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Tracking Control of Fully-actuated port-Hamiltonian Mechanical Systems via Sliding Manifolds and Contraction Analysis

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Abstract: In this paper, we propose a trajectory tracking controller for fully-actuated port-Hamiltonian (pH) mechanical systems, which is based on recent advances in contraction analysis and differential Lyapunov theory. The tracking problem is solved by defining a suitable invariant sliding manifold which provides a desired steady state behavior. The manifold is then made attractive via contraction techniques. Finally, we present numerical simulation results where a SCARA robot is commanded by the proposed tracking control law.

Keywords: Trajectory tracking control, port-Hamiltonian systems, sliding manifold, differential Lyapunov theory, contraction analysis

1. INTRODUCTION

The control of electro-mechanical (EM) systems is a well-studied problem in control theory literature. Using Euler-Lagrange (EL) formalism for describing the dynamics of EM systems, many control design tools have been proposed and studied to solve the stabilization/set-point regulation problem. Recent works exploit the physical structure of the system through passivity-based control methods which are expounded in Ortega et al. (2013) and references therein. However, for motion control/output regulation problem (which includes trajectory tracking and path-following problems), the use of EL formalism in the control design is relatively recent and the problem is solved based on passivity/dissipativity theory for nonlinear systems. We refer interested reader on the early work of tracking control for EL systems in Slotine and Li (1987) and recent works in Kelly et al. (2006); Jayawardhana and Weiss (2008).

As an alternative to the EL formalism for describing EM systems, port-Hamiltonian (pH) framework has been proposed and studied (see also the pioneering work in van der Schaft and Maschke (1995)), which has a nice (Dirac) structure, provides port-based modeling and has physical energy interpretation. For the latter part, the energy function can directly be used to show the dissipativity and stability property of the systems.

Using the pH framework, a number of control design tools have been proposed and implemented for the past two decades. For solving the stabilization and set-point regulation problem of pH systems, we can apply, for instance, the standard proportional-integral (PI) control (Jayawardhana et al. (2007)), Interconnection and Damping Assignment Passivity-Based Control (IDA-PBC) approach (Ortega et al. (2002)) or Control by Interconnection (CbI) method (Ortega et al. (2008)) (among many others).

However, for motion control problem, where the reference signal can be time-varying, it is not straightforward to design control laws for such pH systems that still provides an insightful energy interpretation of the closed-loop system. For example, it is not trivial to obtain an incremental passive system [1] via a controller interconnected with the pH system. One major difficulty is that the external reference signals can induce both the closed-loop system and total energy function to be time-varying. In this case, the closed-loop system may not be dissipative, or if it is a time-varying dissipative system, the usual La-Salle invariance principle argument can no longer be invoked for analyzing the asymptotic behavior.

In order to overcome the loss of passivity in the trajectory tracking control of pH systems, a pH structure preserving error system was introduced in Fujimoto et al. (2003) which is based on generalized canonical transformations. In Fujimoto et al. (2003), necessary and sufficient conditions for passivity preserving are given. Once in the new canonical coordinates, the pH error system can be stabilized with standard passivity-based control methods. In Dirksz and Scherpen (2010), the previous approach

[1] This concept generalizes the usual notion of passivity and is suitable for output regulation problem of non-constant signals, see Jayawardhana (2006), Pavlov and Marconi (2006)

is extended to an adaptive control. In Romero et al. (2015), a generalized canonical transformation is used to obtain a particular pH system which is *partially linear* in the momentum with constant inertia matrix. The control scheme is then proposed to give a pH structure for the closed-loop error system. Although solving partial differential equations that correspond to the existence of such transformation is not trivial, some characterizations of this canonical transformation is presented in Venkatraman et al. (2010) for a specific classes of systems. In Yaghmaei and Yazdanpanah (2015), the so-called *timed-IDA-PBC* was introduced, where the standard IDA-PBC method is adapted in such a way that it can incorporate tracking problem through a modification in the IDA-PBC matching equations; albeit it may easily lead to a non-tractable problem in solving a set of complex PDE. Finally, in a recent paper Zada and Belda (2016), a trajectory tracking control for standard pH systems without dissipation is proposed, using a similar change of coordinates as in Slotine and Li (1987) where the the *Coriolis* term is defined explicitly in the Hamiltonian domain.

As an alternative to the use of passivity-based control method for motion control problems, we propose a contraction-based control for fully-actuated pH systems, which still has an energy-like interpretation [2]. Contraction theory is about convergence among trajectories, and then it is more natural and suitable approach for motion control, rather than other methods.

