the square roots of the edge weights connecting these agents. Subsequently, we consider the network (6.5) with this new output:

$$\begin{aligned}
\begin{pmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{w}} \end{pmatrix} &= \begin{pmatrix} I_N \otimes A & I_N \otimes BF \\ L \otimes GC_1 & I_N \otimes (A - GC_1) + L \otimes BF \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{w} \end{pmatrix} + \begin{pmatrix} I_N \otimes E \\ L \otimes GD_1 \end{pmatrix} \mathbf{d}, \\
\boldsymbol{\zeta} &= \begin{pmatrix} W^{\frac{1}{2}} R^\top \otimes C_2 & W^{\frac{1}{2}} R^\top \otimes D_2 F \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{w} \end{pmatrix}.
\end{aligned} \tag{6.7}$$

Denote

$$\begin{aligned}
A_e &= \begin{pmatrix} I_N \otimes A & I_N \otimes BF \\ L \otimes GC_1 & I_N \otimes (A - GC_1) + L \otimes BF \end{pmatrix}, \\
C_e &= \begin{pmatrix} W^{\frac{1}{2}} R^\top \otimes C_2 & W^{\frac{1}{2}} R^\top \otimes D_2 F \end{pmatrix}, \\
E_e &= \begin{pmatrix} I_N \otimes E \\ L \otimes GD_1 \end{pmatrix}.
\end{aligned}$$

The impulse response matrix from the external disturbance $\mathbf{d}$ to the output $\boldsymbol{\zeta}$ is then equal to

$$T_{F,G}(t) = C_e e^{A_e t} E_e. \tag{6.8}$$

Next, the associated global $\mathcal{H}_2$ cost functional is defined to be the squared $\mathcal{L}_2$-norm of the closed loop impulse response, and is given by

$$J(F, G) := \int_0^\infty \operatorname{tr}\left[T_{F,G}^\top(t) T_{F,G}(t)\right] dt. \tag{6.9}$$

The distributed $\mathcal{H}_2$ optimal control problem by dynamic output feedback is the problem of minimizing (6.9) over all dynamic protocols of the form (6.4) that achieve synchronization for the network. Unfortunately, due to the particular form of the protocol (6.4), this optimization problem is, in general, non-convex and difficult to solve, and a closed form solution has not been provided in the literature up to now. Therefore, instead of trying to find an optimal solution, in this chapter we will address a suboptimality version of the problem. More specifically, we will design synchronizing dynamic protocols (6.4) that guarantee the associated cost (6.9) to be smaller than an a priori given upper bound. More concretely, the problem that we will address is the following:

Problem 6.1. *Let $\gamma > 0$ be a given tolerance. Design local gains $F \in \mathbb{R}^{m \times n}$ and $G \in \mathbb{R}^{n \times r}$ such that the dynamic protocol* (6.4) *achieves $J(F, G) < \gamma$ and synchronizes the network.*

Before we address Problem 6.1, we will first study the $\mathcal{H}_2$ suboptimal control problem by dynamic output feedback for a single linear system. In that way, we will collect the required preliminary results to treat the actual distributed $\mathcal{H}_2$ suboptimal control problem for multi-agent systems.

6.3 $\mathcal{H}_2$ suboptimal control for linear systems by dynamic output feedback

In this section, we will discuss the $\mathcal{H}_2$ suboptimal control problem by dynamic output feedback for a single linear system. This problem has been dealt with before, see e.g. [95], [96], [100] or [25]. In particular, in [25], the *separation principle* for $\mathcal{H}_2$ suboptimal control for discrete-time linear systems was established. Here, we will establish the analogue of that result for the continuous-time case.

Consider the linear system

$$\begin{aligned} \dot{x} &= \bar{A}x + \bar{B}u + \bar{E}d, \\ y &= \bar{C}_1 x + \bar{D}_1 d, \\ z &= \bar{C}_2 x + \bar{D}_2 u, \end{aligned} \tag{6.10}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ the control input, $d \in \mathbb{R}^q$ an unknown external disturbance, $y \in \mathbb{R}^r$ the measured output, and $z \in \mathbb{R}^p$ the output to be controlled. The matrices $\bar{A}$, $\bar{B}$, $\bar{C}_1$, $\bar{D}_1$, $\bar{C}_2$, $\bar{D}_2$ and $\bar{E}$ have compatible dimensions. In this section, we assume that the pair $(\bar{A}, \bar{B})$ is stabilizable and that the pair $(\bar{C}_1, \bar{A})$ is detectable. Moreover, we consider dynamic output feedback controllers of the form

$$\begin{aligned} \dot{w} &= \bar{A}w + \bar{B}u + G\left(y - \bar{C}_1 w\right), \\ u &= Fw, \end{aligned} \tag{6.11}$$

where $w \in \mathbb{R}^n$ is the state of the controller, and $F \in \mathbb{R}^{m \times n}$ and $G \in \mathbb{R}^{n \times r}$ are gain matrices to be designed. By interconnecting the controller (6.11) and the system (6.10), we obtain the controlled system

$$\begin{aligned} \begin{pmatrix} \dot{x} \\ \dot{w} \end{pmatrix} &= \begin{pmatrix} \bar{A} & \bar{B}F \\ G\bar{C}_1 & \bar{A} + \bar{B}F - G\bar{C}_1 \end{pmatrix} \begin{pmatrix} x \\ w \end{pmatrix} + \begin{pmatrix} \bar{E} \\ G\bar{D}_1 \end{pmatrix} d, \\ z &= \begin{pmatrix} \bar{C}_2 & \bar{D}_2 F \end{pmatrix} \begin{pmatrix} x \\ w \end{pmatrix}. \end{aligned} \tag{6.12}$$

Denote

$$A_a = \begin{pmatrix} \bar{A} & \bar{B}F \\ G\bar{C}_1 & \bar{A}+\bar{B}F-G\bar{C}_1 \end{pmatrix}, \quad C_a = \begin{pmatrix} \bar{C}_2 & \bar{D}_2F \end{pmatrix}, \quad E_a = \begin{pmatrix} \bar{E} \\ G\bar{D}_1 \end{pmatrix}.$$

Then the impulse response matrix from the disturbance d to the output z is given by $T_{F,G}(t) = C_a e^{A_a t} E_a$. Next, we introduce the associated $\mathcal{H}_2$ cost functional, given by

$$J(F,G) := \int_0^\infty \operatorname{tr}\left[T_{F,G}^\top(t) T_{F,G}(t)\right] dt. \tag{6.13}$$

We are interested in the problem of finding a controller of the form (6.11) such that the controlled system (6.12) is internally stable and the associated cost (6.13) is smaller than an a priori given upper bound.

Before we proceed, we will first review a well-known result that provides *necessary and sufficient* conditions such that a closed loop system is $\mathcal{H}_2$ suboptimal, see e.g. [95, Proposition 3.13].

Proposition 6.1. *Let $\gamma > 0$. Then the following statements are equivalent:*

(i) the system (6.12) *is internally stable and $J(F,G) < \gamma$.*

(ii) there exists $X_a > 0$ such that

$$\begin{aligned} A_a X_a + X_a A_a^\top + E_a E_a^\top &< 0, \\ \operatorname{tr}\left(C_a X_a C_a^\top\right) &< \gamma. \end{aligned} \tag{6.14}$$

(iii) there exists $Y_a > 0$ such that

$$\begin{aligned} A_a^\top Y_a + Y_a A_a + C_a^\top C_a &< 0, \\ \operatorname{tr}\left(E_a^\top Y_a E_a\right) &< \gamma. \end{aligned} \tag{6.15}$$

The following lemma is an extension of Theorem 6 in [25]. It provides conditions under which the controller (6.11) with gain matrices F and $G = Q\bar{C}_1^\top$ is suboptimal for the continuous-time system (6.10), where Q is a particular real symmetric solution of a given Riccati inequality. The result shows that the separation principle is also applicable in the context of $\mathcal{H}_2$ suboptimal control for continuous-time systems.

Lemma 6.2. *Let $\gamma > 0$ be a given tolerance. Assume that $\bar{D}_1\bar{E}^\top = 0$, $\bar{D}_2^\top\bar{C}_2 = 0$, $\bar{D}_1\bar{D}_1^\top = I_r$ and $\bar{D}_2^\top\bar{D}_2 > 0$. Let $F \in \mathbb{R}^{m\times n}$. Suppose that there exists $P > 0$ satisfying*

$$(\bar{A}+\bar{B}F)^\top P + P(\bar{A}+\bar{B}F) + (\bar{C}_2+\bar{D}_2F)^\top(\bar{C}_2+\bar{D}_2F) < 0. \tag{6.16}$$

Let $Q > 0$ be a solution of the Riccati inequality

$$\bar{A}Q + Q\bar{A}^\top - Q\bar{C}_1^\top \bar{C}_1 Q + \bar{E}\bar{E}^\top < 0. \tag{6.17}$$

If, moreover, the inequality

$$\operatorname{tr}\left(\bar{C}_1 QPQ\bar{C}_1^\top\right) + \operatorname{tr}\left(\bar{C}_2 Q\bar{C}_2^\top\right) < \gamma \tag{6.18}$$

holds, then the controller (6.11) *with the gains F and $G = Q\bar{C}_1^\top$ yields an internally stable closed loop system* (6.12)*, and it is suboptimal, i.e. $J(F,G) < \gamma$.*

Proof. Let $Q > 0$ satisfy (6.17) and gain matrix F be given. Note that (6.18) is equivalent to

$$\operatorname{tr}\left(\bar{C}_1 QPQ\bar{C}_1^\top\right) < \gamma - \operatorname{tr}\left(\bar{C}_2 Q\bar{C}_2^\top\right). \tag{6.19}$$

According to cases (ii) and (iii) in Proposition 6.1, there exists $P > 0$ satisfying (6.16) and (6.19) if and only if there exists $\Delta > 0$ satisfying

$$\operatorname{tr}\left((\bar{C}_2 + \bar{D}_2 F)\Delta(\bar{C}_2 + \bar{D}_2 F)^\top\right) < \gamma - \operatorname{tr}\left(\bar{C}_2 Q\bar{C}_2^\top\right) \tag{6.20}$$

and

$$(\bar{A} + \bar{B}F)\Delta + \Delta(\bar{A} + \bar{B}F)^\top + Q\bar{C}_1^\top \bar{C}_1 Q < 0. \tag{6.21}$$

On the other hand, by applying the state transformation

$$\begin{pmatrix} w \\ e \end{pmatrix} = \begin{pmatrix} 0 & I_n \\ -I_n & I_n \end{pmatrix} \begin{pmatrix} x \\ w \end{pmatrix}.$$

The system (6.12) then becomes

$$\begin{aligned} \begin{pmatrix} \dot{w} \\ \dot{e} \end{pmatrix} &= \begin{pmatrix} \bar{A} + \bar{B}F & -G\bar{C}_1 \\ 0 & \bar{A} - G\bar{C}_1 \end{pmatrix} \begin{pmatrix} w \\ e \end{pmatrix} + \begin{pmatrix} G\bar{D}_1 \\ G\bar{D}_1 - \bar{E} \end{pmatrix} d, \\ z &= \begin{pmatrix} \bar{C}_2 + \bar{D}_2 F & -\bar{C}_2 \end{pmatrix} \begin{pmatrix} w \\ e \end{pmatrix}. \end{aligned} \tag{6.22}$$

Clearly, the system (6.12) is internally stable if and only if $\bar{A} + \bar{B}F$ and $\bar{A} - G\bar{C}_1$ are Hurwitz. Thus, what remains to show is that the controller (6.11) with the gains F and $G = Q\bar{C}_1^\top$ internally stabilizes the system (6.10) and that $J(F,G) < \gamma$.

Note that (6.17) is equivalent to

$$(\bar{A} - Q\bar{C}_1^\top \bar{C}_1)Q + Q(\bar{A} - Q\bar{C}_1^\top \bar{C}_1)^\top + (\bar{E} + Q\bar{C}_1^\top \bar{D}_1)(\bar{E} + Q\bar{C}_1^\top \bar{D}_1)^\top < 0, \tag{6.23}$$

where we use the fact that $\bar{D}_1 \bar{E}^\top = 0$ and $\bar{D}_1 \bar{D}_1^\top = I_r$. Since $G = Q\bar{C}_1^\top$, it then follows that $\bar{A} - G\bar{C}_1$ is Hurwitz. Similarly, it follows from (6.16) that $\bar{A} + \bar{B}F$ is

Hurwitz. Consequently, the system (6.12) is internally stable.

Next, we will show that $J(F,G) < \gamma$. Again consider (6.22) and denote

$$\bar{A}_a = \begin{pmatrix} \bar{A}+\bar{B}F & -G\bar{C}_1 \\ 0 & \bar{A}-G\bar{C}_1 \end{pmatrix}, \quad \bar{C}_a = \begin{pmatrix} \bar{C}_2+\bar{D}_2F & -\bar{C}_2 \end{pmatrix}, \quad \bar{E}_a = \begin{pmatrix} G\bar{D}_1 \\ G\bar{D}_1-\bar{E} \end{pmatrix}.$$

According to Proposition 6.1, in particular the inequalities in (6.14), we have $J(F,G) < \gamma$ if and only if there exists $P_a > 0$ satisfying

$$\begin{aligned} \bar{A}_a P_a + P_a \bar{A}_a^\top + \bar{E}_a \bar{E}_a^\top < 0, \\ \operatorname{tr}(\bar{C}_a P_a \bar{C}_a^\top) < \gamma. \end{aligned} \tag{6.24}$$

We will show that the existence of solutions $Q > 0$ and $\Delta > 0$ to the inequalities (6.17), (6.20) and (6.21) implies that (6.24) has a solution $P_a > 0$. Let

$$P_a = \begin{pmatrix} \Delta & 0 \\ 0 & Q \end{pmatrix}.$$

Clearly, $P_a > 0$. By substituting P_a , $\bar{A}_a$, $\bar{E}_a$ and $\bar{C}_a$ into (6.24), we obtain

$$\begin{pmatrix} R_1 & R_{12} \\ R_{12}^\top & R_2 \end{pmatrix} < 0, \tag{6.25}$$

where

$$\begin{aligned} R_1 &= (\bar{A}+\bar{B}F)\Delta + \Delta(\bar{A}-\bar{B}F)^\top + GG^\top, \\ R_{12} &= -G\bar{C}_1 Q + GG^\top, \\ R_2 &= (\bar{A}-G\bar{C}_1)Q + Q(\bar{A}-G\bar{C}_1)^\top + (G\bar{D}_1-\bar{E})(G\bar{D}_1-\bar{E})^\top, \end{aligned}$$

and

$$\text{tr}\left((\bar{C}_2+\bar{D}_2F)\Delta(\bar{C}_2+\bar{D}_2F)^\top\right) + \text{tr}\left(\bar{C}_2 Q \bar{C}_2^\top\right) < \gamma. \tag{6.26}$$

It then follows from $G = Q\bar{C}_1^\top$, (6.17) and (6.21) that $R_1 < 0$, $R_{12} = 0$ and $R_2 < 0$. Subsequently, $R < 0$. Also, it follows from (6.20) that (6.26) holds. Hence, $J(F,G) < \gamma$. This completes the proof. □

Theorem 6.3. *Let $\gamma > 0$. Assume that $\bar{D}_1\bar{E}^\top = 0$, $\bar{D}_2^\top\bar{C}_2 = 0$ and $\bar{D}_1\bar{D}_1^\top = I_r$, $\bar{D}_2^\top\bar{D}_2 > 0$. Suppose that there exist $P > 0$ and $Q > 0$ satisfying*

$$\bar{A}^\top P + P\bar{A} - P\bar{B}(\bar{D}_2^\top\bar{D}_2)^{-1}\bar{B}^\top P + \bar{C}_2^\top\bar{C}_2 < 0, \tag{6.27}$$

$$\bar{A}Q + Q\bar{A}^\top - Q\bar{C}_1^\top\bar{C}_1 Q + \bar{E}\bar{E}^\top < 0, \tag{6.28}$$

$$\text{tr}\left(\bar{C}_1 QPQ\bar{C}_1^\top\right) + \text{tr}\left(\bar{C}_2 Q\bar{C}_2^\top\right) < \gamma. \tag{6.29}$$

Let $G = Q\bar{C}_1^\top$ *and* $F = -(\bar{D}_2^\top \bar{D}_2)^{-1}\bar{B}^\top P$. *Then the controller* (6.11) *internally stabilizes the system* (6.10), *and it is suboptimal, i.e.* $J(F,G) < \gamma$.

Proof. Substituting $F = -(\bar{D}_2^\top \bar{D}_2)^{-1}\bar{B}^\top P$ into (6.16) gives us the inequality (6.27). The rest follows from Lemma 6.2. □

We are now ready to deal with the distributed $\mathcal{H}_2$ suboptimal control problem by dynamic output feedback for multi-agent systems.

6.4 Distributed $\mathcal{H}_2$ suboptimal protocols by dynamic output feedback

In this section, we will address Problem 6.1. For the multi-agent system (6.1), we will establish a design method for local gains F and G such that the protocol (6.2) achieves $J(F,G) < \gamma$ and synchronizes the network (6.7).

Let U be an orthogonal matrix such that $U^\top LU = \Lambda = \mathrm{diag}(0, \lambda_2, \ldots, \lambda_N)$ with $0 = \lambda_1 < \lambda_2 \leqslant \cdots \leqslant \lambda_N$ the eigenvalues of the Laplacian matrix. We apply the state transformation

$$\begin{pmatrix} \bar{\mathbf{x}} \\ \bar{\mathbf{w}} \end{pmatrix} = \begin{pmatrix} U^\top \otimes I_n & 0 \\ 0 & U^\top \otimes I_n \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{w} \end{pmatrix}. \tag{6.30}$$

Then the controlled network (6.7) is also represented by

$$\begin{aligned} \begin{pmatrix} \dot{\bar{\mathbf{x}}} \\ \dot{\bar{\mathbf{w}}} \end{pmatrix} &= \begin{pmatrix} I_N \otimes A & I_N \otimes BF \\ \Lambda \otimes GC_1 & I_N \otimes (A - GC_1) + \Lambda \otimes BF \end{pmatrix} \begin{pmatrix} \bar{\mathbf{x}} \\ \bar{\mathbf{w}} \end{pmatrix} + \begin{pmatrix} U^\top \otimes E \\ U^\top L \otimes GD_1 \end{pmatrix} \mathbf{d}, \\ \boldsymbol{\zeta} &= \begin{pmatrix} W^{\frac{1}{2}} R^\top U \otimes C_2 & W^{\frac{1}{2}} R^\top U \otimes D_2 F \end{pmatrix} \begin{pmatrix} \bar{\mathbf{x}} \\ \bar{\mathbf{w}} \end{pmatrix}. \end{aligned} \tag{6.31}$$

Denote

$$\begin{aligned} \bar{A}_e &= \begin{pmatrix} I_N \otimes A & I_N \otimes BF \\ \Lambda \otimes GC_1 & I_N \otimes (A - GC_1) + \Lambda \otimes BF \end{pmatrix}, \\ \bar{C}_e &= \begin{pmatrix} W^{\frac{1}{2}} R^\top U \otimes C_2 & W^{\frac{1}{2}} R^\top U \otimes D_2 F \end{pmatrix}, \\ \bar{E}_e &= \begin{pmatrix} U^\top \otimes E \\ U^\top L \otimes GD_1 \end{pmatrix}. \end{aligned}$$

Obviously, the impulse response matrix $T_{F,G}(t)$ given by (6.8) is then equal to $\bar{C}_e e^{\bar{A}_e t} \bar{E}_e$.

