Classification of constrained differential equations embedded in the theory of slow fast systems

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A_k singularities and geometric desingularization

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Chapter 1

Introduction

This thesis is dedicated to the study of ***C**onstrained **D**ifferential **E**quations* (CDEs) and of ***S**low **F**ast **S**ystems* (SFSs). A SFS is a singularly perturbed ordinary differential equation of the form

$$\begin{aligned} \dot{x}(t) &= f(x(t), z(t), \varepsilon) \\ \varepsilon \dot{z}(t) &= g(x(t), z(t), \varepsilon), \end{aligned} \tag{1.1}$$

where $x \in \mathbb{R}^m$, $z \in \mathbb{R}^n$; where ε is a small positive parameter, i.e., $0 < \varepsilon \ll 1$, and where we assume that the functions f and g are of class $\mathcal{C}^\infty$ (all partial derivatives exist and are continuous)[1] . The over dot denotes the derivative with respect to the time parameter t.

In some applications (see e.g. section 1.1), $z(t)$ represents states or measurable quantities of a process while $x(t)$ denotes control parameters. The small parameter ε models the difference of the rates of change between z and x, and that is why systems like (1.1) are often used to model phenomena with two time scales. Observe that the smaller ε is, the faster z evolves with respect to x. Therefore x (resp. z) is called the *slow* (resp. *fast*) variable. The time parameter t is named as the *slow time* and then (1.1) is known as *the slow equation*. Whenever $\varepsilon \neq 0$, a new time parameter τ is defined by $\tau = \frac{1}{\varepsilon}t$. With this new time parametrization (1.1) becomes

$$\begin{aligned} x'(\tau) &= \varepsilon f(x(\tau), z(\tau), \varepsilon) \\ z'(\tau) &= g(x(\tau), z(\tau), \varepsilon), \end{aligned} \tag{1.2}$$

[1] Along this document, smooth means of class $\mathcal{C}^\infty$. Whenever we refer to a finite class of differentiability (in a well chosen neighborhood of a point) we shall write $\mathcal{C}^\ell$ instead, with $\ell < \infty$ but as large as necessary.

where the prime denotes the derivative with respect to τ. In contrast to (1.1), the system (1.2) is known as *the fast equation*. We remark that (1.2) is not a singular perturbation problem anymore, but is an ε-parameter family of smooth vector fields. Since only autonomous systems are considered, we omit to indicate the time dependence of the variables. Observe that as long as $\varepsilon \neq 0$ and f is not identically zero, the systems (1.1) and (1.2) are equivalent up to time re-parametrization.

A good strategy to understand the qualitative behavior of SFSs is to study (1.1) and (1.2) in the limit $\varepsilon \to 0$. Accordingly, when $\varepsilon = 0$, the slow equation (1.1) becomes

$$\begin{aligned} \dot{x} &= f(x,z,0) \\ 0 &= g(x,z,0). \end{aligned} \tag{1.3}$$

A system of the form (1.3) is called *constrained differential equation* (CDE), see chapter 2 where a detailed description of CDEs is provided. An important geometric object to consider is "the set of constraints". To be more precise, the set S, defined as

$$S = \{(x,z) \in \mathbb{R}^m \times \mathbb{R}^n \,|\, g(x,z,0) = 0\}$$

serves as the phase space of (1.3). For generic CDEs (by generic we mean "almost all", but we formalize this notion in chapter 2) this set is an m-dimensional smooth manifold and is called *the constraint manifold* [39]. Note that if the matrix $\frac{\partial g(x,z,0)}{\partial z}$ is regular on S, then S can be expressed as a graph $z = h(x)$. Then, under this condition, the motion along the constraint manifold S is given by

$$\dot{x} = f(x,h(x),0).$$

On the other hand, when $\varepsilon = 0$ the fast equation (1.2) becomes

$$\begin{aligned} x' &= 0 \\ z' &= g(x,z,0), \end{aligned} \tag{1.4}$$

which is called *the layer equation* [46]. In this case, the constraint manifold S serves as the set of equilibrium points of (1.4). A point $p \in S$ which is a hyperbolic equilibrium point of (1.4) is called *regular*; and it is called *singular* otherwise. A qualitative description of the dynamics of a SFS can be obtained from the CDE and from the layer equation as follows.

Qualitative description. The constrained differential equation (1.3) is an approximation of the slow dynamics (1.1), while the layer equation (1.4) approximates the fast dynamics (1.2). Far away from the manifold S, the trajectories of the SFS (1.2) closely follow those of the layer equation (1.4). Observe that, due to $x' = 0$, the phase space of the layer equation is a family of n-dimensional planes parametrized by $x \equiv$ constant. Such a family is called *the fast foliation*. On the other hand, sufficiently close to S, the trajectories of (1.1) closely follow those of the CDE (1.3). The trajectories of (1.3) have to satisfy the constraint $g = 0$.

Suppose that an initial condition (x_0, z_0) does not lie in S. In that case there is an infinitely fast transition towards a regular point of S (either in forward or in backward time) along the fast foliation according to (1.4). If the trajectory arrives at S, then its dynamics are now governed by the CDE (1.3) along the manifold S. If a trajectory of the CDE arrives at a singular point, then such a trajectory follows again the layer equation either indefinitely or until arriving again at a regular point of S. This description is shown in fig. 1.1. In the present document, we are interested in studying the flow of slow fast systems near a singular point as described above. That is, in the situation where a sudden change of the trajectories' behavior occurs.

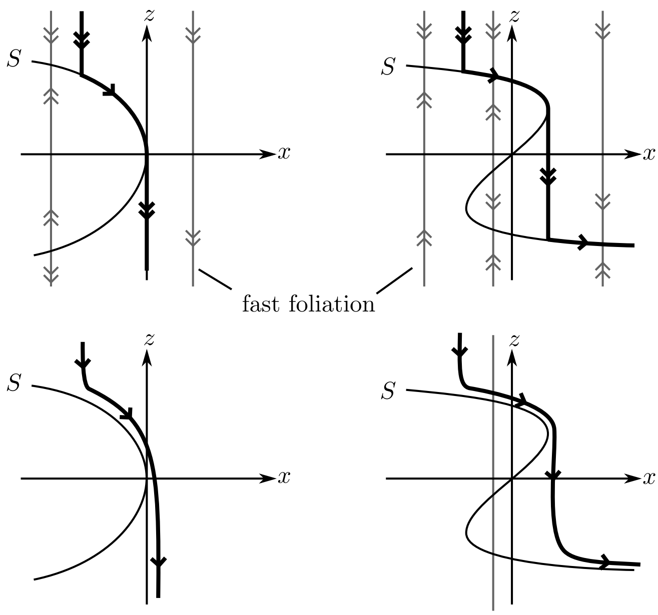


Figure 1.1: Qualitative description of the dynamics of a slow fast system from the corresponding constrained differential equation and layer equation. The manifold S is defined by a constraint $g(x, z) = 0$. Far away from S, the flow goes along the fast foliation according to the layer equation. Near S, the flow follows the constrained dynamics of the CDE. Note that depending on the geometry of the manifold S, there may be sudden changes on the flow's behavior.

Constrained differential equations were thoroughly studied by Takens [39]. One of his main results in this context, is the topological[2] classification of generic CDEs with two or less parameters. In such a classification, the following argument becomes important.

Remark 1.0.1. *We assume that the vector field $z' = g(x,z,0)$ is conservative. This means that there exists an m-parameter family of functions $V_x : \mathbb{R}^n \to \mathbb{R}$ such that*

$$g_0(x,z) = \frac{\partial V_x}{\partial z}(z),$$

where $g_0(x,z) = g(x,z,0)$. This allows us to consider CDEs of the form

$$\begin{aligned} \dot{x} &= f(x,z,0) \\ 0 &= \frac{\partial V}{\partial z}(x,z), \end{aligned}$$

where $V : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$ is a smooth function with $V_x(z) = V(x,z)$, and thus the constraint manifold is the critical set of the family V_x. Accordingly, the classification of critical points of smooth functions [3, 4, 10] shall play an important role in our studies. Recall that for few parameters, i.e. $m \leq 4$, such a classification yields a finite number of equivalence classes, where the preferred representatives are called elementary catastrophes, see appendix A.1.

Motivated by Takens's studies [39], our first contribution is a classification of generic CDEs where the potential function V is related to singularities of type A_k, D_4^+, and D_4^- [3, 4]. Once this classification is obtained, the next natural question is to study SFSs given as ε-perturbation of these generic CDEs. More specifically, we study SFS given by

$$\begin{aligned} \dot{x} &= f(x,z,0) + \varepsilon \tilde{f}(x,z,\varepsilon) \\ \varepsilon \dot{z} &= \frac{\partial V}{\partial z} + \varepsilon \tilde{g}(x,z,\varepsilon), \end{aligned}$$

where $\tilde{f}$ and $\tilde{g}$ are smooth maps and where the corresponding CDE is a member of the aforementioned classification.

To motivate our studies, let us now recall two classical examples of biological phenomena which were modeled by a SFS and where elementary catastrophe theory plays an important role.

[2]Two CDEs are topologically equivalent if there exists a continuous function that maps solutions of one into solutions of the other, see definition 2.1.9.

1.1 Zeeman's Examples

In this section we review two classical examples of natural phenomena modeled by a slow fast system and where elementary catastrophe theory plays an important role. These applications were studied by Zeeman [48]. His interest for using this theory was that it enables a qualitative description of the local dynamics of a biological system instead of modeling the complicated biochemical processes involved.

1.1.1 Zeeman's heartbeat model

The simplified heart is considered to have two (measurable) states. The *diastole* which corresponds to a relaxed state of the heart's muscle fiber, and *systole* which stands for the contracted state. When a heart stops beating it does so in relaxed state, an equilibrium state. There is an electrochemical wave that makes the heart contract into systole. When such a wave reaches a certain threshold, it triggers a sudden contraction of the heart fibers: a catastrophe occurs. After this, the heart remains in systole for a certain amount of time (larger in comparison to the contraction-relaxation time) and then rapidly returns to diastole. A mathematical local representation of the behavior just explained is given by

$$\begin{aligned} \dot{x}_1 &= z - z^*, \\ \varepsilon \dot{z} &= -\left(z^3 - z + x_1\right) \end{aligned} \tag{1.5}$$

where $x_1,\ z \in \mathbb{R}$. Observe the similarity of (1.5) with the Van der Pol oscillator with small damping [43]. The variable z models the length of the muscle fiber, x_1 corresponds to an electrochemical control variable and $z^* > \frac{1}{\sqrt{3}}$ represents the threshold. In the limit $\varepsilon = 0$ we obtain the CDE

$$\begin{aligned} \dot{x}_1 &= z - z^*, \\ 0 &= -\left(z^3 - z + x_1\right) \end{aligned} \tag{1.6}$$

The critical points of the potential function $V(x_1, z) = \frac{1}{4}z^4 - \frac{1}{2}z + x_1 z$ (compare with the cusp catastrophe in appendix A.1) form the S-shaped constraint manifold

$$S_V = \left\{(x_1, z) \in \mathbb{R} \times \mathbb{R} \mid z^3 - z + x_1 = 0\right\}.$$

Observe that there are two singular points of S_V defining the singularity set

$$\begin{aligned} B &= \left\{(x_1, z) \in S_V \,|\, 3z^2 - 1 = 0\right\} \\ &= \left\{\left(\frac{2}{3\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \left(-\frac{2}{3\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)\right\}. \end{aligned}$$

At points of B, the solution of (1.6) suffer a sudden change of behavior, a jump occurs. A schematic of the dynamics of (1.6) is shown in fig. 1.2

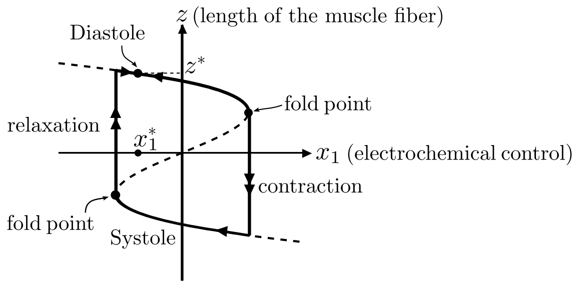


Figure 1.2: Dynamics of the simplified heartbeat model (1.6) in the limit $\varepsilon = 0$. In this limit the S-shaped constrained manifold S_V is the phase space of the solutions. The singularity set B is the union of the two fold points. The interpretation is as follows: a pacemaker controls the value of the parameter x_1 changing its value from x_1^* up to an adequate threshold, beyond the fold point, such that the action of contraction is triggered. Then, the heart remains in a contracted state for a while until the value of x_1 triggers the relaxation action. Contractions and relaxations are modeled by a fast transition between the two stable branches of the curve S_V.

