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A General Criterion for Synchronization of Incrementally Dissipative Nonlinearly Coupled Agents

Anton V. Proskurnikov, Fan Zhang, Ming Cao and Jacquelin M.A. Scherpen

Abstract—Whereas synchronization (consensus, agreement) in linear networks has been thoroughly studied in recent years up to certain exhaustiveness, reaching a synchrony among general nonlinear agents still remains a hard open problem. In this paper, we propose a general criteria of synchronization in undirected networks with nonlinear nodes, based on the idea of *incremental dissipativity*. We show that our result implies a number of existing synchronization criteria, employing incremental Lyapunov functions (such as e.g. the sum of squared deviations between the agents' states). At the same time, unlike most of existing results of synchronization, our criterion is applicable not only to linearly coupling maps, but also to a wide class of *nonlinear* couplings, provided they are anti-symmetric and satisfy a quadratic constraint, which should be in correspondence with the incremental supply rate and the network topology. Our criterion thus allows to extend many existing synchronization criteria to the case of nonlinear couplings; moreover, it can be applied to some classes of agent, uncovered by previous results, among them are nonlinear agents in Lurie form and biochemical oscillators of Goodwin's type.

Index Terms—Synchronization, complex network, robustness, dissipativity, oscillators

I. INTRODUCTION

The fundamental principle of *synchronization* between dynamical processes [1] lies in the heart of many natural phenomena and engineering designs, attracting enormous attention of the research community. In his celebrated experiment with pendulums, suspended on a common beam, C. Huygens discovered that synchronization may be achieved via *coupling* between individual parts of a complex system without any global “orchestrators” or reference signals. During recent decade this phenomenon of self-synchronization has been extensively studied in the frameworks of *complex networks* [2]–[4] and *multi-agent control* [5]–[7]; the interest was mainly focused in achieving the synchrony between the nodes of the network, or *agents*, via local interactions.

In spite of enormous progress in the field, most existing results on synchronization in complex networks are confined to the case of *linear couplings*. The case of linear stationary networks has been exhaustively investigated by using tools from linear control theory: for homogenous agents consensus is reduced to simultaneous stabilization via a linear transformation of the state space [8]–[11]; networks of heterogeneous agents can be tackled via the networked internal

model principle [12] of frequency-domain techniques [13]. In the case where the agents are nonlinear, or topology may switch, synchronization is proved by Lyapunov methods. Most of the results, concerned with such networks, fall into one of two groups. The first group of synchronization criteria deals with *passive* or *feedback passive* agents [11], [14]–[16], which may be heterogeneous. The conventional Lyapunov function in this case is the total energy of the system, obtained via summation of individual agents' storage functions. The second group of results deals with identical agents and *incremental* Lyapunov functions [17], [18]; as will be discussed, these results implicitly use the idea of *incremental dissipativity*, which proved to be a very efficient tool in analysis of oscillator networks [19]–[22].

Whereas the aforementioned results mainly addressed *linearly* coupled agents, many applications deal with *non-linearly* coupled networks. Classic examples are given by pulse or sinusoidally coupled oscillator networks [23] and smart grids, coordination with range-restricted communication [24], cellular neural networks etc. In general, linear couplings may. Most results on synchronization under nonlinear couplings, available in the literature, are based on linearization along some synchronous trajectory and thus provide only *local* convergence [25]–[27]. A global result on synchronization of *passive* agents coupled via passive maps was obtained in [15]. In the recent papers by the first author [28]–[30] criteria for synchronization of LTI agents under nonlinear couplings were obtained; these criteria may be considered as extensions of the classical circle and Popov stability criteria to network synchronization.

In this paper, we combine the results from [28]–[30] with the incremental dissipativity techniques, obtaining a general criterion for synchronization of nonlinear agents under nonlinear couplings and switching undirected topology. Like in the papers [28]–[30], the couplings may be uncertain, they are assumed only to be anti-symmetric and satisfy some *quadratic constraint*. We assume the time-dependent topology to be undirected and connected; at the same time, positive dwell-time between consequent switchings is not required. We show that if the agents are incrementally dissipative with a quadratic supply rate, depending on the quadratic constraint and the interaction topology, then the agents get synchronized. As will be shown, our results not only extend a number of synchronization criteria, previously obtained in the literature, but also allows to derive new criteria for synchronization of nonlinear oscillators.