In order to proceed, we now introduce the $N-1$ auxiliary linear systems

$$\begin{aligned} \dot{\xi}_i &= A\xi_i + \lambda_i B v_i + E\delta_i, \\ \vartheta_i &= C_1\xi_i + D_1\delta_i, \\ \eta_i &= \sqrt{\lambda_i}C_2\xi_i + \lambda_i\sqrt{\lambda_i}D_2 v_i, \end{aligned} \tag{6.32}$$

and associated dynamic output feedback controllers

$$\begin{aligned} \dot{\omega}_i &= A\omega_i + \lambda_i B v_i + G(\vartheta_i - C_1\omega_i), \\ v_i &= F\omega_i, \quad i = 2, 3, \ldots, N \end{aligned} \tag{6.33}$$

with gain matrices F and G. By interconnecting (6.33) and (6.32), we obtain the $N-1$ closed loop systems

$$\begin{aligned} \begin{pmatrix} \dot{\xi}_i \\ \dot{\omega}_i \end{pmatrix} &= \begin{pmatrix} A & \lambda_i BF \\ GC_1 & A - GC_1 + \lambda_i BF \end{pmatrix} \begin{pmatrix} \xi_i \\ \omega_i \end{pmatrix} + \begin{pmatrix} E \\ GD_1 \end{pmatrix} \delta_i, \\ \eta_i &= \begin{pmatrix} \sqrt{\lambda_i}C_2 & \lambda_i\sqrt{\lambda_i}D_2F \end{pmatrix} \begin{pmatrix} \xi_i \\ \omega_i \end{pmatrix}, \end{aligned} \tag{6.34}$$

for $i = 2, 3, \ldots, N$. The impulse response matrix of (6.34) from the disturbance δ_i to the output η_i is equal to

$$T_{i,F,G}(t) = \bar{C}_i e^{\bar{A}_i t}\bar{E}_i \tag{6.35}$$

with

$$\bar{A}_i = \begin{pmatrix} A & \lambda_i BF \\ GC_1 & A - GC_1 + \lambda_i BF \end{pmatrix}, \; \bar{E}_i = \begin{pmatrix} E \\ GD_1 \end{pmatrix}, \; \bar{C}_i = \begin{pmatrix} \sqrt{\lambda_i}C_2 & \lambda_i\sqrt{\lambda_i}D_2F \end{pmatrix}.$$

Furthermore, for each system (6.32) the associated $\mathcal{H}_2$ cost functional is given by

$$J_i(F,G) := \int_0^\infty \operatorname{tr}\left[T_{i,F,G}^\top(t) T_{i,F,G}(t)\right] dt, \quad i = 2, 3, \ldots, N. \tag{6.36}$$

Then we have the following lemma:

Lemma 6.4. *Let $F \in \mathbb{R}^{m\times n}$ and $G \in \mathbb{R}^{n\times r}$. Then the dynamic protocol* (6.2) *with gain matrices F and G achieves synchronization for the network* (6.7) *if and only if for each $i = 2, 3, \ldots, N$ the controller* (6.33) *with gain matrices F and G internally stabilizes the system* (6.32)*. Moreover, we have*

$$J(F,G) = \sum_{i=2}^N J_i(F,G). \tag{6.37}$$

Proof. It follows immediately from [103, Lemmas 3.2 and 3.3] that the dynamic protocol (6.2) achieves synchronization for the network (6.7) if and only if for $i = 2, 3, \dots, N$ the system (6.32) is internally stabilized by the controller (6.33).

Next, we prove (6.37). Let F and G be such that synchronization is achieved. Then we have

$$J(F, G) = \int_0^\infty \text{tr}\left(\bar{E}_e^\top e^{\bar{A}_e^\top t} \bar{C}_e^\top \bar{C}_e e^{\bar{A}_e t} \bar{E}_e\right) dt.$$

Since $U^\top L U = \Lambda$, $L = RWR^\top$, we have $\bar{C}_e^\top \bar{C}_e = \tilde{C}_e^\top \tilde{C}_e$ with

$$\tilde{C}_e := \left(\Lambda^{\frac{1}{2}} \otimes C_2 \;\; \Lambda^{\frac{1}{2}} \otimes D_2 F\right).$$

We also have $\bar{E}_e \bar{E}_e^\top = \tilde{E}_e \tilde{E}_e^\top$ with

$$\tilde{E}_e := \begin{pmatrix} I_N \otimes E \\ \Lambda \otimes G D_1 \end{pmatrix}.$$

Thus we find that

$$\text{tr}\left(\bar{E}_e^\top e^{\bar{A}_e^\top t} \bar{C}_e^\top \bar{C}_e e^{\bar{A}_e t} \bar{E}_e\right) = \text{tr}\left(\tilde{E}_e^\top e^{\bar{A}_e^\top t} \tilde{C}_e^\top \tilde{C}_e e^{\bar{A}_e t} \tilde{E}_e\right). \tag{6.38}$$

We now analyze the matrix function $\tilde{C}_e e^{\bar{A}_e t} \tilde{E}_e$ appearing in (6.38). By applying suitable permutations of the blocks appearing in the matrices $\tilde{C}_e$, $\tilde{E}_e$ and $\bar{A}_e$, it is straightforward to show that

$$\tilde{C}_e e^{\bar{A}_e t} \tilde{E}_e = \text{blockdiag}\left(0, C_2 e^{A_2 t} E_2, \dots, C_N e^{A_N t} E_N\right),$$

where

$$A_i := \begin{pmatrix} A & BF \\ \lambda_i G C_1 & A - G C_1 + \lambda_i B F \end{pmatrix},\; C_i := \begin{pmatrix} \sqrt{\lambda_i} C_2 & \sqrt{\lambda_i} D_2 F \end{pmatrix},\; E_i := \begin{pmatrix} E \\ \lambda_i G D_1 \end{pmatrix}.$$

It is easily seen that for $i = 2, 3, \dots, N$ the systems (A_i, E_i, C_i) and $(\bar{A}_i, \bar{E}_i, \bar{C}_i)$ are isomorphic. Hence they have the same impulse response $T_{i,F,G}(t)$, which is given by (6.35), see e.g., [102, Theorem 3.10]. As a consequence we obtain that

$$\tilde{C}_e e^{\bar{A}_e t} \tilde{E}_e = \text{blockdiag}\left(0, T_{2,F,G}(t), \dots, T_{N,F,G}(t)\right).$$

Thus we find that

$$J(F, G) = \int_0^\infty \sum_{i=2}^N \text{tr}\left[T_{i,F,G}^\top(t) T_{i,F,G}(t)\right] dt.$$

The claim (6.37) then follows immediately. $\square$

By applying Lemma 6.4, we have transformed the distributed $\mathcal{H}_2$ suboptimal control problem by dynamic output feedback for the multi-agent network (6.7) into $\mathcal{H}_2$ suboptimal control problems for the $N-1$ linear systems (6.32) using controllers (6.33) with the same gain matrices F and G. Next, we establish conditions under which the $N-1$ systems (6.32) are internally stabilized by their corresponding controllers (6.33) for $i = 2, 3, \ldots, N$, while achieving $\sum_{i=2}^{N} J_i(F, G) < \gamma$.

Lemma 6.5. *Let $\gamma > 0$ be a given tolerance. Assume that $D_1 E^\top = 0$, $D_2^\top C_2 = 0$, $D_1 D_1^\top = I_r$ and $D_2^\top D_2 = I_m$. For $i = 2, 3, \ldots, N$, let F, $P_i > 0$, and $Q > 0$ be such that the inequalities*

$$
\begin{aligned}
(A + \lambda_i BF)^\top P_i + P_i(A + \lambda_i BF)& \\
+(\sqrt{\lambda_i} C_2 + \lambda_i \sqrt{\lambda_i} D_2 F)^\top (\sqrt{\lambda_i} C_2 + \lambda_i \sqrt{\lambda_i} D_2 F) &< 0, \qquad (6.39)\\
AQ + QA^\top - QC_1^\top C_1 Q + EE^\top &< 0, \qquad (6.40)\\
\sum_{i=2}^{N} \left[\mathrm{tr}\left(C_1 Q P_i Q C_1^\top\right) + \lambda_i \mathrm{tr}\left(C_2 Q C_2^\top\right)\right] &< \gamma \qquad (6.41)
\end{aligned}
$$

hold. Then for each $i = 2, 3, \ldots, N$, the controller (6.33) *with gain matrices F and $G = QC_1^\top$ internally stabilizes the system* (6.32)*, and, moreover, $\sum_{i=2}^{N} J_i(F, G) < \gamma$.*

Proof. By (6.41), for $\epsilon_i > 0$ sufficiently small, we have $\sum_{i=2}^{N} \gamma_i < \gamma$, where $\gamma_i := \mathrm{tr}\left(C_1 Q P_i Q C_1^\top\right) + \lambda_i \mathrm{tr}\left(C_2 Q C_2^\top\right) + \epsilon_i$. Since $\mathrm{tr}\left(C_1 Q P_i Q C_1^\top\right) + \lambda_i \mathrm{tr}\left(C_2 Q C_2^\top\right) < \gamma_i$, by taking $\bar{A} = A$, $\bar{B} = \lambda_i B$, $\bar{C}_1 = C_1$, $\bar{D}_1 = D_1$, $\bar{C}_2 = \sqrt{\lambda_i} C_2$, $\bar{D}_2 = \lambda_i \sqrt{\lambda_i} D_2$, $\bar{C}_1 = C_1$ and $\bar{E} = E$ in Lemma 6.2, it follows that the controller (6.33) internally stabilizes the system (6.32) and $J_i(F, G) < \gamma_i$. Thus, from $\sum_{i=2}^{N} \gamma_i < \gamma$ it follows that $\sum_{i=2}^{N} J_i(F, G) < \gamma$. $\square$

Again, we note that the four conditions $D_1 E^\top = 0$, $D_2^\top C_2 = 0$, $D_1 D_1^\top = I_r$ and $D_2^\top D_2 = I_m$ are made here to simplify notation, and can be replaced by the regularity conditions $D_1 D_1^\top > 0$ and $D_2^\top D_2 > 0$ alone.

By combining Lemma 6.4 and Lemma 6.5 we have established sufficient conditions for given gain matrices F and G to synchronize the network (6.7) and to be suboptimal, i.e. $J(F, G) < \gamma$. In fact, G is taken to be equal to $QC_1^\top$, with $Q > 0$ a solution to the Riccati inequality (6.40). However, no design method has yet been provided to compute a suitable matrix F. In the following theorem, we will establish a design method for computing such gain matrix F. Together with G given above, this will lead to a distributed suboptimal protocol for multi-agent system (6.1) with associated cost functional (6.9).

Theorem 6.6. *Let $\gamma > 0$ be a given tolerance. Assume that $D_1 E^\top = 0$, $D_2^\top C_2 = 0$, $D_1 D_1^\top = I_r$ and $D_2^\top D_2 = I_m$. Let $Q > 0$ satisfy*

$$AQ + QA^\top - QC_1^\top C_1 Q + E^\top E < 0. \tag{6.42}$$

Let c be any real number such that $0 < c < \frac{2}{\lambda_N^2}$. We distinguish two cases:

(i) if

$$\frac{2}{\lambda_2^2 + \lambda_2\lambda_N + \lambda_N^2} \leqslant c < \frac{2}{\lambda_N^2}, \tag{6.43}$$

then there exists $P > 0$ satisfying

$$A^\top P + PA + (c^2\lambda_N^3 - 2c\lambda_N)PBB^\top P + \lambda_N C_2^\top C_2 < 0. \tag{6.44}$$

(ii) if

$$0 < c < \frac{2}{\lambda_2^2 + \lambda_2\lambda_N + \lambda_N^2}, \tag{6.45}$$

then there exists $P > 0$ satisfying

$$A^\top P + PA + (c^2\lambda_2^3 - 2c\lambda_2)PBB^\top P + \lambda_N C_2^\top C_2 < 0. \tag{6.46}$$

In both cases, if in addition P and Q satisfy

$$\operatorname{tr}\left(C_1 QPQC_1^\top\right) + \lambda_N \operatorname{tr}\left(C_2 QC_2^\top\right) < \frac{\gamma}{N-1}. \tag{6.47}$$

Then the protocol (6.2) *with $F := -cB^\top P$ and $G := QC_1^\top$ synchronizes the network* (6.7) *and it is suboptimal, i.e. $J(F,G) < \gamma$.*

Proof. We will only provide the proof for case (i) above. Using the upper and lower bound on c given by (6.43), it can be verified that $c^2\lambda_N^3 - 2c\lambda_N < 0$. Thus the Riccati inequality (6.44) has positive definite solutions. Since $c^2\lambda_i^3 - 2c\lambda_i \leqslant c^2\lambda_N^3 - 2c\lambda_N < 0$ and $\lambda_i \leqslant \lambda_N$ for $i = 2, 3, \ldots, N$, any positive definite solution P of (6.44) also satisfies the $N-1$ Riccati inequalities

$$A^\top P + PA + (c^2\lambda_i^3 - 2c\lambda_i)PBB^\top P + \lambda_i C_2^\top C_2 < 0, \tag{6.48}$$

equivalently,

$$(A - c\lambda_i BB^\top P)^\top P + P(A - c\lambda_i BB^\top P) + c^2\lambda_i^3 PBB^\top P + \lambda_i C_2^\top C_2 < 0, \tag{6.49}$$

for $i = 2, \dots, N$. Using the conditions $D_2^\top C_2 = 0$ and $D_2^\top D_2 = I_m$ this yields

$$\begin{aligned} &(A - c\lambda_i BB^\top P)^\top P + P(A - c\lambda_i BB^\top P) \\ &\quad + (\sqrt{\lambda_i} C_2 + \lambda_i \sqrt{\lambda_i} D_2 B^\top P)^\top (\sqrt{\lambda_i} C_2 + \lambda_i \sqrt{\lambda_i} D_2 B^\top P) < 0, \end{aligned} \tag{6.50}$$

for $i = 2, \dots, N$. Taking $P_i = P$ for $i = 2, 3, \dots, N$ and $F = -cB^\top P$ in (6.50) immediately yields (6.39). Next, it follows from (6.47) that also (6.41) holds. By Lemma 6.5 then, all systems (6.32) are internally stabilized and $\sum_{i=2}^N J_i(F, G) < \gamma$. Subsequently, it follows from Lemma 6.4 that the protocol (6.2) achieves synchronization for the network (6.7) and $J(F, G) < \gamma$. □

Remark 6.7. In Theorem 6.6, in order to select γ, the following should be done:

(i) First compute a solution $Q > 0$ of the Riccati inequality (6.42) and a solution $P > 0$ of the Riccati inequality (6.44) (or (6.46), depending on the choice of parameter c). Note that these solutions exist.

(ii) Let $S(P, Q) := \mathrm{tr}(C_1 QPQC_1^\top) + \lambda_N \mathrm{tr}(C_2 QC_2^\top)$.

(iii) Then choose $\gamma > 0$ such that $(N - 1)S(P, Q) < \gamma$.

Obviously, the smaller $S(P, Q)$, the smaller the feasible upper bound γ. It can be shown that, unfortunately, the problem of minimizing $S(P, Q)$ over all $P, Q > 0$ that satisfy (6.42) and (6.44) is a nonconvex optimization problem. However, since smaller Q leads to smaller tr $\left(C_2 QC_2^\top\right)$ and smaller P and Q leads to smaller tr $\left(C_1 QPQC_1^\top\right)$ and, consequently, smaller feasible γ, we could therefore try to find P and Q as small as possible. In fact, one can find $Q = Q(\epsilon) > 0$ to (6.42) by solving

$$AQ + QA^\top - QC_1^\top C_1 Q + E^\top E + \epsilon I_n = 0. \tag{6.51}$$

with $\epsilon > 0$ arbitrary. By using a standard argument, it can be shown that $Q(\epsilon)$ decreases as ϵ decreases, so ϵ should be taken close to 0 in order to get small Q. Similarly, one can find $P = P(c, \sigma) > 0$ satisfying (6.44) by solving

$$A^\top P + PA - PBR(c)^{-1} B^\top P + \lambda_N C_2^\top C_2 + \sigma I_n = 0 \tag{6.52}$$

with $R(c) = \frac{1}{-c^2\lambda_N^3 + 2c\lambda_N} I_n$, where c is chosen as in (6.43) and $\sigma > 0$ arbitrary. Again, it can be shown that $P(c, \sigma)$ decreases with decreasing σ and c. Therefore, small P is obtained by choosing $\sigma > 0$ close to 0 and $c = \frac{2}{\lambda_2^2 + \lambda_2\lambda_N + \lambda_N^2}$.

Similarly, if c satisfies (6.45) corresponding to case (ii), it can be shown that if we choose $\epsilon > 0$ and $\sigma > 0$ very close to 0 and $c > 0$ very close to $\frac{2}{\lambda_2^2 + \lambda_2\lambda_N + \lambda_N^2}$, we find small solutions to the Riccati inequalities (6.42) and (6.46) in the sense as explained above for case (i).

Remark 6.8. In Theorem 6.6, exact knowledge of the largest and the smallest nonzero eigenvalue of the Laplacian matrix is used to compute the local control gains F and G. We want to remark that our results can be extended to the case that only lower and upper bounds for these eigenvalues are known. In the literature, algorithms are given to estimate λ_2 in a distributed way, yielding lower and upper bounds, see e.g. [2]. Also, an upper bound for λ_N can be obtained in terms of the maximal node degree of the graph, see e.g. [1]. Using these lower and upper bounds on the largest and the smallest nonzero eigenvalue of the Laplacian matrix, results similar to Theorem 6.6 can be formulated, see e.g., [37] or [27].

6.5 Simulation example

In this section, we will give a simulation example to illustrate our design method. Consider a network of $N = 6$ identical agents of the form

$$\begin{aligned} \dot{x}_i &= Ax_i + Bu_i + Ed_i, \\ y_i &= C_1 x_i + D_1 d_i, \qquad\qquad i = 1, 2, \dots, 6, \\ z_i &= C_2 x_i + D_2 u_i, \end{aligned}$$

where $A = \begin{pmatrix} -2 & 2 \\ -1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $E = \begin{pmatrix} 0 & 0 \\ 0.5 & 0 \end{pmatrix}$, $C_1 = \begin{pmatrix} 1 & 0 \end{pmatrix}$, $D_1 = \begin{pmatrix} 0 & 1 \end{pmatrix}$, $C_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $D_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. The pair (A, B) is stabilizable and the pair (C_1, A) is detectable. We also have $D_1 E^\top = \begin{pmatrix} 0 & 0 \end{pmatrix}$, $D_2^\top C_2 = \begin{pmatrix} 0 & 0 \end{pmatrix}$ and $D_1 D_1^\top = 1$, $D_2^\top D_2 = 1$. We assume that the communication among the six agents is represented by the undirected cycle graph. For this graph, the smallest non-zero and largest eigenvalue of the Laplacian are $\lambda_2 = 1$ and $\lambda_6 = 4$. Our goal is to design a distributed dynamic output feedback protocol of the form (6.2) that synchronizes the controlled network and guarantees the associated cost (6.9) to satisfy $J(F, G) < \gamma$. Let the desired upper bound for the cost be $\gamma = 17$.

We adopt the design method given in case (i) of Theorem 6.6. First we compute a positive definite solution P to (6.44) by solving the Riccati equation

$$A^\top P + PA + (c^2 \lambda_6^3 - 2c\lambda_6) PBB^\top P + \lambda_6 C_2^\top C_2 + \sigma I_2 = 0 \tag{6.53}$$

with $\sigma = 0.001$. Moreover, we choose $c = \frac{2}{\lambda_2^2 + \lambda_2 \lambda_6 + \lambda_6^2} = 0.0952$, which is in fact the 'best' choice for c as explained in Remark 6.7. Then, by solving (6.53) in Matlab,

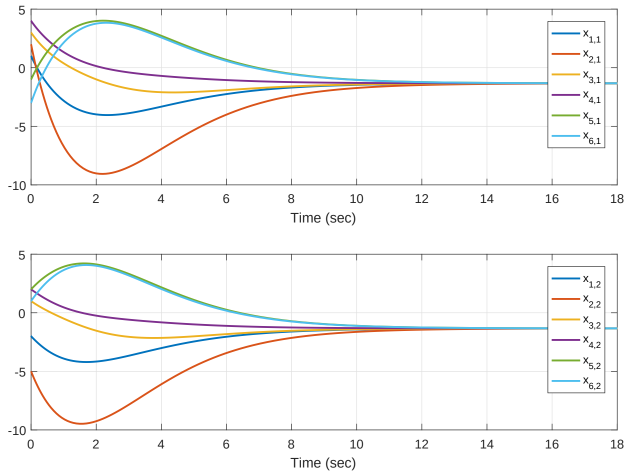

Figure 6.1: Plots of the state vector $\mathbf{x}^1 = (x_{1,1}, x_{2,1}, \ldots, x_{6,1})^\top$ and $\mathbf{x}^2 = (x_{1,2}, x_{2,2}, \ldots, x_{6,2})^\top$ of the controlled network

we compute a positive definite solution

$$P = \begin{pmatrix} 0.9048 & -2.2810 \\ -2.2810 & 6.9779 \end{pmatrix}.$$

Next, by solving the Riccati equation

$$AQ + QA^\top - QC_1^\top C_1 Q + E^\top E + \epsilon I_2 = 0$$

with $\epsilon = 0.001$ in Matlab, we compute a positive definite solution

$$Q = \begin{pmatrix} 0.5000 & 0.5000 \\ 0.5000 & 0.6250 \end{pmatrix}.$$

Accordingly, we compute the associated gain matrices

$$F = \begin{pmatrix} 0.2172 & -0.6646 \end{pmatrix}, \quad G = \begin{pmatrix} 0.5000 & 0.5000 \end{pmatrix}^\top.$$

As an example, we take the initial states of the agents to be $x_{10} = \begin{pmatrix} 1 & -2 \end{pmatrix}^\top$, $x_{20} = \begin{pmatrix} 2 & -5 \end{pmatrix}^\top$, $x_{30} = \begin{pmatrix} 3 & 1 \end{pmatrix}^\top$, $x_{40} = \begin{pmatrix} 4 & 2 \end{pmatrix}^\top$, $x_{50} = \begin{pmatrix} -1 & 2 \end{pmatrix}^\top$ and $x_{60} = \begin{pmatrix} -3 & 1 \end{pmatrix}^\top$, and we take the initial states of the protocol to be zero. In Figure 6.1, we have plotted the controlled state trajectories of the agents. It can be seen that the designed protocol indeed synchronizes the network. The plots of the protocol

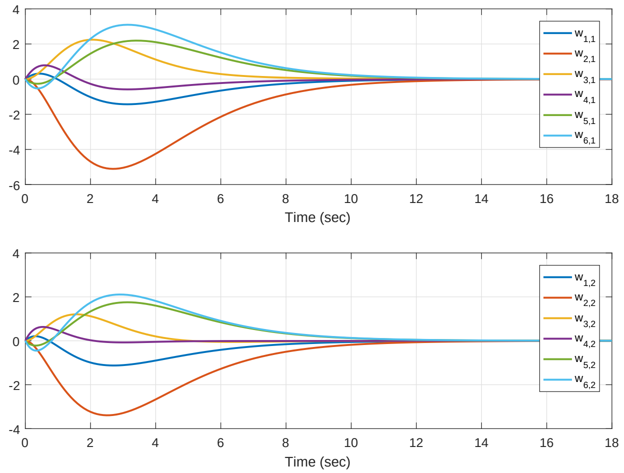

Figure 6.2: Plots of the state vector $\mathbf{w}^1 = (w_{1,1}, w_{2,1}, \dots, w_{6,1})^\top$ and $\mathbf{w}^2 = (w_{1,2}, w_{2,2}, \dots, w_{6,2})^\top$ of the dynamic protocol

states are shown in Figure 6.2. For each i, the state w_i of the local controller is an estimate of the weighted sum of the relative states of agent i, it is seen that the protocol states converge to zero. Moreover, we compute

$$5\left(\text{tr}\left(C_1 QPQC_1^\top\right) + \lambda_6 \text{tr}\left(C_2 QC_2^\top\right)\right) = 16.6509,$$

which is indeed smaller than the desired tolerance $\gamma = 17$.