For sufficiently small $\varepsilon > 0$, the trajectories of (1.5) are close to those of (1.6). See fig. 1.3. It is one of the goals of the theory of slow fast systems to make precise the notion of closeness mentioned above, especially in the neighborhood of singular points (see chapter 3 and e.g., [13, 16, 27]).

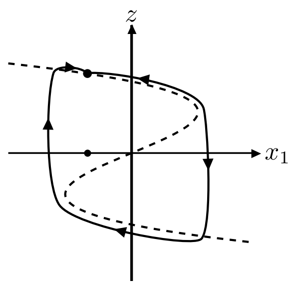


Figure 1.3: Dynamics of the simplified heartbeat model (1.5) for sufficiently small ε. The trajectory shown is a small perturbation of the solution curve of the corresponding CDE in (1.6).

1.1.2 Zeeman's nerve impulse model

This model qualitatively describes the local and simplified behavior of a neuron when transmitting information through its axon, see [48] for details and compare also with the Hodgkin-Huxley model [20]. Qualitatively speaking, there are three important components in this process: a) the concentration of sodium (Na) around the walls of the axon, b) the concentration of potassium (K) around the walls of the axon, and c) the electric potential, or voltage (V), around the wall of the axon. As information is being transmitted, through an electric signal, there is a slow and smooth change of the voltage and of the concentration of potassium but a rather sudden change in the concentration of sodium. Another local characteristic is that the return to the equilibrium state, when there is no transmission of information, is slow and smooth. The three variables aforementioned behave qualitatively as shown in fig. 1.4.

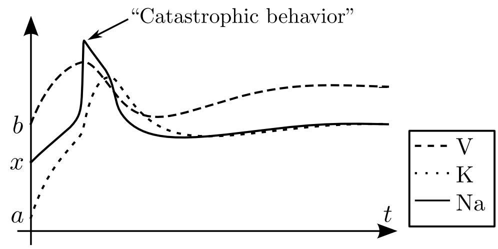


Figure 1.4: A qualitative picture of the three variables involved in the local model of the nerve impulse [48]. The signal V represents the potential around the axon's wall. The signals of Na and K represent the conductance of sodium and potassium respectively. Observe that a characteristic property is the sudden and rapid change of the sodium conductance followed by a smooth and slow return to its equilibrium state. See [48], where a qualitatively similar graph is plotted from measured data.

A mathematical model that roughly describes the nerve impulse process is given by

$$\begin{aligned} \dot{a} &= -1 - b \\ \dot{b} &= -2(b + z) \\ \varepsilon \dot{z} &= -(z^3 + bz + a), \end{aligned}$$

where a, b, and z stand for the concentration of potassium, the voltage and the concentration of sodium respectively [48]. The corresponding constrained differential equation reads as

$$\begin{aligned} \dot{a} &= -1 - b \\ \dot{b} &= -2(b + z) \\ 0 &= -(z^3 + bz + a). \end{aligned} \tag{1.7}$$

The defining potential function is given by the cusp catastrophe $V = \frac{1}{4}z^4 + \frac{1}{2}bz^2 + az$. The constraint manifold is defined as

$$S_V = \left\{(a, b, z) \in \mathbb{R}^2 \times \mathbb{R} \mid z^3 + bz + a = 0\right\},$$

and is the critical set of the 2-parameter family $V_{(a,b)}(z) = V(a, b, z)$. Recall that S_V serves as the phase space of the flow of (1.7). The singular set, points of S_V at which S_V is tangent to the z-direction, is defined by

$$B = \left\{(b, z) \in S_V \,|\, 3z^2 + b = 0\right\}.$$

The attracting part of the manifold S_V, denoted by $S_{V,min}$, is given by points where $D_z^2 V > 0$, i.e. by

$$S_{V,min} = \left\{(a, b, z) \in S_V \,|\, 3z^2 + b > 0\right\}.$$

Let us now describe a procedure under which a smooth vector field, whose solutions are equivalent (in a sense specified below) to the solutions of (1.7), can be obtained. Let π denote the projection $\pi : (a, b, z) \mapsto (a, b)$. Note that $\pi|_{S_V \setminus B}$ is a local diffeomorphism, which means that we can also project solutions curves on S_V onto the parameter space. To be more precise, if we restrict the coordinates to S_V, the projection $\pi|_{S_V}$ reads as

$$(a, b, z) \mapsto (-z^3 - bz, b),$$

which allows us to rewrite (1.7) as the planar system

$$\begin{aligned} \dot{b} &= -2(b + z) \\ \dot{z} &= \frac{1 + b + 2(b + z)z}{3z^2 + b}. \end{aligned} \tag{1.8}$$

The vector field (1.8) is not smooth everywhere; in particular, it is not well defined at the singular set $B = \left\{3z^2 + b = 0\right\}$. However outside B, the function $h(z, b) = 3z^2 + b$ is nonzero and smooth. Furthermore, $h(z, b) > 0$ for all points in $S_{V,min}$. Therefore, we can multiply (1.8) by h. Thus, whenever $h > 0$, we have that (1.8) is smoothly equivalent[3] to

$$\begin{aligned} \dot{b} &= -2(3z^2 + b)(b + z) \\ \dot{z} &= 1 + b + 2(b + z)z. \end{aligned} \tag{1.9}$$

The vector field (1.9) is called *the desingularized vector field*. Note that (1.9) is smooth and is defined for all $(b, z) \in \mathbb{R}^2$. The importance of (1.9) lies in the fact that we obtain the solutions of the CDE (1.7) from the integral curves of (1.9). The general reduction process through which we obtain the desingularized vector field is described in section 2.2.

It is straightforward to show that (1.9) has equilibrium points (b, z) as follows:

- $p_a = (-1, 1) \in S_{V,min}$, which is a regular equilibrium point,
- $p_f = \left(-\frac{3}{4}, \frac{1}{2}\right) \in B$, which receives the name folded singularity.

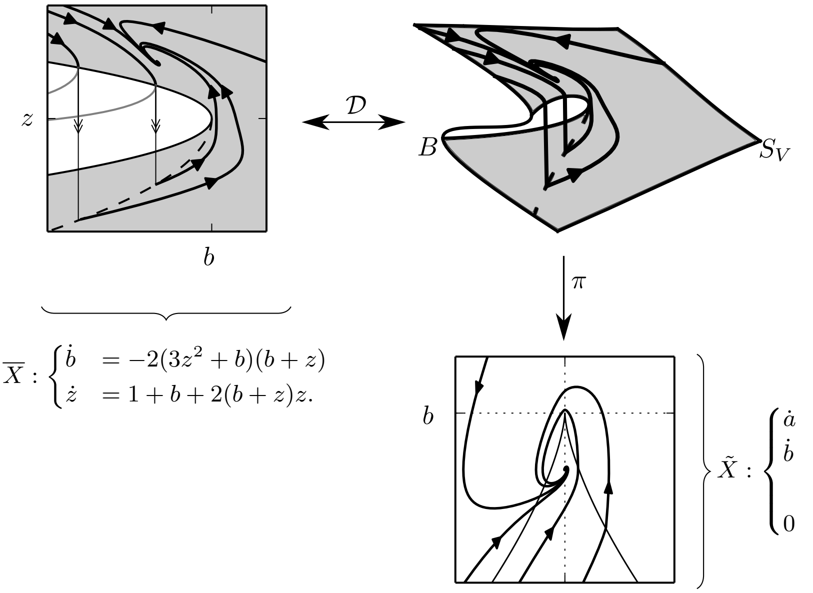
Furthermore, p_f is a saddle point, whence it is called a folded-saddle singularity. Refer to fig. 1.5 for the phase portrait of (1.9) and note the smooth return of some trajectories and compare with the heartbeat model where this effect does not occur.

Once (1.9) is better understood, we can give a qualitative picture of the flow of (1.7) taking back the operations performed above. We show in fig. 1.5 the phase portraits of desingularized vector field (1.9) and of the CDE (1.7).

Remark 1.1.1. *Figure 1.5 graphically shows all the important elements in the theory of constrained differential equations.*

- *The constraint manifold S_V is the phase space of the CDE.*
- *The map $\pi : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$ is a projection from the total space onto the parameter space. The vector field induced in this space is denoted by $\tilde{X}$.*
- *The smooth vector field $\overline{X}$ is obtained by desingularization, which is denoted by $\mathcal{D}$. In the previous example such a process is as follows: first a restriction of the coordinates to S_V allows the change of coordinates $a = -z^3 - bz$. Then the projection π restricted to S_V is performed, that is $(a, b, z)|_{S_V} = (-z^3 - bz, b, z) \mapsto (-z^3 - bz, b)$. By a smooth re-parametrization the smooth vector field $\overline{X}$ is obtained. Observe that for points in $S_{V,min}$, the desingularization process $\mathcal{D}$ can be seen as a one-to-one map between the solution curves of $\overline{X}$ and those of the CDE* (1.7)*. The details of the desingularization procedure are given in section 2.2.*

[3]Two vector fields, X and Y, on a manifold M are smoothly equivalent if there exists a smooth map $\Phi : M \to M$ such that $D\Phi \cdot X(\Phi) = Y$.



$$\overline{X} : \begin{cases} \dot{b} &= -2(3z^2 + b)(b + z) \\ \dot{z} &= 1 + b + 2(b + z)z. \end{cases}$$

$$\tilde{X} : \begin{cases} \dot{a} &= -1 - b, \\ \dot{b} &= -2(b + z) \\ & \text{Satisfying} \\ 0 &= z^3 + bz + a \end{cases}$$

Figure 1.5: Top right: phase portrait of the CDE (1.7). The manifold S_V serves as the phase space of the corresponding flow. The shaded region is the attracting part of the constraint manifold, that is $S_{V,min}$. Top left: phase portrait of the desingularized vector field (1.9). In this picture, and in the rest of the document, the symbol $\mathcal{D}$ denotes the desingularization process, to be detailed in section 2.2. Observe that although the vector field $\overline{X}$ is defined for all $(b, z) \in \mathbb{R}^2$ we are only interested in the region of (b, z)-values corresponding to $S_{V,min}$. When the trajectories reach the singular set B, they jump to another attracting part of $S_{V,min}$. Bottom right: projection of the phase portrait of (1.7) onto the parameter space. The map π is a projection of the total space $(a, b, z) \in \mathbb{R}^3$ onto the parameter space $(a, b) \in \mathbb{R}^2$.

