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Disturbance decoupling problem for multi-agent systems: A graph topological approach

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ABSTRACT

This paper studies the disturbance decoupling problem for multi-agent systems with single integrator dynamics and a directed communication graph. We are interested in topological conditions that imply the disturbance decoupling of the network, and more generally guarantee the existence of a state feedback rendering the system disturbance decoupled. In particular, we will develop a class of graph partitions, which can be described as a "topological translation" of controlled invariant subspaces in the context of dynamical networks. Then, we will derive sufficient conditions in terms of graph partitions such that the network is disturbance decoupled, as well as conditions guaranteeing solvability of the disturbance decoupling problem. The proposed results are illustrated by a numerical example.

1. Introduction

Analysis and design of multi-agent systems and networks of dynamical agents have turned to an extremely popular research target in the last decade. Studying consensus/synchronization and designing feedback protocols to achieve consensus/synchronization have perhaps been the most popular framework in this direction; see e.g. [1–4].

Motivated by the fact that agents are subjected to external disturbances in practice, consensus analysis in the presence of disturbance has been carried out in the literature, and consensus protocols which are robust against external disturbances have been proposed; see e.g. [5,4]. Another example of dynamical networks in which one seeks for disturbance rejection is balancing demand–supply in distribution networks; see e.g. [6,7]. In this framework, storage variables correspond to vertices, flow inputs corresponding to edges, disturbances amount for inflows and outflows at certain vertices, and the objective is to achieve load balancing, that is convergence of storage variables to the same value by regulating the flow inputs.

It is well-known that relative information of the agents play a crucial role in the context of distributed control. In fact, reaching consensus and achieving a desired formation heavily relies on the efficient and accurate transmission of relative information. In this paper, we investigate the conditions under which communication in certain channels is not affected by the external disturbances acting on some of the agents. To achieve this, we consider the possibility of applying state feedback controllers to some agents, called *leaders*.

An important issue in studying networks of dynamical agents is to deduce certain network properties from the "network topology" which is typically given in terms of the so-called "communication graph" of the network. For instance, it is well-known that connectivity of the communication graph plays a crucial role in the consensus problem (see e.g. [1]). Recently, studying network properties from a topological perspective has attracted the attention of many researchers see e.g. [8–11]. A notable instance is controllability analysis; see e.g. [12,11,13,10,14,15]. In this framework, agents are labeled as *leaders* and *followers*. Leaders are agents through which external input signals are injected to the network, and the rest of the agents are called followers. Then, controllability analysis amounts to investigate the possibility of deriving the states of the agents to arbitrary values by appropriate input signals applied to the leaders. Graph partitions, and in particular "almost equitable partitions", has been proven to be a useful tool in controllability analysis [12] and also model order reduction [16]. These partitions can be considered as a topological translation of L-invariant subspaces, with L denoting the Laplacian matrix of the network communication graph; see e.g. [17,12].

In this paper, we study the "disturbance decoupling problem" of diffusively coupled leader–follower networks, where each vertex

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has single integrator dynamics and some agents are affected by external disturbances. As mentioned earlier, roughly speaking, the disturbance decoupling problem (DDP) for a classical linear system with inputs and outputs, amounts to find a feedback (typically, state feedback) such that the output of the closed-loop system is not affected by disturbance signals acting on the states of the system, see e.g. [18]. If such a feedback exists, then we say the DDP for the system is solvable.

Despite being potentially attractive, only very few problems have been formulated in terms of disturbance decoupling problem. An exception is [19], where DDP has been related to formation control of nonholonomic mobile robots. The notion of disturbance decoupling problem has also been studied in the context of a dynamic game contest; see [20].

Our contribution comes from but goes beyond the disturbance decoupling solution in the *geometric approach* for linear systems. The geometric approach to linear system synthesis was inaugurated by the recognition of controlled invariant subspaces, due independently to Basile and Marro [21] and to Wohnam and Morse [22]. Disturbance decoupling problem is in fact an immediate application of the controlled invariance property, see e.g. [23]. To the authors' best knowledge, the current manuscript is the first attempt to study the disturbance decoupling problem for networks of dynamical agents from a topological perspective. Studying network properties such as controllability and DDP from a topological perspective provides valuable insights into the structure/behavior of the network, and it will facilitate the design.

In this paper, we introduce a new class of partitions, namely almost equitable partitions with respect to a cell, in order to provide an appropriate topological translation for controlled invariant subspaces in the context of dynamical networks. Then, by using this extended notion of almost equitability, we derive sufficient (topological) conditions for the network to be disturbance decoupled. More precisely, we consider both open loop and closed loop disturbance decoupling problem. In the first case, we investigate if the network is already disturbance decoupled without applying input signals to the leaders. In the latter case, we consider the solvability of the DDP for the network that amounts to find (if possible) a state feedback controller rendering the network disturbance decoupled. In particular, we establish sufficient topological conditions guaranteeing the network to be disturbance decoupled (open loop) as well as conditions guaranteeing the solvability of DDP (closed loop). A crucial point in the context of distributed control is to exploit the relative (local) information of the states of the agents rather than absolute (global) information of the states. As desired, it will be observed that in case the DDP for the network is solvable then the controller rendering the network disturbance decoupled is indeed using relative information of the states of the agents.

The structure of this paper is as follows. In Section 2, some preliminary materials are provided, and the open loop and closed loop disturbance decoupling problems for multi-agent systems are formulated. In Section 3, the notion of almost equitability with respect to a cell is proposed and characterized in terms of controlled invariant subspaces. In Section 4, we establish sufficient conditions guaranteeing the network to be disturbance decoupled as well as conditions guaranteeing the solvability of DDP. A brief discussion on algorithms verifying the proposed topological conditions is provided in Section 5. To illustrate the proposed results, a numerical example is provided in Section 6. Finally, the paper ends with a summary in Section 7.