6.6 Conclusions

In this chapter, we have studied the distributed $\mathcal{H}_2$ suboptimal control problem by dynamic output feedback for linear multi-agent systems. The interconnection structure between the agents is given by a connected undirected graph. Given a linear multi-agent system with identical agent dynamics and an associated global $\mathcal{H}_2$ cost functional, we have provided a design method for computing distributed protocols that guarantee the associated cost to be smaller than a given tolerance while synchronizing the controlled network. The local gains are given in terms of solutions of two Riccati inequalities, each of dimension equal to that of the agent dynamics. One these Riccati inequalities involves the largest and smallest nonzero eigenvalue of the Laplacian matrix of the network graph.

7 $\mathcal{H}_2$ suboptimal output synchronization of heterogeneous multi-agent systems

This chapter deals with the $\mathcal{H}_2$ suboptimal output synchronization problem for heterogeneous linear multi-agent systems. Given a multi-agent system with possibly distinct agents and an associated $\mathcal{H}_2$ cost functional, the aim is to design output feedback based protocols that guarantee the associated cost to be smaller than a given upper bound while the controlled network achieves output synchronization. A design method is provided to compute such protocols. For each agent, the computation of its two local control gains involves two Riccati inequalities, each of dimension equal to the state space dimension of the agent. We also consider the special case that full relative state information is available for each agent. A simulation example is provided to illustrate the performance of the proposed protocols.

7.1 Introduction

Over the last two decades, the problems of designing protocols that achieve consensus or synchronization in multi-agent systems have attracted much attention in the field of systems and control, see e.g. [85], [7], [54] and [11]. The essential feature of these problems is that, while each agent makes use of only local state or output information to implement its own local controller, the resulting global protocol will achieve consensus or synchronization for the global controlled multi-agent network [94], [103]. One of the challenging problems in this context is the problem of designing protocols that minimize given quadratic cost criteria while achieving consensus or synchronization, see e.g. [37], [39], [10], [75] and [76]. Due to the structural constraints imposed on the protocols, such optimal control problems are non-convex and very difficult to solve. It is also unclear whether in general closed form solutions exist.

In the past, many efforts have been devoted to designing distributed protocols for *homogeneous* multi-agent systems that guarantee suboptimal or optimal performance and achieve *state* synchronization or consensus. In [10], this was done for distributed linear quadratic control of multi-agent systems with *single integrator* agent dynamics, see also [38]. In [76] and [37], multi-agent systems with general agent dynamics and a global linear quadratic cost functional were considered. In [75] and [123], an inverse optimal approach was adopted to address the distributed linear quadratic control problem, see also [77]. For $\mathcal{H}_2$ cost functionals of a particular form, [55] and [53] proposed distributed suboptimal protocols that stabilize the controlled multi-agent network. In [36], a distributed $\mathcal{H}_2$ suboptimal control problem was addressed using static state feedback. The results in [36] were then generalized in [39] to the case of dynamic output feedback.

More recently, *output* synchronization problems for *heterogeneous* multi-agent systems have also attracted much attention. In [115], it was shown that solvability of certain regulator equations is a necessary condition for output synchronization of heterogeneous multi-agent systems, and suitable protocols were proposed, see also [22]. In [46], by embedding an internal model in the local controller of each agent, dynamic output feedback based protocols were proposed for a class of heterogeneous uncertain multi-agent systems. In [59], it was shown that the outputs of the agents can be synchronized by a networked protocol if and only if these agents have certain dynamics in common. Later on, in [72] a linear quadratic control method was adopted for computing output synchronizing protocols. In [42], an $\mathcal{L}_2$-gain output synchronization problem was addressed by casting this problem into a number of $\mathcal{L}_2$-gain stabilization problems for certain linear systems, where the state space dimensions of these systems are equal to that of the agents. For related work, we also mention [51], [120] and [99], to name a few.

Up to now, little attention has been paid in the literature to problems of designing output synchronizing protocols for heterogeneous multi-agent systems that guarantee a certain performance. In the present chapter, we will deal with the problem of $\mathcal{H}_2$ optimal output synchronization for heterogeneous linear multi-agent systems, i.e. the problem of minimizing a given $\mathcal{H}_2$ cost functional over all protocols that achieve output synchronization. Instead of addressing this *optimal* control problem, we will address a version of this problem that requires *suboptimality*. More specifically, we will extend previous results in [39] for homogeneous multi-agent systems to the case of heterogeneous multi-agent systems.

The outline of this chapter is as follows. In Section 7.2, we formulate the $\mathcal{H}_2$ suboptimal output synchronization problem. In order to solve this problem, in Section 7.3 we review some basic material on $\mathcal{H}_2$ suboptimal control by dynamic output feedback for linear systems, and some relevant results on output synchronization of heterogeneous multi-agent systems. In Section 7.4, we solve the problem introduced in Section 7.2 and provide a design method for obtaining $\mathcal{H}_2$ suboptimal protocols. In Section 7.5, we solve the special case of the $\mathcal{H}_2$ suboptimal output synchronization problem in which full state information of the agents is available. To illustrate the performance of our proposed protocols, a simulation example is provided in Section 7.6. Finally, Section 7.7 concludes this chapter.

7.2 Problem formulation

In this chapter, we consider a heterogeneous linear multi-agent system consisting of N possibly distinct agents. The dynamics of the ith agent is represented by the linear time-invariant system

$$\begin{aligned} \dot{x}_i &= A_i x_i + B_i u_i + E_i d_i, \\ y_i &= C_{1i} x_i + D_{1i} d_i, \qquad\qquad i = 1, 2, \ldots, N, \\ z_i &= C_{2i} x_i + D_{2i} u_i, \end{aligned} \tag{7.1}$$

where $x_i \in \mathbb{R}^{n_i}$ is the state, $u_i \in \mathbb{R}^{m_i}$ is the coupling input, $d_i \in \mathbb{R}^{q_i}$ is an unknown external disturbance input, $y_i \in \mathbb{R}^{r_i}$ is the measured output and $z_i \in \mathbb{R}^p$ is the output to be synchronized. The matrices A_i, B_i, C_{1i}, D_{1i}, C_{2i}, D_{2i} and E_i are of suitable dimensions. Throughout this chapter we assume that the pairs (A_i, B_i) are stabilizable and the pairs (C_{1i}, A_i) are detectable. Since in (7.1) the agents may have non-identical dynamics, in particular the state space dimensions of the agents may differ. Therefore, one can not expect to achieve *state* synchronization for the network. Instead, in the context of heterogeneous networks it is natural to consider *output* synchronization, see e.g. [115], [22] and [59].

It was shown in [115] that solvability of certain *regulator equations* is necessary for output synchronization of heterogeneous linear multi-agent systems, see also [22], [42], [99] and [3]. Following up on this, throughout this chapter we make the standard standing assumption that there exists a positive integer r such that the regulator equations

$$\begin{aligned} A_i \Pi_i + B_i \Gamma_i &= \Pi_i S, \\ C_{2i} \Pi_i + D_{2i} \Gamma_i &= R, \quad i = 1, 2, \ldots, N \end{aligned} \tag{7.2}$$

have solutions $\Pi_i \in \mathbb{R}^{n_i \times r}$, $\Gamma_i \in \mathbb{R}^{m_i \times r}$, $R \in \mathbb{R}^{p \times r}$ and $S \in \mathbb{R}^{r \times r}$, where the eigenvalues of S lie on the imaginary axis and the pair (R, S) is observable.

Following [115], we assume that the agents (7.1) should be interconnected by a protocol of the form

$$\begin{aligned} \dot{w}_i &= A_i w_i + B_i u_i + G_i (y_i - C_{1i} w_i), \\ \dot{v}_i &= S v_i + \sum_{i=1}^{N} a_{ij} (v_j - v_i), \\ u_i &= F_i (w_i - \Pi_i v_i) + \Gamma_i v_i, \quad i = 1, 2, \ldots, N, \end{aligned} \tag{7.3}$$

where $v_i \in \mathbb{R}^r$ and $w_i \in \mathbb{R}^{n_i}$ are the states of the ith local controller, the matrices S, Π_i and Γ_i are solutions of (7.2), and the matrices $F_i \in \mathbb{R}^{m_i \times n_i}$ and $G_i \in \mathbb{R}^{n_i \times r_i}$ are control gains to be designed. The coefficients a_{ij} are the entries of the adjacency matrix $\mathcal{A}$ of the communication graph. Throughout this chapter it will be a standing assumption that the communication between the agents of the network is represented by a connected, simple undirected weighted graph. We briefly explain the structure of this protocol. The first equation in (7.3) has the structure of an asymptotic observer for the state of the ith agent. The second equation represents an auxiliary system associated with the ith agent. Each auxiliary system receives the relative state values with respect to its neighboring auxiliary systems. In this way, the network of auxiliary systems will reach state synchronization. The third equation in (7.3) is a static gain, it feeds back the value $w_i - \Pi_i v_i$ and the state v_i of the associated auxiliary system to the ith agent. The idea of the protocol (7.3) is that, as time goes to infinity, the state x_i of the ith agent and its estimate w_i converge to $\Pi_i v_i$ due the first equation in (7.2). Subsequently, as a consequence of the second equation in (7.2), the outputs z_i of the agents will reach synchronization.

Denote by $\mathbf{x} = (x_1^\top, x_2^\top, \ldots, x_N^\top)^\top$ the aggregate state vector and likewise define $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, $\mathbf{y}$, $\mathbf{z}$ and $\mathbf{d}$. Denote by A the block diagonal matrix

$$A = \text{blockdiag}(A_1, A_2, \ldots, A_N) \tag{7.4}$$

and likewise define B, C_1, C_2, D_1, D_2 and E. The multi-agent system (7.1) can then be written in compact form as

$$\begin{aligned} \dot{\mathbf{x}} &= A\mathbf{x} + B\mathbf{u} + E\mathbf{d}, \\ \mathbf{y} &= C_1\mathbf{x} + D_1\mathbf{d}, \\ \mathbf{z} &= C_2\mathbf{x} + D_2\mathbf{u}. \end{aligned} \tag{7.5}$$

Similarly, denote

$$F = \text{blockdiag}(F_1, F_2, \ldots, F_N)$$

and likewise define G, Γ and Π. The protocol (7.3) can be written in compact form as

$$\begin{aligned} \dot{\mathbf{w}} &= A\mathbf{w} + B\mathbf{u} + G(\mathbf{y} - C_1\mathbf{w}), \\ \dot{\mathbf{v}} &= (I_N \otimes S - L \otimes I_r)\mathbf{v}, \\ \mathbf{u} &= F\mathbf{w} + (\Gamma - F\Pi)\mathbf{v}. \end{aligned} \tag{7.6}$$

Next, denote

$$\mathbf{x}_o = (\mathbf{x}^\top, \mathbf{w}^\top, \mathbf{v}^\top)^\top.$$

By interconnecting the system (7.5) and the protocol (7.6), the controlled network is then represented in compact form by

$$\begin{aligned} \dot{\mathbf{x}}_o &= A_o \mathbf{x}_o + E_o \mathbf{d}, \\ \mathbf{z} &= C_o \mathbf{x}_o, \end{aligned} \tag{7.7}$$

where

$$A_o = \begin{pmatrix} A & BF & B\Gamma - BF\Pi \\ GC_1 & A + BF - GC_1 & B\Gamma - BF\Pi \\ 0 & 0 & I_N \otimes S - L \otimes I_r \end{pmatrix},$$

$$C_o = \begin{pmatrix} C_2 & D_2 F & D_2\Gamma - D_2 F\Pi \end{pmatrix}, \quad E_o = \begin{pmatrix} E \\ GD_1 \\ 0 \end{pmatrix}.$$

Foremost, we want the protocol (7.3) to achieve output synchronization for the overall network:

Definition 7.1. *The protocol* (7.3) *is said to achieve* **z***-output synchronization for the network* (7.7) *if, for all* $i, j = 1, 2, \ldots, N$, *we have* $z_i(t) - z_j(t) \to 0$, $v_i(t) - v_j(t) \to 0$ *and* $w_i(t) - w_j(t) \to 0$ *as* $t \to \infty$.

In the context of output synchronization, we are interested in the differences of the output values of the agents in the controlled network. Since the differences of the output values of communicating agents are captured by the incidence matrix R of the communication graph [63], we define a performance output variable as

$$\boldsymbol{\zeta} = (W^{\frac{1}{2}} R^{\top} \otimes I_p)\mathbf{z},$$

where W is the weight matrix defined in (1.2). The output $\boldsymbol{\zeta}$ reflects the weighted disagreement between the outputs of the agents in accordance with the weights of the edges connecting these agents. Subsequently, we have the following equations for the controlled network

$$\begin{aligned} \dot{\mathbf{x}}_o &= A_o \mathbf{x}_o + E_o \mathbf{d}, \\ \mathbf{z} &= C_o \mathbf{x}_o, \\ \boldsymbol{\zeta} &= C_p \mathbf{x}_o, \end{aligned} \tag{7.8}$$

where

$$C_p = (W^{\frac{1}{2}} R^{\top} \otimes I_p) C_o.$$

The impulse response matrix of the disturbance **d** to the performance output $\boldsymbol{\zeta}$ is given by

$$T_d(t) = C_p e^{A_o t} E_o. \tag{7.9}$$

The performance of the network is now quantified by the $\mathcal{H}_2$-norm of this impulse response. Thus we define the associated $\mathcal{H}_2$ cost functional as

$$J := \int_0^\infty \text{tr}\left[T_d^\top(t)T_d(t)\right] dt. \tag{7.10}$$

Note that the cost functional (7.10) is a function of the gain matrices $F_1, F_2, \ldots, F_N$ and $G_1, G_2, \ldots, G_N$.

The $\mathcal{H}_2$ optimal output synchronization problem is now defined as the problem of minimizing the cost functional (7.10) over all protocols (7.3) that achieve output synchronization. Since the protocol (7.3) has a particular structure imposed by the communication topology, the $\mathcal{H}_2$ optimal output synchronization problem is a non-convex optimization problem, and it is unclear whether a closed form solution exists in general. Therefore, in this chapter we will address a version of this problem that only requires *suboptimality*. The aim of this chapter is then to design a protocol of the form (7.3) that guarantees the associated cost (7.10) to be smaller than an a priori given upper bound while achieving **z**-output synchronization for the network. More concretely, the problem we will address is the following:

Problem 7.1. *Let $\gamma > 0$ be a given tolerance. Design gain matrices $F_1, F_2, \ldots, F_N$ and $G_1, G_2, \ldots, G_N$ such that the resulting protocol* (7.3) *achieves* **z***-output synchronization and its associated cost* (7.10) *satisfies $J < \gamma$.*

To solve Problem 7.1, in the next section we will first review some preliminary results on $\mathcal{H}_2$ suboptimal control for linear systems and on output synchronization of heterogeneous linear multi-agent systems. It will become clear later on that these preliminary results are necessary ingredients to address Problem 7.1.

7.3 Preliminary results

7.3.1 $\mathcal{H}_2$ suboptimal control for linear systems by dynamic output feedback

In this subsection, we will review the $\mathcal{H}_2$ suboptimal control problem by dynamic output feedback for linear systems, see e.g. [95], [96], [100], [25] and [39]. In particular, we will review the results from [39] on separation principle based $\mathcal{H}_2$ suboptimal control for continuous-time linear systems.

Consider the system

$$\begin{aligned} \dot{x} &= \bar{A}x + \bar{B}u + \bar{E}d, \\ y &= \bar{C}_1 x + \bar{D}_1 d, \\ z &= \bar{C}_2 x + \bar{D}_2 u, \end{aligned} \tag{7.11}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, $d \in \mathbb{R}^q$ is an unknown external disturbance input, $y \in \mathbb{R}^r$ is the measured output, and $z \in \mathbb{R}^p$ is the output to be controlled. The matrices $\bar{A}$, $\bar{B}$, $\bar{C}_1$, $\bar{C}_2$, $\bar{D}_1$, $\bar{D}_2$ and $\bar{E}$ are of suitable dimensions. We assume that the pair $(\bar{A}, \bar{B})$ is stabilizable and the pair $(\bar{C}_1, \bar{A})$ is detectable. We consider dynamic output feedback controllers of the form

$$\begin{aligned} \dot{w} &= \bar{A}w + \bar{B}u + G\left(y - \bar{C}_1 w\right), \\ u &= Fw, \end{aligned} \tag{7.12}$$

where $w \in \mathbb{R}^n$ is the state of the controller, $F \in \mathbb{R}^{m\times n}$ and $G \in \mathbb{R}^{n\times r}$ are gain matrices to be designed. By interconnecting the controller (7.12) and the system (7.11), we obtain the controlled system

$$\begin{aligned} \begin{pmatrix} \dot{x} \\ \dot{w} \end{pmatrix} &= \begin{pmatrix} \bar{A} & \bar{B}F \\ G\bar{C}_1 & \bar{A} + \bar{B}F - G\bar{C}_1 \end{pmatrix} \begin{pmatrix} x \\ w \end{pmatrix} + \begin{pmatrix} \bar{E} \\ G\bar{D}_1 \end{pmatrix} d, \\ z &= \begin{pmatrix} \bar{C}_2 & \bar{D}_2 F \end{pmatrix} \begin{pmatrix} x \\ w \end{pmatrix}. \end{aligned} \tag{7.13}$$

Denote

$$A_e = \begin{pmatrix} \bar{A} & \bar{B}F \\ G\bar{C}_1 & \bar{A} + \bar{B}F - G\bar{C}_1 \end{pmatrix}, \quad C_e = \begin{pmatrix} \bar{C}_2 & \bar{D}_2 F \end{pmatrix}, \quad E_e = \begin{pmatrix} \bar{E} \\ G\bar{D}_1 \end{pmatrix}.$$

The impulse response matrix of the disturbance d to the output z is given by $T_{F,G}(t) = C_e e^{A_e t} E_e$. We define the $\mathcal{H}_2$ cost functional as

$$J(F, G) := \int_0^\infty \mathrm{tr}\left[T_{F,G}^\top(t) T_{F,G}(t)\right] dt. \tag{7.14}$$

The $\mathcal{H}_2$ suboptimal control problem by dynamic output feedback is the problem of finding a controller of the form (7.12) such that the associated cost (7.14) is smaller than an a priori given upper bound and the controlled system (7.13) is internally stable. The following lemma provides a design method for computing such a controller, see also [39, Theorem 4].

Lemma 7.1. *Let $\gamma > 0$ be a given tolerance. Assume that $\bar{D}_1 \bar{E}^\top = 0$, $\bar{D}_2^\top \bar{C}_2 = 0$ and $\bar{D}_1 \bar{D}_1^\top = I_r$, $\bar{D}_2^\top \bar{D}_2 = I_m$. Let $P > 0$ and $Q > 0$ satisfy the Riccati inequalities*

$$\begin{aligned} &\bar{A}^\top P + P\bar{A} - P\bar{B}\bar{B}^\top P + \bar{C}_2^\top \bar{C}_2 < 0, \\ &\bar{A}Q + Q\bar{A}^\top - Q\bar{C}_1^\top \bar{C}_1 Q + \bar{E}\bar{E}^\top < 0. \end{aligned}$$

If, in addition, such P and Q satisfy

$$\mathrm{tr}\left(\bar{C}_1 QPQ\bar{C}_1^\top\right) + \mathrm{tr}\left(\bar{C}_2 Q\bar{C}_2^\top\right) < \gamma,$$

then the controller (7.12) *with* $F = -\bar{B}^\top P$ *and* $G = Q\bar{C}_1^\top$ *internally stabilizes the system* (7.11) *and is suboptimal, i.e.* $J(F, G) < \gamma$.