- *The solutions of the CDE are obtained from the integral curves of the desingularized vector field $\overline{X}$.*

Remark 1.1.2. *From the examples presented above, it is possible to conclude that understanding CDEs is a good preliminary step in order to study SFSs. However, this is not enough, since we cannot expect that a complete description a singular perturbation problem is obtained from the limiting behavior. In particular, we want to describe the way the solutions of a SFS depend on the small parameter ε as $\varepsilon \to 0$. This problem is well understood, e.g. by Geometric Singular Perturbation Theory [16, 25, 24, 47] in the case where the manifold S is a set of hyperbolic equilibrium points of the layer equation; in other words, when S is regular. In this document we are interested in studying SFSs* *near singular points of the manifold S_V.*

1.2 Outline of this thesis

In chapter 2 we extend Takens's list of singularities of CDEs [39] by obtaining the following classifications.

1. *A topological classification of singularities of generic CDEs with three parameters.* That is, the possible potential functions are given by the Swallowtail, the Hyperbolic and the Elliptic Umbilical catastrophes, and the possible vector fields are given up to topological equivalence. This result is also reported in [22].

2. *A topological classification of generic CDEs with one fast variable.* In this classification the potential functions are given by the A_k catastrophes. A corresponding normal form is given by

$$\begin{aligned} \dot{x}_1 &= 1 \\ \dot{x}_j &= 0 \\ 0 &= z^k + \sum_{i=1}^{k-1} x_i z^{i-1}, \end{aligned} \tag{1.10}$$

where $j = 2, 3, \ldots, k-1$. The constraint manifold of (1.10) may be regarded as the critical set of an A_k catastrophe. Hence we call (1.10) A_k-CDE.

In Chapter 3, we embed the A_k-CDEs into the theory of SFSs. That is, we study SFSs of the form

$$X : \begin{cases} x_1' &= \varepsilon(1 + \varepsilon g_1(x, z, \varepsilon)) \\ x_j' &= \varepsilon^2 g_j(x, z, \varepsilon) \\ z' &= -\left(z^k + \sum_{i=1}^{k-1} x_i z^{i-1}\right) + \varepsilon g_k(x, z, \varepsilon) \\ \varepsilon' &= 0, \end{cases} \tag{1.11}$$

where $k \geq 2$, $j = 2, 3, \ldots, k-1$, $x = (x_1, \ldots, x_{k-1}) \in \mathbb{R}^{k-1}$, $z \in \mathbb{R}$, $0 < \varepsilon \ll 1$; and where g_i's are smooth functions vanishing at the origin. Our investigations are focused on understanding the flow of (1.11) in a small neighborhood of the origin, which is a degenerate equilibrium point of X. The first step in our program is to normalize (1.11). In such a context, we show that there exists a near identity formal diffeomorphism $\hat{\Phi} : \mathbb{R}^{k+1} \to \mathbb{R}^{k+1}$ that conjugates (1.11) to a polynomial vector field F defined by

$$F : \begin{cases} x_1' &= \varepsilon \\ x_j' &= 0 \\ z' &= -\left(z^k + \sum_{i=1}^{k-1} x_i z^{i-1}\right) \\ \varepsilon' &= 0. \end{cases} \tag{1.12}$$

In this sense, we say that F is (formally) stable under ε-perturbations. This result has been reported in [21].

Next, by means of the Borel's lemma [10], the formal normal form F can be realized as a smooth vector field X^N of the form

$$X^N = F + \varepsilon H,$$

where $H = \sum_{i=1}^{k-1} h_i(x,z,\varepsilon)\frac{\partial}{\partial x_i} + h_k(x,z,\varepsilon)\frac{\partial}{\partial z} + 0\frac{\partial}{\partial \varepsilon}$ is a smooth vector field and where all the functions h_j are flat at $(x,z,\varepsilon) = (0,0,0) \in \mathbb{R}^{k+1}$. Next, we conveniently choose two sections Σ^{en} and Σ^{ex} which are transversal to the flow of X^N within a small neighborhood of $0 \in \mathbb{R}^{k+1}$. According to these sections and induced by the flow of X^N, the transition map $\Pi : \Sigma^{\mathrm{en}} \to \Sigma^{\mathrm{ex}}$ is defined. With this transition map we are able to describe the solution curves of X^N.

Our principal contribution, in the context of A_k slow fast systems, is that we provide a general methodology by which the local flow of (1.12) can be understood by means of transition maps. The frequent use of normal form theory allows us to specify the differentiability of Π as $\varepsilon \to 0$. This follows the program of Geometric Singular Perturbation Theory developed for slow fast systems near degenerate singularities, e.g., [8, 13, 14, 26, 27].

In Chapter 4, we show how the program developed in Chapter 3 works for the A_2 (fold), A_3 (cusp, see [23]) and A_4 (swallowtail) cases. We also digress on the A_k, for $k > 4$, case.

Finally, in Chapter 5, concluding remarks and future research direction are discussed.

For readability purposes, the long proofs are gathered at the end of the respective chapter. In the appendix, relevant and supplementary information used along this thesis is provided.

Chapter 2

Constrained Differential Equations

2.1 Definitions

Let us start with the important definitions. Many of these are intrinsic. However, since our studies are local, whenever necessary the corresponding local model is given.

Definition 2.1.1 (CONSTRAINED DIFFERENTIAL EQUATION (CDE))**.** *Let $\mathcal{E}$ and $\mathcal{B}$ be $\mathcal{C}^\infty$-manifolds, and $\pi : \mathcal{E} \to \mathcal{B}$ a $\mathcal{C}^\infty$-projection.* A constrained differential equation on $\mathcal{E}$ *is a pair (V, X), where $V : \mathcal{E} \to \mathbb{R}$ is a $\mathcal{C}^\infty$-function, called* potential function, *that has the following properties:*

CDE.1 *V restricted to any fiber of $\mathcal{E}$ (denoted by $V|\pi^{-1}(\pi(e))$, $e \in \mathcal{E}$) is proper and bounded from below,*

CDE.2 *the set $S_V = \left\{e \in \mathcal{E} : V|\pi^{-1}(\pi(e)) \text{ has a critical point in } e\right\}$, called* the constraint manifold, *is locally compact in the sense: for each compact $K \subset \mathcal{B}$, the set $S_V \bigcap \pi^{-1}(K)$ is compact,*

and X is such that:

CDE.3 *$X : \mathcal{E} \to T\mathcal{B}$ is a $\mathcal{C}^\infty$-map covering $\pi : \mathcal{E} \to \mathcal{B}$.*

Remark 2.1.1.

- *S_V is a smooth manifold of the same dimension as $\mathcal{B}$ [39, 5].*

- *The covering property of X means that for all $e \in \mathcal{E}$, the tangent vector $X(e)$ is an element of $T_{\pi(e)}\mathcal{B}$, the tangent space of $\mathcal{B}$ at the point $\pi(e)$. $T\mathcal{B}$ denotes the tangent bundle of $\mathcal{B}$. The covering property of X defines a vector field $\tilde{X} : \mathcal{B} \to T\mathcal{B}, \tilde{X} = X \circ \pi^{-1}$.*

Now, recall from chapter 1 that the solutions of the CDE are defined in the attracting region of the constraint manifold, thus we have the following.

Definition 2.1.2 (THE SET OF MINIMA). *The set of minima of V is denoted by $S_{V,min}$ and is defined by*

$$S_{V,min} = \left\{ e \in \mathcal{E} : V|\pi^{-1}(\pi(e)) \begin{array}{l}\text{has a critical point in } e \text{ where the Hes-}\\ \text{sian is positive semi-definite}\end{array} \right\}$$

Once that we have defined the region where solutions may exists, let us give the formal definition of what we mean by a solution of a CDE.

Definition 2.1.3 (SOLUTION). *Let (V, X) be as in definition 2.1.1. A curve $\gamma : J \to \mathcal{E}$, J an open interval of $\mathbb{R}$, is a* solution *of (V, X) if*

S1 $\gamma\left(t_0^+\right)=\lim_{t\downarrow t_0}\gamma$ *and* $\gamma\left(t_0^-\right) = \lim_{t\uparrow t_0}\gamma$ *exist for all $t_0 \in J$, satisfying*

- $\pi\left(\gamma\left(t_0^+\right)\right) = \pi\left(\gamma\left(t_0^-\right)\right)$,
- $\gamma\left(t_0^+\right), \gamma\left(t_0^-\right) \in S_{V,min}$.

S2 *For each $t \in J$, $X\left(\gamma\left(t^-\right)\right)$ (resp. $X\left(\gamma\left(t^+\right)\right)$) is the left (resp. right) derivative of $\pi(\gamma)$ at t.*

S3 *Whenever $\gamma\left(t^-\right) \neq \gamma\left(t^+\right),\ t \in J$, there is a curve in $\pi^{-1}\left(\pi\left(\gamma\left(t^+\right)\right)\right)$ from $\gamma\left(t^-\right)$ to $\gamma\left(t^+\right)$ along which V is monotonically decreasing.*

Remark 2.1.2.

- *Solutions are also defined for closed or semiclosed intervals. A curve $\gamma : [\alpha, \beta] \to \mathcal{E}$ ($\gamma : (\alpha, \beta] \to \mathcal{E}$, or $\gamma : [\alpha, \beta) \to \mathcal{E}$) is a solution of (V, X) if, for any $\alpha < \alpha' < \beta' < \beta$, the restriction $\gamma|(\alpha', \beta')$ is a solution and if γ is continuous at α and β (at β, or at α) or if there is a curve from $\gamma(\alpha)$ to $\gamma\left(\alpha^+\right)$ and from $\gamma\left(\beta^-\right)$ to $\gamma(\beta)$ (from $\gamma\left(\beta^-\right)$ to $\gamma(\beta)$, or from $\gamma(\alpha)$ to $\gamma\left(\alpha^+\right)$) as in property S3 above.*

- *The projection $\pi(\gamma)$ is continuous.*

- *The property S3 above describes the jumping process. It basically says that if a jump occurs, it happens along some fiber $\pi^{-1}(\pi(e))$. A jump is an infinitely fast transition along a fiber passing through a singular point of S_V.*

The concept of singularity, as in the context of vector fields, is defined for CDEs as follows.