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II. PRELIMINARIES

We first recall some concepts from the graph theory. A (weighted) *graph* is a triple $G = (V, E, A)$ constituted by the finite set of *nodes* $V = \{v_1, \dots, v_N\}$, the set of *arcs* $E \subset V \times V$ and the *adjacency matrix* $A = (a_{jk})_{j,k=1}^N$ where $a_{jk} > 0$ if $(k, j) \in E$ and $a_{jk} = 0$ otherwise. One can always identify V with the set $V_N := \{1, 2, \dots, N\}$, in this case the graph is fully determined by its adjacency matrix so we denote it with $G[A]$. We confine ourselves to the graph that are *undirected* (i.e. $A = A^\top$) and contain no self-loops: $a_{jj} = 0 \forall j$. A number $D_j(A) := \sum_{k=1}^N a_{jk}$ is referred to as the (weighted) *degree* of the j th node. Given a graph, a sequence of its nodes v_1, v_2, \dots, v_k with $(v_i, v_{i+1}) \in E \forall i$ is called the *walk* between v_1 and v_k ; the (undirected) graph is called *connected* if a walk exists between any two nodes.

For a graph $G = (V, E, A)$ with N nodes we define its *algebraic connectivity* $\lambda_2(A)$ as follows:

$$\lambda_2(A) = N \min_{z \in \Upsilon} \frac{\sum_{i,j=1}^N a_{ij} (z_j - z_i)^2}{\sum_{i,j=1}^N (z_j - z_i)^2}, \quad (1)$$

where $\Upsilon := \{z \in \mathbb{R}^N : z_k \neq z_j \text{ for some } j, k\}$ and \min is necessarily attained [31]. For an undirected graph λ_2 can be defined as the second term in the ascending sequence of eigenvalues $\lambda_1(A) = 0 \leq \lambda_2(A) \leq \dots \leq \lambda_N(A)$ of the *Laplacian* matrix $L(A)$ [8], [32].

We also need a concept of *incremental dissipativity* [19] and detectability. Consider a dynamical system

$$\dot{x}(t) = f(x(t), u(t)), \quad y(t) = h(x_j(t), u_j(t)), \quad (2)$$

where $x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m, y(t) \in \mathbb{R}^l$ stand, respectively, for the state, control input and output. Given two solutions $x^\dagger, u^\dagger, y^\dagger$ and $x^\ddagger(t), u^\ddagger(t), y^\ddagger(t)$, consider the corresponding deviation $\Delta x := x^\ddagger - x^\dagger, \Delta u := u^\ddagger - u^\dagger, \Delta y := y^\ddagger - y^\dagger$.

Definition 1: Suppose there exist functions $V \in C^1(\mathbb{R}^n \rightarrow \mathbb{R}_+)$ and $w : \mathbb{R}^l \times \mathbb{R}^m \rightarrow \mathbb{R}$ such that for any two solutions of (2) the inequality holds

$$\frac{d}{dt} V(\Delta x(t)) \leq w(\Delta y(t), \Delta u(t)). \quad (3)$$

Then system (2) is called *incrementally dissipative*; the functions $V(\Delta x)$ and $w(\Delta y, \Delta u)$ are referred to as respectively the (incremental) storage function and the supply rate.

Definition 2: We say the system (2) is *detectable* (at zero input) if conditions $\Delta y(t) \equiv 0$ and $u^\dagger(t) = u^\ddagger(t) \equiv 0$ imply that $\Delta x(t) \equiv 0$ and, moreover, the latter property is “robust” in the following sense: if $\Delta y(t) \rightarrow 0, u^\dagger(t) \rightarrow 0, u^\ddagger(t) \rightarrow 0$ as $t \rightarrow +\infty$ then also $\Delta x(t) \rightarrow 0$.

A simple example is a Lurie system

$$\dot{x}(t) = Dx(t) + B(t)u(t) + E\psi(y(t)), \quad y(t) = Cx(t), \quad (4)$$

where B, C, D, E are constant matrices, the pair (D, C) is detectable, and $\psi(\cdot)$ is a uniformly continuous mapping $(|\psi(y_1) - \psi(y_2)| \rightarrow 0 \text{ as } |y_1 - y_2| \rightarrow 0)$.

III. PROBLEM SETUP

Throughout the paper we consider a team of $N \geq 2$ agents indexed 1 through N and obeying a nonlinear model

$$\dot{x}_j(t) = f(x_j(t), u_j(t)), \quad y_j(t) = h(x_j(t)), \quad t \geq 0. \quad (5)$$

Here $j \in 1 : N$, and $x_j(t) \in \mathbb{R}^n, u_j(t) \in \mathbb{R}^m, y_j(t) \in \mathbb{R}^l$ stand, respectively, for the state, control and output of the j th agent. We assume the map f to be continuous and h to be C^1 -smooth with globally bounded derivative $h'(x)$.