Generally speaking, we first construct an error system using backstepping method. Such an error system posses a sliding manifold with the desired steady-state behavior. Then, we define a composite control for the stabilization of the sliding manifold in such a way that the closed-loop error system has a pH-like structure. For the convergence analysis, the state space is extended with the incorporation of a *virtual system* where the latter system admits both the error system's trajectory, as well as, the origin as its solution. By using *partial contraction* as in (Wang and Slotine (2005)), the contraction of the virtual system implies that state trajectory converges exponentially to the desired one.

2. PRELIMINARIES

2.1 Control design using sliding manifolds

Let $\mathcal{X}$ be the state-space with tangent bundle $T\mathcal{X}$ of a nonlinear system given by

$$\dot{x} = f(x,t) + G(x,t)u \tag{1}$$

where $G(x,t) = [g_1(x,t), \cdots, g_n(x,t)]$ has full rank and $x \in \mathcal{X} \subset \mathbb{R}^N$ the solution to (1), f, g_k are smooth vector fields on $\mathcal{X} \times \mathbb{R}_{\geq 0}$ for $k \in \{1, \ldots, n\}$ and $u = [u_1, \cdots, u_n]^\top$ the control input. Recall the definitions of invariant and sliding manifolds.

Definition 1. (Ghorbel and Spong (2000)). The set $\mathcal{S} \subset \mathcal{X}$ is said to be an *invariant manifold* for system (1) if whenever $x(t_0) \in \mathcal{S}$, implies that $x(t) \in \mathcal{S}$, for all $t > t_0$.

[2] Other approaches have been developed in the past such as geometric control. However, in many of those approaches, the physical nature of the system is often not taken into account or lost.

Definition 2. (Sira-Ramrez (2015)). A *sliding manifold* for system (1) is a subset of the state space, which is the intersection of n smooth $(N-1)$-dimensional manifolds,

$$\Omega(t) = \{x \in \mathcal{X} : \sigma(x,t) = 0\} \tag{2}$$

where $\sigma(x,t) = [\sigma_1(x,t), \ldots, \sigma_n(x,t)]^\top$ is the *sliding variable* with σ_i a smooth function $\sigma_i : \mathcal{X} \times \mathbb{R}_{\geq 0} \to \mathbb{R}$.

It is assumed that $\Omega(t)$ is locally an $(N-n)$-dimensional, sub-manifold of $\mathcal{X}$. The smooth control vector u_{eq}, known as the *equivalent control*, renders the manifold Ω to an invariant manifold $\mathcal{S}$ of (1) (Sira-Ramrez (2015)). If $\text{rank}\{L_G\sigma\} = n$, the equivalent control is the well defined solution to the following invariance conditions

$$\sigma(x,t) = 0, \quad \dot{\sigma}(x,t) = 0, \tag{3}$$

uniformly in t. The dynamical system $\dot{x} = f(x,t) + G(x,t)u_{eq}(x,t)$ is said to describe the *ideal sliding motion.*

Using sliding manifolds in control design has as goal, designing a suitable control scheme $u = u_{eq} + u_{at}$, such that u_{eq} renders $\Omega(t)$ to an invariant sliding manifold, under invariance conditions (3), and u_{at} makes to the invariant manifold attractive.

2.2 Contraction analysis and differential Lyapunov theory

System (1) in closed-loop with u, denoted by

$$\dot{x} = F(x,t), \tag{4}$$

is be called *contracting*, if initial any pair of solution x_1 and x_2 converges to each other, with respect to a *distance*. In this paper, for contraction analysis, we adopt the approach given in Forni and Sepulchre (2014).

The *prolonged system* (Crouch and van der Schaft (1987)) of (4) corresponds to the original system *together* with its *variational system*, that is the system

$$\begin{cases} \dot{x} = F(x,t) \\ \dot{\delta x} = \dfrac{\partial F}{\partial x}(x,t)\delta x \end{cases} \tag{5}$$

with $(x, \delta x, t) \in T\mathcal{X} \times \mathbb{R}_{\geq 0}$. A *differential Lyapunov function* $V : T\mathcal{X} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ satisfies the bounds

$$c_1 \|\delta x\|_x^p \leq V(x, \delta x, t) \leq c_2 \|\delta x\|_x^p, \tag{6}$$

where $c_1, c_2 \in \mathbb{R}_{>0}$, p is some positive integer and $\|\cdot\|_x$ is a Finsler structure. The role of (6) is to measure the distance of any tangent vector δx from 0. Thus, (6) can be understood as a classical Lyapunov function for the linearized dynamics with respect to the origin in $T\mathcal{X}$.

