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A New Controllability Gramian for Semistable Systems and its Application to Approximation of Directed Networks

Xiaodong Cheng, *Student Member* and Jacquelien M.A. Scherpen, *Senior Member*

***Abstract*— In this paper, we propose a new definition of a controllability Gramian for semistable systems, which is then used for model reduction of network systems. The system under consideration is modeled as single integrators that interconnected with each other according to a connected directed network. In the proposed method, the complexity of the network is reduced through a graph clustering method which aggregates the vertices if they respond similarly with respect to external inputs. Here, the similarity of the vertices is computed based on the new Gramian. The reduced-order model is obtained by a Petrov-Galerkin projection where the projection matrices are constructed from the resulting clustering. The reduced system preserves the network structure, and the approximation error between the full-order and reduced-order models is shown to be always bounded. Finally, the proposed approach is illustrated by an example.**

I. Introduction

The study of network systems has become a popular topic of interdisciplinary research in the recent decades, since they represent complex systems appearing in a wide range of scenarios, including social and ecological networks, chemical reactions and physical networks, see e.g. [1]–[5]. However, large-scale networks usually pose a challenge for both experiment and theory because of their high dimensions. Therefore, it is worth addressing a structure preserving model reduction problem, which aims to find a reduced-order model inheriting a network structure as well as approximating the behavior of the original high-dimensional system.

For general linear systems, methods such as balanced truncation, Hankel norm approximation, and Krylov subspace methods have been extensively studied [6]–[8]. Nevertheless, those approaches do not guarantee to maintain a network configuration of the reduced-order model, i.e., the obtained model cannot be interpreted as a network again. A balanced truncation method based on generalized Gramians is proposed in [9], which yields a reduced network interpreted by a smaller-sized Laplacian matrix. However, there is no direct relation between the reduced and original networks, and the states of the reduced-order system are mixtures of all the original states.

Recently, graph clustering is introduced for model reduction of network systems, and the preservation of a network structure is its major advantage. Related research for undirected networks can be found in [10]–[15]. However, many applications are modeled as Laplacian dynamics on directed networks. Examples can be found in, e.g., chemical reaction networks, food web, and metabolic processes [1], which usually describe mass/energy exchanges among different species. In [16], Markovian processes are considered as semistable positive directed networks. The notion of reducibility is introduced to characterize the uncontrollability of state variables. Then, aggregating reducible variables yields a reduced-order model that still contains the network structural information. However, this method does not take into account the Laplacian matrix of the underlying graph. In contrast, the current paper considers the simplification of a network as the reduction of its associated Laplacian matrix such that the topology of the reduced network is reflected by a lower-dimensional Laplacian matrix.

In our previous work, we have introduced a framework of clustering-based model reduction for undirected networks [14], [17], [18]. The behavior of each vertex is captured by a transfer function from external inputs to an individual state. Thus, the dissimilarity of a pair of vertices can be characterized by the norm of the transfer function discrepancy. Then, some clustering algorithms are proposed to systematically group these vertices with smaller dissimilarities.

In this paper, we extend our method to strongly connected directed networks, which are modeled as Laplacian dynamics. The definition of a semi-controllability Gramian is generalized from [18], which provides us a convenient way to compute the vertex dissimilarities. Besides, the hierarchical clustering algorithm is adapted such that the corresponding model reduction problem finds an appropriate network clustering. In contrast to [16], the resulting reduced-order model always guarantees a bounded approximation error, regardless of the choices of clusters.

The rest of this paper is organized as follows: In Section II, we introduce a notation of semi-controllability Gramian, which is used to propose a scheme of model reduction of directed networks in Section III. Section IV demonstrates the effectiveness of the proposed method by an example. Finally, concluding remarks are made in Section V.

Notation: Denoted $\mathbb{R}$ as the set of real numbers. $\mathbb{R}(s)$ is the rational function field over $\mathbb{R}$ with variable s. For a vector $v \in \mathbb{R}^n$, v_i is its i-th element. The cardinality of a set $\mathcal{V}$ is given by $|\mathcal{V}|$. The $\mathcal{H}_2$ norm of transfer function $G(s)$ is denoted by $\|G(s)\|_{\mathcal{H}_2}$, respectively. The identity matrix of size n is given as I_n, and $\mathbf{1}_n$ denotes a n-entries vector of all ones. The subscript n is omitted when no confusion arises. e_i is the i-th column vector of I_n. Furthermore, $\mathrm{tr}(A)$ denotes the trace of A.