2. Diffusively coupled multi-agent systems and disturbance decoupling

2.1. Leader–follower diffusively coupled multi-agent systems with disturbance

In this paper, we consider a multi-agent system consisting of $n > 1$ agents labeled by the set $V = \{1, 2, \ldots, n\}$. We assign three subsets of V as follows: $V_{\mathrm{L}} = \{\ell_1, \ell_2, \ldots, \ell_m\}$ where $m \leqslant n$, $V_{\mathrm{F}} = V \setminus V_{\mathrm{L}}$ and $V_{\mathrm{D}} = \{w_1, w_2, \ldots, w_r\}$ where $r \leqslant n$.

We associate the dynamics

$$\dot{x}_i(t) = \begin{cases} z_i(t) + u_k(t) + d_l(t) & \text{if } i = w_l \in V_{\mathrm{D}} \\ z_i(t) + u_k(t) & \text{otherwise} \end{cases} \tag{1}$$

to each agent $i = \ell_k \in V_{\mathrm{L}}$, and

$$\dot{x}_i(t) = \begin{cases} z_i(t) + d_l(t) & \text{if } i = w_l \in V_{\mathrm{D}} \\ z_i(t) & \text{otherwise} \end{cases} \tag{2}$$

to each agent $i \in V_{\mathrm{F}}$, where $x_i \in \mathbb{R}$ represents the state of agent $i \in V$, z_i indicates the coupling variable of agent $i \in V$, $u_k \in \mathbb{R}$ is an external control input signal received by agent $i = \ell_k \in V_{\mathrm{L}}$, and $d_l \in \mathbb{R}$ is taken as an external disturbance signal influencing agent $i = w_l \in V_{\mathrm{D}}$.

Considering the roles of the defined subsets of V, we refer to V_{L} as the *leader set*, V_{F} as the *follower set*, and V_D as the *disturbance set*. Correspondingly, we say i is a *leader* if $i \in V_L$, and i is a *follower* if $i \in V_F$.

We consider a simple directed graph $G = (V, E)$, where V is the vertex set and $E \subseteq V \times V$ is the arc set of G. For the sake of simplicity and clarity of the presentation, we restrict ourselves to unweighted graphs. For two distinct vertices $i, j \in V$, we have $(i, j) \in E$ if there is an arc from i to j with i being the *tail* and j being the *head* of the arc. Then i is said to be a *neighbor* of j. The coupling variable z_i admits the following *diffusive coupling* rule:

$$z_i(t) = -\sum_{(j,i)\in E} (x_i(t) - x_j(t)). \tag{3}$$

By defining $x(t) = \mathrm{col}(x_1(t), x_2(t), \ldots, x_n(t))$, $u(t) = \mathrm{col}(u_1(t), u_2(t), \ldots, u_m(t))$ and $d(t) = \mathrm{col}(d_1(t), d_2(t), \ldots, d_r(t))$, we write the above leader–follower diffusively coupled multi-agent system (1)–(3) into a compact form as follows:

$$\dot{x}(t) = -Lx(t) + Mu(t) + Sd(t) \tag{4}$$

where L is the in-degree Laplacian of the simple directed graph G (see e.g. [8, p. 26]), the matrix $M \in \mathbb{R}^{n\times m}$ is defined by

$$M_{ik} = \begin{cases} 1 & \text{if } i = \ell_k \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

and the matrix $S \in \mathbb{R}^{n\times l}$ is defined by

$$S_{il} = \begin{cases} 1 & \text{if } i = w_l \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

To introduce the output variables, we consider another simple directed graph $G_y = (V, E_y)$ and define the output $y(t)$ of the system (4) as follows:

$$y(t) = R^{\top}x(t) \tag{7}$$

where R is the incidence matrix of G_y (see e.g. [8, p. 23]). Observe that the output variables (7) capture the differences between the state components of certain pairs of agents determined by the arc set E_y of G_y. In particular, an arc from i to j in G_y corresponds to the output variable $x_i - x_j$ in (7).

In this paper, we study the so-called *disturbance decoupling problem* for multi-agent system (4) by establishing graph topological conditions. Roughly speaking, our aim is to investigate the effect of the disturbance signal d on the output y, given by (7).

For a formal description of the problem and discussing the proposed results, we first review the disturbance decoupling problem and its solution for ordinary linear systems.

2.2. Review: disturbance decoupling problem of linear systems

Consider the linear system

$$\dot{x}(t) = Ax(t) + Ed(t) \quad (8a)$$

$$y(t) = Cx(t) \quad (8b)$$

where $x \in \mathbb{R}^n$ is the state, $d \in \mathbb{R}^r$ is the disturbance, $y \in \mathbb{R}^q$ is the output, and all involved matrices are of appropriate sizes. We denote the state trajectory of the system (8) for the initial state $x(0) = x_0$ and the disturbance d by $x^{x_0,d}$ and the corresponding output trajectory by $y^{x_0,d}$. We quote the following definition for later use.

Definition 1. The linear system (8) is said to be *disturbance decoupled* if $y^{x_0,d_1}(t) = y^{x_0,d_2}(t)$ for all $x_0 \in \mathbb{R}^n$, all locally-integrable disturbances d_1, d_2, and all $t \in \mathbb{R}$. Due to linearity, this is equivalent to the condition $y^{0,d_1}(t) = y^{0,d_2}(t)$ for all locally-integrable disturbances d_1, d_2, and all $t \in \mathbb{R}$.