Definition 2.1.4 (SINGULARITY). *We say that a CDE (V, X) has a* singularity *at $e \in \mathcal{E}$ if*

1. $X(e) = 0$, *or*
2. $V|\pi^{-1}(\pi(e))$ *has a degenerate critical point at e.*

In the rest of this chapter, we are interested in singularities of CDEs defined by degenerate critical points of the potential function V. Thus we shall assume in the following that $X(e) \neq 0$, which is a generic property.

From definition 2.1.4 we see that the classification of critical points of smooth functions plays an important role in the classification of singularities of CDEs. Let us now recall some useful definitions.

Definition 2.1.5 (JET SPACE). *Let $\pi : \mathcal{E} \to \mathcal{B}$ be a fiber bundle as before. We define $J^k_V(\mathcal{E}, \mathbb{R})$ as the space of $k-$jets of functions $V : \mathcal{E} \to \mathbb{R}$. Similarly $J^k_X(\mathcal{E}, T\mathcal{B})$ is defined to be the space of $k-$jets of smooth maps $X : \mathcal{E} \to T\mathcal{B}$ covering π. Finally $J^k(\mathcal{E}) = J^k_V(\mathcal{E}, \mathbb{R}) \oplus J^k_X(\mathcal{E}, T\mathcal{B})$ is the space of $k-$jets of constrained equations. For a given (V, X), the smooth map $j^k(V, X) : \mathcal{E} \to J^k(\mathcal{E})$ assigns to each $e \in \mathcal{E}$ the corresponding $k-$jets of V and X at e.*

Remark 2.1.3. *The elements of $J^k_V(\mathcal{E}, \mathbb{R})$ are equivalence classes of pairs (V, e), $V \in \mathcal{C}^\infty(\mathcal{E}, \mathbb{R})$, $e \in \mathcal{E}$; where $(V, e) \sim (V', e')$ if $e = e'$ and all partial derivatives of $(V - V')$ up to order k vanish at e. The same idea holds for $J^k_X(\mathcal{E}, T\mathcal{B})$ and thus for $J^k(\mathcal{E})$. This equivalence relation is independent of the choice of coordinates.*

Definition 2.1.6 (THE SET Σ^I). *Let $I = (i_1, i_2, \ldots, i_k)$ be a sequence of positive integers such that $i_1 \geq i_2 \geq \cdots \geq i_k$. The set $\Sigma^I \subset J^\ell(\mathcal{E})$ ($\ell \geq k$) is the set of CDEs (V, X) for whose restriction $V|\pi^{-1}(\pi(e))$ has in e a critical point of Thom Boardman symbol I (see appendix A.2 for details).*

The following statements are shown, for example, in [4]:

- $J^\ell(\mathcal{E})$ can be stratified since the closure of Σ^I is an algebraic subset of $J^\ell(\mathcal{E})$,
- Σ^I is a submanifold of $J^\ell(\mathcal{E})$.

It is useful now to state Thom's transversality theorem in the context of constrained differential equations.

Theorem 2.1.1 (THOM'S STRONG TRANSVERSALITY THEOREM [41]). *Let $Q \subset J^k(\mathcal{E})$ be a stratified subset of codimension p. Then there is an open and dense subset $\mathcal{O}_Q \subset \mathcal{C}^\infty(\mathcal{E}, \mathbb{R}) \times \mathcal{C}^\infty(\mathcal{E}, T\mathcal{B})$ such that for each $(V, X) \in \mathcal{O}_Q$, $j^k(V, X)$ is transversal to Q. Therefore $\left(j^k(V, X)\right)^{-1}(Q)$ is a codimension p stratified subset of $\mathcal{E}$.*

Definition 2.1.7 (GENERIC CDE)**.** *Let $I = (i_1, i_2, \ldots, i_k)$ be a sequence of positive integers such that $i_1 \geq i_2 \geq \cdots \geq i_k$. We say that a CDE (V, X) is* generic *if $j^k(V, X)$ is transversal to $\Sigma^I \subset J^\ell(\mathcal{E})$, with $(\ell \geq k)$.*

In the rest of this document, the term *generic* refers to definition 2.1.7.

Remark 2.1.4. *The analysis of the present document is local. Therefore, we identify the fibre bundle $\pi : \mathcal{E} \to \mathcal{B}$ with the trivial fibre bundle $\pi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$. Moreover, by definition 2.1.7, let $e \in \mathbb{R}^n \times \mathbb{R}^m$ be a point such that $V|\pi^{-1}(\pi(e))$ has a degenerate critical point at e. Then, for $m \leq 4$, there are local coordinates such that V can be written as one of the seven elementary catastrophes of table A.1. Hence, let without loss of generality $e = 0$. A generic CDE is a pair (V, X) where V is an elementary catastrophe and the constraint manifold is indeed a smooth manifold. Furthermore, the local normal form of the pair (V, X) can be given as a polynomial expression.*

Definition 2.1.8 (THE SINGULARITY AND CATASTROPHE SETS)**.** *The* singularity set*, also called bifurcation set, is locally defined as*

$$B = \left\{ (x, z) \in S_V \mid \det \frac{\partial^2 V}{\partial z^2} = 0 \right\}.$$

The projection of B into the parameter space $\pi(B)$ is called the catastrophe set*, and shall be denoted by Δ.*

As can be seen from the definitions of this section, many of the topological characteristics of a generic CDE are given by the form of the potential function V. It is especially important to know how the critical set of V is stratified. The following example is intended to give a qualitative idea of the geometric objects that we have to consider.

Example 2.1.1 (STRATA OF THE SWALLOWTAIL CATASTROPHE)**.** *Consider a CDE (V, X) where the potential function V is given by the swallowtail catastrophe, that is*

$$V(x, z) = \frac{1}{5} z^5 + \frac{1}{3} c z^3 + \frac{1}{2} b z^2 + a z.$$

See appendix A.1 and table A.1 for more details. Then we have the following sets:

Σ^I	$\Sigma^I(V) = (j^k(V, X))^{-1}\left(\Sigma^I\right)$
Σ^1	S_V
$\Sigma^{1,1}$	*B, the singularity set*
$\Sigma^{1,1,0}$	*The set of* only *fold points*
$\Sigma^{1,1,1,0}$	*The set of* only *cusp points*
$\Sigma^{1,1,1,1}$	*The swallowtail point*

The sets $\Sigma^i(V)$ above are formed as follows (see appendix A.2 for the generalization)

$$\begin{aligned}
&\Sigma^1(V) = \left\{(x,z) \in \mathbb{R}^4 \,|\, D_z V = 0\right\} \\
&\Sigma^{1,1}(V) = \left\{(x,z) \in \mathbb{R}^4 \,|\, D_z V = D_z^2 V = 0\right\} \\
&\vdots
\end{aligned}$$

The strata are manifolds of certain dimension formed by points of the same degeneracy. In our particular example we have

$\Sigma^{1,0}(V) = \Sigma^1(V) \backslash \Sigma^{1,1}(V)$	*A 3-dim. manifold of regular points of S_V*
$\Sigma^{1,1,0}(V) = \Sigma^{1,1}(V) \backslash \Sigma^{1,1,1}(V)$	*A 2-dim. manifold of fold points*
$\Sigma^{1,1,1,0}(V) = \Sigma^{1,1,1}(V) \backslash \Sigma^{1,1,1,1}(V)$	*A 1-dim. manifold of cusp points*

Note the inclusion $S_V \supset B \supset \Sigma^{1,1,1} \supset \Sigma^{1,1,1,1}$, which is a generic situation [4, 17].

Following example 2.1.1, the critical set of the codimension 3 catastrophes are stratified as shown at the end of this section in figures 2.1, 2.2 and 2.3 respectively.

Definition 2.1.9 (TOPOLOGICAL EQUIVALENCE [39]). *Let (V, X) and (V', X') be two constrained differential equations. Let $e \in S_{V,min}$ and $e' \in S_{V',min}$. We say that (V, X) at e is* topologically equivalent *to (V', X') at e' if there exists a* local homeomorphism *h form a neighborhood U of e to a neighborhood U' of e', such that if γ is a solution of (V, X) in U, $h \circ \gamma$ is a solution of (V', X') in U'.*

Observe that definition 2.1.9 does not require preservation of the time parametrization, only of direction.

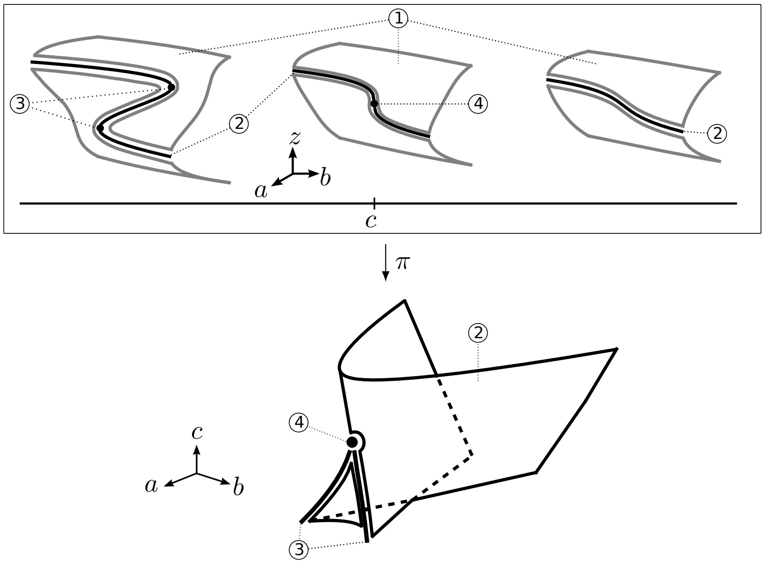


Figure 2.1: Stratification of the swallowtail catastrophe. Top: The total space is $\mathbb{R}^4$, therefore, we show some representative tomograms, in particular we show the stratification of the set of critical points of the swallowtail catastrophe (refer to example 2.1.1). For various constant values of c, ① represents the 3-dimensional set of regular points of S_V, this is $S_V \backslash B$. ② indicates a 2-dimensional surface of folds. ③ denotes a 1-dimensional curve of cusps. ④ represents the central singularity (at the origin) which is the swallowtail point. Note that with this notation $B =$ ② $\cup$ ③ $\cup$ ④. By π we represent the projection onto the parameter space. In the bottom picture we present the projection of the singularity set, this is $\Delta = \pi(B)$. The same numbered notation is used to indicate the different strata.