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II. Semi-Controllability Gramian

In this section, we introduce the concept of a *semi-controllability Gramian*, which is extended from the definition of network controllability Gramian in [18]. The new Gramian is the generalization of standard Gramians in [7] and well-defined for all semistable systems.

Consider a general linear time-invariant system

$$\mathbf{\Sigma}_s : \dot{x} = Ax + Bu, \tag{1}$$

with states $x \in \mathbb{R}^n$ and inputs $u \in \mathbb{R}^k$. The formal definition of a semistable system is as follows.

Definition 1: [19] The system $\mathbf{\Sigma}_s$ is **semistable** if $\lim_{t\to\infty} x(t)$ exists for all initial conditions $x(0)$ when $u = 0$.

Furthermore, a necessary and sufficient condition for the semistability of $\mathbf{\Sigma}_s$ is given by the following lemma.

Lemma 1: [19] The system $\mathbf{\Sigma}_s$ is semistable if and only if the zero eigenvalues of A are **semisimple** (i.e., the geometric multiplicity of each eigenvalue coincides with its algebraic multiplicity), and all the other eigenvalues have negative real parts.

Recall from [7] the standard definition of controllability Gramian.

$$\mathcal{P}_s = \int_0^\infty e^{At} B B^T e^{A^T t} dt. \tag{2}$$

It is clear that this definition is not applicable for the semistable system $\mathbf{\Sigma}_s$, since e^{At} does not necessarily converge to zero as $t \to \infty$, and the integral in (2) may be unbounded. Hence, we propose a new definition of controllability Gramian as follows.

Notice that all the zero eigenvalues of A in (1) are semisimple. Assume that the algebraic multiplicity of the zero eigenvalues is m. Then, there exists a decomposition of A as follows.

$$A = \mathcal{U} D \mathcal{U}^{-1} = \begin{bmatrix} U, \bar{U} \end{bmatrix} \begin{bmatrix} \mathbf{0}_{m\times m} & \\ & \bar{D} \end{bmatrix} \begin{bmatrix} V^T \\ \bar{V}^T \end{bmatrix} \tag{3}$$

where $\bar{D} \in \mathbb{R}^{(n-m)\times(n-m)}$ is Hurwitz. The columns of $U \in \mathbb{R}^{n\times m}$ and $V \in \mathbb{R}^{n\times m}$ are, respectively, the *right eigenvectors* and *left eigenvectors* of A corresponding to the zero eigenvalues. $\bar{U}$ is a matrix such that $\mathcal{U} = \begin{bmatrix} U, \bar{U} \end{bmatrix}$ is a unitary matrix.

Denote the *convergence matrix* of the system $\mathbf{\Sigma}_s$ by

$$\mathcal{J} := \lim_{t\to\infty} e^{At} = UV^T. \tag{4}$$

Then, the *semi-controllability Gramian* of $\mathbf{\Sigma}_s$ is defined as follows.

Definition 2: Consider a semistable system $\mathbf{\Sigma}_s$ as in (1). The **semi-controllability Gramian** is defined by

$$\mathcal{P} = \int_0^\infty (e^{At} - \mathcal{J}) B B^T (e^{A^T t} - \mathcal{J}^T) dt \in \mathbb{R}^{n\times n}, \tag{5}$$

where $\mathcal{J}$ is a constant matrix defined in (4).

Remark 1: Let $h(t) := e^{At}B$ be the impulse response of $\mathbf{\Sigma}_s$ and $\tilde{h}(t) := h(t) - \mathcal{J}B$. Equation (4) implies that $\tilde{h}(t)$ is an *absolutely integrable function*. Therefore, the integral in (5) is finite.

Remark 2: The integral in (5) can be approximated by

$$\mathcal{P} \approx \int_0^{T_c} (e^{At} - \mathcal{J}) B B^T (e^{A^T t} - \mathcal{J}^T) dt, \tag{6}$$

where T_c, satisfying $e^{AT_c}A \approx 0$, is similar to the definition of convergence time in the context of consensus networks. Moreover, the Lyapunov characteristic of $\mathcal{P}$ will be discussed in the further version.

Under some conditions, the $\mathcal{H}_2$-norm of a semistable system can be characterized by the semi-controllability Gramian, as explained in the following lemma.