For a proof of Lemma 7.1, we refer to [39, Theorem 4].

7.3.2 Output synchronization of heterogeneous linear multi-agent systems

In this subsection, we will review some relevant results on output synchronization of heterogeneous linear multi-agent systems, see also [115], [22], [46] and [59].

Consider a heterogeneous linear multi-agent system consisting of N possibly distinct agents. The dynamics of the ith agent is represented by the linear time-invariant system

$$\begin{aligned} \dot{x}_i &= A_i x_i + B_i u_i, \\ y_i &= C_{1i} x_i, \qquad\qquad i = 1, 2, \ldots, N. \\ z_i &= C_{2i} x_i + D_{2i} u_i, \end{aligned} \tag{7.15}$$

The agents (7.15) will be interconnected by a protocol of the form (7.3), where the matrices S, Γ_i and Π_i are assumed to satisfy the regulator equations (7.2). The multi-agent system (7.15) can be written in compact form as

$$\begin{aligned} \dot{\mathbf{x}} &= A\mathbf{x} + B\mathbf{u}, \\ \mathbf{y} &= C_1 \mathbf{x}, \\ \mathbf{z} &= C_2 \mathbf{x} + D_2 \mathbf{u}, \end{aligned} \tag{7.16}$$

and the protocol (7.3) can be written as (7.6). By interconnecting the system (7.16) and the protocol (7.6), the controlled network is then given by

$$\begin{aligned} \dot{\mathbf{x}}_o &= A_o \mathbf{x}_o, \\ \mathbf{z} &= C_o \mathbf{x}_o. \end{aligned} \tag{7.17}$$

The following lemma yields conditions under which the controlled network (7.17) achieves **z**-output synchronization.

Lemma 7.2. *Consider the multi-agent system* (7.15) *and the protocol* (7.3). *Let gain matrices* F_i *and* G_i *be such that the matrices* $A_i + B_i F_i$ *and* $A_i - G_i C_{1i}$ *are Hurwitz. Then the associated protocol* (7.3) *achieves* **z***-output synchronization for the network.*

A proof of Lemma 7.2 can be given along the lines of the proof of [115, Theorem 5].

We are now ready to deal with the $\mathcal{H}_2$ suboptimal output synchronization problem formulated in Problem 7.1.

7.4 $\mathcal{H}_2$ suboptimal output synchronization protocols by dynamic output feedback

In this section, we will resolve Problem 7.1. More specifically, we will establish a design method for computing gain matrices $F_1, F_2, \ldots, F_N$ and $G_1, G_2, \ldots, G_N$ such that the associated protocol (7.3) achieves **z**-output synchronization and guarantees $J < \gamma$.

In the sequel, we will first show that this problem can be simplified by transforming it into $\mathcal{H}_2$ suboptimal control problems for N auxiliary systems. The suboptimal gains F_i and G_i for these N separate problems will turn out to also yield a suboptimal protocol for the heterogeneous network.

To this end, we introduce the following N auxiliary systems

$$\begin{aligned}\dot{\xi}_i &= A_i\xi_i + B_i\nu_i + E_i\delta_i,\\ \vartheta_i &= C_{1i}\xi_i + D_{1i}\delta_i,\\ \eta_i &= C_{2i}\xi_i + D_{2i}\nu_i, \quad i = 1, 2, \ldots, N,\end{aligned} \tag{7.18}$$

where $\xi_i \in \mathbb{R}^{n_i}$ is the state, $\nu_i \in \mathbb{R}^{m_i}$ is the coupling input, $\delta_i \in \mathbb{R}^{q_i}$ is an unknown external disturbance input, $\vartheta_i \in \mathbb{R}^{r_i}$ is the measured output and $\eta_i \in \mathbb{R}^p$ is the output to be controlled. For given gain matrices F_i and G_i, consider the dynamic output feedback controllers

$$\begin{aligned}\dot{\omega}_i &= A_i\omega_i + B_i\nu_i + G_i(\vartheta_i - C_{1i}\omega_i),\\ \nu_i &= F_i\omega_i, \quad i = 1, 2, \ldots, N,\end{aligned} \tag{7.19}$$

where $\omega_i \in \mathbb{R}^n$ is the state of the ith controller.

By interconnecting the systems (7.18) and the controllers (7.19), we obtain the N controlled auxiliary systems

$$\begin{aligned}\begin{pmatrix}\dot{\xi}_i\\ \dot{\omega}_i\end{pmatrix} &= \begin{pmatrix}A_i & B_iF_i\\ G_iC_{1i} & A_i + B_iF_i - G_iC_{1i}\end{pmatrix}\begin{pmatrix}\xi_i\\ \omega_i\end{pmatrix} + \begin{pmatrix}E_i\\ G_iD_{1i}\end{pmatrix}\delta_i,\\ \eta_i &= \begin{pmatrix}C_{2i} & D_{2i}F_i\end{pmatrix}\begin{pmatrix}\xi_i\\ \omega_i\end{pmatrix}, \quad i = 1, 2, \ldots, N.\end{aligned} \tag{7.20}$$

For $i = 1, 2, \ldots, N$, denote

$$\bar{A}_i = \begin{pmatrix}A_i & B_iF_i\\ G_iC_{1i} & A_i + B_iF_i - G_iC_{1i}\end{pmatrix}, \quad \bar{C}_i = \begin{pmatrix}C_{2i} & D_{2i}F_i\end{pmatrix}, \quad \bar{E}_i = \begin{pmatrix}E_i\\ G_iD_{1i}\end{pmatrix}.$$

The impulse response matrix of the disturbance δ_i to the output η_i is equal to

$$T_{\delta i}(t) = \bar{C}_i e^{\bar{A}_i t} \bar{E}_i,$$

and an associated $\mathcal{H}_2$ cost functional is defined as

$$J_i = \int_0^\infty \text{tr}[T_{\delta i}^\top(t) T_{\delta i}(t)] dt. \tag{7.21}$$

The following lemma holds.

Lemma 7.3. *Let $\gamma > 0$ be a given tolerance. Assume, for $i = 1, 2, \ldots, N$, the systems* (7.20) *are internally stable and the costs* (7.21) *satisfy*

$$\sum_{i=1}^N J_i < \frac{\gamma}{\lambda_N}, \tag{7.22}$$

where λ_N is the largest eigenvalue of the Laplacian matrix L. Then the protocol (7.3) *achieves* **z***-output synchronization for the network* (7.8) *and the associated cost* (7.10) *satisfies $J < \gamma$.*

Proof. First, note that the systems (7.20) are internally stable if and only if the matrices $A_i + B_i F_i$ and $A_i - G_i C_{1i}$ are Hurwitz, see e.g. [102, Section 3.12]. Hence, by Lemma 7.2, if the systems (7.20) are internally stable, then the network controlled using the protocol (7.3) reaches **z**-output synchronization.

Next, we will show that if (7.22) holds, then $J < \gamma$. Note that (7.22) is equivalent to

$$\lambda_N \sum_{i=1}^N \int_0^\infty \text{tr}[T_{\delta i}^\top(t) T_{\delta i}(t)] dt < \gamma. \tag{7.23}$$

In turn, the inequality (7.23) holds if and only if

$$\lambda_N \int_0^\infty \text{tr}[\bar{T}_d^\top(t) \bar{T}_d(t)] dt < \gamma \tag{7.24}$$

holds, where

$$\bar{T}_d = \bar{C}_o e^{\bar{A}_o t} \bar{E}_o$$

with

$$\bar{A}_o = \begin{pmatrix} A & BF \\ GC_1 & A + BF - GC_1 \end{pmatrix}, \quad \bar{C}_o = \begin{pmatrix} C_2 & D_2 F \end{pmatrix}, \quad \bar{E}_o = \begin{pmatrix} E \\ GD_1 \end{pmatrix}.$$

Recall that the matrix A is the block diagonal matrix defined in (7.4), similarly for the matrices B, C_1, C_2, D_1, D_2, E, F and G. Using the fact that $\lambda_N I_{pN} - L \otimes I_p \geqslant 0$, it can be shown that (7.24) implies

$$\int_0^\infty \text{tr}[\bar{T}_d^\top(t)(L \otimes I_p)\bar{T}_d(t)]dt < \gamma. \tag{7.25}$$

On the other hand,

$$\int_0^\infty \text{tr}[\bar{T}_d^\top(t)(L \otimes I_p)\bar{T}_d(t)]dt = \int_0^\infty \text{tr}\left[T_d^\top(t)T_d(t)\right] dt \tag{7.26}$$

with $T_d(t)$ given by (7.9). Note that the right hand side of (7.26) is exactly the cost J given by (7.10) associated with the network (7.8). It follows that $J < \gamma$. This completes the proof. □

By the previous, if the gain matrices F_i and G_i are such that $A_i + B_iF_i$ and $A_i - G_iC_{1i}$ are Hurwitz and (7.22) holds, then the protocol (7.3) using these F_i and G_i yields **z**-output synchronization and $J < \gamma$. In the next theorem, we will provide a method for computing gain matrices F_i and G_i such that the above holds.

Theorem 7.4. *Let $\gamma > 0$ be a given tolerance. For $i = 1, 2, \ldots, N$, assume that $D_{1i}E_i^\top = 0$, $D_{2i}^\top C_{2i} = 0$, $D_{1i}D_{1i}^\top = I_{r_i}$ and $D_{2i}^\top D_{2i} = I_{m_i}$. Let $P_i > 0$ satisfy*

$$A_i^\top P_i + P_iA_i^\top - P_iB_iB_i^\top P_i + C_{2i}^\top C_{2i} < 0. \tag{7.27}$$

Let $Q_i > 0$ satisfy

$$A_iQ_i + Q_iA_i^\top - Q_iC_{1i}^\top C_{1i}Q_i + E_iE_i^\top < 0. \tag{7.28}$$

If, in addition, such P_i and Q_i satisfy

$$\text{tr}(C_{1i}Q_iP_iQ_iC_{1i}^\top) + \text{tr}(C_{2i}Q_iC_{2i}^\top) < \frac{\gamma}{N\lambda_N}, \tag{7.29}$$

then the protocol (7.3) *with $F_i := -B_i^\top P_i$ and $G_i := Q_iC_{1i}^\top$ achieves **z**-output synchronization for the network* (7.8) *and guarantees $J < \gamma$.*

Proof. Note that (7.27) is equivalent to

$$(A_i - B_iB_i^\top P_i)^\top P_i + (A_i - B_iB_i^\top P_i) + P_iB_iB_i^\top P_i + C_{2i}^\top C_{2i} < 0 \tag{7.30}$$

and (7.28) is equivalent to

$$(A_i - Q_iC_{1i}^\top C_{1i})Q_i + Q_i(A_i - Q_iC_{1i}^\top C_{1i})^\top + Q_iC_{1i}^\top C_{1i}Q_i + E_iE_i^\top < 0. \tag{7.31}$$

Taking $F_i := -B_i^\top P_i$ and $G_i := Q_i C_{1i}^\top$, it then follows that $A_i + B_i F_i$ and $A_i - G_i C_{1i}$ are Hurwitz.

Next, by (7.29), it follows from Lemma 7.1 that

$$J_i < \frac{\gamma}{N\lambda_N}, \quad i = 1, 2, \ldots, N.$$

Thus we have (7.22), and the conclusion then follows from Lemma 7.3. □

We note that the conditions $D_{1i}E_i^\top = 0$, $D_{2i}^\top C_{2i} = 0$, $D_{1i}D_{1i}^\top = I_{r_i}$ and $D_{2i}^\top D_{2i} = I_{m_i}$ are made here to simplify notation, and can be relaxed to the regularity conditions $D_{1i}D_{1i}^\top > 0$ and $D_{2i}^\top D_{2i} > 0$ alone.

Remark 7.5. In Theorem 7.4, in order to select γ, the following steps could be taken. For $i = 1, 2 \ldots, N$:

(i) Compute positive definite solutions P_i and Q_i of the Riccati inequalities (7.27) and (7.28). Such solutions exist.

(ii) Denote $S_i = \text{tr}(C_{1i}Q_iP_iQ_iC_{1i}^\top) + \text{tr}(C_{2i}Q_iC_{2i}^\top)$.

(iii) Choose γ such that $N\lambda_N S_i < \gamma$.

Note that the smaller S_i or λ_N is, the smaller such feasible γ is allowed to be. Unfortunately, the problem of minimizing S_i over all $P_i > 0$ and $Q_i > 0$ that satisfy (7.27) and (7.28) is a non-convex optimization problem. However, since smaller Q_i leads to smaller $\text{tr}(C_{2i}Q_iC_{2i}^\top)$ and smaller P_i and Q_i lead to smaller $\text{tr}(C_{1i}Q_iP_iQ_iC_{1i}^\top)$, and consequently smaller feasible γ, we could try to find P_i and Q_i as small as possible. In fact, one can find $P_i = P_i(\epsilon_i) > 0$ to (7.27) by solving the Riccati equation

$$A_i^\top P_i + P_iA_i^\top - P_iB_iB_i^\top P_i + C_{2i}^\top C_{2i} + \epsilon_i I_{n_i} = 0$$

with $\epsilon_i > 0$ arbitrary. Similarly, one can find $Q_i = Q_i(\sigma_i) > 0$ to (7.28) by solving the dual Riccati equation

$$A_iQ_i + Q_iA_i^\top - Q_iC_{1i}^\top C_{1i}Q_i + E_iE_i^\top + \sigma_i I_{n_i} = 0$$

with $\sigma_i > 0$ arbitrary. By using a standard argument, it can be shown that $P_i(\epsilon_i)$ and $Q_i(\sigma_i)$ decrease as ϵ_i and σ_i decrease, respectively. So ϵ_i and σ_i should be taken close to 0 to get smaller P_i and Q_i.

7.5 $\mathcal{H}_2$ suboptimal output synchronization protocols by state feedback

In this section, we consider a special case of Problem 7.1, namely, the case that full state information of the agents is available. More specifically, we consider the $\mathcal{H}_2$ suboptimal output synchronization problem by relative state feedback for heterogeneous linear multi-agent systems.

Consider a heterogeneous multi-agent system consisting of N possibly distinct agents. The dynamics of the ith agent is represented by

$$\begin{aligned} \dot{x}_i &= A_i x_i + B_i u_i + E_i d_i, \\ z_i &= C_i x_i + D_i u_i, \end{aligned} \qquad i = 1, 2, \ldots, N, \tag{7.32}$$

where $x_i \in \mathbb{R}^{n_i}$ is the state, $u_i \in \mathbb{R}^{m_i}$ the control input, $d_i \in \mathbb{R}^{q_i}$ the external disturbance and $z_i \in \mathbb{R}^p$ the output to be synchronized. The matrices A_i, B_i, C_i, D_i and E_i are of suitable dimensions. We assume that the pairs (A_i, B_i) are stabilizable. Also, as before, we assume that, for some positive integer r, the regulator equations

$$\begin{aligned} A_i \Pi_i + B_i \Gamma_i &= \Pi_i S, \\ C_i \Pi_i + D_i \Gamma_i &= R, \quad i = 1, 2, \ldots, N \end{aligned} \tag{7.33}$$

have solutions $\Pi_i \in \mathbb{R}^{n_i \times r}$, $\Gamma_i \in \mathbb{R}^{m_i \times r}$, $S \in \mathbb{R}^{r \times r}$ and $R \in \mathbb{R}^{p \times r}$, where all eigenvalues of S are on the imaginary axis and the pair (R, S) is observable.

Following [115] and [43], the agents will be interconnected by a protocol of the form

$$\begin{aligned} \dot{v}_i &= S v_i + \sum_{i=1}^{N} a_{ij}(v_j - v_i), \\ u_i &= K_i(x_i - \Pi_i v_i) + \Gamma_i v_i, \quad i = 1, 2, \ldots, N, \end{aligned} \tag{7.34}$$

where $v_i \in \mathbb{R}^r$ is the state of the ith local controller, the matrices Π_i and Γ_i are solutions of (7.33) and the matrices $K_i \in \mathbb{R}^{m_i \times n_i}$ are control gains to be designed. The coefficients a_{ij} are the entries of the adjacency matrix $\mathcal{A}$ of the communication graph.

Denote by $\mathbf{x} = (x_1^\top, x_2^\top, \ldots, x_N^\top)^\top$ the aggregate state vector and likewise define $\mathbf{u}$, $\mathbf{z}$ and $\mathbf{d}$. As before, denote by A the block diagonal matrix

$$A = \text{blockdiag}(A_1, A_2, \ldots, A_N)$$

and likewise define B, C, D and E. The multi-agent system (7.32) can be written in compact form as

$$\begin{aligned}\dot{\mathbf{x}} &= A\mathbf{x} + B\mathbf{u} + E\mathbf{d},\\ \mathbf{z} &= C\mathbf{x} + D\mathbf{u}.\end{aligned} \tag{7.35}$$

Denote

$$K = \text{blockdiag}(K_1, K_2, \ldots, K_N)$$

and likewise define Γ and Π. The protocol (7.34) can be written in compact form as

$$\begin{aligned}\dot{\mathbf{v}} &= (I_N \otimes S - L \otimes I_r)\mathbf{v},\\ \mathbf{u} &= K\mathbf{x} + (\Gamma - K\Pi)\mathbf{v}.\end{aligned} \tag{7.36}$$

Denote

$$\mathbf{x}_s = (\mathbf{x}^\top, \mathbf{v}^\top)^\top.$$

By interconnecting the system (7.35) and the protocol (7.36), the controlled network is given by

$$\begin{aligned}\dot{\mathbf{x}}_s &= A_s\mathbf{x}_s + E_s\mathbf{d},\\ \mathbf{z} &= C_s\mathbf{x}_s,\end{aligned} \tag{7.37}$$

where

$$A_s = \begin{pmatrix} A + BK & B\Gamma - BK\Pi \\ 0 & I \otimes S - L \otimes I_r \end{pmatrix},\ C_s = \begin{pmatrix} C + DK & D\Gamma - DK\Pi \end{pmatrix},\ E_s = \begin{pmatrix} E \\ 0 \end{pmatrix}.$$

Similar to Definition 7.1, we have the following definition for output synchronization by relative state feedback:

Definition 7.2. *The protocol* (7.34) *is said to achieve* **z***-output synchronization for the network* (7.37) *if, whenever the disturbance is equal to zero, i.e.,* $\mathbf{d} = 0$*, then for all* $i, j = 1, 2, \ldots, N$ *we have* $z_i(t) - z_j(t) \to 0$ *and* $v_i(t) - v_j(t) \to 0$ *as* $t \to \infty$.

By introducing the performance output

$$\boldsymbol{\zeta}_s = (W^{\frac{1}{2}} R^\top \otimes I_p)\mathbf{z},$$

we have the following equations for the controlled network

$$\begin{aligned}\dot{\mathbf{x}}_s &= A_s\mathbf{x}_s + E_s\mathbf{d},\\ \mathbf{z} &= C_s\mathbf{x}_s,\\ \boldsymbol{\zeta}_s &= C_c\mathbf{x}_s,\end{aligned} \tag{7.38}$$

where

$$C_c = (W^{\frac{1}{2}} R^\top \otimes I_p)C_s.$$

The impulse response from the disturbance $\mathbf{d}$ to the performance output $\boldsymbol{\zeta}$ is then given by $T_s(t) = C_c e^{A_s t} E_s$. Subsequently, we define the associated $\mathcal{H}_2$ cost functional as

$$J := \int_0^\infty \text{tr}\left[T_s^\top(t) T_s(t)\right] dt. \tag{7.39}$$

Note that the cost functional (7.39) is a function of the gain matrices $K_1, K_2, \ldots, K_N$. The problem we want to address in this section is the following:

Problem 7.2. *Let $\gamma > 0$ be a given tolerance. Design gain matrices $K_1, K_2, \ldots, K_N$ such that the associated protocol* (7.34) *achieves* **z***-output synchronization and $J(K) < \gamma$.*

In the following proposition, we provide a design method for computing a protocol (7.34) such that the controlled network (7.38) achieves **z**-output synchronization and the associated cost is smaller than an a priori given upper bound.