Theorem 1. Consider the prolonged system (5), a connected and forward invariant set $\mathcal{D}$, and a strictly increasing function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$. Let V be a differential Lyapunov function satisfying

$$\dot{V}(x, \delta x, t) \leq -\alpha(V(x, \delta x, t)) \tag{7}$$

for each $(x, \delta x) \in T\mathcal{X}$ and uniformly in $t \in \mathbb{R}_{\geq 0}$. Then, (4) contracts V in $\mathcal{D}$. V is called the *contraction measure*, and $\mathcal{D}$ *the contraction region.*

Remark 1. Contraction of (4) is guaranteed by (6) and (7), with respect to the distance induced by the Finsler measure $\|\cdot\|_x$, through integration. As direct consequence (Forni and Sepulchre (2014)), system (4) is incrementally

- *stable* on $\mathcal{D}$ if $\alpha(s) = 0$ for each $s \geq 0$;
- *asymptotically stable* on $\mathcal{D}$ if α is a strictly increasing;

- *exponentially stable* on $\mathcal{D}$ if $\alpha(s) = \beta s, \forall s > 0$.

Remark 2. By taking as differential Lyapunov function to $V(x, \delta x) = \frac{1}{2}\delta x^\top \Pi(x,t)\delta x$, with $\Pi(x,t)$ a smooth Riemannian metric and uniform in t, expression (7) results in the so-called *generalized contraction analysis* in Lohmiller and Slotine (1998), i.e.,

$$\frac{\partial \Pi}{\partial x}F(x,t) + \frac{\partial F^\top}{\partial x}\Pi(x,t) + \Pi(x,t)\frac{\partial F}{\partial x} < -2\beta\Pi. \quad (8)$$

If the interest is convergence with respect to a specific behavior [3], the concept introduced by Wang and Slotine, 2005, with the name of *partial contraction*, gives a solution

Theorem 2. Consider a *virtual system* of the form

$$\dot{x}_v = h(x_v, x, t), \quad (9)$$

such that a trajectory $x_v = x_d(t)$ is a particular solution. Assume that the *actual system* (4) can be written as

$$\dot{x} = h(x, x, t). \quad (10)$$

If the virtual system (9) is contracting, uniformly in x and t, then x converges to $x_d(t)$ as $t \to \infty$. System (10) is said to be *partially contracting*.

Remark 3. In case the interest is to prescribe a desired system's behavior, in Wang and Slotine (2005) and Jouffroy and Fossen (2010), Theorem 2 was reformulated for control design purposes in the open system (1) as follows.

Consider now as virtual system to

$$\dot{x}_v = h(x_v, x, u, t). \quad (11)$$

Suppose the actual system (1) is rewritten as

$$\dot{x} = h(x, x, u, t), \quad (12)$$

and assume that the control input $u = u(x, x_d, \dot{x}_d)$ can be chosen such that

$$\dot{x}_d = h(x_d, x, u, t), \quad (13)$$

with x_d a desired trajectory. If (11) is contracting uniformly in x, u and t, then conclusion of Theorem 2 holds.

2.3 Port-Hamiltonian mechanical systems

Consider the *input-state port-Hamiltonian* (van der Schaft and Jeltsema (2014)) representation of a fully-actuated mechanical system of the form

$$\begin{bmatrix}\dot{q}\\ \dot{p}\end{bmatrix} = \begin{bmatrix}0_n & I_n\\ -I_n & -D(q)\end{bmatrix}\begin{bmatrix}\frac{\partial H}{\partial q}(q,p)\\ \frac{\partial H}{\partial p}(q,p)\end{bmatrix} + \begin{bmatrix}0_n\\ I_n\end{bmatrix}u, \quad (14)$$

where $x = [q,p]^\top \in T^*\mathcal{Q} = \mathcal{X}$, and the generalized momentum is $p = M(q)\dot{q}$, with $\dot{q}$ the generalized velocity; $m_1 I_n \leq M(q) = M^\top(q) \leq m_2 I_n$ is the bounded inertia matrix, where $m_1, m_2 \in \mathbb{R}_{>0}$; $D(q) = D^\top(q) \geq 0_n$ is the damping matrix, the matrices identity I_n and zero 0_n have dimension $n = \dim\mathcal{Q}$; $u \in \mathbb{R}^n$ the control input and the Hamiltonian function is given by the total energy

$$H(x) = \frac{1}{2}p^\top M^{-1}(q)p + V(q), \quad (15)$$

with $V(q)$ the potential energy. In Arimoto (1996) it was proven that the matrix $S(q,\dot{q})$ (which is skew-symmetric, homogeneous and linear in $\dot{q}$), defined by

$$S_{ij}(q,\dot{q}) := \frac{1}{2}\left[\sum_{k=1}^{n}\dot{q}_k\left(\frac{\partial M_{ik}}{\partial q_j}(q) - \frac{\partial M_{jk}}{\partial q_i}(q)\right)\right], \quad (16)$$

where $S_{ij} = -S_{ji}$, fulfills the property

$$\left[S(q,\dot{q}) - \frac{1}{2}\dot{M}(q)\right]\dot{q} = -\frac{\partial}{\partial q}\left[\frac{1}{2}\dot{q}^\top M(q)\dot{q}\right]. \quad (17)$$