The agents are coupled in accordance with the rule

$$u_j(t) = \sigma \sum_{k=1}^N a_{jk}(t) \varphi_{jk}(y_k(t) - y_j(t)). \quad (6)$$

This coupling protocol involves continuous nonlinearities $\varphi_{jk} : \mathbb{R}^l \rightarrow \mathbb{R}^m$, referred to as *couplings* and describing how the agents interact to each other. In general, they may be uncertain, assumed only to be *antisymmetric* $\varphi_{jk}(y) = -\varphi_{kj}(-y)$ and constrained by some known quadratic cone (e.g. to a sector in the scalar case); no extra knowledge about them is available. The matrix $A(t) = (a_{jk}(t))$ characterizes the “intensities” of these interactions and defines the time-varying *interaction graph* $G[A(t)]$, where an arc $k \mapsto j$ exists if the output $y_k(t)$ directly affects $y_j(t)$ at time $t \geq 0$. Finally, $\sigma > 0$ stands for the coupling strength.

The goal of the paper is to find out conditions under which the network (5),(6) achieves *output synchronization*.

Definition 3: The protocol (6) establishes *output synchronization* among the agents (5) if the following relation holds:

$$y_j(t) - y_k(t) \xrightarrow[t \rightarrow +\infty]{} 0 \quad \forall j, k \in 1 : N \quad \forall x_1(0), \dots, x_N(0). \quad (7)$$

Remark 1: If the gains $a_{jk}(t)$ are bounded and agents are detectable, then output synchronization (7) implies full (state) synchronization $x_j(t) - x_k(t) \rightarrow 0$ since $u_j(t) \rightarrow 0$ for any $j \in 1 : N$. This holds, in particular, for Lurie agents (4) whenever $\psi(\cdot)$ is uniformly continuous.

IV. MAIN RESULTS

We start with the main assumptions about the protocol (6), where both couplings and interaction graph may be uncertain. We start with the assumption about the graph.

Assumption 1: The matrix-valued function $A(t) : \mathbb{R}_+ \rightarrow \mathcal{A}$ is Lebesgue measurable and attains its values in a *compact* set of $N \times N$ -matrices $\mathcal{A} = \{A\}$, where all matrices $A \in \mathcal{A}$ are symmetric $A = A^\top$, have non-negative entries $a_{ij} \geq 0$ and correspond to *connected* graphs $G[A]$.

Assumption 1 implies that the time-varying interaction graph is undirected and retains its connectivity. A natural choice of \mathcal{A} is the set of all matrices $A = A^\top$ with $G[A]$ connected and $a_{jk} \in \{0\} \cup [\underline{M}; \overline{M}]$, $0 < \underline{M} \leq \overline{M} < \infty$.

Assumption 1 obviously implies the existence of $\min_{A \in \mathcal{A}} \lambda_2(A) > 0$ and $\max_{A \in \mathcal{A}} \max_{j \in 1:N} D_j(A)$. Although their direct computation may be troublesome, usually a lower and upper bounds respectively for this quantities are available. In the case of non-weighted graph $a_{jk} \in \{0; 1\}$ estimates for the algebraic connectivity can be found in [31], [32]; for sparse

graphs an elegant lower bound for λ_2 , relating weights a_{jk} and paths in the graph, was found in [18, section 3.2].

For instance, to estimate the algebraic connectivity the Fiedler inequality [31] and other estimates may be put in use [32], whereas $\max_j D_j(A) \leq (N-1) \max_{j,k} a_{jk}$, where a priori maximal value for the gain is usually known. So we always assume such constants λ_2^0 and D_0 to be known that

$$\lambda_2(A) \geq \lambda_2^0 > 0, \quad D_j(A) \leq D_0 \quad \forall A \in \mathcal{A} \forall j \in 1 : N. \quad (8)$$

It should be noticed Assumption 1 *does not* assume the existence of positive dwell-time between switchings of the graph, nor even requires $A(t)$ to be piecewise-continuous.

Our next assumptions are concerned with the couplings, requiring them to be antisymmetric and satisfy a constraint, generalizing conventional sector inequalities.

Assumption 2: For any $y \in \mathbb{R}^l$ and $j, k \in 1 : N$ one has $\varphi_{jk}(y) = -\varphi_{kj}(-y)$ and $\varphi_{jk}(0) = 0$.

In the case of homogenous couplings $\varphi_{jk} = \varphi$ Assumption 2 means that the map φ is odd. Treating the coupling as a force, the relation $\varphi_{jk}(y_k - y_j) = -\varphi_{kj}(y_j - y_k)$ implied by Assumption 2 expresses the Newtons Third Law and usually holds if the nodes are coupled via a physical interaction, like in oscillator networks, power grids, etc.