To further illustrate the notion of disturbance decoupling, note that for any given initial x_0 and locally integrable disturbance d we have $y^{x_0,d}(t) = y^{x_0,0}(t) + y^{0,d}(t)$ where $y^{x_0,0}(t) = Ce^{At}x_0$, and $y^{0,d}(t) = C\int_0^t e^{A(t-\tau)}Ed(\tau)$. Hence, disturbance decoupling of (1) means that the forced response of the system to the disturbance d is zero, i.e. $y^{0,d}(t) = 0$.

In what follows, we quickly review the geometric approach for DDP. For more details, we refer to [18,24].

Let $\langle A \mid \text{im}\, E\rangle$ denote the controllable subspace corresponding to the matrix pair (A, E), that is $\langle A \mid \text{im}\, E\rangle = \text{im}\, E + A\, \text{im}\, E + \cdots + A^{n-1}\text{im}\, E$. As it is well-known, the subspace $\langle A \mid \text{im}\, E\rangle$ is the smallest A-invariant subspace that contains im E. Note that we call a subspace $\mathcal{V} \subseteq \mathbb{R}^n$ *A-invariant* if $A\mathcal{V} \subseteq \mathcal{V}$ where $A : \mathbb{R}^n \to \mathbb{R}^n$. For the matrix pair, (A, C), the unobservable subspace is denoted by $\langle \ker C \mid A\rangle$, that is $\langle \ker C \mid A\rangle = \ker C \cap A^{-1} \ker C \cap \cdots \cap A^{-n+1} \ker C$. Here, for a given subspace $\mathcal{X}$, $A^{-1}\mathcal{X}$ denotes the subspace $\{x : Ax \in \mathcal{X}\}$. It is well-known that the unobservable subspace $\langle \ker C \mid A\rangle$ is the largest A-invariant subspace that is contained in ker C.

Necessary and sufficient conditions for the system (8) to be disturbance decoupled is well-known and are recapped in the following lemma.

Lemma 2. *The following conditions are equivalent.*

(1) *The system (8) is disturbance decoupled.*
(2) *There exists an A-invariant subspace $\mathcal{V}$ such that* $\text{im}\, E \subseteq \mathcal{V} \subseteq \ker C$.
(3) *The inclusion* $\text{im}\, E \subseteq \langle \ker C \mid A\rangle$ *holds.*
(4) *The inclusion* $\langle A \mid \text{im}\, E\rangle \subseteq \ker C$ *holds.*

Note that the equivalence between the first three statements is quite standard and can be, for instance, found in [18, Ch. 4]. The fourth statement immediately follows from the first two and will be employed in the context of multi-agent systems later.

Now, suppose that the linear system (8) is not disturbance decoupled. Then, one may think of applying control inputs to manipulate the system dynamics such that the closed loop system will be disturbance decoupled. This is discussed next.

Consider the linear system

$$\dot{x}(t) = Ax(t) + Bu(t) + Ed(t) \quad (9a)$$

$$y(t) = Cx(t) \quad (9b)$$

where $u \in \mathbb{R}^m$ is the input and $B \in \mathbb{R}^{n\times m}$. The disturbance decoupling problem by state feedback is defined as follows.

Definition 3. The *disturbance decoupling problem* by state feedback for the system (9) amounts to finding a state feedback of the form $u = Kx$ such that the resulting closed loop system

$$\dot{x}(t) = (A + BK)x(t) + Ed(t) \quad (10a)$$

$$y(t) = Cx(t) \quad (10b)$$

is disturbance decoupled. Moreover, if such a state feedback exists, then we say the disturbance decoupling problem for system (9) is *solvable*.

Necessary and sufficient conditions for solvability of disturbance decoupling problem are among the classical results of the geometric approach. In order to state these classical results, we need to review a few more notions of geometric approach. We say a subspace $\mathcal{V} \subseteq \mathbb{R}^n$ is *controlled invariant* for the pair (A, B) if there exists K such that $(A + BK)\mathcal{V} \subseteq \mathcal{V}$. Moreover, we have

$$\mathcal{V} \text{ is controlled invariant for } (A, B) \Leftrightarrow A\mathcal{V} \subseteq \mathcal{V} + \text{im}\, B. \quad (11)$$

For the pair (A, B), we denote the set of all controlled invariant subspaces which are contained in ker C by $\mathcal{V}(A, B, C)$. Let $\mathcal{V}^*(A, B, C)$ denote the maximal element of the set $\mathcal{V}(A, B, C)$ with respect to the partial order induced by the subspace inclusion, that is $\mathcal{V} \subseteq \mathcal{V}^*(A, B, C)$ for all $\mathcal{V} \in \mathcal{V}(A, B, C)$. The existence and uniqueness of such an element immediately follow from finite-dimensionality. It is well-known that $\mathcal{V}^*(A, B, C) \in \mathcal{V}(A, B, C)$. Now, the following lemma states the necessary and sufficient condition for the solvability of the disturbance decoupling problem for system (10).

Lemma 4. *Considering the system (10), the following statements are equivalent:*

(1) *The disturbance decoupling problem for system (10) is solvable.*
(2) *There exists a controlled invariant subspace $\mathcal{V}$ for the pair (A, B) such that* $\text{im}\, E \subseteq \mathcal{V} \subseteq \ker C$.
(3) *The inclusion* $\text{im}\, E \subseteq \mathcal{V}^*(A, B, C)$ *holds.*

2.3. Problem formulation

In this subsection, we formally state the disturbance decoupling problem for multi-agent systems, which we will study in this paper. Now, recall the multi-agent system (4) together with the output (7). Similar to Section 2.2, we first consider the open loop case where no external control input is applied to the agents. Consequently, we propose the following problem.