Lemma 2: Consider a semistable system $\mathbf{\Sigma}_s$ in (1) with output $y = Cx$. The $\mathcal{H}_2$-norm of $\mathbf{\Sigma}_s$ exists if and only if $C\mathcal{J}B = 0$. Furthermore, if the $\mathcal{H}_2$-norm of $\mathbf{\Sigma}_s$ exists, then

$$\|\mathbf{\Sigma}_s\|_{\mathcal{H}_2}^2 = \mathsf{tr}(C\mathcal{P}C^T). \tag{7}$$

Proof: Recall the following equation from e.g. [7]

$$\|\mathbf{\Sigma}_s\|_{\mathcal{H}_2}^2 = \mathsf{tr}\left(\int_0^\infty C e^{At} B B^T e^{A^T t} C^T dt \right). \tag{8}$$

Let $g(t) := Ce^{At}B$. The above integral is finite if and only if $g(t)$ is absolutely integrable. Since $g(t)$ is a smooth function on $\mathbb{R}$, the integrability id then equivalent to

$$\lim_{t\to\infty} g(t) = 0, \text{ i.e., } C\left(\lim_{t\to\infty} e^{At}\right) B = C\mathcal{J}B = 0. \tag{9}$$

Hence,

$$\begin{aligned} \mathsf{tr}(C\mathcal{P}C^T) &= \mathsf{tr}\left[\int_0^\infty C(e^{At} - \mathcal{J}) B B^T (e^{A^T t} - \mathcal{J}^T) C^T dt \right] \\ &= \mathsf{tr}\left(\int_0^\infty C e^{At} B B^T e^{A^T t} C^T dt \right) = \|\mathbf{\Sigma}_s\|_{\mathcal{H}_2}^2. \end{aligned}$$

That completes the proof. ■

Directed networks with the so called Laplacian dynamics [2] are special semistable systems. In the next section, we apply the results developed in this section to the model reduction problem of network systems.

III. Model Reduction of Directed Network Systems

A. Problem Formulation

This section provides the mathematical model of a network system evolving over a strongly connected digraph. Then, it formulates its model reduction problem.

In this paper, we consider an edge-weighted digraph, which is defined by a triplet $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$ with $\mathcal{V}$ and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ as the sets of vertices and edges, respectively. $\mathcal{W} \in \mathbb{R}^{n\times n}$ ($n = |\mathcal{V}|$) is called the *weighted adjacency matrix*. The (i, j) entry of $\mathcal{W}$, denoted by w_{ij}, is positive if the directed edge $(i, j) \in \mathcal{E}$, and $w_{ij} = 0$ otherwise. In general, $w_{ij} \neq w_{ji}$ holds for weighted digraphs. The *Laplacian matrix* of the digraph $\mathcal{G}$ is then defined by $L \in \mathbb{R}^{n\times n}$ whose (i, j) entry is written as

$$L_{ij} = \begin{cases} \sum_{j=1, j\neq i}^n w_{ij}, & i = j \\ -w_{ij}, & \text{otherwise.} \end{cases} \tag{10}$$

The Laplacian dynamics on $\mathcal{G}$ can be modeled as the following linear time-invariant system

$$\mathbf{\Sigma}: \dot{x} = -Lx + Fu, \tag{11}$$

where L is the weighted digraph Laplacian matrix, and F is the input matrix.

A typical example can be found in the chemical dynamics [2], [20], where chemical species are the vertex states. The directed edges represent a series of chemical reactions converting source species to target species, and the edge weights are the rate constants of the corresponding reactions. As in [16], we assume L is irreducible (i.e., L is not similar via a permutation to a block upper triangular matrix). Then, the following lemma is given by [21].

Lemma 3: [21] If Laplacian matrix L is irreducible then the following hold:

- The associated digraph $\mathcal{G}$ is (strongly) connected;
- L has a simple zero eigenvalue, whose associated left eigenvector has all positive entries.
- $\mathbf{1}$ is a right eigenvector of L associated to the zero eigenvalue.

Clearly, the network system $\mathbf{\Sigma}$ is semistable. Now, we formulate the model reduction problem of system $\mathbf{\Sigma}$ in the framework of Petrov-Galerkin projection as follows.

Problem 1: Given a network system $\mathbf{\Sigma}$, find a projection $\Gamma = \mathbb{V}\mathbb{W}^T$ with $\mathbb{W}, \mathbb{V} \in \mathbb{R}^{n\times r}$, $r \ll n$ and $\mathbb{W}^T\mathbb{V} = I$ such that the obtained reduced-order model

$$\hat{\mathbf{\Sigma}}: \begin{cases} \dot{z} = -\hat{L}z + \hat{F}u, \\ \hat{x} = \mathbb{V}z, \end{cases} \tag{12}$$

where $\hat{L} := \mathbb{W}^T L\mathbb{V}$ and $\hat{F} := \mathbb{W}^T F$, approximates the original system $\mathbf{\Sigma}$ in a way that $\|\mathbf{\Sigma} - \hat{\mathbf{\Sigma}}\|$ is small enough in terms of $\mathcal{H}_2$- or $\mathcal{H}_\infty$-norms, and $\hat{L}$ is a Laplacian matrix representing a smaller-sized connected digraph.