To proceed with our last assumption, we introduce a class $\mathfrak{S}(F)$, where $F : \mathbb{R}^m \times \mathbb{R}^l \rightarrow \mathbb{R}$ is a continuous map with $F(0,0)$, as the set $\{\varphi(\cdot)\}$ of all continuous mappings $\varphi : \mathbb{R}^l \rightarrow \mathbb{R}^m$, such that $\varphi(0) = 0$ and

$$F(y, \varphi(y)) > 0 \quad \forall y \neq 0. \quad (9)$$

Assumption 3: For any j, k the coupling φ_{jk} belong to $\mathfrak{S}(F)$ defined by a mapping F as follows:

$$F(y, u) = y^\top Q u - y^\top R y - u^\top S u, \quad y \in \mathbb{R}^l, u \in \mathbb{R}^m, \quad (10)$$

where $R = R^\top \geq 0$, $S = S^\top \geq 0$ and Q are constant.

In other words, for any $\varphi = \varphi_{jk}$ one has $y^\top Q \varphi(y) \geq y^\top R y + \varphi(y)^\top S \varphi(y)$. The set $\{(y, \varphi) : \mathcal{F}(y, \varphi) > 0\}$ defines a cone (possibly, non-convex) in the space $\{(y, \varphi) : \mathbb{R}^l \times \mathbb{R}^m$, and $S(\mathcal{F})$ consists of maps whose graphs are confined to this cone (except for the origin). In the scalar case ($l = m = 1$) a typical example is the *sector constraint* [33]:

$$0 \leq \alpha < \frac{\varphi(y)}{y} < \beta \leq +\infty \quad \forall y \neq 0, \quad \varphi(0) = 0. \quad (11)$$

Introducing a quadratic form $F_{\alpha,\beta} := \frac{1}{1+\alpha\beta} (u - \alpha y)(y - \beta^{-1}u)$, (11) is rewritten as $\varphi \in \mathfrak{S}(F_{\alpha,\beta})$.

Now we are in position to formulate our main result, giving a criterion for output synchronization under any protocol, satisfying Assumptions 1-3.

Theorem 1: Suppose the matrix-valued function $A(\cdot)$ and the couplings φ_{jk} satisfy Assumptions 1-3. Assume that agents (5) are incrementally dissipative, that is (3) holds for some storage function $V(\Delta x) \geq 0$ and the supply rate

$$w(\Delta y, \Delta u) = \begin{bmatrix} \Delta y \\ \Delta u \end{bmatrix}^\top \begin{bmatrix} R^0 & Q/2 \\ Q/2 & S^0 \end{bmatrix} \begin{bmatrix} \Delta y \\ \Delta u \end{bmatrix}, \quad (12)$$

where the matrices R^0, S^0 are defined by

$$R^0 := \sigma \lambda_2^0 R, \quad S^0 := \frac{1}{2\sigma D^0} S. \quad (13)$$

Then the outputs get synchronized $y_j(t) - y_k(t) \xrightarrow[t \rightarrow +\infty]{} 0 \forall j, k$ for any *bounded* solution of the system (5),(6).

If additionally $V(\Delta x) \rightarrow \infty$ as $|\Delta x| \rightarrow \infty$ and for some continuous function $c(\cdot)$ one has

$$|f(x_1, u_1) - f(x_2, u_2)| \leq c(|x_1 - x_2| + |u_1| + |u_2|), \quad (14)$$

then protocol (6) establishes output synchronization (7).

Remark 2: Condition (14) holds, for instance, if the derivatives $f'_x(x, u)$, $f'_u(x, u)$ are bounded.

The proof of Theorem 1 is based on the quadratic constraint for the solutions, derived from Assumptions 1-3 (see [28, Lemma 12], [29, Lemma 19] and [30, Lemma 1] for details). Given a sequence of vectors ξ_1, \dots, ξ_N , we denote the stack vector constituted by them with $\bar{\xi} := (\xi_1^\top, \dots, \xi_N^\top)^\top$. We define an auxiliary quadratic form

$$\mathfrak{F}(\bar{y}, \bar{u}) := - \sum_{j,k=1}^N w(y_k - y_j, u_k - u_j), \quad (15)$$

where w is given by (12) and $y_1, \dots, y_N \in \mathbb{R}^l$ and $u_1, \dots, u_N \in \mathbb{R}^m$.

Lemma 1: Let the maps φ_{jk} satisfy Assumptions 2 and 3 for all $j, k \in 1 : N$, and $A \in \mathcal{A}$, where \mathcal{A} is a class of matrices from Assumption 1. Consider a sequence $y_1, \dots, y_N \in \mathbb{R}^l$ and let $u_j[A, \bar{y}] := \sum_{k=1}^N a_{jk} \varphi_{jk}(y_k - y_j)$. Then $\mathfrak{F}(\bar{y}, \bar{u}[A, \bar{y}]) \geq 0$, and the latter inequality is strict unless $y_1 = y_2 = \dots = y_N$.