Such a property is for the Euler-Lagrange realization of mechanical systems with state variables $(q,\dot{q})$. However, as was shown in Zada and Belda (2016), by applying the *Legendre transformation*, property (17) can be expressed in state x of the Hamiltonian realization (14) as

$$\left[S(q,p) - \frac{1}{2}\dot{M}(q)\right]M^{-1}(q)p = \frac{\partial}{\partial q}\left[\frac{1}{2}p^\top M^{-1}(q)p\right]. \quad (18)$$

Using (18), system (14) can be rewritten as

$$\begin{bmatrix}\dot{q}\\ \dot{p}\end{bmatrix} = \begin{bmatrix}0_n & I_n\\ -I_n & -E(q,p)\end{bmatrix}\begin{bmatrix}\frac{\partial V}{\partial q}(q)\\ \frac{\partial H}{\partial p}(q,p)\end{bmatrix} + \begin{bmatrix}0\\ G(q)\end{bmatrix}u \quad (19)$$

with $E(q,p) := S(q,p) - \frac{1}{2}\dot{M}(q) + D(q)$.

3. TRAJECTORY TRACKING CONTROLLER

Control objective: Design a control law for system (14) such that x converges to a smooth desired trajectory $x_d(t)$.

To solve the control problem, we to construct a suitable error system for (14) as in Fujimoto et al. (2003). Consider a twice differentiable desired trajectory $x_d(t) = [q_d(t), p_d(t)]^\top$, with $p_d(t) = M(q_d(t))\dot{q}_d(t)$ and the change of coordinates

$$\tilde{x} := \begin{bmatrix}\tilde{q}\\ \sigma\end{bmatrix} = \begin{bmatrix}q - q_d(t)\\ p - p_r(t)\end{bmatrix}, \quad (20)$$

where p_r is an auxiliary momentum reference to be defined. The dynamics of $\tilde{q}$ in (20) is

$$\dot{\tilde{q}} = M^{-1}(\tilde{q} + q_d)p - M^{-1}(q_d)p_d. \quad (21)$$

We define the following notation $M(\tilde{q} + q_d) = M_t(\tilde{q})$ to indicate that the inertia depends on the exosignal $q_d(t)$.

Like in *backstepping*, assume $p = \sigma + p_r$ is a control input to (21), with σ as new state and p_r as a stabilizing term. After substitution of p in (21), p_r is defined as

$$p_r = p_{d\sigma} - \Lambda\tilde{q}, \quad (22)$$

where $p_{d\sigma} = M_t(\tilde{q})\dot{q}_d$ and $-\Lambda = -\Lambda^\top$ is a Hurwitz matrix. It results in the position error dynamics

$$\dot{\tilde{q}} = M_t^{-1}(\tilde{q})\left(\sigma - \Lambda\tilde{q}\right), \quad (23)$$

with σ as input. When $\sigma = 0$ in (23), the origin $\tilde{q} = 0$ is asymptoticly stable, since $-M_t^{-1}(\tilde{q})\Lambda$ is a Hurwitz matrix. Above implies $q \to q_d$ as $t \to \infty$. Simultaneously, from (22), $p_r \to p_d$ as $t \to \infty$.

The dynamics of σ is $\dot{\sigma} = \dot{p} - \dot{p}_r$ evaluated in the change of coordinates (20). Then, an error system for (14) is

$$\begin{aligned}\dot{\tilde{q}} &= M_t^{-1}(\tilde{q})\left(\sigma - \Lambda\tilde{q}\right)\\ \dot{\sigma} &= -\left[\frac{\partial H}{\partial q}(x) + D(q)\frac{\partial H}{\partial p}(x) - u + \dot{p}_r\right]_{\substack{q\,=\,\tilde{q}\,+\,q_d\\ p\,=\,\sigma\,+\,p_r}}\end{aligned}. \quad (24)$$

The following result gives a solution to the control problem. For sake of space, some arguments are left out.

Proposition 1. Consider a twice differentiable desired trajectory $x_d \in T^*\mathcal{Q}$, together with the change of coordinates

[3] For instance an equilibrium point or a steady state.