B. Projection by Graph Clustering

This subsection constructs the reduced network system by the Petrov-Galerkin projection where the characteristic matrix of graph clustering is used as the projection matrix. First, some notions related to clustering are provided [14].

Definition 3: Consider a connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with $\mathcal{V} = \{1, 2, \cdots, n\}$ the index set of vertices. A nonempty subset of $\mathcal{V}$, denoted by $\mathcal{C}$, is called a **cluster** of graph $\mathcal{G}$. Then, **graph clustering** is defined as the partition of $\mathcal{V}$ into r disjoint clusters which cover all the elements in $\mathcal{V}$.

Definition 4: Consider a graph clustering $\{\mathcal{C}_1, \mathcal{C}_2, \cdots, \mathcal{C}_r\}$ of $\mathcal{V}$ with $|\mathcal{V}| = n$. The **characteristic vector** of the cluster $\mathcal{C}_i$ is denoted by a binary vector $\pi(\mathcal{C}_i) \in \mathbb{R}^n$ where $\mathbf{1}_n^T\pi(\mathcal{C}_i) = |\mathcal{C}_i|$, and the k-th element of $\pi(\mathcal{C}_i)$ is 1 when $k \in \mathcal{C}_i$ and 0 otherwise. The **characteristic matrix** of the clustering is a binary matrix defined by

$$\Pi := [\pi(\mathcal{C}_1), \pi(\mathcal{C}_2), \cdots, \pi(\mathcal{C}_r)] \in \mathbb{R}^{n\times r}. \tag{13}$$

Next, we construct a reduced-order network system using the clustering-based projection. By Lemma 3, the connectedness of the digraph $\mathcal{G}$ implies L has a simple eigenvalue equal to zero. There fore Correspondingly, we denote the right and left eigenvectors of this zeros eigenvalue by μ and ν, respectively, which satisfy

$$L\mu = 0, \text{ and } \nu^T L = 0. \tag{14}$$

Notice that all the entries of ν are real positive. Furthermore, we choose $\mu = \mathbf{1}_n/\sqrt{n}$ and $\nu^T\mu = 1$. Notice that the matrices U and V in the algebraic result of Section II need to be replaced in this section by vectors μ and ν, respectively.

Now, we consider the system $\mathbf{\Sigma}$ on digraph $\mathcal{G}$ of n vertices. To formulate a reduced model of dimension r, we find a graph clustering that groups the vertices of $\mathcal{G}$ into r clusters. The following projection matrices are utilized.

$$\mathbb{W}^T = (\Pi^T N\Pi)^{-1}\Pi^T N, \ \mathbb{V} = \Pi \tag{15}$$

with $N := \mathsf{diag}(\nu)$ and Π a characteristic matrix of the graph clustering. Consequently, the r-dimensional simplified network system in (12) is specified as

$$\hat{\mathbf{\Sigma}}: \begin{cases} \dot{z} = -\hat{L}z + \hat{F}u, \\ \hat{x} = \Pi z, \end{cases} \tag{16}$$

where

$$\hat{L} := (\Pi^T N\Pi)^{-1}\Pi^T NL\Pi, \text{ and } \hat{F} = (\Pi^T N\Pi)^{-1}\Pi^T NF.$$

Remark 3: NL is a Laplacian matrix associated to a *balanced digraph*, i.e. in-degree and out-degree of each vertex are equal [21]. For a balanced digraph, $\mathbf{1}_n$ is also a left eigenvector of NL associated with the zero eigenvalue, i.e., $\mathbf{1}_n^T NL = 0$.

Remark 4: The reduced model $\hat{\mathbf{\Sigma}}$ preserves the network structure since $\hat{L}$ is a Laplacian matrix, which can be interpreted as a reduced digraph. Moreover, $\mathbf{1}_r$ and $\Pi^T\nu$ are right and left eigenvectors of $\hat{L}$ associated to its zero eigenvalue. We treat this further in Theorem 1.