Problem 5. Consider the input/state/output system given by

$$\dot{x}(t) = -Lx(t) + Sd(t) \quad (12a)$$

$$y(t) = R^\top x(t) \quad (12b)$$

where the matrices L, S, and R are defined as before. Determine (topological) conditions under which the system (12) is disturbance decoupled.

Note that if the system (12) is disturbance decoupled, then we have $y^{x_0,d}(t) = y^{x_0,0}(t)$ for any given initial state x_0 and locally integrable disturbance d. This means that the output trajectories of the system, which correspond to relative information of certain states of the system, are not influenced by disturbance signals in this case.

This problem together with the proposed solutions will be discussed in detail in Sections 4.1 and 5. In case the system (12) is not disturbance decoupled, similar to the idea in Section 2.2, we investigate the possibility of rendering the system disturbance decoupled by choosing some agents as leaders and apply appropriate inputs to these agents. This leads us to the following problem.

Problem 6. Consider the input/state/output system given by

$$\dot{x}(t) = -Lx(t) + Mu(t) + Sd(t) \quad (13a)$$

$$y(t) = R^\top x(t) \quad (13b)$$

where the matrices L, M, S, and R are defined as before. Determine (topological) conditions under which the disturbance decoupling problem for system (13) is solvable in the sense of Definition 3.

The corresponding results and discussions for Problem 6 are provided in Sections 4.2 and 5.

3. Graph partitions and almost equitability with respect to a cell

Before discussing solutions for the aforementioned problems, in this section we review some notions from graph theory, including graph partitions and, in particular, almost equitable partitions. Moreover, we define an extended version of almost equitability. This new notion together with the results established in this section provides the main foundation for the subsequent results which will be developed in Section 4.

Let $G=(V,E)$ be a simple (unweighted) directed graph where $V=\{1,2,\dots,n\}$, $E\subseteq V\times V$, and $(i,i)\notin E$. By $L(G)$, we denote the in-degree Laplacian of G (see [8, p. 26]). We simply use L to denote the Laplacian matrix when the underlying graph is clear from the context.

We call any subset of V a *cell* of V. We call a collection of cells, given by $\rho=\{C_1,C_2,\dots,C_k\}$, a *partial partition* of V if $C_i\cap C_j=\varnothing$ whenever $i\neq j$. In addition, we call ρ a *partition* of V if it is a partial partition and $\cup_i C_i=V$. In some occasions, to clarify the underlying graph we say ρ is a (partial) partition of $G=(V,E)$, or shortly G, meaning that ρ is a (partial) partition of V.

For a cell $C\subseteq V$, we define the *characteristic vector* of C as

$$p_i(C)=\begin{cases}1 & \text{if } i\in C\\ 0 & \text{otherwise.}\end{cases} \tag{14}$$

For a partial partition $\rho=\{C_1,C_2,\dots,C_k\}$, we define the *characteristic matrix* of π as $P(\rho)=\begin{bmatrix}p(C_1) & p(C_2) & \cdots & p(C_k)\end{bmatrix}$.

Finally, the notion of partial ordering for partitions is defined as follows. We say that a partition π_1 is *finer* than another partition π_2, or alternatively π_2 is *coarser* than π_1, if each cell of π_1 is a subset of some cell of π_2 and we write $\pi_1\leqslant\pi_2$. Also we write as $\pi_1\nleq\pi_2$ meaning that π_1 is not finer than π_2. It is a direct consequence of the definition that

$$\pi_1\leqslant\pi_2 \iff \operatorname{im}P(\pi_2)\subseteq\operatorname{im}P(\pi_1). \tag{15}$$

3.1. Almost equitable partitions

Here, we adopt the notion of almost equitability (see e.g. [17]) for directed graphs. For a given cell $C\subseteq V$, we write

$$N(j,C)=\{i\in C:(i,j)\in E\}.$$

We call a partition $\pi=\{C_1,C_2,\dots,C_k\}$ an *almost equitable partition* (AEP) of G if for each $i,j\in\{1,2,\dots,k\}$ with $i\neq j$ there exists an integer d_{ij} such that $|N(v,C_j)|=d_{ij}$ for all $v\in C_i$, where $|N|$ denotes the cardinality of N.

For a given matrix A, we denote its (i,j)th element by A_{ij}. Then, associated to an almost equitable partition $\pi=\{C_1,C_2,\dots,C_k\}$, we define the matrix $\mathcal{L}_\pi$ as:

$$(\mathcal{L}_\pi)_{ij}=\begin{cases}-d_{ij} & \text{if } i\neq j\\ s_i & \text{otherwise}\end{cases} \tag{16}$$

where $s_i=\sum_{j\neq i}d_{ij}$.

For undirected graphs, characterization of almost equitable partitions in terms of invariant subspaces has been provided in [17, 12]. In particular, it is shown that a partition is almost equitable if and only if the image of its characteristic matrix is L-invariant. This result can be extended to the case of directed graphs as stated in the following lemma.

Lemma 7. *A partition $\pi=\{C_1,C_2,\dots,C_k\}$ is an AEP of G if and only if $\operatorname{im}P(\pi)$ is L-invariant.*

Proof. See [25, Lem. 7.2.2]. ∎

Note that, if π is an AEP, then based on the proof of Lemma 7 we have

$$LP(\pi)=P(\pi)X \tag{17}$$

for $X=\mathcal{L}_\pi$ where $\mathcal{L}_\pi$ is given by (16). Moreover, $X=\mathcal{L}_\pi$ is the unique solution of (17) as $P(\pi)$ has full column rank.