Theorem 1 can be derived from Lemma 1, using the Lyapunov function $\mathfrak{V}(\bar{x}) := \sum_{j,k=1}^N V(x_k - x_j)$, similarly to [29], [34].

Theorem 1 reduces the synchronization problem to a problem of incremental dissipativity with a given quadratic supply rate. Although the verification of this condition in general case remains an open problem, for some important types of agents it is known to be easily checkable, as discussed in Section V. In some situations (e.g. for linear agents, see Subsect.V-C) one can proof the existence of a storage function $V(\Delta x)$ satisfying (3) but not its nonnegativity. However, the condition $V \geq 0$ follows from the remaining conditions if at least one synchronizing protocol (6) exists.

Theorem 2: Let agents are detectable and satisfy (3), where $w(\Delta y, \Delta u)$ is defined by (12) and $V(\Delta x)$ is *not* assumed to be non-negative. Suppose *there exists* a function $A(\cdot)$ and a set of couplings $\varphi_{jk}(\cdot)$ such that Assumptions 1-3 hold and protocol (6) establishes output synchronization (7). Then one has $V(\Delta x) > 0$ whenever $\Delta x \neq 0$.

Proof: Consider a synchronizing protocol (6), satisfying Assumptions 1-3, whose existence is supposed. Since the output synchronization (7) is established, one has also $u_j(t) \rightarrow 0$ and thus $x_j(t) - x_k(t) \rightarrow 0$ as $t \rightarrow +\infty$ due to detectability property. This entails $\mathfrak{V}(\bar{x}(T)) \rightarrow 0$ as $T \rightarrow +\infty$ along any solution of (5),(6) which means, thanks to Lemma 1, that $\mathfrak{V}(\bar{x}(0)) \geq 0$ for any initial conditions.

Moreover, if $\mathfrak{W}(\bar{x}(0)) = 0$ then one has $\mathfrak{F}(\bar{y}(t), \bar{u}(t)) \equiv 0$ and thus $y_1(t) \equiv y_2(t) \equiv \dots \equiv y_N(t)$ and $u_j(t) \equiv 0$ which implies that $x_1(t) \equiv x_2(t) \equiv \dots \equiv x_N(t)$ due to detectability, so $x_k(0) - x_j(0) = 0$ for any j, k . Taking $x_1(0) = \dots = x_{N-1}(0) = 0$ and $x_N(0) = x^0$, one obtains that $V(x^0) \geq 0$ for any $x^0 \in \mathbb{R}^n$ and $V(x^0) > 0$ whenever $x^0 \neq 0$. ■

Being formally non-constructive, in practice the second condition from Theorem 2 may be easily verifiable if $\mathfrak{S}(F)$ contains at least one linear mapping $\varphi(y) = Ky$, since synchronization under linear couplings is a well-studied problem, especially for the case of linear agents [9].

V. APPLICATIONS AND EXAMPLES

In this section, we consider several applications of our general results.

A. Fully actuated agents with Lipschitz condition

In this subsection, we consider agents as follows

$$\dot{x}_j(t) = f(x_j(t)) + u_j(t), \quad y_j(t) = x_j(t). \quad (16)$$

Here $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a Lipschitz map: $|f(x) - f(x')| \leq \alpha|x - x'|$, and hence, due to the Cauchy-Schwartz inequality

$$(f(x) - f(y))^\top (x - y) \leq \alpha|x - y|^2 \quad \forall x, y \in \mathbb{R}^n.$$

A straightforward computation shows that the agent (16) is incrementally dissipative with the storage function $V(\Delta x) = |\Delta x|^2$ and the supply rate $w(\Delta y, \Delta u) = \Delta y^\top \Delta u + \alpha|\Delta y|^2$. Theorem 1, applied to $F(y, u) = y^\top u - \varepsilon|y|^2$, yields in the following synchronization criterion.

Theorem 3: Suppose that Assumption 1 and 2 hold and $\varphi_{ij}(y)^\top y > \varepsilon|y|^2$ for any $y \neq 0$, where $\sigma\lambda_2^0\varepsilon \geq \alpha$. Then protocol (6) establishes synchronization (7).

It should be noticed that for *linear* couplings $\varphi_{jk}(x) = x$ one can take $\varepsilon \in (0; 1)$ arbitrary and for synchronization it thus suffices that $\sigma > \alpha/\lambda_2^0$. This criterion was obtained in [17] (Theorem 4) by using contraction principle under fixed topology and additional assumption that $\sigma > \alpha$. Besides discarding the latter conditions, Theorem 3 also extends the result from [17] to *nonlinear* anti-symmetric couplings.