Denote the transfer functions of $\mathbf{\Sigma}$ and $\hat{\mathbf{\Sigma}}$ by

$$\eta(s) = (sI_n + L)^{-1}F, \text{ and } \hat{\eta}(s) = \Pi(sI_r + \hat{L})^{-1}\hat{F}, \tag{17}$$

respectively. As mentioned above, $\mathbf{\Sigma}$ is not asymptotically stable, which means $\|\eta(s)\|_{\mathcal{H}_2}$ and $\|\eta(s)\|_{\mathcal{H}_\infty}$ may be unbounded. The following theorem guarantees that the error system $\mathbf{\Sigma} - \hat{\mathbf{\Sigma}}$ is stable.

Theorem 1: Consider the network system $\mathbf{\Sigma}$ in (11) and the reduced-order model $\hat{\mathbf{\Sigma}}$ in (16). For an arbitrary network clustering, $\|\eta(s) - \hat{\eta}(s)\|_{\mathcal{H}_2}$ is always bounded.

Proof: Consider the impulse responses of $\mathbf{\Sigma}$ and $\hat{\mathbf{\Sigma}}$, respectively.

$$\xi(t) = e^{-Lt}F, \text{ and } \hat{\xi}(t) = \Pi e^{-\hat{L}t}\hat{F}. \tag{18}$$

From (4), we have

$$\lim_{t\to\infty} \xi(t) = \mu\nu^T F = \sqrt{1/n}\mathbf{1}_n\nu^T F. \tag{19}$$

Observe that $\Pi\mathbf{1}_r = \mathbf{1}_n$ and $\nu = N\mathbf{1}_n$. Then, we let

$$\hat{\nu} = c\Pi^T\nu = c\cdot(\Pi^T N\Pi)\mathbf{1}_r, \text{ and } \hat{\mu} = \mathbf{1}_r/\sqrt{r}, \tag{20}$$

with c a constant scaler to be determined, which satisfy

$$\begin{aligned}\hat{\nu}^T\hat{L} &= c\mathbf{1}_r^T(\Pi^T N\Pi)\cdot(\Pi^T N\Pi)^{-1}\Pi^T NL\Pi\\ &= c\mathbf{1}_r^T\Pi^T NL\Pi = c\mathbf{1}_n^T NL\Pi = 0.\\ \hat{L}\hat{\mu} &= (\Pi^T N\Pi)^{-1}\Pi^T NL\Pi\mathbf{1}_r/\sqrt{r}\\ &= (\Pi^T N\Pi)^{-1}\Pi^T NL\mathbf{1}_n/\sqrt{r} = 0.\end{aligned} \tag{21}$$

Therefore, ν and μ are the left and right eigenvectors of $\hat{L}$ corresponding to the zero eigenvalue, respectively. Furthermore, since $\nu^T\mu = 1$, it follows that

$$c = \frac{\sqrt{r}}{\mathbf{1}_n^T N\mathbf{1}_n} = \sqrt{r/n}. \tag{22}$$

Then the following equation follows from (4) as well.

$$\begin{aligned}\lim_{t\to\infty}\hat{\xi}(t) &= \lim_{t\to\infty}\Pi e^{-\hat{L}t}\hat{F} = \Pi\hat{\mu}\hat{\nu}^T\hat{F}\\ &= \Pi\frac{\mathbf{1}_r}{\sqrt{r}}\frac{\sqrt{r}\mathbf{1}_r^T(\Pi^T N\Pi)}{\sqrt{n}}(\Pi^T N\Pi)^{-1}\Pi^T NF\\ &= \sqrt{1/n}\mathbf{1}_n\nu^T F = \lim_{t\to\infty}\xi(t).\end{aligned} \tag{23}$$

By definition, the $\mathcal{H}_2$-norm of $\eta_i(s) - \hat{\eta}_j(s)$ is given by

$$\|\eta(s) - \hat{\eta}(s)\|_{\mathcal{H}_2}^2 = \int_0^\infty \|\xi(t) - \hat{\xi}(t)\|_2^2 dt, \tag{24}$$

Since $\xi(t)$ and $\hat{\xi}(t)$ are bounded smooth functions of t, which exponentially converge to the same value, the integral in (24) is bounded. ■

Remark 5: Unlike the undirected case, L in (11) is asymmetric in general, which means $\nu \in \mathsf{im}(\mu)$ does not necessarily hold. The projections used in [11], [14] for undirected networks may yield an unstable error system $\mathbf{\Sigma} - \hat{\mathbf{\Sigma}}$. Besides, in contrast to the model reduction approach of directed networks in [16], our method always guarantees the boundedness of the approximation error for any clustering.