3.2. Almost equitability with respect to a cell

Next, we define almost equitability with respect to a given cell as follows. Given a cell C and a partition $\pi=\{C_1,C_2,\dots,C_k\}$, we call π an AEP with respect to C if for each $i,j\in\{1,2,\dots,k\}$ with $i\neq j$ there exists an integer d_{ij} such that $|N(v,c_j)|=d_{ij}$ for all $v\in C_i\backslash C$.

Observe that if π is an AEP, then the number of neighbors that a vertex in C_i has in C_j is independent of the choice of the vertex in C_i, for all $i,j\in\{1,2,\dots,k\}$ with $i\neq j$. The notion of almost equitability with respect to a cell C is obtained by exempting the nodes in C from satisfying neighborhood constraints of the ordinary almost equitability. Clearly, π is an AEP of G if and only if it is an AEP with respect to the empty cell. Moreover, if π is an AEP of G, then it is an AEP with respect to any arbitrary cell of V.

Note that if π is an almost equitable partition with respect to C, then, similar to ordinary almost equitable partitions, we can define the matrix $\mathcal{L}_\pi$ as in (16). In this case we use the notation $\mathcal{L}_\pi^C$ to distinguish with the case of ordinary almost equitability.

Our aim, here, is to characterize the property of almost equitability with respect to a cell. To state this characterization, we need some additional notation and auxiliary results.

For a given matrix $M\in\mathbb{R}^{m\times m}$, let M^α with $\alpha\subseteq\{1,2,\dots,m\}$ denote the submatrix of M obtained by collecting the rows of M indexed by α. Then, the following result holds.

Lemma 8. *A partition π is an AEP with respect to cell C if and only if*

$$L^\alpha\operatorname{im}P(\pi)\subseteq\operatorname{im}P^\alpha(\pi) \tag{18}$$

where $\alpha=V\backslash C$.

Proof. The proof is analogous to that of Lemma 7 by restricting the rows of L and P to those indexed by α. ∎

Note that, if π is an AEP with respect to C, we have

$$L^\alpha P(\pi)=P^\alpha(\pi)\mathcal{L}_\pi^C \tag{19}$$

where $\alpha=V\backslash C$, and $\mathcal{L}_\pi^C$ is given by (16) with the elements d_{ij} obtained from the definition of almost equitability with respect to C. Now, we have the following characterization for almost equitability with respect to a cell.

Theorem 9. *Let $C=\{\ell_1,\ell_2,\dots,\ell_m\}$ be a cell of V and $\pi=\{C_1,C_2,\dots,C_k\}$ be a partition of G. Then the following statements are equivalent:*

(1) *The partition π is an AEP of G with respect to C.*
(2) *$L\operatorname{im}P(\pi)\subseteq\operatorname{im}P(\pi)+\operatorname{im}P(\rho)$ where $\rho=\{\{\ell_1\},\{\ell_2\},\dots,\{\ell_m\}\}$.*
(3) *There exists a simple (unweighted) directed graph $H=(V,F)$ obtained from $G=(V,E)$ by adding some non-existing or removing some existing arcs from a vertex in V to a vertex in C such that π is an almost equitable partition of H.*

Proof. First we show that the first two statements are equivalent. It is easy to observe that the second statement is equivalent to:

$$\begin{bmatrix}L_{\bar\alpha}\\ L^\alpha\end{bmatrix}\operatorname{im}P(\pi)\subseteq\operatorname{im}\begin{bmatrix}P^{\bar\alpha}(\pi)\\ P^\alpha(\pi)\end{bmatrix}+\operatorname{im}\begin{bmatrix}I_m\\ 0\end{bmatrix}, \tag{20}$$

where $\alpha=V\backslash C$ and $\bar\alpha=C$. This holds if and only if $L^\alpha\operatorname{im}P(\pi)\subseteq\operatorname{im}P^\alpha(\pi)$, which is equivalent to almost equitability of π with respect to C by Lemma 8.

Now, by assuming that the first two statements hold, we prove the third statement as follows. Since π is an AEP with respect to C, the equality (19) holds. Let the matrices X and Y be defined as $X = \mathcal{L}^C_\pi$ and $Y = L^{\bar\alpha}P(\pi) - P^{\bar\alpha}(\pi)\mathcal{L}^C_\pi$. Then, clearly, we have

$$\begin{bmatrix} L^{\bar\alpha} \\ L^{\alpha} \end{bmatrix} P(\pi) = \begin{bmatrix} P^{\bar\alpha}(\pi) \\ P^{\alpha}(\pi) \end{bmatrix} X + \begin{bmatrix} I_m \\ 0 \end{bmatrix} Y.$$

Now, for each $i = \{1, 2, \ldots, m\}$, let r_i be an integer such that $\ell_i \in C_{r_i}$. Then, it is easy to observe that the matrix Y is obtained as

$$Y_{ij} = -|N(\ell_i, C_j)| - (\mathcal{L}^C_\pi)_{r_i j} \tag{21}$$

for each $i \in \{1, 2, \ldots, m\}$, $j = \{1, 2, \ldots, k\}$, and $j \neq r_i$. The remaining m elements of Y are such that $Y\mathbb{1} = 0$. By (16), the equality (21) can be rewritten as

$$Y_{ij} = -|N(\ell_i, C_j)| + d_{r_i j}$$

where $d_{r_i j}$ are obtained from the definition of almost equitability with respect to C.