(20) and (22). Consider also the pH system (14) in closed-loop with the composite control law [4] $u = u_{eq} + u_{at}$ with

$$\begin{aligned} u_{eq} &= \dot{p}_r + \frac{\partial H}{\partial q}(q, p_r) + D(q)\frac{\partial H}{\partial p}(q, p_r) \\ u_{at} &= -K_d \frac{\partial H}{\partial p}(q, \sigma) - M^{-1}(q)\Lambda\tilde{q} + \frac{\partial}{\partial q}\left(p_r^\top M^{-1}(q)\sigma\right), \end{aligned} \tag{25}$$

where K_d fulfills

$$D + K_d + \frac{1}{2}I_n - \frac{1}{4}(M^{-1} + M) > 0. \tag{26}$$

Then,

(1) The closed-loop system in error coordinates (20) is

$$\dot{\tilde{x}} = \begin{bmatrix} -I_n & I_n \\ -I_n & -E_t(\tilde{q}, \sigma) - K_d \end{bmatrix} \begin{bmatrix} M_t^{-1}(\tilde{q})\Lambda\tilde{q} \\ M_t^{-1}(\tilde{q})\sigma \end{bmatrix}. \tag{27}$$

(2) The origin of (27) is exponentially stable with rate

$$\beta = \lambda_{\min}\left(P^{1/2}(\tilde{x})\Upsilon(\tilde{x})P^{1/2}(\tilde{x})\right). \tag{28}$$

where $\lambda_{\min}(\cdot)$ denotes the minimum eigenvalue of the matrix in its argument and the matrices

$$P(x) = \begin{bmatrix} \Lambda & 0 \\ 0 & M^{-1}(q) \end{bmatrix}, \tag{29}$$

$$\Upsilon(x) = \begin{bmatrix} 2M^{-1} & (M^{-1} - I_n) \\ (M^{-1} - I_n) & 2(D + K_d) \end{bmatrix}. \tag{30}$$

(3) The sliding manifold

$$\Omega(t) = \{x \in T^*\mathcal{Q} : \sigma(x, t) = \tilde{p}_\sigma + \Lambda\tilde{q} = 0\}, \tag{31}$$

where $\tilde{p}_\sigma := M_t(\tilde{q})\dot{\tilde{q}}$, is invariant and attractive, for system (27), with ideal sliding motion

$$\dot{q} = \frac{\partial H}{\partial p}(q, p_r). \tag{32}$$

Proof:

(1) Straight forward computations after substitution of (25) in the error system (24) gives the closed-loop system (27)

(2) To prove this item, we will use partial contraction [5] Theorem 2. Consider the following virtual system with state $\tilde{x}_a = [\tilde{q}_a^\top, \sigma_a^\top]^\top$

$$\dot{\tilde{x}}_a = \begin{bmatrix} -I_n & I_n \\ -I_n & -(E(t, \sigma) + K_d) \end{bmatrix} \begin{bmatrix} M^{-1}\Lambda\tilde{q}_a \\ M^{-1}\sigma_a \end{bmatrix}. \tag{33}$$

Notice $\tilde{x}_a = \tilde{x}$ and $\tilde{x}_a = 0$ are two particular solutions of system (33). The variational dynamics of the virtual system (33) is

$$\delta\dot{\tilde{x}}_a = -\begin{bmatrix} M^{-1}\Lambda & -M^{-1} \\ M^{-1}\Lambda & (E + K_d)M^{-1} \end{bmatrix} \delta\tilde{x}_a. \tag{34}$$

For the prolonged system (33)-(34), let the candidate differential Lyapunov function be

$$V(\tilde{x}_a, \delta\tilde{x}_a, t) = \frac{1}{2}\delta\tilde{x}_a^\top P(\tilde{x})\delta\tilde{x}_a. \tag{35}$$

The time derivative of (35) is

$$\begin{aligned} \dot{V} &= -\delta\tilde{x}_a^\top \begin{bmatrix} \Lambda M^{-1}\Lambda & -\Lambda M^{-1} \\ M^{-2}\Lambda & M^{-1}(E + K_d)M^{-1} \end{bmatrix} \delta\tilde{x}_a \\ &\quad + \frac{1}{2}\delta\tilde{x}_a^\top \begin{bmatrix} \dot{\Lambda} & 0 \\ 0 & \dot{M}^{-1}(q) \end{bmatrix} \delta\tilde{x}_a, \\ &= -\delta\tilde{x}_a^\top \underbrace{\begin{bmatrix} \Lambda M^{-1}\Lambda & -\Lambda M^{-1} \\ M^{-2}\Lambda & M^{-1}(D + K_d)M^{-1} \end{bmatrix}}_{\Xi(\tilde{x})} \delta\tilde{x}_a, \end{aligned} \tag{36}$$

where the symmetric part of $\Xi(\tilde{x})$ is expressed as

$$\mathrm{Sym}(\Xi(\tilde{x})) = \frac{1}{2}P_t(\tilde{x})\Upsilon_t(\tilde{x})P_T(\tilde{x}) \tag{37}$$