Proposition 7.6. *Let $\gamma > 0$ be a given tolerance. For $i = 1, 2 \ldots, N$, assume that $D_i^\top C_i = 0$ and $D_i^\top D_i = I_{m_i}$. Let $P_i > 0$ satisfy*

$$A_i^\top P_i + P_i A_i^\top - P_i B_i B_i^\top P_i + C_i^\top C_i < 0. \tag{7.40}$$

If, in addition, such P_i satisfy

$$\text{tr}(E_i^\top P_i E_i) < \frac{\gamma}{N \lambda_N}, \tag{7.41}$$

then the protocol (7.34) *with $K_i := -B_i^\top P_i$ achieves* **z***-output synchronization for the network* (7.38) *and guarantees $J < \gamma$.*

A proof of Proposition 7.6 can be given along the lines of the proof of Theorem 7.4, and is hence omitted here. The conditions $D_i^\top D_i = I_{m_i}$ are made here to simplify notation, and can be relaxed to the regularity conditions $D_i^\top D_i > 0$ alone.

7.6 Simulation example

In this section, we will give a simulation example based on the example in [115] to illustrate the design method of Theorem 7.4.

Consider a network of $N = 6$ heterogeneous agents. The dynamics of the agents are given by

$$\begin{aligned}
\dot{x}_i &= A_i x_i + B_i u_i + E_i d_i, \\
y_i &= C_{1i} x_i + D_{1i} d_i, \qquad\qquad i = 1, 2, \ldots, 6, \\
z_i &= C_{2i} x_i + D_{2i} u_i,
\end{aligned}$$

where $A_i = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & c_i \\ 0 & -f_i & -a_i \end{pmatrix}$, $B_i = \begin{pmatrix} 0 \\ 0 \\ b_i \end{pmatrix}$, $E_i = \begin{pmatrix} 0 & 0.2 \\ 0 & 0 \\ 0 & 0.2 \end{pmatrix}$, $C_{1i} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$, $D_{1i} = \begin{pmatrix} 1 & 0 \end{pmatrix}$, $C_{2i} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $D_{2i} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. The parameters a_i, b_i, c_i and f_i are chosen to be

$$
\begin{aligned}
&a_i = 2,\ c_i = 1, \quad i = 1, 2, \ldots, 6, \\
&b_1 = b_4 = 1,\ b_2 = b_5 = 2,\ b_3 = b_6 = 3, \\
&f_1 = f_4 = 1,\ f_2 = f_5 = 2,\ f_3 = f_6 = 3.
\end{aligned}
$$

The pairs (A_i, B_i) are stabilizable and the pairs (C_{1i}, A_i) are detectable. We also have that $D_{1i}E_i^\top = 0$, $D_{2i}^\top C_{2i} = 0$, $D_{1i}D_{1i}^\top = 1$ and $D_{2i}^\top D_{2i} = 1$. The communication graph between the six agents is assumed to be an undirected cycle graph. The largest eigenvalue of the corresponding Laplacian matrix L is $\lambda_6 = 4$.

We choose the matrices S and R in the regulator equations (7.2) to be

$$
S = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.
$$

The eigenvalues of S are on the imaginary axis and the pair (R, S) is observable. We solve the equations (7.2) and compute

$$
\Pi_i = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \Gamma_i = \begin{pmatrix} 0 & 1 \end{pmatrix}, \quad i = 1, 2, \ldots, 6.
$$

The objective is to design a protocol of the form (7.3) such that the associated cost (7.10) satisfies $J < \gamma$ while achieving **z**-output synchronization. Let the desired upper bound be $\gamma = 18$.

Following the design method in Theorem 7.4, for $i = 1, 2, \ldots, 6$, we compute a positive definite solution P_i to (7.27) by solving the Riccati equation

$$
A_i^\top P_i + P_i A_i^\top - P_i B_i B_i^\top P_i + C_{2i}^\top C_{2i} + \epsilon I_{n_i} = 0
$$

with $\epsilon = 0.001$. We also compute a positive definite solution Q_i to (7.27) by solving the dual Riccati equation

$$
A_i Q_i + Q_i A_i^\top - Q_i C_{1i}^\top C_{1i} Q_i + E_i E_i^\top + \sigma I_{n_i} = 0
$$

with $\sigma = 0.001$. Accordingly, we compute the associated gain matrices F_i and G_i

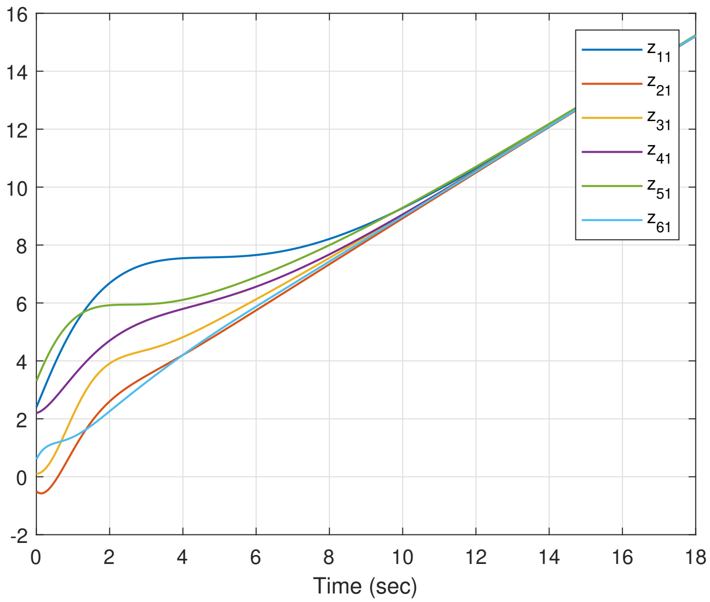

Figure 7.1: Plots of trajectories of the first component of the output vectors $z_1, z_2, \ldots, z_6$

to be

$$
\begin{aligned}
F_1 = F_4 &= \begin{pmatrix} -1.0005 & -1.7329 & -0.7326 \end{pmatrix}, \\
F_2 = F_5 &= \begin{pmatrix} -1.0005 & -1.2345 & -0.4951 \end{pmatrix}, \\
F_3 = F_6 &= \begin{pmatrix} -1.0005 & -1.0327 & -0.3982 \end{pmatrix},
\end{aligned}
$$

and

$$
\begin{aligned}
G_1 = G_4 &= \begin{pmatrix} 0.3290 & 0.0341 & 0.0028 \end{pmatrix}^\top, \\
G_2 = G_5 &= \begin{pmatrix} 0.2804 & 0.0193 & 0.0007 \end{pmatrix}^\top, \\
G_3 = G_6 &= \begin{pmatrix} 0.2578 & 0.0132 & 0.0002 \end{pmatrix}^\top.
\end{aligned}
$$

As an example, we take the initial states of the agents to be $x_{10} = \begin{pmatrix} 1.0 & 1.4 & 1.6 \end{pmatrix}^\top$, $x_{20} = \begin{pmatrix} 1.2 & -1.7 & 0.5 \end{pmatrix}^\top$, $x_{30} = \begin{pmatrix} 1.3 & -1.2 & 1.3 \end{pmatrix}^\top$, $x_{40} = \begin{pmatrix} 0.6 & 1.6 & -1.3 \end{pmatrix}^\top$, $x_{50} = \begin{pmatrix} 1.8 & 1.5 & 1.6 \end{pmatrix}^\top$, $x_{60} = \begin{pmatrix} -1.1 & 1.7 & 0.9 \end{pmatrix}^\top$. We take the initial states w_i to be zero, and the initial states v_i to be $v_{10} = \begin{pmatrix} 0.9 & 1.1 \end{pmatrix}^\top$, $v_{20} = \begin{pmatrix} 0.8 & 1.4 \end{pmatrix}^\top$,

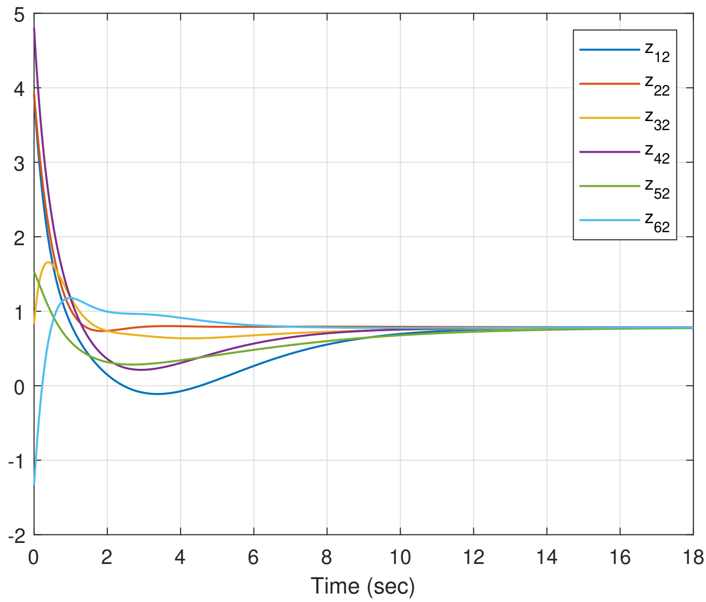

Figure 7.2: Plots of trajectories of the second component of the output vectors $z_1, z_2, \dots, z_6$

$v_{30} = \begin{pmatrix}-1.0 & 0.9\end{pmatrix}^\top$, $v_{40} = \begin{pmatrix}1.8 & 1.1\end{pmatrix}^\top$, $v_{50} = \begin{pmatrix}-1.6 & 1.4\end{pmatrix}^\top$, $v_{60} = \begin{pmatrix}1.1 & -1.2\end{pmatrix}^\top$. In Figures 7.1 and 7.2, we have plotted the trajectories of the output vectors z_i, $i = 1, 2 \dots, 6$ of the controlled network. The proposed protocol indeed achieves **z**-output synchronization for the network.

Moreover, for $i = 1, 2, \dots, 6$, we compute

$$S_i = \text{tr}(C_{1i}Q_iP_iQ_iC_{1i}^\top) + \text{tr}(C_{2i}Q_iC_{2i}^\top),$$

and obtain that

$$S_1 = S_4 = 0.6621,\ S_2 = S_5 = 0.4379,\ S_3 = S_6 = 0.3637.$$

Note that, for all $i = 1, 2, \dots, 6$, we have

$$S_i < \frac{\gamma}{N\lambda_N} = 0.75,$$

it then follows from Theorem 7.4 that the designed protocol is suboptimal, i.e. the associated cost is indeed smaller than the desired tolerance $\gamma = 18$.

7.7 Conclusions

In this chapter, we have studied the $\mathcal{H}_2$ suboptimal output synchronization problem for heterogeneous linear multi-agent systems. Given a heterogeneous multi-agent system and an associated $\mathcal{H}_2$ cost functional, we have provided a design method for computing dynamic output feedback based protocols that guarantee the associated cost to be smaller than a given upper bound while the controlled network achieves output synchronization. For each agent, its two local control gains are given in terms of solutions of two Riccati inequalities, each of dimension equal to that of the agent dynamics. The computation of the local control gains involves the largest eigenvalue of the Laplacian matrix of the communication graph. We have also considered the special case that full state information of the agents is available.

8 $\mathcal{H}_2$ and $\mathcal{H}_\infty$ suboptimal distributed filter design for linear systems

This chapter investigates the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ suboptimal distributed filtering problems for continuous time linear systems. Consider a linear system monitored by a number of local filters, where each of the filters receives only part of the measured output of the system. Each filter can communicate with the other filters according to an a priori given strongly connected weighted directed graph. The aim is to design filter gains that guarantee the $\mathcal{H}_2$ or $\mathcal{H}_\infty$ norm of the transfer matrix from the disturbance input to the output estimation error to be smaller than an a priori given upper bound, while all local filters reconstruct the full system state asymptotically. We provide a centralized design method for obtaining such suboptimal distributed $\mathcal{H}_2$ and $\mathcal{H}_\infty$ filters. The proposed design method is illustrated by a simulation example.

8.1 Introduction

Recent years have witnessed an increasing interest in problems of state estimation for spatially constrained large-scale systems. Such problems are relevant in applications, such as power grids [32], industrial plants [107] and wireless sensor networks [86]. Due to the physical constraints, the measured output of these systems is often monitored by a sensor network, consisting of a number of local sensors. Each of these local sensors makes use of its local measurements and then communicates with the other local sensors. In this way, all of these sensors together are able to estimate the state of the system asymptotically. In this problem setting, one of the main challenges is that none of the local sensors by itself is able to estimate the system state by using its own local measurements. Consequently, standard estimation methods do not directly apply anymore.

The distributed estimation problem has been mainly studied in two research directions, namely, distributed observer design and distributed Kalman filtering. In [88], an augmented state observer was proposed to cast the distributed observer design problem into a decentralized control problem for linear systems, using the notion of 'fixed modes' [92]. Later on, in [113], the results in [88] were extended and a more general form of distributed observers was provided, allowing the rate of convergence of the observer to be freely assignable. In [114], for time-varying communication graphs, a hybrid observer was introduced to distributedly estimate the state of a linear system. Based on observability decompositions, the problem of distributed observer design was also investigated in [27, 49, 65]. In [66], an attack resilient algorithm was introduced to address the distributed estimation problem when certain nodes are compromised by adversaries.

On the other hand, much attention in the literature has also been devoted to distributed filtering problems. A Kalman-filter-based distributed filter was proposed in [81, 83, 84]. There, the proposed methods employ a two-step strategy: a state update rule based on a Kalman-filter and a data fusion step based on consensus. In [105], a distributed robust filtering problem was addressed using dissipativity theory. Later on in [106], the results of [105] were generalized to the case that the communication graph is allowed to randomly change. Recently, in [47], a distributed Kalman-Bucy filtering problem was studied, using the idea of averaging the dynamics of heterogeneous multi-agent systems [48].

Different from the existing work, in this chapter, we will consider two deterministic versions of the distributed optimal filtering problem for linear systems, i.e., the distributed $\mathcal{H}_2$ and $\mathcal{H}_\infty$ filtering problems. Given a linear system and a network of local filters, each local filter receives a portion of the measured output of the system and then exchanges its state with that of its neighboring local filters. Together, these local filters form a distributed filter. We introduce $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performances to quantify the influence of the disturbances on the output estimation error. The distributed optimal filtering problem is then to find suitable filter gain matrices such that the associated $\mathcal{H}_2$ or $\mathcal{H}_\infty$ performance is minimized, while the states of all local filters asymptotically track the system state. However, due to *non-convexity*, this problem is difficult to solve in general. Therefore, in this chapter we will address a *suboptimality* version of this problem. The objective of this chapter is then to design suitable filter gain matrices such that the $\mathcal{H}_2$ or $\mathcal{H}_\infty$ performance is smaller than an a priori given tolerance. The main contributions of this chapter are the following:

1. We establish conditions for the existence of suitable filter gains in terms of solvability of LMI's for both the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ suboptimal distributed filtering problem. For the $\mathcal{H}_2$ filtering problem, all except one of these LMI's will always turn out to be solvable.

2. We provide conceptual algorithms for obtaining suitable distributed $\mathcal{H}_2$ and $\mathcal{H}_\infty$ suboptimal filters, respectively.

This chapter is organized as follows. In Section 8.2, we review some basic results on graph theory, detectability properties of linear systems, and the $\mathcal{H}_2$ and

$\mathcal{H}_\infty$ performance of linear systems. Subsequently, in Section 8.3 we formulate the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ suboptimal distributed filtering problems. We then provide design methods for obtaining such distributed filters in Section 8.4. In Section 8.5 we provide a simulation example to illustrate our design method. Finally, in Section 8.6 we formulate our conclusions.

8.2 Preliminaries

8.2.1 Detectability and detectability decomposition

In this subsection, we will review detectability and the detectability decomposition of linear systems. Consider the linear system

$$\begin{aligned} \dot{x} &= Ax, \\ y &= Cx, \end{aligned} \tag{8.1}$$

where $x \in \mathbb{R}^n$ represents the state and $y \in \mathbb{R}^p$ the measured output. The matrices A and C are of suitable dimensions.

Let $p(s)$ be the characteristic polynomial of A. Then $p(s)$ can be factorized as

$$p(s) = p_-(s)p_+(s),$$

where $p_-(s)$ and $p_+(s)$ have roots in the open left half-plane and the closed right half-plane, respectively. The undetectable subspace of the pair (C, A) is defined as

$$\mathcal{S} := \mathcal{N} \cap \ker\big(p_+(A)\big),$$

where

$$\mathcal{N} := \ker \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix}.$$

The pair (C, A) is detectable if and only if $\mathcal{S} = \{0\}$, see e.g. [116].

There exists an orthogonal matrix $T \in \mathbb{R}^{n\times n}$ such that the pair (C, A) is transformed into the detectability decomposition form

$$T^\top AT = \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix}, \quad CT = \begin{pmatrix} C_1 & 0 \end{pmatrix},$$

where $A_{11} \in \mathbb{R}^{v\times v}$, $A_{21} \in \mathbb{R}^{(n-v)\times v}$, $A_{22} \in \mathbb{R}^{(n-v)\times(n-v)}$, $C_1 \in \mathbb{R}^{p\times v}$ and the pair (C_1, A_{11}) is detectable. In addition, if we partition $T = (T_1\ T_2)$, where T_1 contains the first v columns, then the undetectable subspace is given by

$$\mathrm{im}(T_2) = \mathcal{S}.$$

Since T is orthogonal, we also have

$$\mathrm{im}(T_1) = \mathcal{S}^\perp.$$

8.2.2 $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance of linear systems

In this subsection, we will review the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance of a linear system with external disturbances. Consider the linear system

$$\begin{aligned} \dot{x} &= Ax + Ed, \\ y &= Cx, \end{aligned} \tag{8.2}$$

where $x \in \mathbb{R}^n$ is the state, $d \in \mathbb{R}^q$ the external disturbance and $y \in \mathbb{R}^p$ the measured output. The matrices A, C and E are of suitable dimensions.

We will first review the $\mathcal{H}_2$ performance of the system (8.2). Let $T_d(t) = Ce^{At}E$ be the impulse response of (8.2). Then the associated $\mathcal{H}_2$ performance is defined to be the square of its $\mathcal{L}_2$-norm, given by

$$J = \int_0^\infty \mathrm{tr}\left[T_d^\top(t)T_d(t)\right] dt. \tag{8.3}$$

Note that the performance (8.3) is finite if the system (8.2) is internally stable, i.e., A is Hurwitz.

The following well-known result provides a necessary and sufficient condition under which (8.2) is internally stable and (8.3) is smaller than a given upper bound (see e.g. [33], [96]):

Lemma 8.1. *Let $\gamma > 0$. Then the system* (8.2) *is internally stable and $J < \gamma$ if and only if there exists $P > 0$ satisfying*

$$\begin{aligned} A^\top P + PA + C^\top C &< 0, \\ \mathrm{tr}\left(E^\top PE\right) &< \gamma. \end{aligned}$$

Next, we will review the $\mathcal{H}_\infty$ performance of the system (8.2). Let $T_d(s) = C(sI_n - A)^{-1}E$ be the transfer matrix of (8.2). If A is Hurwitz, then the $\mathcal{H}_\infty$ performance of (8.2) is defined as the $\mathcal{H}_\infty$ norm of $T_d(s)$, given by

$$||T_d||_\infty := \sup_{\omega\in\mathbb{R}} \sigma(T(j\omega)), \tag{8.4}$$

where $\sigma(T_d(j\omega))$ is the maximum singular value of the complex matrix $T_d(j\omega)$.

The well-known bounded real lemma provides a necessary and sufficient condition under which (8.2) is stable and (8.4) is smaller than a given upper bound (see e.g. [124], [102]):

Lemma 8.2. *Let $\gamma > 0$. Then the system* (8.2) *is internally stable and $||T_d||_\infty < \gamma$ if and only if there exists $P > 0$ such that*

$$A^\top P + PA + \frac{1}{\gamma^2} PEE^\top P + C^\top C < 0.$$

In the next section, we will formulate the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ distributed filter design problems that will be addressed in this chapter.

8.3 Problem formulation

Consider the finite-dimensional linear time-invariant system

$$\begin{aligned} \dot{x} &= Ax + Ed, \\ y &= Cx + Dd, \\ z &= Hx, \end{aligned} \tag{8.5}$$

where $x \in \mathbb{R}^n$ is the state, $d \in \mathbb{R}^q$ the external disturbance, $y \in \mathbb{R}^r$ the measured output and $z \in \mathbb{R}^p$ the output to be estimated. The matrices A, C, D, E and H are of suitable dimensions.

The standard optimal filtering problem for the system (8.5) is to find a filter that takes y as input and returns an optimal estimate ζ of z, while the filter state asymptotically tracks the state x of (8.5). Here, 'optimal' means that the $\mathcal{H}_2$ or $\mathcal{H}_\infty$ norm of the transfer matrix from d to the estimation error $z - \zeta$ is minimized over all such filters. In that problem setting, however, a standing assumption is that one single filter is able to acquire the complete measured output y of the system.