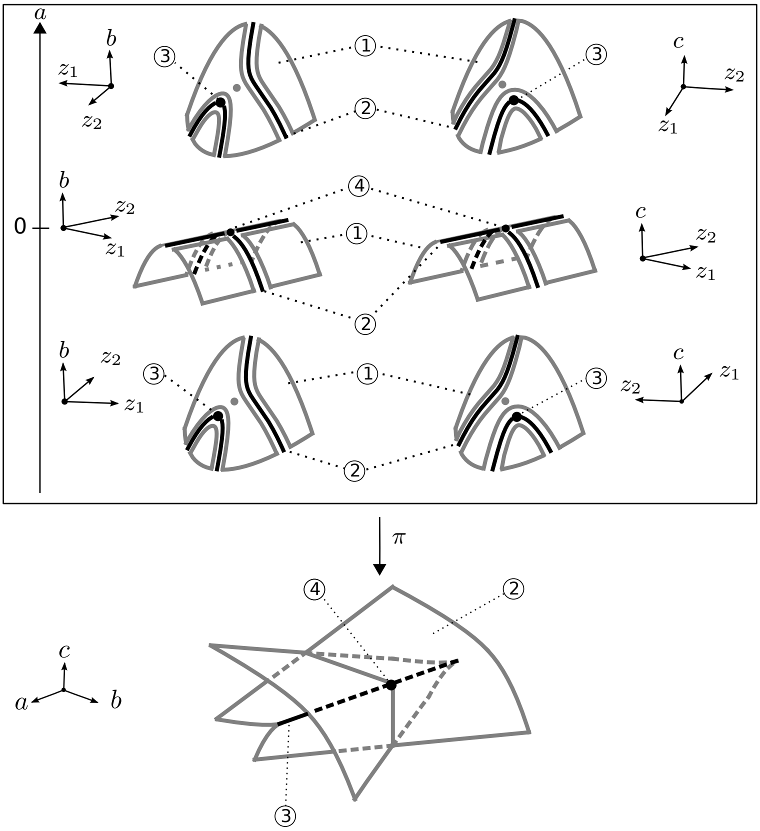


Figure 2.2: Stratification of the hyperbolic umbilic catastrophe. We follow the same numbered notation as in figure 2.1. ① The 3-dimensional manifold of regular points of S_V, this is $S_V \backslash B$. ② 2-dimensional surface of folds. ③ 1-dimensional curve of cusps. ④ The central singularity corresponding to the hyperbolic umbilic.

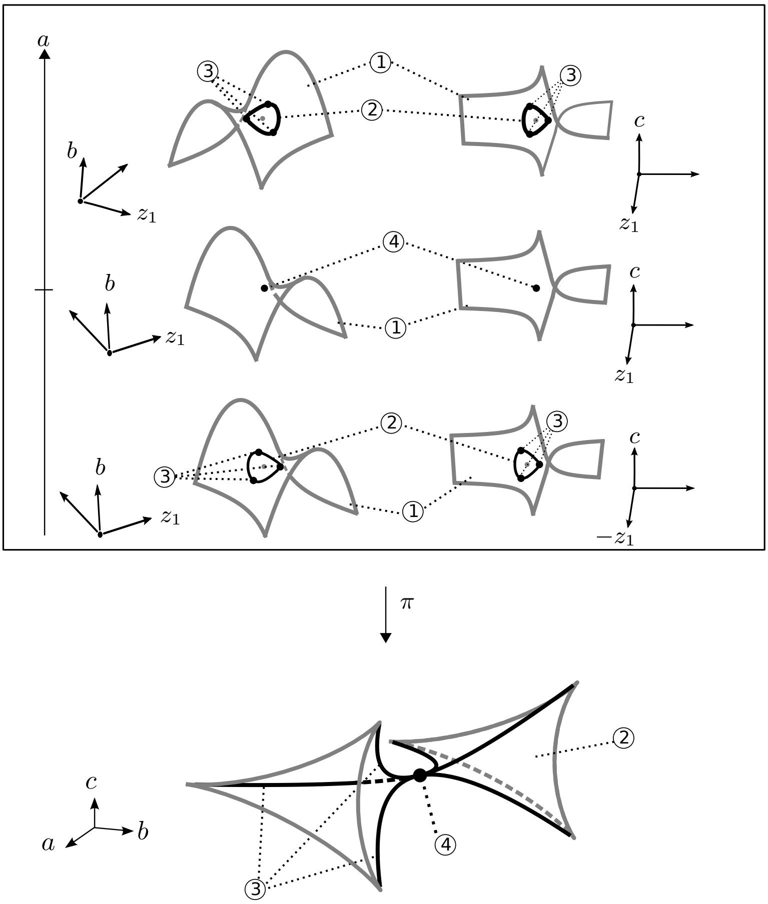


Figure 2.3: Stratification of the elliptic umbilic catastrophe. We follow the same numbered notation as in figures 2.1 and 2.2. ① The 3-dimensional manifold of regular points of S_V, this is $S_V \backslash B$. ② 2-dimensional surface of folds. ③ 1-dimensional curve of cusps. ④ The central singularity corresponding to the elliptic umbilic.

Remark 2.1.5. *Figures 2.1, 2.2 and 2.3 play an important role in understanding the behavior of the solutions of generic $CDEs$ with potential function corresponding to a codimension 3 catastrophe. In each figure, the solution curves are contained in the attracting part of S_V. By the generic conditions of X, we have that for each point $p \in \Delta$, the tangent vector $X(p)$ is transverse to Δ at p. When a solution curve reaches a point in B we generically expect to see a catastrophic change in the behavior of the solutions.*

2.2 Desingularization

In the context of constrained differential equations (CDEs), *desingularization* is a process in which one obtains a *smooth* vector field which is related to the CDE. This process plays a key role when finding topological normal forms of CDEs. The desingularized vector field $\overline{X}$ of a CDE (V, X) is constructed in such a way that we can relate its integral curves to the solutions of (V, X). Recall the example given in section 1.1.2. The general process to obtain such a vector field is described in the following lines.

Lemma 2.2.1 (DESINGULARIZATION [39]). *Consider a constrained differential equation (V, X) with V one of the elementary catastrophes. Then the induced* smooth *vector field, called* the desingularized vector field *is given by*

$$\overline{X} = \det(d\tilde{\pi})(d\tilde{\pi})^{-1}X(\tilde{\pi}, z), \tag{2.1}$$

where $\tilde{\pi} = \pi|S_V$. Furthermore, given the integral curves of the vector field $\overline{X}$ and the map $\tilde{\pi}$, it is possible to obtain the solution curves of (V, X).

For a proof and details see section 2.5.1. Once the desingularized vector field (2.1) is well understood, the solutions of (V, X) are obtained from the integral curves of $\overline{X}$. Note that in the case where $\det(d\tilde{\pi}) < 0$, the solutions of $\overline{X}$ and the corresponding solutions of the CDE go in opposite directions.

Remark 2.2.1. *Let (V, X) and (V', X') be topologically equivalent CDEs. From definition 2.1.9 the homeomorphism h also maps $S_{V,min}$ to $S_{V',min}$. On the other hand, it is straightforward to see that right equivalent functions have diffeomorphic critical sets. This means that we can choose and fix a representative of generic potential functions. The natural choice for a low number of parameters is one of the seven elementary catastrophes. Thus, our problem reduces to study the topological equivalence of CDEs (V, X) and (V, X'), that is with the same (up to right equivalence) potential function. Denote by $\overline{X}$ and $\overline{X}'$ the corresponding desingularized vector fields. Then if $\overline{X}$ and $\overline{X}'$ are topologically equivalent, so are the CDEs (V, X) and (V, X').*

Now, let us write the CDEs with three parameters in local coordinates. Following the list of catastrophes in appendix A.1 we have the following list of desingularized vector fields.

Corollary 2.2.1. *Let $(a, b, c, z_1, z_2) \in \mathbb{R}^3 \times \mathbb{R}^2$ be local coordinates. The parameters correspond to (a, b, c) while the state variables are (z_1, z_2). Let (V, X) be a constrained differential equation with the potential function V given by a codimension 3 catastrophe (see table A.1). Let the map $X : \mathcal{E} \to T\mathcal{B}$ be given in general form as*

$$X = f_a \frac{\partial}{\partial a} + f_b \frac{\partial}{\partial b} + f_c \frac{\partial}{\partial c},$$

where f_a, f_b, f_c are smooth functions on the total space $\mathcal{E}$. Then the corresponding desingularized vector fields $\overline{X}$ read as

- *Swallowtail:*

$$\overline{X} =(z^2 f_c + z f_b + f_a)\frac{\partial}{\partial z} - (4z^3 + 2cz + b) f_b \frac{\partial}{\partial b} - (4z^3 + 2cz + b) f_c \frac{\partial}{\partial c}.$$

- *Elliptic Umbilic:*

$$\begin{aligned}\overline{X} = &\left((12z_1^2 - 4az_2 - 12y^2) f_a + (6z_1 - 2a) f_b - 6z_2 f_c\right) \frac{\partial}{\partial z_1} + \\ &(-4z_2(a + 6z_1) f_a - 6z_2 f_b - (2a + 6z_1) f_c) \frac{\partial}{\partial z_2} + \left(4a^2 - 36z_1^2 - 36z_2^2\right) f_a \frac{\partial}{\partial a}\end{aligned}$$

- *Hyperbolic Umbilic:*

$$\begin{aligned}\overline{X} = &(36z_1 z_2 - a^2) f_a \frac{\partial}{\partial a} + ((az_1 - 6z_2^2) f_a - 6z_2 f_b + a f_c) \frac{\partial}{\partial z_1} + \\ &\left((az_2 - 6z_1^2) f_a + a f_b - 6z_1 f_c\right) \frac{\partial}{\partial z_2}.\end{aligned}$$

Proof. Straightforward computations following lemma 2.2.1. □

The desingularized vector fields of corollary 2.2.1 are to be used later to compute the list of topological normal forms of CDEs with three parameters.

We end this section with Takens's theorem on normal forms of constrained differential equations with two parameters.

Theorem 2.2.1 (TAKENS'S NORMAL FORMS OF CDEs [39]). *Let $\pi : \mathcal{E} \to \mathcal{B}$ be as in definition 2.1.1 and let* $\dim(\mathcal{B}) = 2$. *Then there are* 12 normal forms *(under topological equivalence, definition 2.1.9) of generic constrained differential equations, which are given by*

Regular	
$V(a,b,z)$	$X(a,b,z)$
$\frac{1}{2}z^2$	$\frac{\partial}{\partial a}$
	$a\frac{\partial}{\partial a} + b\frac{\partial}{\partial b}$
	$a\frac{\partial}{\partial a} - b\frac{\partial}{\partial b}$
	$-a\frac{\partial}{\partial a} - b\frac{\partial}{\partial b}$

Fold	
$V(a,b,z)$	$X(a,b,z)$
$\frac{1}{3}z^3 + az$	$\frac{\partial}{\partial a}$
	$-\frac{\partial}{\partial a}$
	$(a+3z)\frac{\partial}{\partial a} + \frac{\partial}{\partial b}$
	$(a-3z)\frac{\partial}{\partial a} + \frac{\partial}{\partial b}$
	$-b\frac{\partial}{\partial a} + \frac{\partial}{\partial b}$
	$(b+z)\frac{\partial}{\partial a} + \frac{\partial}{\partial b}$

Cusp	
$V(a,b,z)$	$X(a,b,z)$
$\frac{1}{4}z^4 + bz^2 + az$	$\frac{\partial}{\partial a}$
$-\left(\frac{1}{4}z^4 + bz^2 + az\right)$	$\frac{\partial}{\partial a}$

Remark 2.2.2.