C. Vertex Dissimilarity & Cluster Selection

In this subsection, the notation of semi-controllability Gramian is applied to define the dissimilarity of vertices. Based on that, hierarchical clustering algorithm is adapted to find a suitable clustering.

First, the transfer functions from the external input to the states of an individual vertex are denoted by

$$\eta_i(s) := \mathbf{e}_i^T\eta(s),\ i = \{1, 2, \cdots, n\}. \tag{25}$$

Then, the definition of dissimilarity is given as follows.

Definition 5: [17] Consider the network system $\mathbf{\Sigma}$ in (11). The **dissimilarity matrix** is denoted by $\mathcal{D}$, whose (i, j)-entry represents the dissimilarity of vertices i and j,

$$\mathcal{D}_{ij} := \|\eta_i(s) - \eta_j(s)\|_{\mathcal{H}_2}. \tag{26}$$

Particularly, if $\mathcal{D}_{ij} = 0$, vertices i and j are 0-dissimilar.

Clearly, $\mathcal{D}$ is nonnegative, symmetric, and has zero diagonal elements. When the scale of the directed network is large, the computation of $\mathcal{D}$ entry by entry may become a formidable task. The proposed semi-controllability Gramian provides an efficient computational method as stated in the following theorem.

Theorem 2: For a connected directed network, each entry in the dissimilarity matrix of $\mathbf{\Sigma}$ is finite, which is given by

$$\mathcal{D}_{ij} = \sqrt{(\mathbf{e}_i^T - \mathbf{e}_j^T)\mathcal{P}(\mathbf{e}_i - \mathbf{e}_j)}, \tag{27}$$

where $\mathcal{P}$ is the semi-controllability Gramian of $\mathbf{\Sigma}$.

Proof: Let $C := \mathbf{e}_i^T - \mathbf{e}_j^T \in \mathbb{R}^{1\times n}$. For a connected directed network,

$$\mathcal{J} = \mu\nu^T = \sqrt{1/n}\mathbf{1}_n\nu^T. \tag{28}$$

Thus, $C\mathcal{J}F = 0$. Lemma 2 then implies that $\mathcal{D}_{ij}$ is bounded for any i, j and $\mathcal{D}_{ij}^2 = C\mathcal{P}C^T$. ■

Each entry of $\mathcal{D}$ quantifies the similarity of two vertices' behaviors, which can be regarded as a special definition of "distance". With such a notion, we are able to build a bridge from model reduction to the clustering problems in data mining or computer graphics [22], [23]. In Algorithm 1, we adapt a ready-made algorithm in computer graphics, the hierarchical clustering [22], [23], into a cluster selection problem for the purpose of model reduction. First, the dissimilarity between clusters $\mathcal{C}_p$ and $\mathcal{C}_q$ is approximated by

$$\delta(\mathcal{C}_p, \mathcal{C}_q) = \frac{1}{|\mathcal{C}_p|\cdot|\mathcal{C}_q|}\sum_{i\in\mathcal{C}_p}\sum_{j\in\mathcal{C}_q}\mathcal{D}_{ij}. \tag{29}$$

The hierarchical clustering links the pairs of vertices that are in close proximity and place them into binary clusters. Then, the newly formed clusters can be merged into larger clusters according to the cluster dissimilarity in (29).

Algorithm 1 Hierarchical Clustering

Input: L and F, model order n, desired order r
Output: $\hat{L}$, $\hat{F}$

```
Compute the semistable Gramian 𝒫
Use Theorem 2 to calculate the dissimilarity matrix 𝒟
Place each vertex into its own singleton cluster, that is
  𝒞_i ← {i} for all 1 ≤ i ≤ n
k ← n
while k > r do
    Set δ_m to be an arbitrary large number
    for i = 1 : k − 1 and j = 2 : i − 1 do
        Compute δ(𝒞_i, 𝒞_j) by (29)
        if δ_m > δ(𝒞_i, 𝒞_j) then
            p ← i, q ← j, δ_m ← δ(𝒞_i, 𝒞_j)
        end if
    end for
    Merge cluster p and q into a single cluster
    k ← k − 1
end while
Compute Π ∈ ℝ^{n×r} and obtain L̂ and F̂ in (16).
```

Remark 6: Note that Algorithm 1 is a greedy method. We can also adapt other clustering strategies (e.g. K-means clustering [22]) to our problem. Besides, the current clustering algorithm does not focus on manipulating individual edges. Nevertheless, it can be easily modified for the consideration of edges. We just set the dissimilarity of vertices i and j to be ∞ if they are not neighbors. As a result, the clustering only aggregates vertices linked by edges.