Now, we construct the graph $H = (V, F)$ by adding some non-existing arcs or removing some existing arcs of G as follows. For each $i \in \{1, 2, \ldots, m\}$ and $j = \{1, 2, \ldots, k\}$, we add a total number of Y_{ij} arcs from some available nodes in C_j to ℓ_i if $Y_{ij} > 0$. Note that multiple arcs between two vertices is not allowed. This is always possible since $d_{r_i j} \leqslant |C_j|$, and hence $Y_{ij} \leqslant |C_j| - |N(\ell_i, C_j)|$. Similarly, if $Y_{ij} < 0$, we remove a total number of $|Y_{ij}|$ existing arcs which are from some nodes in C_j to ℓ_i. This is also always implementable, as $-Y_{ij} \leqslant |N(\ell, C_j)|$. Denoting the arc set obtained in this way by F, it is easy to observe that the partition π is an AEP of $H = (V, F)$ by construction.

It remains to show that the third statement implies either of the other two. Assume that there exists a simple graph $H = (V, F)$ obtained from $G = (V, E)$ by adding some non-existing or removing some existing arcs from some vertices in V to vertices in C such that π is an almost equitable partition of H. Let $L(H)$ denote the Laplacian matrix of H. Then, by Lemma 7, we have

$$L(H)P(\pi) = P(\pi)X \tag{22}$$

for some matrix X. Hence, $L^\alpha(H)P(\pi) = P^\alpha(\pi)X$ for $\alpha = V\backslash C$. Now, since the head of all arcs which are added or removed from G are all in C, we have $L^\alpha(H) = L^\alpha(G)$. Consequently, π is an AEP of G with respect to C by Lemma 8. ∎

4. Solution to the disturbance decoupling problem of multi-agent systems

In this section, we propose solutions for Problems 5 and 6. Recall that $V_D = \{w_1, w_2, \ldots, w_r\}$. We assume without loss of generality that leaders are not affected by the disturbance signals, i.e. $V_L \cap V_D = \varnothing$. Indeed, it is easy to show that if $V_L \cap V_D$ is nonempty then one can redefine the leader set as $V'_L = V_L \backslash (V_L \cap V_D)$, and solve the DDP with respect to the leader set V'_L. Now, let the partition π_S of V be defined as

$$\pi_S = \{\{w_1\}, \{w_2\}, \ldots, \{w_r\}, V \setminus V_D\}. \tag{23}$$

Obviously, we have

$$\operatorname{im} S \subseteq \operatorname{im} P(\pi_S). \tag{24}$$

Moreover, it is easy to observe that there exists a partition of G, say π_R such that

$$\operatorname{im} P(\pi_R) = \ker R^\top. \tag{25}$$

Then, the following result holds.

Lemma 10. *The multi-agent system (12) is disturbance decoupled only if $\pi_R \leqslant \pi_S$. Similarly, the disturbance decoupling problem for system (13) is solvable only if $\pi_R \leqslant \pi_S$ holds.*

Proof. Suppose that the system (12) is disturbance decoupled, or the DDP for system (13) is solvable. Then by Lemmas 2 and 4, it follows that $\operatorname{im} S \subseteq \ker R^\top = \operatorname{im} P(\pi_R)$. Hence, by (23) and the structure of S in (6), we have $\pi_R \leqslant \pi_S$. ∎

Next, we discuss the open loop and the closed loop disturbance decoupling, and establish sufficient conditions for Problems 5 and 6.

4.1. Open loop disturbance decoupling

The following theorem gives a sufficient (topological) condition for multi-agent system (12) to be disturbance decoupled.

Theorem 11. *Let π_S and π_R be given by (23) and (25), respectively. Then the multi-agent system (12) is disturbance decoupled if there exists a partition π such that both of the following conditions hold:*

(1) *π is an AEP of G*
(2) *$\pi_R \leqslant \pi \leqslant \pi_S$.*

Proof. Suppose that conditions 1 and 2 hold. Then, by Lemma 7 and (15), we obtain that $\operatorname{im} P(\pi)$ is L-invariant and $\operatorname{im} S \subseteq \operatorname{im} P(\pi) \subseteq \ker R^\top$. Hence, it follows from Lemma 2 that (12) is disturbance decoupled. ∎

Based on Theorem 11, the DDP for (12) is solvable if there exists an almost equitable partition which is finer than π_S and coarser than π_R. In principle, this requires searching for all almost equitable partitions of G to find one which satisfy the partial ordering constraint of Theorem 11. However, similar to the idea of largest/smallest invariant subspaces for ordinary linear systems (see Section 2.2), one may try to find a partition, say π^*, which is extremal in certain sense. Then, providing that such a partition exists and can be efficiently computed, disturbance decoupling of (12) can be guaranteed upon a satisfaction of a single and easily verifiable condition. This will be discussed in Section 5.

4.2. Closed loop disturbance decoupling

Now suppose that the DDP is not solvable for (12). Then, similar to the case of general linear systems in Section 2.2, one may try to make the system (12) disturbance decoupled by applying a control input. This brings us to Problem 6, the solution of which is discussed in this subsection.

Recall the notion of controlled invariant subspaces in Section 2.2. As we are dealing with graph topological conditions, we are not interested in all subspaces but only those which can be written as an image of a partition. As observed in the previous subsection, almost equitable partitions corresponds to L-invariant subspaces. Now, the following Lemma establishes the relationship between almost equitability with respect to a cell and controlled invariance of the pair (L, M).

Lemma 12. *For a given graph G, let V_L, M, and L be defined as before. Let π be a partition of G. Then $\operatorname{im} P(\pi)$ is controlled invariant for the pair (L, M) if and only if π is an almost equitable partition with respect to V_L.*

Proof. Recall that $V_L = \{\ell_1, \ell_2, \ldots, \ell_m\}$. Note that $\operatorname{im} M = \operatorname{im} P(\rho)$ where $\rho = \{\{\ell_1\}, \{\ell_2\}, \ldots, \{\ell_m\}\}$. The result now immediately follows by using (11) together with Theorem 9. ∎

Now we are at the position to apply the results of Section 3.2 to disturbance decoupling problem of multi-agent system (13). This is discussed in the following theorem.