- *In the cusp case, a distinction between V and $-V$ is made. This is due to the fact that the corresponding phase portraits are topologically different. This does not happen in the regular and fold cases. See for reference figs. 2.12a, 2.12b, 2.13, 2.14, 2.15, 2.16 and 2.17.*
- *In the fold case of theorem 2.2.1, one extra parameter is considered (see the catastrophes list in section A.1). Due to this fact, instead of having a fold point at $(z,a) = (0,0)$, there is a fold line given by $\{(z,a,b) = (0,0,b)\}$. In contrast, if $\mathcal{E}$ is 2-dimensional, this is, $(V,X) = \left(\frac{z^3}{3} + az, f(a,z)\frac{\partial}{\partial a}\right)$, then there are only two topological normal forms which read as*

$$V(z,a) = \frac{z^3}{3} + az \quad , \qquad X = \frac{\partial}{\partial a},$$
$$V(z,a) = \frac{z^3}{3} + az \quad , \qquad X = -\frac{\partial}{\partial a}.$$

- *Although the classification under topological equivalence may seem too coarse, it is the simplest one. Recall the well-known fact [2, 9] that there is no topological difference between the phase portraits shown in figure 2.4.*

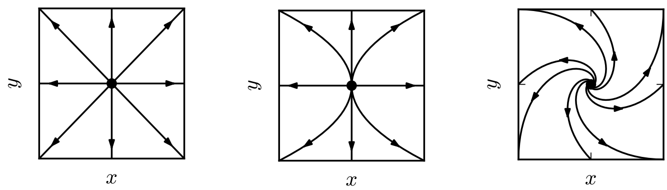


Figure 2.4: Topologically equivalent sources.

2.3 Normal form of CDEs with three parameters

In this section we provide our first result phrased in theorem 2.3.1. We give 16 local normal forms of generic constrained differential equations with three parameters. Thereby, we extend the existing Takens's list [39].

For clarity purposes, before stating the main result of the present chapter, we provide a geometric description of the constraint manifold S_V and how a generic vector field would look like. After this, the results stated in theorem 2.3.1 will seem natural.

2.3.1 Geometry of the codimenion 3 catastrophes.

In this section we review some of the geometrical aspects of the codimension 3 catastrophes to have an idea of what is their influence in the type of the generic desingularized vector fields.

The Swallowtail

We recall that the swallowtail catastrophe is given by the potential function

$$V(a,b,c.z) = \frac{1}{5}z^5 + \frac{1}{3}cz^3 + \frac{1}{2}bz^2 + az. \tag{2.2}$$

The constraint manifold, this is the phase space of the constrained differential equation (V, X) with potential function given by (2.2), is the critical set of V.

$$S_V = \left\{(a,b,c,z) \in \mathbb{R}^4 \mid z^4 + cz^2 + bz + a = 0\right\}. \tag{2.3}$$

Within the constraint manifold, there are two important sets. The set $S_{V,min}$ is the attracting region of S_V. The set B consists of singular point of S_V, that is where S_V is tangent to the one dimensional fast foliation. The previous sets read as

$$S_{V,min} = \left\{ (a,b,c,z) \in S_V \,|\, 4z^3 + 2cz + b > 0 \right\},$$

and

$$B = \left\{ (a,b,c,z) \in S_V \,|\, 4z^3 + 2cz + b = 0 \right\}.$$

The projection of the singular set B into the parameter space is the catastrophe set, denoted by Δ, and given by $\Delta = \pi(B)$. As it is readily seen, the set S_V is 3-Dimensional. In figure 2.5 we show tomographies of S_V as well as sections of Δ (see also figure 2.1 for the stratification of the swallowtail catastrophe).

Recall also from corollary 2.2.1 that the desingularized vector field reads as

$$\overline{X} = (z^2 f_c + x f_b + f_a)\frac{\partial}{\partial z} - (4z^3 + 2cz + b) f_b \frac{\partial}{\partial b} - (4z^3 + 2cz + b) f_c \frac{\partial}{\partial c}.$$

Note that we have the generic condition $\overline{X}(0) = f_a(0)\frac{\partial}{\partial z} \neq 0$. This is, we expect that $\overline{X}$ is given by a flow-box in a neighborhood of the central singularity. From figure 2.6 we can see that a flow-box in the direction of the a-axis is transversal to Δ in a neighborhood of the swallowtail point.

On the other hand, the fast fibers are parallel lines to the z-axis. If a trajectory jumps, it does so along such a fiber. A jump of a trajectory from a singular point of S to a stable branch of S_V is expected only when $c < 0$ as this is the only case where (2.3) may have more than two distinct real roots. We show in figure 2.7 the projections of the singular set B into the manifold S_V, representing the possible jumps to be encountered.

The Hyperbolic Umbilic

We proceed as in the previous section with a geometric description of the hyperbolic umbilic singularity. The corresponding catastrophe (see table A.1) reads

$$V(z_1, z_2, a, b, c) = -\left(z_1^3 + z_2^3 + a z_1 z_2 + b z_1 + c z_2 \right).$$

Now we have two constraint variables (z_1, z_2) (as opposed to the swallowtail singularity where the constraint variable is z). This means that the fast foliation is a family of planes parallel to $(z_1, z_2, 0, 0, 0) \in \mathbb{R}^5$. The critical set of V is given by

$$S_V = \left\{ (z_1, z_2, a, b, c) \in \mathbb{R}^5 \mid b = -3z_1^2 - a z_2,\ c = -3z_2^2 + a z_1 \right\}.$$

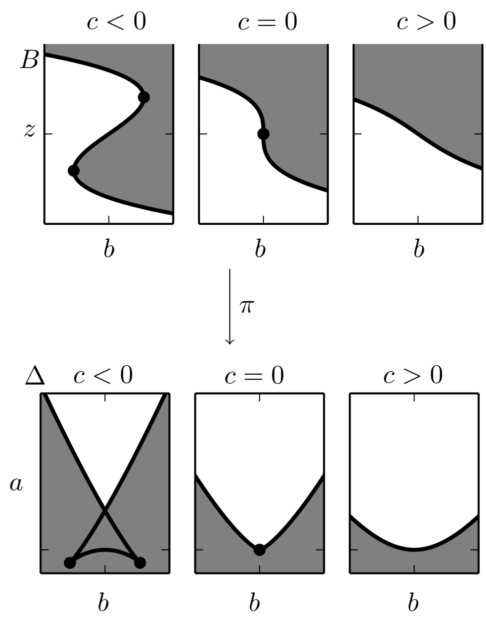


Figure 2.5: From left to right we show a tomography of the 3-dimensional manifold S_V for different values of a and parametrized by different coordinates. Compare with figure 2.1. The shaded region represents the stable part of S_V, that is $S_{V,min}$. In each figure the thick curve represents the 2-dimentional set of folds. For $a < 0$ the dots stand for the 1-dimensional set of cusps. For $a = 0$ the dot represents the central singularity, the swallowtail point. Note that for $a > 0$ the only singularities of S_V are fold points. The projection π occurs along a one dimensional fast foliation.

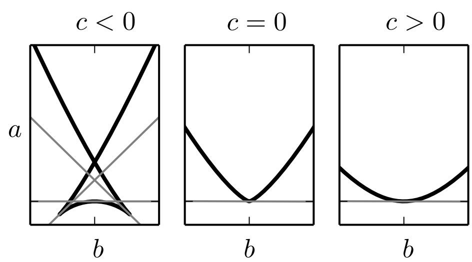


Figure 2.6: The thick curve represents section of the catastrophe set Δ. We show some tangent planes to Δ in a neighborhood of the Swallowtail point. A generic condition of the map X is to be transversal to Δ. So, observe that a flow-bow in the direction of the c-axis would have this property.

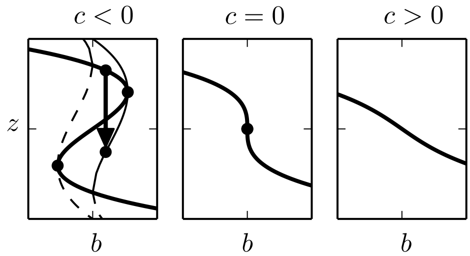


Figure 2.7: For values of $c < 0$ a trajectory may jump. A jump is an infinitely fast transition from a singular point of the manifold S_V to a stable part of S_V. The transition occurs along a one dimensional fiber. The thick lines represent the singularity set B, and the thin lines represent the projection of B into S_V. Such lines represent possible arriving points when a jump occurs. We show also a possible jump situation represented as an arrow starting in B and arriving at the projection of B in to $S_{V,min}$ (the attracting part of S_V).

There are attracting points within S_V defined as

$$S_{V,min} = \left\{ (z_1, z_2, a, b, c) \in S_V \mid \begin{bmatrix} 6z_1 & a \\ a & 6z_2 \end{bmatrix} \geq 0 \right\}.$$

The singular set of S_V is formed by all the points which are tangent to the fast fibers. Such a singular set reads

$$B = \left\{ (z_1, z_2, a, b, c) \in S_V \mid 36z_1z_2 - a^2 = 0 \right\}.$$

We show in figure 2.8 some tomographies of the constraint manifold S_V as well as sections of the singular set B.

Now, recall from corollary 2.2.1 that the desingularized vector field reads

$$\begin{aligned} \overline{X} =& (36z_1z_2 - a^2) f_a \frac{\partial}{\partial a} + \left(\left(-6z_2^2 + az_1\right) f_a - 6z_2 f_b + a f_c \right) \frac{\partial}{\partial z_1} + \\ & \left(\left(-6z_1^2 + az_2\right) f_a + a f_b - 6z_1 f_c \right) \frac{\partial}{\partial z_2}. \end{aligned}$$

The vector field $\overline{X}$ has generically an equilibrium point at the origin. It can also be shown that such a point is isolated within a sufficiently small neighborhood of the origin. Therefore, in contrast with the swallowtail case, we do not expect that a generic vector field $\overline{X}$ has the form of a flow-box. Note however, from the Jacobian of $\overline{X}$ evaluated at the origin, that the hyperbolic eigenspace is two dimensional and the center eigenspace is one dimensional (see

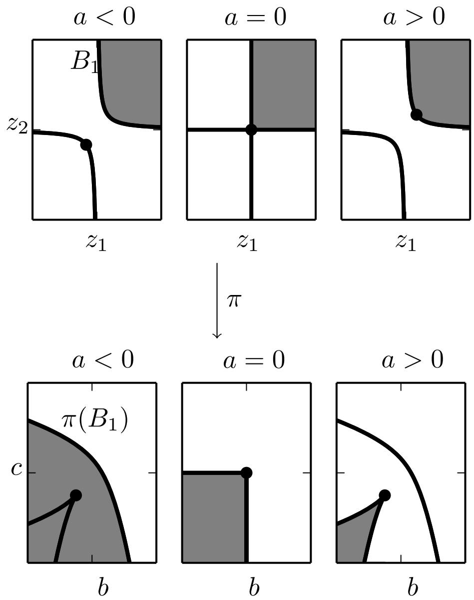


Figure 2.8: From left to right we show a tomography of the 3-dimensional manifold S_V for different values of a and parametrized by different coordinates. Compare with figure 2.2. The shaded region represents the stable part of S_V, that is $S_{V,min}$. For reference purposes, the singularity set B is divided into two components B_1 and B_2. In each figure the thick curve represents the 2-dimentional set of folds. For $a \neq 0$ the dots stand for the 1-dimensional set of cusps. For $a = 0$ the dot represents the central singularity, the hyperbolic umbilic point, which correspond to the intersection of the cusp lines. Recall that π is a projection from the total space to the parameter space, and occurs along the two dimensional fast foliation.

section 2.5.2 for details). So, we expect to have a 1-dimensional center manifold and a 2-dimensional hyperbolic invariant manifold intersecting at the origin. Such manifolds arrange the whole dynamics in a small neighborhood of the the hyperbolic umbilic point. Moreover, we expect that $\overline{X}$ meets transversally the set $\pi(B)$, this means that $\overline{X}$ is also transversal to B. Such transversality property is depicted in figure 2.9.