Thus, (36) will be negative definite if and only if the Schur complement of matrix (30) with respect to $2M^{-1}$ fulfills (26). Which is always possible by choosing a big enough K_d. Therefore, the prolonged system (33)-(34) contracts (35) with respect to the metric (29) in $T\mathcal{X}$. With (28), the time derivative (36) satisfies

$$\dot{V}(\tilde{x}_a, \delta\tilde{x}_a, t) < -2\beta V(\tilde{x}_a, \delta\tilde{x}_a, t) \tag{38}$$

uniformly in t and $\tilde{x}_a$, or equivalently (8) for the matrix (29). By Remark 1, the virtual system (33) is incrementally exponentially stable with rate (28). Therefore, $\tilde{x}$ converges to 0 exponentially as $t \to \infty$.

(3) The existence of u_{eq} in (25), guarantees that the sliding manifold (31) is rendered invariant. From the previous item, $\sigma \to 0$, which means the invariant manifold is attractive. Finally, system (14) has *regular canonical form*, and definitions of p_r and σ imply that the *reduced-order* ideal sliding motion

$$\dot{q} = \frac{\partial H}{\partial p}(q, p_r). \tag{39}$$

■

In Sanfelice and Praly (2015), an observer was designed for shrinking a Riemannian distance, instead of designing a contracting observer. The following proposition shows that the controller (25) has the same property.

Proposition 2. Consider system (14). The control law (25) shrinks the Riemannian distance $d(x, x_d)$ induced by

$$\Pi(x) = \begin{bmatrix} \Lambda + \Lambda M^{-1}(q)\Lambda & \Lambda M^{-1}(q) \\ M^{-1}(q)\Lambda & M^{-1}(q) \end{bmatrix}. \tag{40}$$

Proof: We will show that both, the actual system (14) and the trajectory driven by controller (25), are partially contracting to a virtual system with respect to the metric (40), in the sense of Remark 3. This means, that in particular, the distance induced by the metric (40), between the actual state x and the desired state x_d shrinks.

To that end, fist we express the controller (25) in implicit form for the state $x_{d\sigma} = [q_d, p_{d\sigma}]^\top$ in original coordinates x. Using (22) and $\sigma = \tilde{p}_\sigma + \Lambda\tilde{a}$, we have

$$\begin{aligned} u = &\frac{\partial V}{\partial q}(q) + \left[E(q, p_{d\sigma}) + S(q, \tilde{p}_\sigma)\right] M^{-1}(q)p_{d\sigma} \\ &- A(\tilde{q}, \tilde{p}_\sigma)M^{-1}(q)\Lambda\tilde{q} - B(\tilde{q}, \tilde{p}_\sigma)M^{-1}(q)\tilde{p}_\sigma + \dot{p}_{d\sigma}. \end{aligned} \tag{41}$$

where

$$\begin{aligned} A(\tilde{q}, \tilde{p}_\sigma) &:= E(q, \tilde{p}_\sigma + \Lambda\tilde{q}) + K_d + I_n, \\ B(\tilde{q}, \tilde{p}_\sigma) &:= \Lambda + K_d - S(q, p_{d\sigma} - \Lambda\tilde{q}). \end{aligned} \tag{42}$$

Considering the fact that $\dot{q}_d = M^{-1}(q)p_{d\sigma}$, we can rewrite the controller implicitly as

[4] With $\frac{\partial}{\partial q}\left(p_r^\top M^{-1}(q)\sigma\right) = S(q, \sigma)\frac{\partial H}{\partial p}(q, p_r) + S(q, p_r)\frac{\partial H}{\partial p}(q, \sigma)$.

[5] By defining $e = \tilde{x}_a$ and $x = \tilde{x}$, properties labeled as TULES-NL, UES-TL and ULMTE in Andrieu et al. (2016), are also verified.

$$\dot{x}_{d\sigma} = \begin{bmatrix} 0 & I_n \\ -I_n & -E(q, p_{d\sigma} + \tilde{p}_\sigma) \end{bmatrix} \begin{bmatrix} \frac{\partial V}{\partial q}(q) \\ M^{-1}(q) p_{d\sigma} \end{bmatrix} + \begin{bmatrix} 0 \\ I_n \end{bmatrix} u + \begin{bmatrix} 0 & 0 \\ A(\tilde{q}, \tilde{p}_\sigma) & B(\tilde{q}, \tilde{p}_\sigma) \end{bmatrix} \begin{bmatrix} M^{-1}(q)\Lambda\tilde{q} \\ M^{-1}(q)\tilde{p}_\sigma \end{bmatrix}. \tag{43}$$