Theorem 13. *Let V_L, π_R, and π_S be defined as before. Then the disturbance decoupling problem for multi-agent system (13) is solvable* *if there exists a partition π of G such that both of the following conditions hold:*

(1) *π is almost equitable with respect to V_L*
(2) *$\pi_R \leqslant \pi \leqslant \pi_S$.*

Proof. Suppose that the conditions 1 and 2 hold. Then, by Lemma 12, im $P(\pi)$ is controlled invariant for the pair (L, M). Moreover, we have im $P(\pi_S) \subseteq$ im $P(\pi) \subseteq$ im $P(\pi_R)$. Hence, by (24) and (25), we obtain that im $S \subseteq$ im $P(\pi) \subseteq \ker R^\top$. Consequently, the DDP for system (13) is solvable by Lemma 4. ■

Suppose that the conditions of Theorem 13 hold, and hence the DDP for system (13) is solvable. This means that there exists a state feedback $u(t) = Kx(t)$ such that the resulting closed loop system is disturbance decoupled. To see the structure of the controller, note that π is an AEP of $G = (V, E)$ with respect to the leader set V_L. Hence, π is an AEP of $H = (V, F)$ where H is obtained from G by adding or removing arcs from vertices in V to vertices in V_L, as discussed in the proof of Theorem 9. These adding and removing of the arcs are indeed associated with the state feedback controller which makes the system (13) disturbance decoupled. In particular, it is easy to observe that adding an arc from a vertex in V, say i, to a vertex in V_L, say ℓ_k, corresponds to the control signal $x_i(t) - x_{\ell_k}(t)$ which is to be applied to the leader vertex ℓ_k. Similarly, removing an arc from i to ℓ_k corresponds to the term $x_{\ell_k}(t) - x_i(t)$ in the control signal. Consequently, the controller can be expressed as

$$u_k(t) = \sum_{(j,\ell_k)\in F\setminus E} (x_j(t) - x_{\ell_k}(t)) - \sum_{(j,\ell_k)\in E\setminus F} (x_j(t) - x_{\ell_k}(t)) \qquad (26)$$

for each $k = \{1, 2, \ldots, m\}$. This shows that, as demanded in the context of distributed control, the controller only uses the relative information of the states of the agents to achieve disturbance decoupling for system (13). Observe that, by applying the controller (26) to system (13), we obtain the following input/state/output system:

$$\dot{x}(t) = -L(H)x(t) + Sd(t) \qquad (27a)$$

$$y(t) = R^\top x(t), \qquad (27b)$$

where $L(H)$ denotes the Laplacian matrix of the graph $H = (V, F)$. The system (27) is indeed disturbance decoupled by Theorem 11, as π is an AEP of H and $\pi_R \leqslant \pi \leqslant \pi_S$. This is in accordance with the fact that the DDP for system (13) is solvable.

Remark 14. It is worth mentioning that, in general, a full characterization of geometrical/structural properties of dynamical networks in terms of graph partitions is very unlikely. The reason lies in the gap between invariant subspaces and image of graph partitions. For instance, as illustrated by the example provided in [12, Rem. 2], the controllable subspace of a multi-agent system with Laplacian-based dynamics cannot always be associated to an image of a partition. Likewise, we postulate that the existence of a necessary and sufficient condition for solvability of the disturbance decoupling problem is very unlikely in terms of ordinary graph partitions, and it would require an appropriate extended notion of graph partitions.

5. Algorithms

Let Π denote the set of all partitions of V. With the partial order "$\leqslant$", the set Π becomes a complete lattice [26], meaning that every subset of Π has both its greatest lower bound and least upper bound within Π. We use $\vee\Pi'$ to denote the least upper bound of a subset $\Pi' \in \Pi$. By definition, $\vee\Pi'$ has the following property:

$$\vee\,\Pi' \geqslant \pi, \quad \text{for all } \pi \in \Pi' \qquad (28a)$$

$$\exists \tilde{\pi} \in \Pi \text{ s.t. } \pi \leqslant \tilde{\pi} \quad \text{for all } \pi \in \Pi' \Longrightarrow \vee\Pi' \leqslant \tilde{\pi}. \qquad (28b)$$

Let $\Pi_{\rm AEP}$ denote the set of all almost equitable partitions of G. For a given partition π_0 of G, we define

$$\Pi_{\rm AEP}(\pi_0) = \{\pi \in \Pi_{\rm AEP} : \pi \leqslant \pi_0\}. \qquad (29)$$

Then, the following result holds.

Lemma 15. *Let π_R and π_S be given as before. Then the following two statements are equivalent.*

(1) *There exists a partition $\pi \in \Pi_{AEP}$ such that $\pi_R \leqslant \pi \leqslant \pi_S$*
(2) *$\pi_R \leqslant \vee\Pi_{AEP}(\pi_S)$.*

Proof. Suppose that the first statement holds. Then, $\pi \in \Pi_{\rm AEP}(\pi_S)$. Hence, by definition, $\pi \leqslant \vee\Pi_{\rm AEP}(\pi_S)$ which yields $\pi_R \leqslant \vee\Pi_{\rm AEP}(\pi_S)$.

Conversely, suppose that the second statement holds. Then, obviously $\pi_R \leqslant \vee\Pi_{\rm AEP}(\pi_S) \leqslant \pi_S$. Besides, it is shown in [12] that $\vee\Pi_{\rm AEP}(\pi_0) \in \Pi_{\rm AEP}(\pi_0)$ for any given partition π_0. Therefore, $\vee\Pi_{\rm AEP}(\pi_S)$ serves as a partition satisfying the conditions in Statement 1 of the lemma. ■

By Lemma 15 and Theorem 11, we obtain the following result.