D. Approximation Error

This subsection provides an efficient method to compute the approximation error between the original and the reduced-order network systems. To this end, an error system is defined as follows.

$$\mathbf{\Sigma_e}: \begin{cases} \dot{\omega} = \mathcal{A}\omega + \mathcal{B}u, \\ \delta = \mathcal{C}\omega, \end{cases} \tag{30}$$

where

$$\mathcal{A} = -\begin{bmatrix} L & 0 \\ 0 & \hat{L} \end{bmatrix}, \mathcal{B} = \begin{bmatrix} F \\ \hat{F} \end{bmatrix}, \mathcal{C} = \begin{bmatrix} I_n & -\Pi \end{bmatrix}.$$

Theorem 1 implies $\|\mathbf{\Sigma_e}\|_{\mathcal{H}_2}$ is always bounded, regardless of the choice of clusters. Furthermore, $\|\mathbf{\Sigma_e}\|_{\mathcal{H}_2}$ is computed as follows.

Theorem 3: The approximation error between $\mathbf{\Sigma}$ and $\hat{\mathbf{\Sigma}}$ is always bounded as

$$\|\mathbf{\Sigma_e}\|_{\mathcal{H}_2} = \|\mathbf{\Sigma} - \hat{\mathbf{\Sigma}}\|_{\mathcal{H}_2} = \sqrt{\mathsf{tr}(\mathcal{C}\mathcal{P}_\mathsf{e}\mathcal{C}^T)}. \tag{31}$$

Proof: Due to the structure of $\mathcal{A}$, $\mathbf{\Sigma_e}$ is also a semistable system, whose semi-controllability Gramian $\mathcal{P}_\mathsf{e}$ is defined by (5). Moreover, from (19) and (20), the convergence matrix is given by

$$\mathcal{J}_\mathsf{e} = -\begin{bmatrix} \mu\nu^T & \\ & \hat{\mu}\hat{\nu}^T \end{bmatrix} = -\frac{1}{\sqrt{n}}\begin{bmatrix} \mathbf{1}_n\nu^T & \\ & \mathbf{1}_r\nu^T\Pi \end{bmatrix}, \tag{32}$$

with ν the left eigenvector of L associated to the zero eigenvalue.

Note that

$$\nu = N\mathbf{1}_n = N\Pi\mathbf{1}_r. \tag{33}$$

Then, we obtain

$$\begin{aligned} \mathcal{C}\mathcal{J}_\mathsf{e}\mathcal{B} &= -\frac{1}{\sqrt{n}}\begin{bmatrix} I_n & -\Pi \end{bmatrix}\begin{bmatrix} \mathbf{1}_n\nu^T & \\ & \mathbf{1}_r\nu^T\Pi \end{bmatrix}\begin{bmatrix} F \\ \hat{F} \end{bmatrix} \\ &= -\frac{1}{\sqrt{n}}\mathbf{1}_n\left[\nu^T - \nu^T\Pi(\Pi^T NP)^{-1}\Pi^T N\right]F \\ &= -\frac{1}{\sqrt{n}}\mathbf{1}_n\left[\nu^T - \mathbf{1}_r^T\Pi^T N\right]F = 0. \end{aligned}$$

Hence, we can apply Lemma 2, which gives (31). ■

IV. Illustrative Example

In this section, we adopt a sensor network example from [24] to illustrate the feasibility of the proposed method. Fig. 1 depicts the topology of the directed network, which is connected. We choose node 2 and node 7 as controlled nodes, which are influenced by the external input signals. We apply the hierarchical clustering algorithm to reduce the dimension of the full-order network system to a 6-dimensional reduced-order model. The result of the hierarchical clustering is given as

$$\begin{aligned} &\mathcal{C}_1 = \{10, 11, 12, 13, 14\} \\ &\mathcal{C}_2 = \{1, 4, 5\},\ \mathcal{C}_3 = \{6, 8, 9\}, \\ &\mathcal{C}_4 = \{3\},\ \mathcal{C}_5 = \{2\},\ \mathcal{C}_6 = \{7\}. \end{aligned}$$

Then, the corresponding simplified network is shown in Fig. 2, which indicates that the network structure is preserved in the new model. Now, to compare the behavior of the reduced-order network to the original one, we use unit step signals with 1s step time as external inputs for both systems. The state trajectories of (11) and (16) are compared in Fig. 3.