Let a virtual system with state $x_v = [q_v, p_v]^\top$ be

$$\dot{x}_v = \begin{bmatrix} 0 & I_n \\ -I_n & -E(q, p_{d\sigma} + \tilde{p}_\sigma) \end{bmatrix} \begin{bmatrix} \frac{\partial V}{\partial q}(q) \\ M^{-1}(q) p_v \end{bmatrix} + \begin{bmatrix} 0 \\ G \end{bmatrix} u + \begin{bmatrix} 0 & 0 \\ A(\tilde{q}, \tilde{p}_\sigma) & B(\tilde{q}, \tilde{p}_\sigma) \end{bmatrix} \begin{bmatrix} M^{-1}(q)\Lambda\tilde{q}_v \\ M^{-1}(q)\tilde{p}_v \end{bmatrix}, \tag{44}$$

with $\tilde{q}_v = q - q_v$, $\tilde{p}_v = p - p_v$ and $\tilde{x}_v^\top = [\tilde{q}_v . \tilde{p}_v]$. Notice that system (44) has as particular solutions to both, $x_v = x_{d\sigma}$ and $x_v = x$. The variational system of (44) is

$$\delta\dot{x}_v = -\begin{bmatrix} 0 & -M^{-1} \\ AM^{-1}\Lambda & (E + B) M^{-1} \end{bmatrix} \delta x_v. \tag{45}$$

Now, consider the following change of coordinates

$$\delta\tilde{x}_a = -\begin{bmatrix} I_n & 0 \\ \Lambda & I_n \end{bmatrix} \delta x_v = -\Theta\delta x_v. \tag{46}$$

Then, system (45) in the new coordinates is

$$\delta\dot{\tilde{x}}_a = -\Theta \begin{bmatrix} 0 & -M^{-1} \\ AM^{-1}\Lambda & (E + B) M^{-1} \end{bmatrix} \Theta^{-1}\delta\tilde{x}_a \tag{47}$$

which is nothing but (34). Then, the virtual system (44) is contracting with rate 2β, with respect to the differential Lyapunov function or contraction measure

$$\overline{V}(x_v, \delta x_v) = \frac{1}{2}\delta x_v^\top \Theta^\top P(x)\Theta\delta x_v \tag{48}$$

where $\Pi(x) = \Theta^\top P(x)\Theta$. Thus, $\overline{V}(x_v, \delta x_v) < e^{-2\beta t}$.

Now, as in Forni and Sepulchre (2014), consider the set $\Gamma(x, x_{d\sigma})$ of all normalized paths $\gamma : [0, 1] \to \mathcal{X}$ connecting x with x_d such that $\gamma(0) = x$ and $\gamma(1) = x_d$. Function (48) defines a *Finsler* structure in $T\mathcal{X}$, which by integration induces the distance

$$d(x, x_d) = \inf_{\Gamma(x, x_d)} \int_\gamma \sqrt{\overline{V}(\gamma(s), \frac{\partial\gamma}{\partial s}(s))} ds < e^{-\beta t}. \tag{49}$$

Therefore, $d(x, x_d) \to 0$ as $t \to \infty$ with rate β. ■

4. CASE OF STUDY: 3 DOF SCARA ROBOT

Consider a SCARA robot with position $q^\top = [\theta_1, \theta_2, z]$, generalized momentum $p^\top = [p_{\theta_1}, p_{\theta_2}, p_z]$ and input force $u^\top = [\tau_1, \tau_2, f]$. Such system can be represented in pH form (14), with inertia matrix

$$M(q) = \begin{bmatrix} M_{11} & M_{12} & 0 \\ M_{12} & m_3 l_2^2 & 0 \\ 0 & 0 & (m_1 + m_2 + m_3)g \end{bmatrix}, \tag{50}$$

where $M_{11} = (m_2 + m_3)l_1^2 + m_3 l_2^2 + 2m_3 l_1 l_2 \cos\theta_2$ and $M_{12} = m_3 l_2^2 + m_3 l_1 l_2 \cos\theta_2$. The potential energy is $V(q) = (m_1 + m_2 + m_3)gz$, and the dissipation matrix is $D = 0.2I_n$.

The goal is to track to $q_d = [\sin(t) + 1, \sin(t), \sin(t)]^\top$, by closing the loop with the control scheme (25), with gain matrices $\Lambda = \text{diag}\{15, 15, 15\}$ and $K_d = \text{diag}\{30, 60, 90\}$.