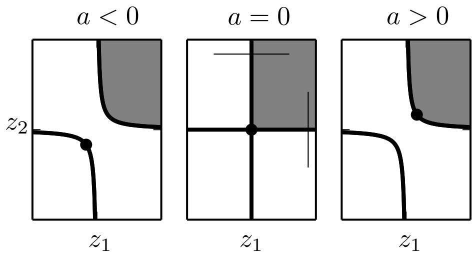


Figure 2.9: The transversality property of $\overline{X}$ with respect to B means that the integral curves of $\overline{X}$ are tangent to the thin lines depicted. Recall that if $\overline{X}$ is transversal to $B|(a = 0)$ (center picture), then $\overline{X}$ is also transversal to a small perturbation of $B|(a = 0)$ (left and right pictures).

It is worth to take a closer look at figure 2.8, specially at the case $a < 0$. Observe, in the parameter space (a, b, c), that within the shaded region $S_{V,min}$, there appear to be a set of singularities $\pi(B_2)$. However this is only a visual effect due to the projection map π. We can note from the the same picture in the space (z_1, z_2, a), that the trajectories in $S_{V,min}$ cannot meet the set B_2.

The jumping behavior is now more complicated. Mainly because a jump may occur along a plane parallel to the $(z_1, z_2, 0, 0, 0)$ space. However, two important facts can be seen from figure 2.8. First, the set $S_{V,min}$ is one connected component. Second, as explained in the previous paragraph, we can see that there is no superposition (along the fibers) of points in $S_{V,min}$ and points in B (compare with the diagram of the swallowtail given in figure 2.5). This means that along the projection π it is not possible to join a point in B with a point in $S_{V,min}$. These facts lead us to conjecture that there are no jumps for generic CDEs with a hyperbolic umbilic singularity. Such an idea is proved in section 2.3.4.

The Elliptic Umbilic

Now we provide some insight on the geometry of the elliptic umbilic catastrophe, which is given by

$$V(z_1, z_2, a, b, c) = z_1^3 - 3z_1 z_2^2 + a(z_1^2 + z_2^2) + bz_1 + cz_2.$$

As in the hyperbolic umbilic case, the fast fibration is now two dimensional. The constraint manifold, the set of critical points of V, reads as

$$S_V = \left\{(z_1, z_2, a, b, c) \in \mathbb{R}^5 \mid b = -3z_1^2 - 3z_2^2 - 2az_1,\ c = -6z_1z_2 - 2az_2\right\}.$$

As before, within S_V there is a set of attracting points and a set of singular points. So we have that the set of regular points reads as

$$S_{V,min} = \left\{(z_1, z_2, a, b, c) \in S_V \mid \det \begin{bmatrix} 6z_1 + 2a & 6z_2 \\ 6z_2 & 6z_1 + 2a \end{bmatrix} \geq 0\right\},$$

which is equivalent to the condition $36z_1^2 + 36z_2^2 - 4a^2 \geq 0$ and $a > 0$. The set of singular points is given by

$$B = \left\{(z_1, z_2, a, b, c) \in S_V \mid 36z_1^2 + 36z_2^2 - 4a^2 = 0\right\}.$$

We show in figure 2.10 some tomographies of the constraint manifold S_V as well as sections of the singular set B.

The desingularized vector field in this case reads as

$$\begin{aligned} \overline{X} =& (4a^2 - 36z_1^2 - 36z_2^2)f_a \frac{\partial}{\partial a} + \\ & \left((12z_1^2 - 4az_1 - 12z_2^2)f_a + (6z_1 - 2a)f_b - 6z_2 f_c\right) \frac{\partial}{\partial z_1} + \\ & (-4z_2(a + 6z_1)f_a - 6z_2 f_b - (2a + 6z_1)f_c) \frac{\partial}{\partial z_2}, \end{aligned}$$

and as in the Hyperbolic Umbilic case, there is generically an equilibrium point at the origin. Similar arguments as before then apply. Namely, we expect that the vector field has a 1-dimensional center manifold and 2-dimensional hyperbolic invariant manifold intersecting at the origin. A qualitative picture of the transversality of $\overline{X}$ with respect to B is shown in figure 2.11

Regarding the jumps, the same arguments as for the hyperbolic umbilic catastrophe apply. Observe from figure 2.10 that it is not possible to join points in B with points in $S_{V,min}$ along the fibers.

2.3.2 Classification of CDEs with three parameters

In this section we provide a list of generic CDEs with three parameters. In contrast with Takens's list of normal forms [39], the result in this sections includes CDEs with two dimensional fast fibers. As it was mentioned in section 2.1 folds and cusps (lower codimension singularities) also appear as generic singularities of CDEs with three parameters. However the qualitative behavior of the solutions,

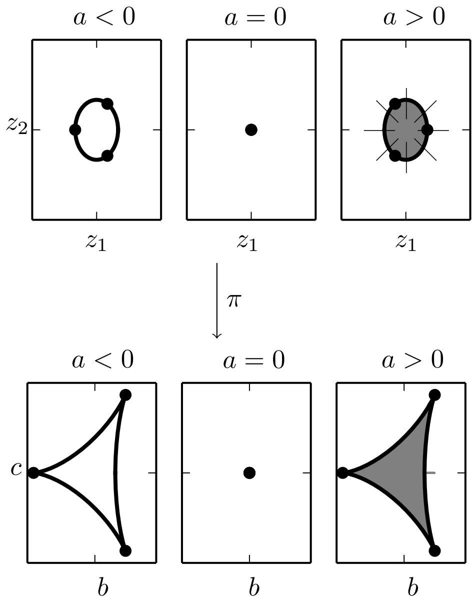


Figure 2.10: From left to right we show a tomography of the 3-dimensional manifold S_V for different values of a and parametrized by different coordinates. Compare with figure 2.3. The shaded region represents the stable part of S_V, that is $S_{V,min}$. In each figure the thick curve represents the 2-dimentional set of folds. For $a \neq 0$ the dots stand for the 1-dimensional set of cusps. For $a = 0$ the dot represents the central singularity, the hyperbolic umbilic point, which correspond to the intersection of the cusp lines. Recall that π is a projection from the total space to the parameter space.

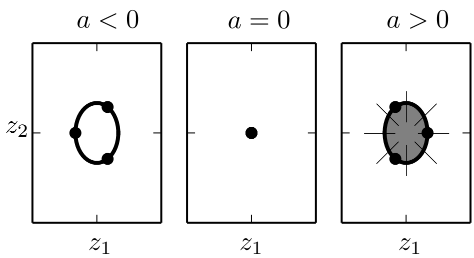


Figure 2.11: The transversality property of $\overline{X}$ with respect to B means that the integral curves of $\overline{X}$ are tangent to the thin lines depicted in the right picture.

in a small neighborhood of folds and cusps, can be understood from Takens's list [39]. The novelty of theorem 2.3.1 is that it provides an understanding of the solutions of generic CDEs in a small neighborhood of the swallowtail, of the elliptic and of the hyperbolic umbilical singularities.

Theorem 2.3.1. *Let (V, X) be a generic constrained differential equation with three parameters. Then (V, X) is topologically equivalent to one of the following* 16 *polynomial local normal forms.*

Regular		
$V(z,a,b,c)$	$X(z,a,b,c)$	Type
$\frac{1}{2}z^2$	$\frac{\partial}{\partial a}$	Flow-box
	$a\frac{\partial}{\partial a} + b\frac{\partial}{\partial b} + c\frac{\partial}{\partial c}$	Source
	$a\frac{\partial}{\partial a} + b\frac{\partial}{\partial b} - c\frac{\partial}{\partial c}$	Saddle-1
	$a\frac{\partial}{\partial a} - b\frac{\partial}{\partial b} - c\frac{\partial}{\partial c}$	Saddle-2
	$-a\frac{\partial}{\partial a} - b\frac{\partial}{\partial b} - c\frac{\partial}{\partial c}$	Sink

Fold		
$V(z,a,b,c)$	$X(z,a,b,c); \quad (\rho = \pm 1,\ \delta \in \mathbb{R})$	Type
$\frac{1}{3}z^3 + az$	$\frac{\partial}{\partial a}$	Flow-box-1
	$-\frac{\partial}{\partial a}$	Flow-box-2
	$\left(3x + \frac{1}{2}b + \frac{1}{2}c\right)\frac{\partial}{\partial a} + P(\rho,\delta,a,b)$	Source
	$\left(-3x + \frac{1}{2}b + \frac{1}{2}c\right)\frac{\partial}{\partial a} + P(\rho,\delta,a,b)$	Sink
	$-\left(\frac{1}{2}b + \frac{1}{2}c\right)\frac{\partial}{\partial a} + P(\rho,\delta,a,b)$	Saddle

where

$$P(\rho,\delta,a,b) = (c-b)^2\left(\rho + \delta(c-b)\right)\left(-\frac{\partial}{\partial b} + \frac{\partial}{\partial c}\right) + \frac{1}{2}\left(\frac{\partial}{\partial b} + \frac{\partial}{\partial c}\right)$$

Remark 2.3.1. *If $b = c$, these fold normal forms reduce to those of theorem 2.2.1.*

Cusp		
$V(z,a,b,c)$	$X(z,a,b,c)$	Type
$\frac{1}{4}z^4 + bz^2 + az$	$\frac{\partial}{\partial a}$	Flow-box
$-\left(\frac{1}{4}z^4 + bz^2 + az\right)$	$\frac{\partial}{\partial a}$	(Dual) Flow-box

Swallowtail		
$V(z,a,b,c)$	$X(z,a,b,c)$	Type
$\frac{1}{5}z^5 + \frac{1}{3}cz^3 + \frac{1}{2}bz^2 + az$	$\frac{\partial}{\partial a}$	Flow-box

Hyperbolic Umbilic		
$V(z_1,z_2,a,b,c)$	$X(z_1,z_2,a,b,c)$	Type
$z_1^3 + z_2^3 + az_1z_2 + bz_1 + cz_2$	$6\Phi(a)\frac{\partial}{\partial a} - \Psi\left(\frac{\partial}{\partial b} + \frac{\partial}{\partial c}\right)$	Center-Saddle
	$\left(\frac{a^2}{6} - 6z_1z_2\right)\frac{\partial}{\partial b} + \left(-\frac{a^2}{6} - 6z_1z_2\right)\frac{\partial}{\partial c} + Q$	Center