In this chapter, we relax this assumption. More specifically, we assume that the measured output y of (8.5) is not available to one single filter, but is observed by N local filters. Moreover, each local filter only acquires a certain portion of the measured output, namely,

$$y_i = C_i x + D_i d,$$

where $y_i \in \mathbb{R}^{r_i}$, $C_i \in \mathbb{R}^{r_i \times n}$ and $D_i \in \mathbb{R}^{r_i \times q}$, for $i = 1, 2, \ldots, N$. Here, the matrices

C_i and D_i are obtained by partitioning

$$C = \begin{pmatrix} C_1 \\ C_2 \\ \vdots \\ C_N \end{pmatrix}, \quad D = \begin{pmatrix} D_1 \\ D_2 \\ \vdots \\ D_N \end{pmatrix}.$$

Clearly, the original output y of (8.5) has then been partitioned as

$$y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{pmatrix}$$

and $\sum_{i=1}^{N} r_i = r$. In this chapter it will be a standing assumption that the pair (C, A) is detectable. We will also assume that none of the pairs (C_i, A) is detectable itself. If, for at least one i, the pair (C_i, A) is detectable, the distributed filtering problem boils down to the standard optimal filtering problem.

In our distributed case, each local filter makes use of the portion of the measured output that it acquires and will then communicate with its neighboring local filters by exchanging filter state information. In this way, the local filters will together form a distributed filter. Following [27] and [49], we propose a distributed filter of the form

$$\begin{aligned} \dot{w}_i &= Aw_i + G_i(y_i - C_i w_i) + F_i \sum_{j=1}^{N} a_{ij}(w_j - w_i), \\ \zeta_i &= Hw_i, \quad i = 1, 2, \ldots, N, \end{aligned} \tag{8.6}$$

where $w_i \in \mathbb{R}^n$ is the state of the ith local filter and $\zeta_i \in \mathbb{R}^p$ is the associated output. The matrices $G_i \in \mathbb{R}^{n \times r_i}$ and $F_i \in \mathbb{R}^{n \times n}$ are local filter gains to be designed. The coefficients a_{ij} are the entries of the adjacency matrix $\mathcal{A}$ of the communication graph. In this chapter, it will be a standing assumption that this graph is a strongly connected weighted directed graph.

For the ith local filter, we introduce the associated local state estimation error e_i and local output estimation error η_i as

$$\begin{aligned} e_i &:= x - w_i, \\ \eta_i &:= z - \zeta_i, \quad i = 1, 2, \ldots, N. \end{aligned}$$

The dynamics of the ith local error system is then given by

$$\dot{e_i} = (A - G_iC_i)e_i + F_i \sum_{j=1}^{N} a_{ij}(e_j - e_i) + (E - G_iD_i)d,$$
$$\eta_i = He_i, \quad i = 1, 2, \ldots, N.$$

Denote $e = (e_1^\top, e_2^\top, \ldots, e_N^\top)^\top$, $\eta = (\eta_1^\top, \eta_2^\top, \ldots, \eta_N^\top)^\top$ and

$$\bar{A} := \text{blockdiag}(A - G_iC_i) \in \mathbb{R}^{nN \times nN},$$
$$\bar{F} := \text{blockdiag}(F_i) \in \mathbb{R}^{nN \times nN},$$
$$\bar{E} := \text{col}(E - G_iD_i) \in \mathbb{R}^{nN \times q}.$$

The global error system is then given by

$$\begin{aligned} \dot{e} &= \left(\bar{A} - \bar{F}(L \otimes I_n)\right) e + \bar{E}d, \\ \eta &= (I_N \otimes H)e, \end{aligned} \tag{8.7}$$

where $L \in \mathbb{R}^{N \times N}$ is the Laplacian matrix of the communication graph. The impulse response of the system (8.7) from the disturbance d to the output estimation error η is equal to

$$T_d(t) = (I_N \otimes H)e^{(\bar{A} - \bar{F}(L \otimes I_n))t}\bar{E}.$$

We introduce the global $\mathcal{H}_2$ cost functional

$$J = \int_0^\infty \text{tr}\left[T_d^\top(t)T_d(t)\right] dt. \tag{8.8}$$

The distributed $\mathcal{H}_2$ optimal filtering problem is then the problem of minimizing the $\mathcal{H}_2$ cost functional (8.8) over all distributed filters (8.6) such that the global error system (8.7) is internally stable. Note that (8.8) is a function of the local gain matrices $F_1, F_2, \ldots, F_N$ and $G_1, G_2, \ldots, G_N$.

Unfortunately, due to the particular form of (8.6), this optimization problem is, in general, non-convex and it is unclear whether a closed-form solution exists. Therefore, instead of trying to find an *optimal* solution, we will address a version of this problem that only requires *suboptimality*. More concretely, we aim at designing a distributed filter such that the error system (8.7) is internally stable and the $\mathcal{H}_2$ performance (8.8) is smaller than an a priori given tolerance γ. In that case, we say that the distributed filter (8.6) is $\mathcal{H}_2$ γ-suboptimal:

Definition 8.1. *Let $\gamma > 0$. The distributed filter* (8.6) *is called $\mathcal{H}_2$ γ-suboptimal if:*

1. *for all $i = 1, 2, \ldots, N$, whenever $d = 0$, we have that $\lim_{t\to\infty} \big(x(t) - w_i(t)\big) \to 0$ for all initial conditions on* (8.5) *and* (8.6).

2. *the associated performance* (8.8) *satisfies* $J < \gamma$.

Correspondingly, the $\mathcal{H}_2$ suboptimal distributed filtering problem that we will address is the following:

Problem 8.1. *Let* $\gamma > 0$. *For* $i = 1, 2, \dots, N$, *find gain matrices* $G_i \in \mathbb{R}^{n\times r_i}$ *and* $F_i \in \mathbb{R}^{n\times n}$ *such that the distributed filter* (8.6) *is* $\mathcal{H}_2$ γ*-suboptimal.*

In addition to the distributed filtering problem with $\mathcal{H}_2$ performance, in this chapter we will also consider the version of this problem with $\mathcal{H}_\infty$ performance. Obviously, the transfer matrix of the system (8.7) from the disturbance d to the output estimation error η is equal to

$$T_d(s) = (I_N \otimes H)\left(sI_{nN} - (\bar{A} - \bar{F}(L \otimes I_n))\right)^{-1} \bar{E}.$$

The $\mathcal{H}_\infty$ performance of the distributed filter (8.6) is given by the $\mathcal{H}_\infty$ norm $||T_d||_\infty$ of $T_d(s)$. The problem that we will then consider is to design a distributed filter (8.6) such that the error system (8.7) is internally stable and its $\mathcal{H}_\infty$ performance is smaller than an a priori given tolerance γ. In that case, we say that the distributed filter (8.6) is $\mathcal{H}_\infty$ γ-suboptimal:

Definition 8.2. *Let* $\gamma > 0$. *The distributed filter* (8.6) *is called* $\mathcal{H}_\infty$ γ*-suboptimal if:*

1. *for all* $i = 1, 2, \dots, N$, *whenever* $d = 0$, *we have that* $\lim_{t\to\infty}\big(x(t) - w_i(t)\big) \to 0$ *for all initial conditions on* (8.5) *and* (8.6).
2. $||T_d||_\infty < \gamma$.

Correspondingly, the $\mathcal{H}_\infty$ suboptimal distributed filtering problem that we will address is the following:

Problem 8.2. *Let* $\gamma > 0$. *For* $i = 1, 2, \dots, N$, *find gain matrices* $G_i \in \mathbb{R}^{n\times r_i}$ *and* $F_i \in \mathbb{R}^{n\times n}$ *such that the distributed filter* (8.6) *is* $\mathcal{H}_\infty$ γ*-suboptimal.*

8.4 $\mathcal{H}_2$ and $\mathcal{H}_\infty$ suboptimal distributed filter design

In this section, we will address Problems 8.1 and 8.2 introduced above and provide design methods for obtaining suboptimal distributed filters.

As we have explained before, the ith local filter (8.6) receives only a certain portion of the measured output, namely,

$$y_i = C_i x + D_i d, \quad i = 1, 2, \dots, N.$$

In order to proceed, we first apply orthogonal transformations to the pairs (C_i, A). For $i = 1, 2, \dots, N$, let T_i be an orthogonal matrix such that the pair (C_i, A) is transformed into the detectability decomposition form

$$T_i^\top A T_i = \begin{pmatrix} A_{i11} & 0 \\ A_{i21} & A_{i22} \end{pmatrix}, \quad C_i T_i = \begin{pmatrix} C_{i1} & 0 \end{pmatrix}, \tag{8.9}$$

where $A_{i11} \in \mathbb{R}^{v_i \times v_i}$, $A_{i21} \in \mathbb{R}^{(n-v_i)\times v_i}$, $A_{i22} \in \mathbb{R}^{(n-v_i)\times(n-v_i)}$, $C_{i1} \in \mathbb{R}^{r_i \times v_i}$ and the pair (C_{i1}, A_{i11}) is detectable. The integer v_i is equal to the dimension of the othogonal complement of the undetectable subspace of the pair (C_i, A). Accordingly, partition

$$T_i^\top E = \begin{pmatrix} E_{i1} \\ E_{i2} \end{pmatrix}, \quad H T_i = \begin{pmatrix} H_{i1} & H_{i2} \end{pmatrix}, \tag{8.10}$$

where $E_{i1} \in \mathbb{R}^{v_i \times q}$, $E_{i2} \in \mathbb{R}^{(n-v_i)\times q}$, $H_{i1} \in \mathbb{R}^{p \times v_i}$ and $H_{i1} \in \mathbb{R}^{p \times (n-v_i)}$.

Using the fact that (C_{i1}, A_{i11}) is detectable, let Q_{i1} be any positive definite solution to

$$A_{i11} Q_{i1} + Q_{i1} A_{i11}^\top - Q_{i1} C_{i1}^\top C_{i1} Q_{i1} < 0. \tag{8.11}$$

Then, by defining

$$G_{i1} := Q_{i1} C_{i1}^\top, \tag{8.12}$$

the matrix $A_{i11} - G_{i1} C_{i1}$ is Hurwitz.

In the sequel, we will make use of the transformed matrices (8.9) and (8.10) and the gain matrix (8.12) to obtain filter gains that solve Problems 8.1 and 8.2. Before presenting the main results of this chapter, we will first provide a lemma that will be essential for later use. This lemma is a generalization of [27, Lemma 4], and connects the Laplacian matrix of the communication graph with detectability properties of the system (8.5).

Lemma 8.3. *Let $\mathcal{L} := \Theta L + L^\top \Theta$, where Θ is defined as in Lemma 1.2. Define $T :=$ blockdiag$(T_i) \in \mathbb{R}^{nN \times nN}$, where the T_i are the orthogonal matrices introduced in* (8.9) *and* (8.10)*. Let $m_i > 0$ and*

$$M_i := \begin{pmatrix} m_i I_{v_i} & 0 \\ 0 & 0_{n-v_i} \end{pmatrix}, \quad i = 1, 2, \dots, N.$$

Define $M :=$ blockdiag(M_i). Then,

$$T^\top (\mathcal{L} \otimes I_n) T + M > 0. \tag{8.13}$$

Proof. Note that the inequality (8.13) holds if and only if the inequality

$$\mathcal{L} \otimes I_n + T M T^\top > 0 \tag{8.14}$$

holds. Let $U \in \mathbb{R}^{N\times N}$ be an orthogonal matrix that diagonalizes the matrix $\mathcal{L}$, i.e.,

$$U^\top \mathcal{L} U = \Lambda = \mathrm{diag}(0, \lambda_2, \dots, \lambda_N).$$

Then, the inequality (8.14) holds if and only if

$$(U \otimes I_n)(\Lambda \otimes I_n)(U \otimes I_n)^\top + TMT^\top > 0 \tag{8.15}$$

holds. The inequality (8.15) holds if

$$(U \otimes I_n)\left(\lambda_2 I_{nN} - \begin{pmatrix} \lambda_2 I_n & 0 \\ 0 & 0_{n(N-1)} \end{pmatrix}\right)(U \otimes I_n)^\top + TMT^\top > 0 \tag{8.16}$$

holds, where we use the fact that $\lambda_2 \leqslant \lambda_i$ for $i = 2, 3, \dots, N$. Note that (8.16) is equal to

$$\lambda_2 I_{nN} - (U \otimes I_n)\begin{pmatrix} \lambda_2 I_n & 0 \\ 0 & 0_{n(N-1)} \end{pmatrix}(U \otimes I_n)^\top + TMT^\top > 0. \tag{8.17}$$

Next, recall that

$$U = \begin{pmatrix} \frac{1}{\sqrt{N}}\mathbf{1}_N & U_2 \end{pmatrix} \quad \text{and} \quad U^\top = \begin{pmatrix} \frac{1}{\sqrt{N}}\mathbf{1}_N^\top \\ U_2^\top \end{pmatrix},$$

the inequality (8.17) is equivalent to

$$\lambda_2 I_{nN} - \frac{\lambda_2}{N}(\mathbf{1}_N \otimes I_n)(\mathbf{1}_N \otimes I_n)^\top + TMT^\top > 0. \tag{8.18}$$

Now, by pre- and post-multiplying $T^\top$ and T, the inequality (8.18) holds if and only if

$$\lambda_2 I_{nN} + M - \frac{\lambda_2}{N}\begin{pmatrix} T_1 & \dots & T_N \end{pmatrix}^\top \begin{pmatrix} T_1 & \dots & T_N \end{pmatrix} > 0. \tag{8.19}$$

Notice that, according to detectability decompositions, the matrix T_i can be partitioned as $T_i = \begin{pmatrix} T_{i1} & T_{i2} \end{pmatrix}$ with $T_{i1}T_{i1}^\top + T_{i2}T_{i2}^\top = I_n$, for $i = 1, 2, \dots, N$. By applying Schur complement, the inequality (8.19) holds if and only if

$$\begin{pmatrix} R_1 & \dots & 0 & T_1^\top \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & R_N & T_N^\top \\ T_1 & \dots & T_N & \frac{N}{\lambda_2} I_n \end{pmatrix} > 0 \tag{8.20}$$

holds, where

$$R_i = \begin{pmatrix} (\lambda_2 + m_i) I_{v_i} & 0 \\ 0 & \lambda_2 I_{n-v_i} \end{pmatrix}, \quad i = 1, 2, \ldots, N.$$

Again by using Schur complement, the inequality (8.20) is equivalent to

$$\begin{pmatrix} \lambda_2 I_{n-v_1} & 0 & \ldots & 0 & T_{12}^\top \\ 0 & R_2 & \ldots & 0 & T_2^\top \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & R_N & T_N^\top \\ T_{12} & T_2 & \ldots & T_N & \frac{N}{\lambda_2} I_n - \frac{1}{\lambda_2 + m_1} T_{11} T_{11}^\top \end{pmatrix} > 0. \tag{8.21}$$

Now, by repeatedly applying Schur complement, we obtain that the inequality (8.19) holds if and only if

$$\frac{N}{\lambda_2} I_n - \sum_{i=1}^{N} \frac{1}{\lambda_2 + m_i} T_{i1} T_{i1}^\top - \sum_{i=1}^{N} \frac{1}{\lambda_2} T_{i2} T_{i2}^\top > 0.$$

holds. Since

$$\begin{aligned} &\frac{N}{\lambda_2} I_n - \sum_{i=1}^{N} \frac{1}{\lambda_2 + m_i} T_{i1} T_{i1}^\top - \sum_{i=1}^{N} \frac{1}{\lambda_2} T_{i2} T_{i2}^\top \\ =&\frac{N}{\lambda_2} I_n - \sum_{i=1}^{N} \frac{1}{\lambda_2} T_{i1} T_{i1}^\top - \sum_{i=1}^{N} \frac{1}{\lambda_2} T_{i2} T_{i2}^\top \\ &\quad + \sum_{i=1}^{N} \frac{1}{\lambda_2} T_{i1} T_{i1}^\top - \sum_{i=1}^{N} \frac{1}{\lambda_2 + m_i} T_{i1} T_{i1}^\top \\ =&\sum_{i=1}^{N} \left(\frac{1}{\lambda_2} - \frac{1}{\lambda_2 + m_i} \right) T_{i1} T_{i1}^\top \end{aligned}$$

Let $m_{\min} = \min\{m_1, m_2, \ldots, m_N\}$, then we have

$$\begin{aligned} &\frac{N}{\lambda_2} I_n - \sum_{i=1}^{N} \frac{1}{\lambda_2 + m_i} T_{i1} T_{i1}^\top - \sum_{i=1}^{N} \frac{1}{\lambda_2} T_{i2} T_{i2}^\top \\ \geqslant&\sum_{i=1}^{N} \left(\frac{1}{\lambda_2} - \frac{1}{\lambda_2 + m_{\min}} \right) T_{i1} T_{i1}^\top \\ =&\left(\frac{1}{\lambda_2} - \frac{1}{\lambda_2 + m_{\min}} \right) \begin{pmatrix} T_{11} & \ldots & T_{N1} \end{pmatrix} \begin{pmatrix} T_{11} & \ldots & T_{N1} \end{pmatrix}^\top \end{aligned}$$

Since $m_{\min} > 0$, then $\frac{1}{\lambda_2} - \frac{1}{\lambda_2 + m_{\min}} > 0$. Therefore, the inequality (8.13) holds if

$$\operatorname{rank}\begin{pmatrix} T_{11} & \dots & T_{N1} \end{pmatrix} = n,$$

i.e., the matrix $\begin{pmatrix} T_{11} & \dots & T_{N1} \end{pmatrix}$ has full row rank. In the sequel, we will show that indeed $\operatorname{rank}\begin{pmatrix} T_{11} & \dots & T_{N1} \end{pmatrix} = n$.

Define

$$\mathcal{N}_i := \ker \begin{pmatrix} C_i \\ C_i A \\ \vdots \\ C_i A^{n-1} \end{pmatrix} \in \mathbb{R}^n$$

and

$$\mathcal{S}_i := \mathcal{N}_i \cap \ker\big(p_+(A)\big),$$

where $p_+(A)$ is defined in Subsection 8.2.1. Note that

$$\operatorname{im}(T_{i1}) = \mathcal{S}_i^{\perp}.$$

Furthermore, if the pair (C, A) is detectable, then

$$\bigcap_{i=1}^{N} \mathcal{S}_i = 0.$$

Next, we find that

$$\begin{aligned} &\big(\operatorname{im}\begin{pmatrix} T_{11} & \dots & T_{N1} \end{pmatrix}\big)^{\perp} \\ =& (\operatorname{im}(T_{11}) + \ldots + \operatorname{im}(T_{N1}))^{\perp} \\ =& \bigcap_{i=1}^{N} (\operatorname{im}(T_{i1}))^{\perp} \\ =& \bigcap_{i=1}^{N} \mathcal{S}_i \\ =& 0, \end{aligned}$$

where the last step is due to the assumption that the pair (C, A) is detectable. It then follows from $\big(\operatorname{im}\begin{pmatrix} T_{11} & \dots & T_{N1} \end{pmatrix}\big)^{\perp} = 0$ that $\operatorname{rank}\begin{pmatrix} T_{11} & \dots & T_{N1} \end{pmatrix} = n$. Consequently, the inequality (8.3) holds. This completes the proof. □

The lines of the proof of Lemma 8.3 is analogous to that of [27, Lemma 4], replacing the observability decomposition by the detectability decomposition. We provide the detailed proof here to make this chapter self-contained.

In the next two subsections, we will deal with the design of $\mathcal{H}_2$ and $\mathcal{H}_\infty$ suboptimal distributed filters, respectively.

8.4.1 $\mathcal{H}_2$ suboptimal distributed filter design

In this subsection, we will provide a design method for obtaining $\mathcal{H}_2$ suboptimal distributed filters. More specifically, we aim at finding a distributed filter such that the global error system (8.7) is stable and the associated $\mathcal{H}_2$ performance (8.8) is less than an a priori given tolerance.

The next lemma expresses the existence of suitable gain matrices F_i and G_i, $i = 1, 2, \dots, N$ in terms of solvability of LMI's.