In Figure 1, the time responses of the error variables are shown. All converge to zero exponentially after transients. Notice the zero steady-state value of time response of $\tilde{q}$ is guaranteed hierarchically by $\sigma = 0$. Above is the reason why $\tilde{q}$ and $\tilde{p}$ converge slower than the sliding variable.

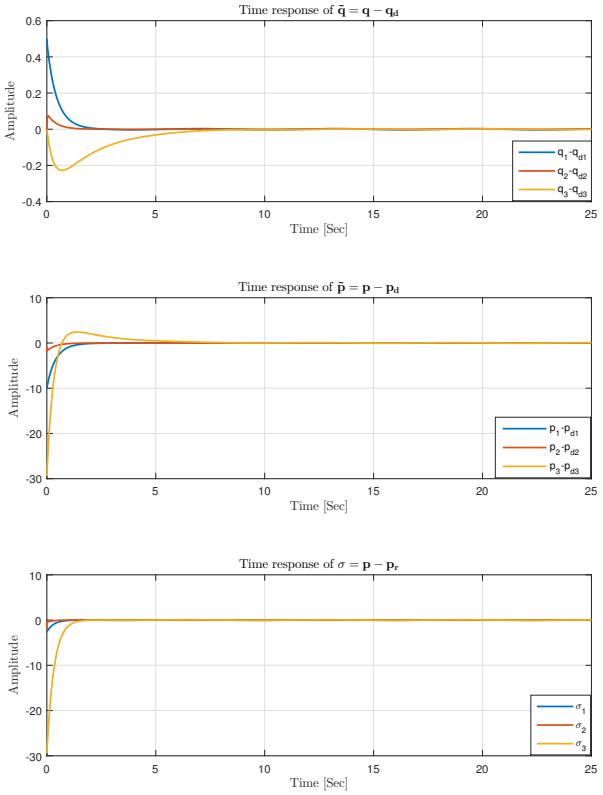

Fig. 1. Position and momentum error and sliding variable

The upper plot of Figure 2 shows the time response of the contraction measure with respect to the desired trajectory (assuming γ is a straight line), which after an overshoot transient, in fact shrinks. This is reflected in the lower plot where it is shown that the actual Hamiltonian converges to the desired Hamiltonian function.

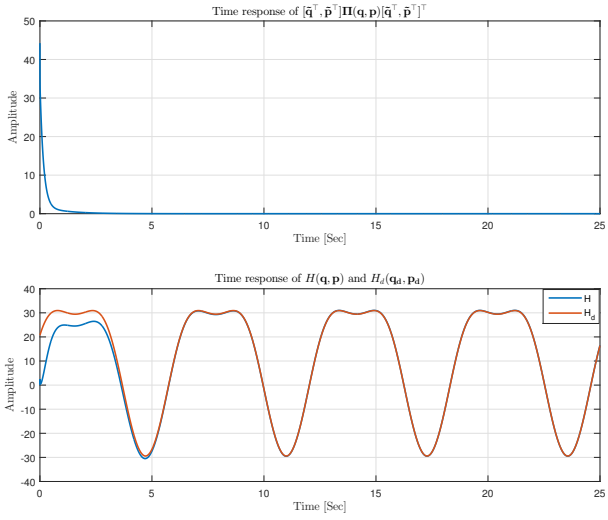

Fig. 2. Contraction measure and Hamiltonian functions

The control effort is shown in Figure 3. It can be seen that the control signals u_2 and u_3 have a big overshoot. This because of the term (18) has a big (but fast) transient, and the controller was designed to compensate it.

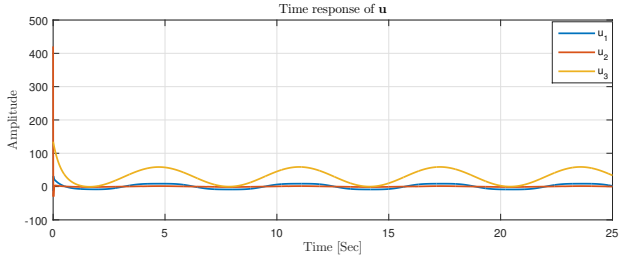

Fig. 3. Control signal time response

5. CONCLUSIONS

In this paper we presented a trajectory tracking controller for fully-actuated port-Hamiltonian mechanical systems. The control law is composed by the equivalent control, which renders the sliding surface to an invariant set; and a feedback controller which ensures attractivity to the invariant set by making the closed-loop error system to be partially contracting. Moreover, the controller contracts exponentially a Riemannian distance as result of incremental stability properties of the virtual system. Simulations showed the good performance of the proposed controller.

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