Similar, however more subtle arguments allow to derive the Belykh synchronization criterion (Theorem 1 in [18]), dealing with underactuated agents

$$\dot{x}_j(t) = f(x_j(t)) + Pu_j(t), \quad y_j(t) = Px_j(t). \quad (17)$$

Here $P = \text{diag}(p_{11}, \dots, p_{nn})$ stands for the diagonal matrix, where $p_{ii} \in \{0; 1\}$ and hence $P^2 = P$. Assumptions adopted in [18] imply the incremental dissipativity of agent (17) with a quadratic positive definite storage function and a quadratic supply rate. It can be checked that condition from [18, Theorem 1], requiring couplings to be sufficiently strong, is equivalent to our Theorem 1.

B. Cyclic Feedback Systems and Biochemical Oscillators

Probably, one of the most interesting class of incrementally dissipative systems is given by oscillators, arising in biology, chemistry and neurosciences [19], [20], [22]. A wide class of such oscillators, including the celebrated Goodwin model [35], may be represented as a cyclic feedback system (CFS) [22].

In the paper [22] a CFS consisting of n blocks was considered that is governed by nonlinear equations

$$\begin{aligned} \dot{x}^1 &= f_1(x^1, u - y^n), & y^1 &= \varrho_1(x^1, u - y^n), \\ \dot{x}^2 &= f_2(x^2, y^1), & y^2 &= \varrho_2(x^2, y^1), \\ &\vdots & & \\ \dot{x}^n &= f_n(x^n, y^{n-1}), & y_n &= \varrho_n(x^n, y^{n-1}). \end{aligned} \quad (18)$$

where $x^i(t) \in \mathbb{R}^{r_i}$ and $y^i(t) \in \mathbb{R}$ are the state and output respectively of the i th constituent nonlinear block,

$$\dot{x}^i = f_i(x^i, u^i), \quad y^i = \varrho(x^i, u^i), \quad (19)$$

whose input $u^i(t)$ is $u(t) - y^n(t)$ for $i = 1$ (being sum of an external control input $u(t)$ and a negative feedback from the last block in the cascade) and $y^{i-1}(t)$ for the remaining blocks. The maps $f_i : \mathbb{R}^{r_i} \times \mathbb{R} \mapsto \mathbb{R}^{r_i}$ and $\varrho_i : \mathbb{R}^{r_i} \times \mathbb{R} \mapsto \mathbb{R}$ are Lipschitz functions. The authors of [22] mainly focus on the case where each block (19) is *incrementally output strictly passive* (iOSP). The latter property is a special case of the incremental dissipativity.

Definition 4: The system (2) with $\dim y = \dim u$ is called incrementally output feedback passive [22] with a gain $\gamma \neq 0$ (possibly, negative), written iOFP(γ^{-1}), if it is incrementally dissipative with the supply rate $w(\Delta y, \Delta u) := \Delta y^\top \Delta u - \gamma^{-1}|\Delta y|^2$ and a radially unbounded storage function V . If $\gamma > 0$, an iOFP block is said to be incrementally output strictly passive with gain γ , written iOSP(γ^{-1}).

A fundamental result of [22, Theorem 1] states that if each block (19) is iOSP(γ_i), $\gamma_i > 0$, then their cyclic interconnection is iOFP($-K$) with respect to the input u and output $y = y^1$ whenever the *secant condition* holds

$$K > \bar{K} := -\frac{1}{\gamma_1} + \gamma_2\gamma_3 \dots \gamma_n \left(\cos \frac{\pi}{n} \right)^n. \quad (20)$$

As was demonstrated in [22] (see also analogous results in [19], [20]), the team of coupled iOFP oscillators is synchronized under sufficiently strong *linear* coupling, where the “strength” is measured by the algebraic connectivity of the fixed interaction topology. The next theorem extends this result to the case of time-varying topology and anti-symmetric nonlinear couplings from the special class $\mathfrak{S}(F)$.

Given a system of N identical CFS (18), indexed 1 through N , we denote by u_j the external input of the j th system and its output is $y_j = y_j^1$. The j th state vector is $x_j = ((x_j^1)^\top, \dots, (x_j^n)^\top)^\top$.

Theorem 4: Consider N identical CFS (18) coupled via a protocol (6), where $\varphi_{jk} : \mathbb{R} \rightarrow \mathbb{R}$. Suppose that each block (19) is iOSP(γ_i^{-1}). Let Assumptions 1-3 hold, and F from Assumption 3 is given by $F(y, u) = yu - \alpha y^2$, $y, u \in \mathbb{R}$.