Lemma 8.4. *Let $\gamma > 0$. Let the matrices T, M and $\mathcal{L}$ be as introduced in Lemma 8.3. Let $\epsilon > 0$ be such that*

$$T^\top(\mathcal{L} \otimes I_n)T + M > \epsilon I_{nN}. \tag{8.22}$$

Let G_{i1} be as defined in (8.12). *For $i = 1, 2, \dots, N$, assume there exist $\kappa > 0$, $P_{i1} > 0$ and $P_{i2} > 0$ satisfying*

$$\begin{pmatrix} \Phi_i + H_{i1}^\top H_{i1} + \kappa(m_i - \epsilon)I_{v_i} & A_{i21}^\top P_{i2} + H_{i1}^\top H_{i2} \\ P_{i2}A_{i21} + H_{i2}^\top H_{i1} & \Psi_i \end{pmatrix} < 0 \tag{8.23}$$

and

$$\sum_{i=1}^{N} \operatorname{tr}\left[(E_{i1} - G_{i1}D_i)^\top P_{i1}(E_{i1} - G_{i1}D_i) + E_{i2}^\top P_{i2}E_{i2}\right] < \gamma, \tag{8.24}$$

where

$$\Phi_i := A_{i11}^\top P_{i1} + P_{i1}A_{i11} - C_{i1}^\top G_{i1}^\top P_{i1} - P_{i1}G_{i1}C_{i1}, \tag{8.25}$$

$$\Psi_i := P_{i2}A_{i22} + A_{i22}^\top P_{i2} + H_{i2}^\top H_{i2} - \kappa\epsilon I_{n-v_i}. \tag{8.26}$$

For $i = 1, 2, \dots, N$, define gain matrices F_i and G_i by

$$F_i := \kappa\theta_i T_i \begin{pmatrix} P_{i1}^{-1} & 0 \\ 0 & P_{i2}^{-1} \end{pmatrix} T_i^\top \tag{8.27}$$

and

$$G_i := T_i \begin{pmatrix} G_{i1} \\ 0 \end{pmatrix}. \tag{8.28}$$

Then the corresponding distributed filter (8.6) *is $\mathcal{H}_2$ γ-suboptimal.*

Proof. First, it follows from (8.13) in Lemma 8.3 that there exists $\epsilon > 0$ such that

(8.22) holds. Next, note that (8.23) is equivalent to

$$\text{blockdiag}\begin{pmatrix} \Phi_i + H_{i1}^\top H_{i1} & A_{i21}^\top P_{i2} + H_{i1}^\top H_{i2} \\ P_{i2}A_{i21} + H_{i2}^\top H_{i1} & P_{i2}A_{i22} + A_{i22}^\top P_{i2} + H_{i2}^\top H_{i2} \end{pmatrix} + \kappa(M - \epsilon I_{nN}) < 0. \tag{8.29}$$

Using (8.22), it follows from (8.29) that

$$\text{blockdiag}\begin{pmatrix} \Phi_i + H_{i1}^\top H_{i1} & A_{i21}^\top P_{i2} + H_{i1}^\top H_{i2} \\ P_{i2}A_{i21} + H_{i2}^\top H_{i1} & P_{i2}A_{i22} + A_{i22}^\top P_{i2} + H_{i2}^\top H_{i2} \end{pmatrix} - \kappa T^\top(\mathcal{L} \otimes I_n)T < 0. \tag{8.30}$$

Let

$$P := \text{blockdiag}(P_i), \quad P_i := T_i \begin{pmatrix} P_{i1} & 0 \\ 0 & P_{i2} \end{pmatrix} T_i^\top. \tag{8.31}$$

Clearly, $P > 0$. By using (8.27), (8.28), (8.31), (8.9) and (8.10), then (8.30) holds if and only if

$$\bar{A}^\top P + P\bar{A} - (L^\top \otimes I_n)\bar{F}^\top P + P\bar{F}(L \otimes I_n) + I_N \otimes H^\top H < 0 \tag{8.32}$$

holds, where $\bar{F} :=$ blockdiag(F_i) and F_i is defined by (8.27). Therefore, there exist $\kappa > 0$, $P_{i1} > 0$ and $P_{i2} > 0$ such that (8.23) holds for $i = 1, 2, \dots, N$ if and only if there exists $P > 0$ of the form (8.31) such that (8.32) holds. Since the solutions of (8.23) also satisfy (8.24), we obtain

$$\text{tr}\left(\bar{E}^\top P \bar{E}\right) < \gamma. \tag{8.33}$$

Finally, since (8.32) and (8.33) have a solution $P > 0$, it follows from Lemma 8.1 that the error system (8.7) is internally stable and $J < \gamma$. Thus the distributed filter (8.6) with (8.28) and (8.27) is $\mathcal{H}_2$ γ-suboptimal. □

Remark 8.5. In Lemma 8.4, the choice of the parameters $m_i > 0$ is arbitrary. The parameter $\epsilon > 0$ should be chosen sufficiently small so that (8.22) holds. The gain G_i is defined by (8.28). Then, of course, the question arises: for chosen $m_i > 0$, $\epsilon > 0$ and G_i, how can we find the smallest $\gamma > 0$ such that the corresponding distributed filter (8.6) is $\mathcal{H}_2$ γ-suboptimal? This requires to find the smallest γ such that the LMI's (8.23) and (8.24) are solvable. It is well known that this can be done by using a standard bisection algorithm, see e.g. [124, page 115].

Remark 8.6. Lemma 8.4 states that if there exist solutions $\kappa > 0$, $P_{i1} > 0$ and $P_{i2} > 0$ satisfying (8.23) and (8.24), then the distributed filter (8.6) with gain matrices (8.27) and (8.28) is $\mathcal{H}_2$ γ-suboptimal. There, the inequality (8.24) is a global condition for checking suboptimality. In fact, such suboptimality condition can also be checked locally. Indeed, if for $i = 1, 2, \ldots, N$ there exist solutions satisfying (8.23) and

$$\operatorname{tr}\left[(E_{i1} - G_{i1}D_i)^\top P_{i1}(E_{i1} - G_{i1}D_i) + E_{i2}^\top P_{i2}E_{i2}\right] < \frac{\gamma}{N},$$

then the corresponding distributed filter (8.6) with (8.27) and (8.28) is $\mathcal{H}_2$ γ-suboptimal.

Lemma 8.4 provides a condition for the existence of suitable gain matrices F_i and G_i in terms of solvability of LMI's. In the next theorem, we show that, in fact, the LMI's (8.23) in Lemma 8.4 always have solutions. In fact, we can take P_{i2} to be the identity matrix of dimension $n - v_i$ and P_{i1} to be the unique solution of a given Lyapunov equation. In this way we obtain the following conceptual algorithm for computing suitable gain matrices.

Theorem 8.7. *Let $\gamma > 0$. Then an $\mathcal{H}_2$ γ-suboptimal distributed filter of the form* (8.6) *is obtained as follows:*

(i) Compute $\theta = (\theta_1, \theta_2, \ldots, \theta_N)$ with $\theta_i > 0$ such that $\theta L = 0$ and $\theta \mathbf{1}_N = N$.

Then, for $i = 1, 2, \ldots, N$:

(ii) Compute orthogonal matrices T_i that put A, C_i, E and H into the form (8.9) *and* (8.10).

(iii) Take $m_i = 1$ and compute $\epsilon > 0$ such that

$$T^\top(\mathcal{L} \otimes I_n)T + M > \epsilon I_{nN}. \tag{8.34}$$

(iv) Compute $Q_{i1} > 0$ satisfying (8.11). *Define $G_{i1} := Q_{i1}C_{i1}^\top$.*

(v) Take $\kappa > 0$ sufficiently large such that

$$\begin{aligned} A_{i22} + A_{i22}^\top + H_{i2}^\top H_{i2} - \kappa\epsilon I_{n-v_i} \\ + \frac{1}{\kappa\epsilon}(A_{i21} + H_{i2}^\top H_{i1})(A_{i21} + H_{i2}^\top H_{i1})^\top < 0. \end{aligned} \tag{8.35}$$

(vi) Compute $P_{i1} > 0$ satisfying the Lyapunov equation

$$(A_{i11} - G_{i1}C_{i1})^\top P_{i1} + P_{i1}(A_{i11} - G_{i1}C_{i1}) + H_{i1}^\top H_{i1} + \kappa I_{v_i} = 0. \tag{8.36}$$

(vii) Define gain matrices F_i and G_i by

$$F_i := \kappa\theta_i T_i \begin{pmatrix} P_{i1}^{-1} & 0 \\ 0 & I_{n-v_i} \end{pmatrix} T_i^\top, \quad G_i := T_i \begin{pmatrix} G_{i1} \\ 0 \end{pmatrix}. \tag{8.37}$$

Then for all $\gamma > 0$ *satisfying*

$$\sum_{i=1}^{N} \text{tr}\left[(E_{i1} - G_{i1}D_i)^\top P_{i1}(E_{i1} - G_{i1}D_i) + E_{i2}^\top E_{i2}\right] < \gamma, \tag{8.38}$$

the corresponding distributed filter (8.6) *with gain matrices* (8.37) *is* $\mathcal{H}_2$ γ*-suboptimal.*

Proof. Using Lemma 8.3, by choosing $m_i = 1$ for $i = 1, 2, \ldots, N$, there exists $\epsilon > 0$ such that (8.34) holds. Next, for $i = 1, 2, \ldots, N$, we choose $\kappa > 0$ sufficiently large such that (8.35) holds . Since Q_{i1} is a positive definite solution of (8.11) and $G_{i1} := Q_{i1}C_{i1}^\top$, then $A_{i11} - G_{i1}C_{i1}$ is Hurwitz. Consequently, for $i = 1, 2, \ldots, N$, the Lyapunov equation (8.36) has unique solution $P_{i1} > 0$. Since (8.35) holds and $-\kappa\epsilon I_{v_i} < 0$, by using the Schur complement, we obtain

$$\begin{pmatrix} -\kappa\epsilon I_{v_i} & A_{i21}^\top + H_{i1}^\top H_{i2} \\ A_{i21} + H_{i2}^\top H_{i1} & \tilde{\Psi}_i \end{pmatrix} < 0, \quad i = 1, 2, \ldots, N, \tag{8.39}$$

where $\tilde{\Psi}_i := A_{i22} + A_{i22}^\top + H_{i2}^\top H_{i2} - \kappa\epsilon I_{n-v_i}$. Using (8.36) and $P_{i2} = I_{n-v_i}$, it then follows that (8.23) holds.

On the other hand, by taking $P_{i2} = I_{n-v_i}$ in (8.24), we obtain (8.38). It then follows from Lemma 8.4 that the corresponding distributed filter is $\mathcal{H}_2$ γ-suboptimal. □

Remark 8.8. Note that, in step (i) of Theorem 8.7, we need to compute the left eigenvector θ of the Laplacian matrix corresponding to the eigenvalue 0. This requires so-called global information on the communication graph. This dependency on global information can be removed using algorithms that compute left eigenvectors of the Laplacian matrix in a distributed fashion, see e.g. [12] or [24]. On the other hand, in step (iii) we need to compute ϵ. To do so, we need knowledge of the orthogonal matrices T_i, the matrix M and the Laplacian matrix $\mathcal{L}$, which is global information. Also in step (v), we need to find one κ that satisfy (8.35) for $i = 1, 2, \ldots, N$. Note that, however, we can always take $\epsilon > 0$ sufficiently small and $\kappa > 0$ sufficiently large such that (8.34) and (8.35) hold, respectively. This might however lead to an achievable tolerance γ that is very large, giving poor suboptimality of the corresponding distributed filter.

In general, the computation of our suboptimal filters requires global information, so cannot be performed in a decentralized fashion. This is in contrast with the decentralized computation of distributed state observers as described in [50].

8.4.2 $\mathcal{H}_\infty$ suboptimal distributed filter design

In this subsection, we will provide a method for obtaining $\mathcal{H}_\infty$ suboptimal distributed filters. More concretely, we aim at finding, for a given tolerance $\gamma > 0$, a distributed filter such that the global error system (8.7) is stable and $||T_d||_\infty < \gamma$.

The next lemma expresses the existence of suitable gain matrices F_i and G_i, $i = 1, 2, \ldots, N$ in terms of solvability of N nonlinear matrix inequalities.

Lemma 8.9. *Let $\gamma > 0$. Let the matrices T, M and $\mathcal{L}$ be as introduced in Lemma 8.3. Let $\epsilon > 0$ be such that*

$$T^\top(\mathcal{L} \otimes I_n)T + M > \epsilon I_{nN}. \tag{8.40}$$

Let G_{i1} be as defined in (8.12)*. For $i = 1, 2, \ldots, N$, assume there exist $\kappa > 0$, $P_{i1} > 0$ and $P_{i2} > 0$ satisfying*

$$\begin{pmatrix} \Phi_i + \kappa(m_i - \epsilon)I_{v_i} & \Omega_i \\ \Omega_i^\top & \Psi_i - \kappa\epsilon I_{n-v_i} \end{pmatrix} < 0, \tag{8.41}$$

where

$$\begin{aligned} \Phi_i &= (A_{i11} - G_{i1}^\top C_{i1})^\top P_{i1} + P_{i1}(A_{i11} - G_{i1}^\top C_{i1}) \\ &\quad + \frac{1}{\gamma^2} P_{i1}(E_{i1} - G_{i1}D_i)(E_{i1} - G_{i1}D_i)^\top P_{i1} + H_{i1}^\top H_{i1}, \end{aligned} \tag{8.42}$$

$$\Omega_i = A_{i21}^\top P_{i2} + H_{i1}^\top H_{i2} + \frac{1}{\gamma^2} P_{i1}(E_{i1} - G_{i1}D_i)E_{i2}^\top P_{i2}, \tag{8.43}$$

$$\Psi_i = P_{i2}A_{i22} + A_{i22}^\top P_{i2} + \frac{1}{\gamma^2} P_{i2}E_{i2}E_{i2}^\top P_{i2} + H_{i2}^\top H_{i2}.$$

For $i = 1, 2, \ldots, N$, define gain matrices F_i and G_i by

$$F_i := \kappa\theta_i T_i \begin{pmatrix} P_{i1}^{-1} & 0 \\ 0 & P_{i2}^{-1} \end{pmatrix} T_i^\top \tag{8.44}$$

and

$$G_i := T_i \begin{pmatrix} G_{i1} \\ 0 \end{pmatrix}. \tag{8.45}$$

Then the corresponding distributed filter (8.6) *is $\mathcal{H}_\infty$ γ-suboptimal.*

Proof. First, it follows from (8.13) in Lemma 8.3 that there exists $\epsilon > 0$ such that (8.22) holds. Next, note that (8.41) is equivalent to

$$\text{blockdiag} \begin{pmatrix} \Phi_i & \Omega_i \\ \Omega_i^\top & \Psi_i \end{pmatrix} + \kappa(M - \epsilon I_{nN}) < 0. \tag{8.46}$$

Using (8.22), it then follows from (8.46) that

$$\text{blockdiag}\begin{pmatrix}\Phi_i & \Omega_i \\ \Omega_i^\top & \Psi_i\end{pmatrix} - \kappa T^\top(\mathcal{L}\otimes I_n)T < 0. \tag{8.47}$$

Let

$$P := \text{blockdiag}(P_i), \quad P_i := T_i\begin{pmatrix}P_{i1} & 0 \\ 0 & P_{i2}\end{pmatrix}T_i^\top. \tag{8.48}$$

Clearly, $P > 0$. By using (8.44), (8.45), (8.48), (8.9) and (8.10), then (8.47) holds if and only if

$$\begin{aligned}\bar{A}^\top P + P\bar{A} - (L^\top\otimes I_n)\bar{F}^\top P - P\bar{F}(L\otimes I_n)\\ +\frac{1}{\gamma^2}P\bar{E}\bar{E}^\top P + I_N\otimes H^\top H < 0\end{aligned} \tag{8.49}$$

holds, where $\bar{F} := \text{blockdiag}(F_i)$ and F_i is defined by (8.44). Therefore, there exist $\kappa > 0$, $P_{i1} > 0$ and $P_{i2} > 0$ such that (8.41) holds for $i = 1, 2, \ldots, N$ if and only if there exists $P > 0$ of the form (8.31) such that (8.49) holds. Finally, since (8.49) has a solution $P > 0$, it follows from Lemma 8.2 that the error system (8.7) is internally stable and $||T_d||_\infty < \gamma$. Thus the distributed filter (8.6) with (8.45) and (8.44) is $\mathcal{H}_\infty$ γ-suboptimal. □

Lemma 8.9 provides a condition for the existence of suitable gain matrices F_i and G_i in terms of solvability of the nonlinear matrix inequalities (8.41). However, these inequalities are not LMI's. However, by using suitable Schur complements, we can transform the inequalities (8.41) into LMI's. In this way we obtain the following conceptual algorithm for computing suitable gain matrices.

Theorem 8.10. *Let $\gamma > 0$. Then an $\mathcal{H}_\infty$ suboptimal distributed filter of the form* (8.6) *is obtained as follows:*

(i) Compute $\theta = (\theta_1, \theta_2, \ldots, \theta_N)$ with $\theta_i > 0$ such that $\theta L = 0$ and $\theta\mathbf{1}_N = N$.

For $i = 1, 2, \ldots, N$:

(ii) Compute an orthogonal matrix T_i that puts A, C_i, E and H into the form (8.9) *and* (8.10).

(iii) Take arbitrary $m_i > 0$ and compute $\epsilon > 0$ such that

$$T^\top(\mathcal{L}\otimes I_n)T + M > \epsilon I_{nN}. \tag{8.50}$$

(iv) Compute $Q_{i1} > 0$ satisfying (8.11). *Define $G_{i1} := Q_{i1}C_{i1}^\top$.*

(v) *Compute $P_{i1} > 0$, $P_{i2} > 0$ and $\kappa > 0$ such that the inequality*

$$\begin{pmatrix} \Delta_i & A_{i21}^\top P_{i2} + H_{i1}^\top H_{i2} & P_{i1}(E_{i1} - G_{i1}D_i) \\ P_{i2}A_{i21} + H_{i2}^\top H_{i1} & P_{i2}A_{i22} + A_{i22}^\top P_{i2} & P_{i2}E_{i2} \\ (E_{i1} - G_{i1}D_i)^\top P_{i1} & E_{i2}^\top P_{i2} & -\gamma^2 I_q \end{pmatrix} < 0 \quad (8.51)$$

with $\Delta_i = (A_{i11} - G_{i1}^\top C_{i1})^\top P_{i1} + P_{i1}(A_{i11} - G_{i1}^\top C_{i1}) + \kappa(m_i - \epsilon)I_{v_i} + H_{i1}^\top H_{i1}$ *holds.*

(vi) *Define gain matrices F_i and G_i by*

$$F_i := \kappa\theta_i T_i \begin{pmatrix} P_{i1}^{-1} & 0 \\ 0 & I_{n-v_i} \end{pmatrix} T_i^\top, \quad G_i := T_i \begin{pmatrix} G_{i1} \\ 0 \end{pmatrix}. \tag{8.52}$$

Then the corresponding distributed filter (8.6) *is $\mathcal{H}_\infty$ γ-suboptimal.*

Proof. By taking the appropriate Schur complements in (8.51), it follows that (8.51) hold if and only if (8.41) hold. The rest follows from Lemma 8.9. □

We conclude this section by noting that remarks similar to Remark 8.5 and Remark 8.8 hold in the $\mathcal{H}_\infty$ case.

8.5 Simulation example

In this section, we will use a simulation example borrowed from [49] to illustrate the conceptual algorithm in Theorem 8.7 for designing $\mathcal{H}_2$ suboptimal distributed filters. Consider the linear time-invariant system

$$\begin{aligned} \dot{x} &= Ax + Ed, \\ y &= Cx + Dd, \\ z &= Hx, \end{aligned} \tag{8.53}$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 \end{pmatrix}, \quad E = \begin{pmatrix} 0.1 \\ 0.1 \\ 0 \\ 0.1 \end{pmatrix}, \quad C = I_4,$$

$$D = \begin{pmatrix} 0.1 \\ 0 \\ 0.1 \\ 0.1 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

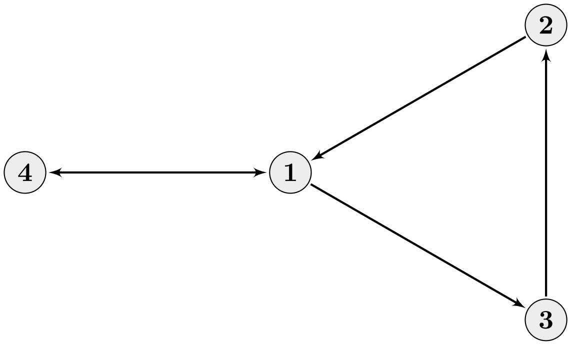

Figure 8.1: The communication graph between the local filters.

The system (8.53) is monitored by four local filters, and each local filter acquires a portion of the measured output y, namely,

$$y_i = C_i x + D_i d, \quad i = 1, 2, 3, 4,$$

where the matrices C_i and D_i are obtained by partitioning

$$C = \begin{pmatrix} C_1 \\ \hline C_2 \\ \hline C_3 \\ \hline C_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} D_1 \\ \hline D_2 \\ \hline D_3 \\ \hline D_4 \end{pmatrix} = \begin{pmatrix} 0.1 \\ \hline 0 \\ \hline 0.1 \\ \hline 0.1 \end{pmatrix}.$$

The pair (C, A) is detectable but none of the pairs (C_i, A) is detectable.

We assume the four local filters to be of the form (8.6).

The communication graph between the four local filters is depicted in Figure 8.1. The graph is strongly connected and the associated Laplacian matrix is given by

$$L = \begin{pmatrix} 2 & -1 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}.$$

The normalized left eigenvector θ of L associated with eigenvalue 0 is computed to be $\theta = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}$.