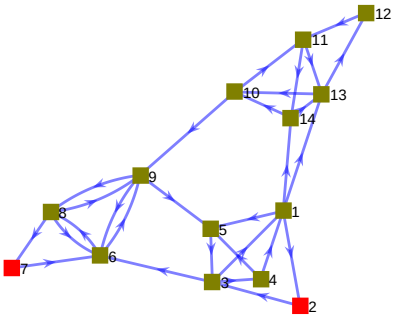

Fig. 1. A directed sensor network which contains 14 nodes. The controlled nodes are labeled as red blocks.

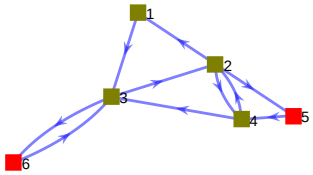

Fig. 2. The reduced sensor network where the controlled nodes are labeled as red blocks.

Fig. 4 is called a *dendrogram*, which is in the shape of a binary, hierarchical tree. From the dendrogram, the result of graph clustering can be interpreted straightforwardly. The bottom vertical lines are called leaves, which represent the vertices on graph. Each fusion of two clusters is indicated by the splitting of a vertical line into two branches, and the horizontal position of the split, shown by the short horizontal bar, reads the similarities between the two clusters. In Fig. 4, we can see there are two pairs of 0-dissimilar vertices, namely 1 and 4, 13 and 14. Our algorithm first aggregates these 0-dissimilar vertices, which generates a 13th order reduced model without any approximation error, and then apply Algorithm 1 to further reduce the order, see Fig. 5.

We also make a comparison between the hierarchical clustering and a random clustering strategy in order to demonstrate the effectiveness of hierarchical clustering. Here, the random clustering randomly assigns n vertices into r nonempty disjoint subsets. Fig. 5 shows the comparison of their approximation errors for different reduced orders r, where the curve of random clustering is plotted based on an average of 50 times random cluster selection. The result indicates that the hierarchical clustering algorithm has better performance. When $r = 6$, the approximation error $\|\mathbf{\Sigma} - \hat{\mathbf{\Sigma}}\|_{\mathcal{H}_2}$ is less than 0.2. The example shows that the proposed method is feasible for model reduction of directed network systems.

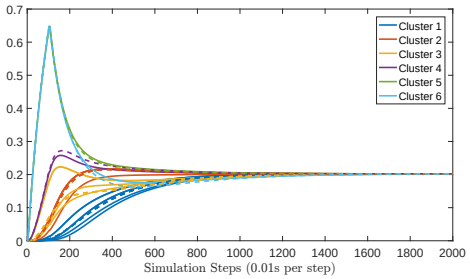

Fig. 3. The state trajectories of the original and reduced directed networks, which are depicted by solid and dash lines respectively. We use the same color to mark the vertices within the same clusters.

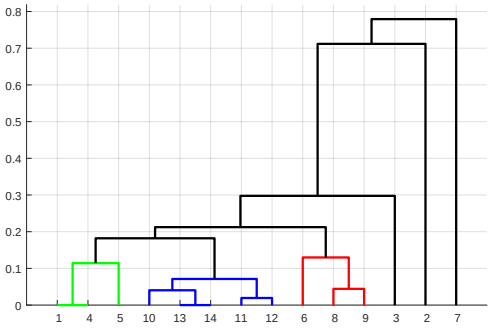

Fig. 4. Dendrogram shows the hierarchical of the graph clustering, where the horizontal axis are labeled by vertex numberings, and the vertical axis indicates the dissimilarity between clusters.

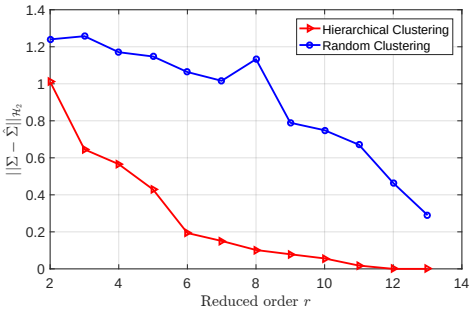

Fig. 5. Approximation error comparisons of hierarchical clustering algorithm with random clustering.

V. Conclusion

In this paper, we propose a mathematical framework for model reduction of directed network systems. This method is based on graph clustering where the vertices having similar behaviors are aggregated. The clustering-based projection maintains the structure of the Laplacian matrix in the reduced model such that it can be interpreted as a network system. Through an example of sensor networks, the efficiency of the proposed method is verified.

Despite that we assume the strong connectedness of the network, the result in this paper can be extended to the weakly connected case. In such case, the network system is still a semistable system, but the multiplicity of the zero eigenvalues of the Laplacian matrix will be larger than one.

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