If $\sigma\lambda_2^0\alpha > \bar{K}$, where \bar{K} is given by (20), then the protocol establishes output synchronization.

The proof follows from [22, Theorem 1] and Theorem 1.

Remark 3: Although conditions of [22, Theorem 1] and our Theorem 4 are valid for some biological oscillators [20], [22] and formally they guarantee synchronization, the solutions are not guaranteed to be bounded, which is required for their biological feasibility. Moreover, for general Goodwin-type oscillators [35] where each of the blocks (19) involves a Mikhaelis-Menten nonlinearity [35], one cannot guarantee finite gains γ_i globally, but only for solutions staying in some a priori known bounded domain. A way to overcome these difficulty is to consider nonlinear control algorithm, which guarantees that solutions are attracted to some known compact, see [36] for details.

C. Linear agents

In this subsection, we deal with LTI agents

$$\dot{x}_j(t) = Dx_j(t) + Bu_j(t), y_j(t) = Cx_j(t). \quad (21)$$

For such an agent, the incremental dissipativity (3) coincides with conventional Willems dissipativity with the storage function $V(x)$ and supply rate $w(u, y)$, defined by (12). Since w is quadratic and the agent dynamics is linear, the storage function can also be found in the class of quadratic forms $V(x) = x^\top Hx$, and the incremental dissipativity condition (3) shapes into the following LMI:

$$\begin{bmatrix} HD + D^\top H & HB \\ * & 0 \end{bmatrix} \leq \begin{bmatrix} C^\top R^0 C & C^\top Q \\ * & S^0 \end{bmatrix}. \quad (22)$$

The second inequality in (22) is a matrix form of the inequality (3). Applying Theorem 1, one arrives at the following.

Theorem 5: Under Assumptions 1-3, suppose that LMIs (22) have a solution $H = H^\top > 0$. Then protocol (6) establishes output synchronization among agents (21). If (D, C) is a detectable pair, state synchronization is also achieved. If a solution $H \geq 0$ exists, these claims remain valid for any solution along which $x_j(t) - x_k(t)$ are bounded for all j, k .

Proof: Under assumption $H \geq 0$, the storage function $V(\Delta x) = (\Delta x)^\top H(\Delta x)$ is non-negative, being radially unbounded for $H > 0$. Agents (21) obviously satisfy (14), so the statements follow from Theorem 1 and Remark 1. ■

Besides efficient numerical tools for LMI solving, analytic criteria based on the Kalman-Yakubovich-Popov Lemma may be put in use [37]. Namely, if the system (21) is controllable, the inequality (3) holds if and only if for any $\omega \in \mathbb{R}$ one has

$$W(i\omega)^* Q + Q^\top W(i\omega) + W(i\omega)^* R^0 W(i\omega) + S^0 \geq 0, \quad (23)$$

where $W(\lambda) = C(\lambda I - D)^{-1}B$ is the transfer matrix of the agent (21) and $*$ stands for the Hermitian conjugate transpose. It should be noticed, however, that inequality (23) guarantees only the existence of a quadratic storage function $V(x) = x^\top Hx$, whereas its non-negativity requires extra assumptions. In this case Theorem 2 may be helpful which allows to discard the condition $V(x) \geq 0$, assuming the

existence of at least one synchronizing protocol, satisfying Assumptions 1-3. Suppose that the class $\mathfrak{S}(F)$ contains at least one linear map $y \mapsto Ky$, i.e. $F(y, Ky) > 0$ whenever $y \neq 0$. Given a constant gain matrix $A(t) \equiv A_0 \in \mathcal{A}$, the condition for synchronization in the network (21),(6) is well known [9] and boils down to the Hurwitz stability of the matrices $D - \lambda BKC$, where λ is arbitrary nonzero eigenvalue of the Laplacian matrix $L[A_0]$.

Theorem 6: Let (23) hold, the pair (D, B) is controllable and the pair (D, C) is observable. Suppose there exist matrices $K \in \mathbb{R}^{m \times l}$ and $A_0 \in \mathcal{A}$ such that $F(y, Ky) > 0 \forall y \neq 0$ and the matrix $D - \lambda BKC$ is Hurwitz, whenever $\lambda \neq 0$ is an eigenvalue of $L[A_0]$. Then any protocol, satisfying Assumptions 1-3, establishes output synchronization between agents (21).