Corollary 16. *The multi-agent system (12) is disturbance decoupled if $\pi_R \leqslant \vee\Pi_{AEP}(\pi_S)$.*

As observed, verification of the conditions provided in Theorem 11 boils down to the computation of $\vee\Pi_{\rm AEP}(\pi_S)$. An algorithm to compute $\vee\Pi_{\rm AEP}(\pi_S)$ is provided in [12].

In Theorem 13, sufficient condition in terms of graph topological conditions have been provided for the multi-agent system (13) to be disturbance decoupled. In principle, one needs to perform an exhaustive search to find a promising partition satisfying the proposed constraints. Hence, it is of interest to have an efficient algorithm to verify the conditions provided in Theorem 13. Again an idea here would be to try to compute an extremal partition, and derive a similar result to that of Corollary 16. However, unfortunately, one can show that the semi-lattice structure of (29) is lost when ordinary almost equitability is replaced by almost equitability with respect to a cell. Therefore, additional assumption/treatments are needed for extending the result of the previous subsection to the case of almost equitability with respect to a cell. It turns out that under an additional assumption on the structure of π_R, namely that each cell of π_R contains at least one follower, i.e. one vertex in V_F, we can construct an extremal partition π^* such that the conditions proposed in Theorem 13 boil down to $\pi_R \leqslant \pi^* \leqslant \pi_S$. We refer the interested reader to [25, Ch. 7] for further details.

6. A numerical example

To illustrate the proposed results, consider the multi-agent system (4) with the communication graph G as shown in Fig. 1 (left). For this system, let black vertices denote the leaders, i.e. $V_L = \{2\}$. Also let the square vertices correspond to the agents affected by disturbance signals, i.e. $V_D = \{3, 5\}$. We are interested in decoupling the outputs $x_1(t) - x_2(t)$ and $x_4(t) - x_6(t)$ from the disturbance. Hence, the output variables in this case are given by $y = R^\top x$ where

$$R^\top = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \end{bmatrix}$$

and $x \in \mathbb{R}^8$. Then π_R and π_S are given by:

$$\pi_R = \{\{1, 2\}, \{3\}, \{4, 6\}, \{5\}, \{7\}, \{8\}\}$$

$$\pi_S = \{\{1, 2, 4, 6, 7, 8\}, \{3\}, \{5\}\}.$$

By [12], the partition $\vee\Pi_{\rm AEP}(\pi_S)$ in Lemma 15 is obtained as

$$\vee\Pi_{\rm AEP}(\pi_S) = \{\{1\}, \{7, 8\}, \{4, 6\}, \{2\}, \{3\}, \{5\}\}.$$

Now since $\pi_R \not\leqslant \vee\Pi_{\rm AEP}(\pi_S)$, the disturbance decoupling of the open loop system (12) is not guaranteed by Corollary 16.

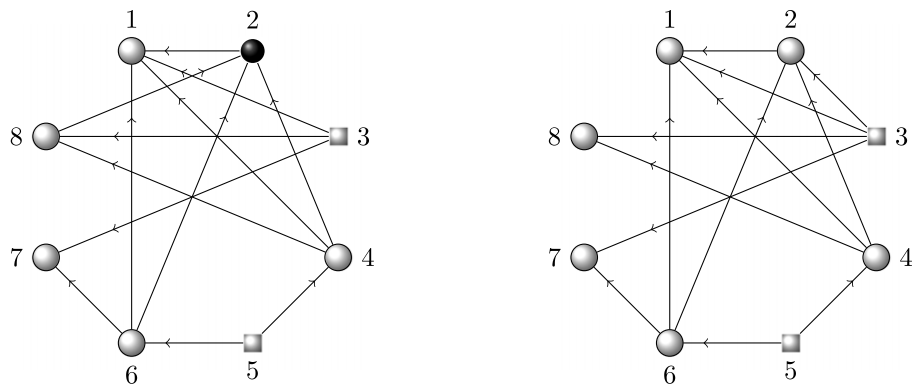

Fig. 1. The simple directed graph G (left) and H (right) of a diffusively coupled multi-agent system.

Next, we consider solvability of the disturbance decoupling problem for this system. Let the partition π^* be given as

$$\pi^* = \{\{1, 2\}, \{7, 8\}, \{4, 6\}, \{3\}, \{5\}\}.$$

It is easy to see that π^* is an almost equitable partition of G with respect to V_L. In fact, π^* becomes almost equitable in the graph H by removing the arc from vertex 8 to 2, and adding an arc from 3 to 2. Now, since $\pi_R \leqslant \pi^* \leqslant \pi_S$, the disturbance decoupling problem in this case is solvable by Theorem 13. Consequently, by (26), the state feedback which renders the system disturbance decoupled is given by

$$u(t) = (x_3(t) - x_2(t)) - (x_8(t) - x_2(t)) = x_3(t) - x_8(t).$$

7. Conclusions

We have studied the disturbance decoupling problem for multi-agent systems. By extending the notion of almost equitability to almost equitability with respect to a cell, an appropriate topological translation for controlled invariant subspaces is provided. We have considered disturbance decoupling of both the open loop and the closed loop system. In the open loop case, we have established sufficient conditions ensuring the system is disturbance decoupled with no input applied to the leaders. In the case of closed loop, we have derived sufficient conditions guaranteeing the solvability of the disturbance decoupling problem. The proposed sufficient conditions are in terms of existence of certain almost equitable partitions with respect to a cell containing leaders. In case the DDP is solvable, an admissible controller rendering the system disturbance decoupled has been provided. As desired, this controller uses the relative information of the states of the agents.

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