Next, for $i = 1, 2, 3, 4$, we compute an orthogonal matrix T_i such that the matrices A, C_i, E and H are transformed into the form (8.9) and (8.10). For $i = 1, 2, 3, 4$, we take $m_i = 1$. We also compute $\epsilon = 0.42$ such that (8.34) holds. Subsequently, for $i = 1, 2, 3, 4$, we solve (8.35) and compute $\kappa = 9.6$. Following the

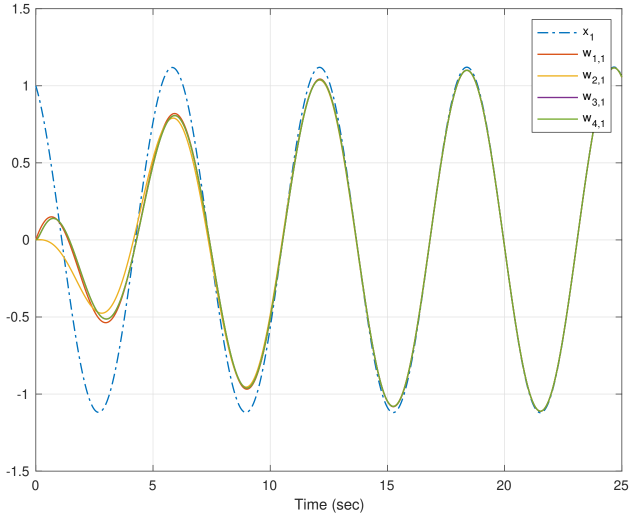

Figure 8.2: Plots of trajectories of x_1 (dashed lines) and the corresponding filter state component (solid lines).

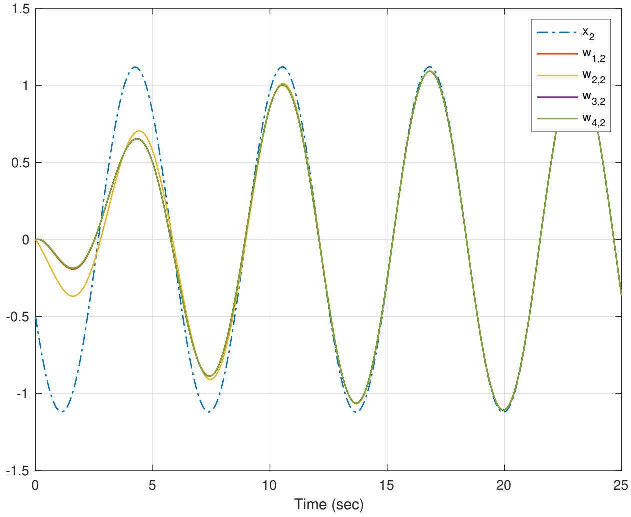

Figure 8.3: Plots of trajectories of x_2 (dashed lines) and the corresponding filter state component (solid lines).

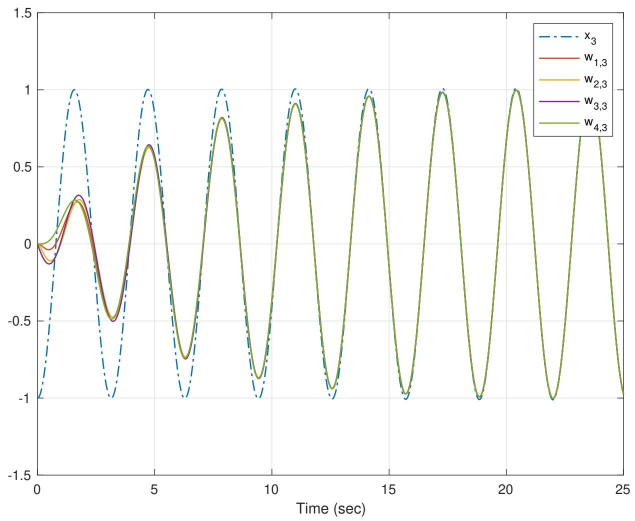

Figure 8.4: Plots of trajectories of x_3 (dashed lines) and the corresponding filter state component (solid lines).

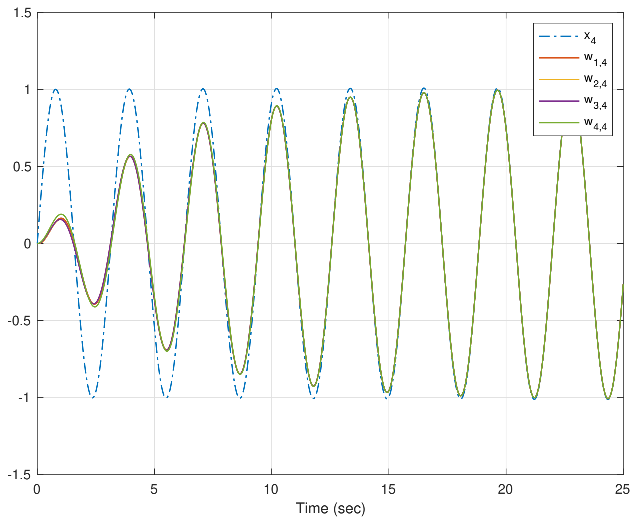

Figure 8.5: Plots of trajectories of x_4 (dashed lines) and the corresponding filter state component (solid lines).

steps in Theorem 8.7, gain matrices F_i and G_i are then computed as

$$F_1 = \begin{pmatrix} 0.3636 & 0.0837 & 0 & 0 \\ 0.0837 & 0.3442 & 0 & 0 \\ 0 & 0 & 9.6000 & 0 \\ 0 & 0 & 0 & 9.6000 \end{pmatrix},$$

$$F_2 = \begin{pmatrix} 0.3460 & -0.0660 & 0 & 0 \\ -0.0660 & 0.3579 & 0 & 0 \\ 0 & 0 & 9.6000 & 0 \\ 0 & 0 & 0 & 9.6000 \end{pmatrix},$$

$$F_3 = \begin{pmatrix} 9.6000 & 0 & 0 & 0 \\ 0 & 9.6000 & 0 & 0 \\ 0 & 0 & 0.4274 & 0.0491 \\ 0 & 0 & 0.0491 & 0.4217 \end{pmatrix},$$

$$F_4 = \begin{pmatrix} 9.6000 & 0 & 0 & 0 \\ 0 & 9.6000 & 0 & 0 \\ 0 & 0 & 0.4220 & -0.0445 \\ 0 & 0 & -0.0445 & 0.4266 \end{pmatrix},$$

and

$$G_1 = \begin{pmatrix} 0.4445 \\ 0.0488 \\ 0 \\ 0 \end{pmatrix}, \quad G_2 = \begin{pmatrix} -0.0488 \\ 0.4445 \\ 0 \\ 0 \end{pmatrix},$$

$$G_3 = \begin{pmatrix} 0 \\ 0 \\ 0.4465 \\ 0.0248 \end{pmatrix}, \quad G_4 = \begin{pmatrix} 0 \\ 0 \\ -0.0248 \\ 0.4465 \end{pmatrix}.$$

As an example, we take the initial state of the system (8.53) to be $x_0 = \begin{pmatrix} 1 & -0.5 & -1 & 0 \end{pmatrix}^\top$ and the initial state of the distributed filter to be zero. In Figures 8.2, 8.3, 8.4 and 8.5, we have plotted the state trajectories of the system and that of the distributed filter in absence of external disturbances. It can be seen that the states of the local filters asymptotically track the state of the system (8.53). Moreover, we compute

$$\sum_{i=1}^{N} \mathrm{tr}\left[(E_{i1}^\top - D_i^\top G_{i1}^\top) P_{i1} (E_{i1} - G_{i1} D_i) + E_{i2}^\top E_{i2}\right] = 1.3717.$$

Thus, for all $\gamma > 1.3717$, the distributed filter (8.6) with gain matrices F_i and G_i is $\mathcal{H}_2$ γ-suboptimal.

8.6 Conclusions

In this chapter, we have studied the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ suboptimal distributed filtering problems for linear systems. We have established conditions for the existence of suitable filter gains. These are expressed in terms of solvability of LMI's. Based on these conditions, we have provided conceptual algorithms for obtaining $\mathcal{H}_2$ and $\mathcal{H}_\infty$ suboptimal distributed filters, respectively.

9 Conclusions and future research

In this thesis, we have studied the distributed linear quadratic suboptimal control problem, the distributed $\mathcal{H}_2$ suboptimal control problem, and the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ suboptimal distributed filtering problems. In the present chapter we will discuss the main results and contributions presented in Chapters 2 - 8. We will also provide directions for possible future research.

9.1 Conclusions

In Chapter 2, we have studied a suboptimality version of the distributed linear quadratic optimal control for leaderless homogeneous multi-agent systems. Given a number of homogeneous agents, we have introduced a global linear quadratic cost functional. The communication graph between the agents is represented by a connected undirected graph. We have provided a *centralized* design method for computing distributed control laws, whose cost is bounded by a given upper bound for all initial states in a closed ball of a given radius, such that the controlled network achieves synchronization. The computation of the local control gain uses the exact knowledge of the smallest nonzero and largest eigenvalue of the Laplacian matrix. As an extension, we have also provided a design method for computing distributed control laws which does not depend on exact knowledge of the smallest nonzero and largest eigenvalue of the Laplacian matrix.

In Chapter 3, we have extended the results in Chapter 2 on distributed linear quadratic optimal control for leaderless homogeneous multi-agent systems to the case of distributed linear quadratic optimal tracking control for leader-follower homogeneous multi-agent systems. Given one autonomous leader and a number of followers, we have introduced an associated global linear quadratic cost functional. We have established a *centralized* design method for computing distributed control laws, whose cost is smaller than an upper bound for all initial states bounded in a norm by given radius, such that the controlled followers reach tracking synchronization. The computation of the local control gain depends on the smallest and largest eigenvalue of a given positive definite matrix associated with the communication graph between the agents.

The computation of the local control gains in Chapters 2 and 3 requires exactly knowledge of eigenvalues of the Laplacian matrix or of a given positive definite matrix associated with the communication graph, which is *global information*. Consequently, the design methods for obtaining such distributed control laws are in a *centralized* fashion. In Chapter 4, we have aimed at removing this dependence on global information. More specifically, we have considered the distributed linear quadratic optimal control problem for leaderless multi-agent systems with *single integrator* dynamics. We have first shown that the optimal local control gain can *only* be obtained in a centralized fashion, i.e. the computation of the local gain depends on global information. We have then established a *decentralized* computation method for obtaining distributed control laws, where each agent computes its local control gain using sampled information of its neighboring agents. This decentralized computation leads to a suboptimal overall network behavior.

In Chapter 5, we have studied the distributed $\mathcal{H}_2$ suboptimal control problem for homogeneous multi-agent systems by *static relative state feedback*. Given a number of identical agents, an associated $\mathcal{H}_2$ cost functional and a connected weighted undirected communication graph interconnecting the agents. We have provided a centralized design method for obtaining distributed protocols such that the associated $\mathcal{H}_2$ cost is smaller than an a priori given upper bound while the controlled network achieves state synchronization.

In Chapter 6, we have generalized the results in Chapter 5 on the distributed $\mathcal{H}_2$ suboptimal control problem for homogeneous multi-agent systems by static state feedback to the case of *dynamic relative output feedback*. We have first solved an open problem of finding, for a single continuous-time linear system, a *separation principle* based $\mathcal{H}_2$ suboptimal dynamic output feedback controller. We have then made use of these results to establish a centralized design method for computing distributed $\mathcal{H}_2$ suboptimal protocols such that the controlled network reaches state synchronization.

In Chapter 7, we have further generalized the results in Chapters 5 and 6 on distributed $\mathcal{H}_2$ suboptimal control of homogeneous multi-agent systems to the case of *heterogeneous* multi-agent systems. For heterogeneous systems, since the agent dynamics is possibly non-identical, in particular the state space dimensions of the agents may differ. Therefore, instead of state synchronization as considered in Chapters 5 and 6, it is more interesting and natural to consider *output synchronization*. Given a heterogeneous multi-agent system and an $\mathcal{H}_2$ cost functional, we have proposed a centralized method for computing distributed protocols such that the associated $\mathcal{H}_2$ cost is smaller than an a priori given upper bound while the controlled agents reach output synchronization.

In Chapter 8, we have studied the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ suboptimal distributed filtering problem for linear systems. Given a linear system monitored by a number of local filters, where each local filter receives only a part of the system output according to a given strongly connected directed graph. We have established conceptual algorithms for obtaining $\mathcal{H}_2$ and $\mathcal{H}_\infty$ suboptimal distributed filters, in the sense that the $\mathcal{H}_2$ or $\mathcal{H}_\infty$ norm of the transfer matrix of the disturbance input to the output estimation error is smaller than an a priori given upper bound while all local filters reconstruct the system state asymptotically.

9.2 Future research

The results presented in this thesis can be extended in several directions.

- In this thesis, we have investigated *suboptimality* versions of the distributed linear quadratic optimal control problem, the distributed $\mathcal{H}_2$ optimal control problem, and the distributed $\mathcal{H}_2$ and $\mathcal{H}_\infty$ optimal filtering problem. Future research could investigate whether, in general, there exist closed form solutions for the genuine *optimal* control and filtering problems.

- The computation of the local control gains in Chapters 2, 3, 5 - 7 requires exact knowledge of the eigenvalues of the Laplacian matrix or of a given positive definite matrix associated with the communication graph, which is global information. As a possibility for future research, we mention the development of methods for *decentralized* computation of the control gains.

- In Chapter 4, the model of the agents is represented by single integrator dynamics. One possibility for future research is to generalize the results in Chapter 4 to the case of *general higher dimensional* agent dynamics. Another possibility for future research is to extend the results in Chapter 4, in which the time clock of sampling is synchronized for all agents, to the case that the sampling takes place in an *asynchronous* way, using results, for example, in [9, 58].

- Chapters 5 - 7 are concerned with the distributed $\mathcal{H}_2$ suboptimal control problem. It would be interesting to extend the results in these chapters to that of distributed $\mathcal{H}_\infty$ suboptimal control.

- In Chapter 8, the computation of the distributed filters requires centralized computation. As a possibility for future research, we mention the extension of the results in this chapter to the case that filter gains are computed in a *decentralized* fashion, see for example [50]. Another possibility for future research is to allow the local filters to estimate, instead of the complete state, only part of the state of the monitored system.

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Summary

The thesis is concerned with several problems related to distributed linear quadratic control and filtering. In particular, we consider the distributed linear quadratic optimal control problem, the distributed $\mathcal{H}_2$ optimal control problem, and the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ optimal distributed filtering problems. Due the the *non-convex* nature of these problem, the *actual* optimization problems are in general very challenging and it is not clear whether closed form solutions exist. Therefore, in this thesis, instead we consider *suboptimality* versions of these problems.

We first consider the distributed linear quadratic suboptimal control problem for *leaderless* linear multi-agent systems. Given a multi-agent system with identical agent dynamics and a global linear quadratic cost functional, we establish a *centralized* design method for computing distributed control laws such that the associated cost is smaller than a given upper bound while the controlled network achieves state synchronization. We then extend these results to the problem of distributed linear quadratic suboptimal tracking control for *leader-follower* multi-agent systems. We provide a *centralized* design method for computing distributed suboptimal control laws such that the states of the controlled followers track that of the leader asymptotically.

In the two problems above, the proposed design methods are centralized, i.e. the computation of the local gains depends on so-called *global information*. To remove the dependence on this global information, as the third problem, we aim at *decentralized computation* of the local gains. For multi-agent systems with *single integrator* agent dynamics, we establish a decentralized computation method for obtaining distributed suboptimal control laws, where each agent computes its own local control gains using sampled state information of its neighboring agents.

As the fourth problem, we consider the distributed $\mathcal{H}_2$ suboptimal control problem for leaderless *homogeneous* multi-agent systems by *static relative state feedback*. Given a multi-agent system and an associated $\mathcal{H}_2$ cost functional, we provide a centralized design method for obtaining distributed protocols such that the associated $\mathcal{H}_2$ cost functional is smaller than a given upper bound while the controlled network achieves *state* synchronization. We then generalize the results on distributed $\mathcal{H}_2$ suboptimal control using static state feedback to the case of

dynamic relative output feedback. The results on distributed $\mathcal{H}_2$ suboptimal control for state synchronization of homogeneous multi-agent systems are then further generalized to the case of *output* synchronization of *heterogeneous* multi-agent systems.

Finally, we study the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ suboptimal distributed filtering problems. Given is a linear system monitored by a number of local filters, where each local filter receives only a certain part of the system output. The local filters exchange information about their estimates of the system state with their neighbors according to a given communication graph. We establish conceptual algorithms for obtaining $\mathcal{H}_2$ and $\mathcal{H}_\infty$ suboptimal distributed filters, in the sense that the $\mathcal{H}_2$ or $\mathcal{H}_\infty$ norm of the transfer matrix of the disturbance input to the output estimation error is smaller than an a priori given upper bound while all local filters reconstruct the system state asymptotically.

Samenvatting

In dit proefschrift worden verschillende gedistribueerde lineair-kwadratische regelproblemen en filterproblemen bestudeerd. In het bijzonder bestuderen we het gedistribueerde lineair-kwadratische optimale regelprobleem, het gedistribueerde $\mathcal{H}_2$ optimale regelprobleem en de $\mathcal{H}_2$ en $\mathcal{H}_\infty$ optimale gedistribueerde filterproblemen. Vanwege de *niet-convexe* aard van deze problemen zijn de *eigenlijke* optimalisatieproblemen over het algemeen erg uitdagend en is het niet duidelijk of er oplossingen in gesloten vorm bestaan. Daarom bestuderen we in dit proefschrift *suboptimale* versies van deze problemen.

Het eerste probleem waar we ons op richten is het gedistribueerde lineair-kwadratische suboptimale regelprobleem voor lineaire multi-agent systemen *zonder leiders*. Voor een multi-agent systeem met identieke agent-dynamica en een globale lineair-kwadratische kostenfunctionaal stellen we een *gecentraliseerde* ontwerpmethode vast om gedistribueerde regelwetten te berekenen, zodanig dat de geassocieerde kosten kleiner zijn dan een gegeven bovengrens, terwijl het geregelde netwerk toestandsynchronisatie bereikt. Vervolgens breiden we deze resultaten uit naar het gedistribueerde lineair-kwadratische suboptimale tracking regelprobleem voor *leider-volger* multi-agent systemen. We bieden een *gecentraliseerde* ontwerpmethode om gedistribueerde suboptimale regelwetten te berekenen zodanig dat de toestanden van de geregelde volgers die van de leider asymptotisch tracken.

In de twee hierboven beschreven problemen zijn de voorgestelde methoden gecentraliseerd, dat wil zeggen dat de berekening van de lokale versterkingsfactoren afhankelijk is van zogenaamde *globale informatie*. Om de afhankelijkheid van deze globale informatie te omzeilen, streven we bij wijze van het derde probleem naar *gedecentraliseerde berekeningen* van de lokale versterkingsfactoren. Voor multi-agent systemen met *enkelvoudige integrator* agent-dynamica stellen we een gedecentraliseerde methode vast om gedistribueerde suboptimale regelwetten te verkrijgen, waar elke agent zijn eigen lokale regel versterkingsfactoren berekent met behulp van bemonsterde toestandinformatie van de aangrenzende agenten.

Het vierde probleem dat wordt onderzocht is het gedistribueerde $\mathcal{H}_2$ suboptimale regelprobleem voor *homogene* multi-agent systemen zonder leiders door middel van *statische relatieve toestandsterugkoppeling*. Voor een multi-agent systeem en een bijbehorende $\mathcal{H}_2$ kostenfunctionaal bieden we een gecentraliseerde ontwerpmethode om gedistribueerde protocollen te verkrijgen zodanig dat de bijbehorende $\mathcal{H}_2$ kostenfunctionaal kleiner is dan een gegeven bovengrens, terwijl het geregelde netwerk *toestandssynchronisatie* bereikt. Vervolgens generaliseren we de resultaten van gedistribueerde $\mathcal{H}_2$ suboptimale regeling met behulp van statische toestandsterugkoppeling naar het geval van *dynamische relatieve uitgangsterugkoppeling*. De resultaten van gedistribueerde $\mathcal{H}_2$ suboptimale regeling voor toestandssynchronisatie van homogene multi-agent systemen worden vervolgens verder gegeneraliseerd naar het geval van *uitgangssynchronisatie* van *heterogene* multi-agent systemen.

Tenslotte bestuderen we de $\mathcal{H}_2$ en $\mathcal{H}_\infty$ suboptimale gedistribueerde filterproblemen. Gegeven is een lineair systeem dat gemonitord wordt door een aantal lokale filters, waar elk lokaal filter slechts een bepaald gedeelte van de uitgang van het systeem ontvangt. De lokale filters wisselen informatie over hun schatting van de toestand van het systeem uit met die van de aangrenzende filters volgens een gegeven communicatiegraaf. We stellen conceptuele algoritmes vast voor het verkrijgen van $\mathcal{H}_2$ en $\mathcal{H}_\infty$ suboptimale gedistribueerde filters, in die zin dat de $\mathcal{H}_2$ of $\mathcal{H}_\infty$ norm van de transfermatrix van de verstoringsingang naar de uitgangsschattingsfout kleiner is dan een a priori gegeven bovengrens, terwijl alle lokale filters de systeemtoestand asymptotisch reconstrueren.