Proof: In accordance with the KYP lemma [37], the frequency-domain condition (23) implies the existence of a solution $H = H^\top$ to (22). Theorem 2 now implies that $H > 0$ due to detectability, so the claim follows from Theorem 5. ■

As discussed in [34], Theorem 6 extends the *quadratic criterion* for absolute stability, established by V.A. Yakubovich [38], to the networked case. In the special case where $\sigma = 1$ and the set \mathcal{A} consists of such matrices A that $a_{jk} \in \{0; 1\}$ and $\lambda_2[A] \geq \theta > 0$, it was established in [29] and for SISO case in [28]. In this case, one can take $\lambda_2^0 := \theta, D^0 := N - 1$ and A_0 corresponding to the complete graph (so $L[A_0]$ has the only non-zero eigenvalue $\lambda = (N - 1)$).

D. Lurie-type agents

A natural extension of the model (21) is a Lurie model

$$\dot{x}_j(t) = Dx_j(t) + Bu_j(t) + Eg(x_j(t)), y_j(t) = Cx_j(t). \quad (24)$$

Here B, C, D, E are constant matrices, $g : \mathbb{R}^n \rightarrow \mathbb{R}^p$ is a nonlinear map, supposed to satisfy some *incremental quadratic constraint* $\Theta(g(\xi_2) - g(\xi_1), \xi_2 - \xi_1) \geq 0$ for any $\xi_1, \xi_2 \in \mathbb{R}^n$. Here $\Theta : \mathbb{R}^p \times \mathbb{R}^n \rightarrow \mathbb{R}$ is a quadratic form. A typical example of such a constraint is Lipschitz condition: $|g(\xi_2) - g(\xi_1)|^2 \leq c|\xi_2 - \xi_1|^2$, where one can take $\Theta(\Delta g, \Delta \xi) = c|\Delta \xi|^2 - |\Delta g|^2$. Another example is a “one-sided” QUAD condition [17]:

$$(g(\xi_2) - g(\xi_1))^T (\xi_2 - \xi_1) - (\xi_2 - \xi_1)^\top \Xi (\xi_2 - \xi_1) \leq -\omega |\xi_2 - \xi_1|^2,$$

where $\omega > 0$ and $\Xi = \Xi^\top$ is some matrix.

Whereas necessary and sufficient conditions for incremental dissipativity of nonlinear agent (24) are unknown, one can easily derive a sufficient condition for the existence of a *quadratic storage function* $V(\Delta x) = (\Delta x)^\top H(\Delta x)$. Putting $\Delta g := g(x^\ddagger) - g(x^\dagger)$, the inequality (3) shapes into

$$2(\Delta x)^\top H(D\Delta x + B\Delta u + E\Delta g) \leq w(\Delta y, \Delta u). \quad (25)$$

Since the only available information about Δf is given by our quadratic constraint $\Theta(\Delta g, \Delta x) \geq 0$, this constraint should imply (25). This holds if $\tau \geq 0$ exists such that

$$2(\Delta x)^\top H(D\Delta x + B\Delta u + E\Delta g) + \tau\Theta(\Delta g, \Delta x) \leq w(\Delta y, \Delta u). \quad (26)$$

Assuming that $\Theta(g, x) = 2x^\top \Theta_{xgg} + g^\top \Theta_{ggg} + x^\top \Theta_{xx}$, one can get rid of the dummy variables $\Delta x, \Delta u, \Delta g$ in (26), transforming it into conventional LMI form:

$$\begin{bmatrix} HD + D^\top H + \tau \Theta_{xx} & HB & HE + \tau \Theta_{xg} \\ * & 0 & 0 \\ * & * & \tau \Theta_{gg} \end{bmatrix} \leq \begin{bmatrix} C^\top R^0 C & C^\top Q & 0 \\ * & S^0 & 0 \\ * & * & 0 \end{bmatrix}. \quad (27)$$

The following result immediately follows from Theorem 1.

Theorem 7: Let Assumptions 1-3 hold. Suppose there exists a solution (H, τ) to LMI (27) with $H > 0, \tau \geq 0$. Then protocol (6) establishes output synchronization between the agents (24). If a solution (H, τ) exists with $\tau \geq 0, H \geq 0$, then any *bounded* solution is output synchronized.

Using the KYP lemma, it is possible to give a frequency-domain condition for solvability of LMI (27), which however is not easily verifiable and remains beyond the present paper.

VI. CONCLUSION

In the present paper, we consider the problem of synchronization for networks of homogeneous nonlinear and nonlinearly coupled agents. The only available information about the couplings comes to the anti-symmetry condition and a quadratic constraint, e.g. a sector inequality in the case of scalar input and output. The topology of the network is time-varying, however assumed to be undirected and retain its connectivity. Under assumption of incremental dissipativity of the agent with a special supply rate, we prove the synchronization under *any* uncertain protocol of the just described type.

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