Where

$$
\begin{aligned}
\Phi(a) &= \pm\frac{a^2}{36} + \frac{\delta a^3}{216}, \qquad \delta \in \mathbb{R} \\
\Psi &= \left(\Phi(a)(6z_1 + 6z_2 - a) - 6z_1 z_2 + \frac{a^2}{6}\right) \\
Q &= 6\sum_{\ell=2}^{k}\sum_{j=0}^{2j=\ell} \rho_{\ell,j} A_{\ell,j}\frac{\partial}{\partial a} + \\
&\sum_{\ell=2}^{k}\left((6z_1 + a - 6z_2)\sum_{j=0}^{2j=\ell}\rho_{\ell,j}A_{\ell,j} + \sum_{j=0}^{2j+1=\ell}\left(\frac{a}{6}\right)^{-1} A_{\ell,j}B_{\ell,j}\right)\frac{\partial}{\partial b} + \\
&\sum_{\ell=2}^{k}\left((6z_2 + a - 6z_1)\sum_{j=0}^{2j=\ell}\rho_{\ell,j}A_{\ell,j} + \sum_{j=0}^{2j+1=\ell}\left(\frac{a}{6}\right)^{-1} A_{\ell,j}\overline{B}_{\ell,j}\right)\frac{\partial}{\partial c} \\
A_{\ell,j} &= \left(\frac{a}{6}\right)^{\ell-j}\Delta^j, \qquad \Delta = \left(\frac{a}{108}\right)\left(a^2 + 18(z_1^2 + z_2^2) + 6(az_1 + az_2)\right) \\
B_{\ell,j} &= -6z_1 C_{\ell,j} - a\overline{C}_{\ell,j} \\
\overline{B}_{\ell,j} &= -aC_{\ell,j} - 6z_2\overline{C}_{\ell,j} \\
C_{\ell,j} &= \eta_{\ell,j}\left(\frac{a}{6} + z_1\right) + \sigma_{\ell,j}\left(\frac{a}{6} + z_2\right) \\
\overline{C}_{\ell,j} &= \eta_{\ell,j}\left(\frac{a}{6} + z_2\right) - \sigma_{\ell,j}\left(\frac{a}{6} + z_1\right),
\end{aligned}
$$

with $\rho_{\ell,j}, \eta_{\ell,j}, \sigma_{\ell,j} \in \mathbb{R}$.

Elliptic Umbilic		
$V(z_1, z_2, a, b, c)$	$X(z_1, z_2, a, b, c)$	Type
$z_1^3 - 3z_1 z_2^2 + a(z_1^2 + z_2^2) + bz_1 + cz_2$	$A\frac{\partial}{\partial a} + \frac{B}{\sqrt{2}}\left(\frac{\partial}{\partial b} + \frac{\partial}{\partial c}\right) - \frac{1}{\sqrt{2}}\left(2xA\frac{\partial}{\partial b} + 2yA\frac{\partial}{\partial c}\right)$	Center-Saddle

Where $A = \frac{1}{9}\left(\pm 3a^2 + \delta a^3\right)$, $\delta \in \mathbb{R}$, and $B = -6z_1^2 - 6z_2^2 + \frac{2}{3}a^2$.

Proof. See section 2.5.2

□

We show in section 2.3.3 a number of phase portraits of the CDEs of theorem 2.3.1. Recall that in figures 2.1, 2.2 and 2.3, we show the phase space (S_V) of each of the codimension 3 CDEs.

2.3.3 Phase portraits of generic CDEs with three parameters

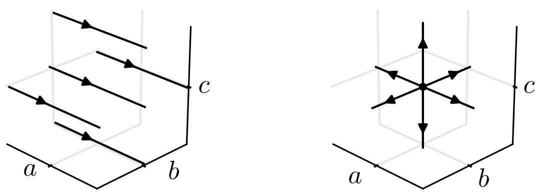
In this section we present the phase portraits of some of the normal forms of theorem 2.3.1. Recall that S_V is the phase space, this is, the solution curves belong to the manifold S_V. Such manifolds are as depicted in figures 2.1, 2.2 and 2.3. At the bifurcation sets B, the solution curves have a sudden change of behavior, and then it is said that a catastrophe occurs.

Briefly speaking, a generic constrained differential equation with three parameters is likely to have solutions that resemble those of the pictures presented in this section.

Remark 2.3.2. *In all our pictures below, the symbol $\mathcal{D}$ represents the desingularization of the corresponding constrained differential equation. The name of each section refers to the tables of theorem 2.3.1.*

Regular

In this case the constraint manifold S_V has no singularities. So the constraint manifold S_V is the whole $\mathbb{R}^3$. In figures 2.12a and 2.12b we show the phase portraits of the flow-box and source case. The pictures of the saddle-1, saddle-2 and sink are similar to figure 2.12b just changing accordingly the directions of the invariant manifolds.



(a) Flow-box phase portrait **(b)** Source phase portrait

Figure 2.12: Phase portraits corresponding to the regular case. We show only two examples corresponding to the flow-box (left) and the source (right) case. As the constraint manifold S_V is regular, the only singularities that may happen are equilibrium points, this is $X(0) = 0$. For to the same reason, there are no jumps. The remaining cases can be obtained by reversing the direction of the flow according to the corresponding spectra.

Fold

In this case the potential function is $V(a,b,c,z) = \frac{1}{3}z^3 + az$. The constraint manifold $S_V = \{(a,b,c,z) \in \mathbb{R}^4 | z^2 + a = 0\}$ is 3-dimensional. The attracting part of S_V is given by

$$S_{V,min} = \{(a,b,c,z) \in S_V \mid z \geq 0\}.$$

The projection $\tilde{\pi} = \pi|S_V$ is given by

$$\tilde{\pi}(a,b,c,z) = \pi(-z^2,b,c,z) = (-z^2,b,c).$$

Note that the determinant of $\tilde{\pi}$ is negative for points in $S_{V,min}$. Therefore, the trajectories of the desingularized vector field $\overline{X}$ and of the CDE (V,X) have opposite directions. Due to the presence of 3 parameters, the singularity set is the plane

$$B = \{(z,a,b,c) \in \mathbb{R}^4 | (z,a) = (0,0)\}.$$

It is important to note that all phase portraits of the Fold case have projections matching figure 3 of [39].

- Flow-box-1. By recalling the normal form in theorem 2.3.1 we see that the integral curves are as depicted in figure 2.13.
- Flow-box-2.

  The phase portrait in this case is as in figure 2.13, just the direction of the trajectories is reversed.
- Sink and Saddle.

  In all the following cases, a 1-dimensional center manifold W^C appears within the fold surface. The choice of $\rho = \pm 1$ changes the direction of W^C. In all the following pictures we set $\rho = 1$. The the integral curves of $\overline{X}$ are in opposite direction with respect to the solutions of (V,X) since $\det(D\tilde{\pi})$ is negative in $S_{V,min}$.

Cusp

In this case the constraint manifold is given by

$$S_V = \left\{(a,b,c,z) \in \mathbb{R}^4 \mid z^3 + bz + a = 0\right\},$$

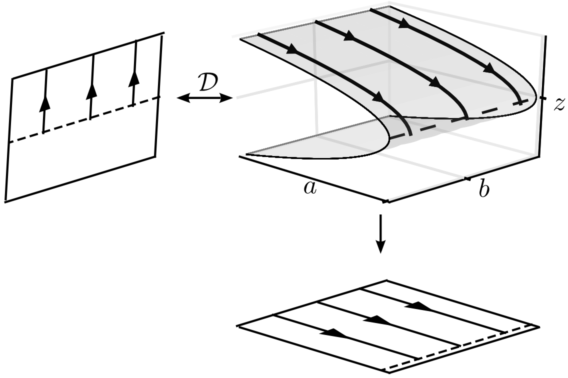


Figure 2.13: Phase portrait and projections of the flow-box-1 case with the variable c suppressed. The shown folded surface is a tomography of the 3-dimensional constraint manifold S_V. The dotted line corresponds to the 2-dimensional bifurcation set. Observe that since we are suppressing the variable c, this phase portrait is also shown in figure 3 of [39]

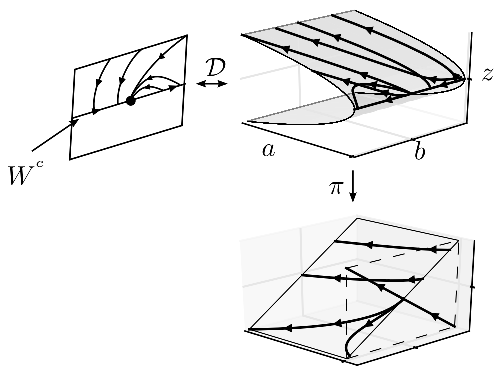


Figure 2.14: Projections of the solutions curves of the sink case. The folded surface is a tomography (fixed value of c) of the 3 dimensional manifold S_V. The hyperplane $\{z, a, b, c \,|\, b = c\}$ is invariant. In such space, the dynamics are reduced to the 2-parameter fold listed in [39] and in theorem 2.2.1. Observe that there exists a 1-dimensional manifold which is locally tangent to the fold surface.

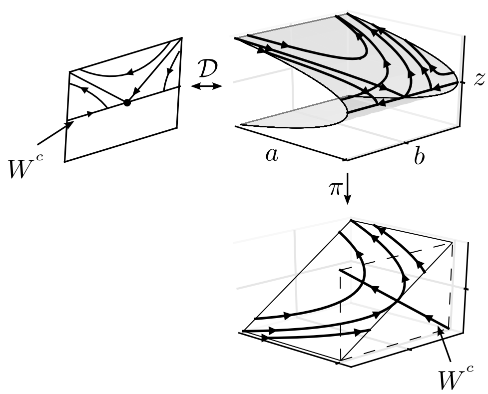


Figure 2.15: Projections of the solutions curves of the saddle case. The folded surface is a tomography (fixed value of c) of the 3 dimensional manifold S_V. The hyperplane $\{z, a, b, c \,|\, b = c\}$ is invariant. In such space, the dynamics are reduced to the 2-parameter fold listed in [39] and in theorem 2.2.1. Observe that there exists a 1-dimensional manifold which is locally tangent to the fold surface.

and the attracting part of S_V reads as

$$S_{V,min} = \left\{(b, c, z) \in S_V \,|\, z^3 + bz > 0\right\}.$$

The projection $\tilde{\pi} = \pi|_{S_V}$ reads as

$$\tilde{\pi}(a, b, c, z) = \pi(-z^3 - bz, b, c, z) = (-z^3 - bz, b, c).$$

Note that $\det(D\tilde{\pi})$ is negative in $S_{V,min}$. This means that the integral curves of the desingularized vector field $\overline{X}$ go in opposite direction with respect to the solutions of (V, X).

- The flow-box and the (dual) flow-box cases. Since in this case the generic vector field $\overline{X}$ is a flow box, the phase portraits that we obtain are just the same as in Takens's list [39]. Just one more artificial variable, the c-coordinate, is considered.

Swallowtail

In this section we present the phase portrait of a generic CDE in a neighborhood of a swallowtail singularity. This is, we consider the potential function

$$V(a, b, c, z) = \frac{1}{5}z^5 + \frac{1}{3}cz^3 + \frac{1}{2}bz^2 + az.$$

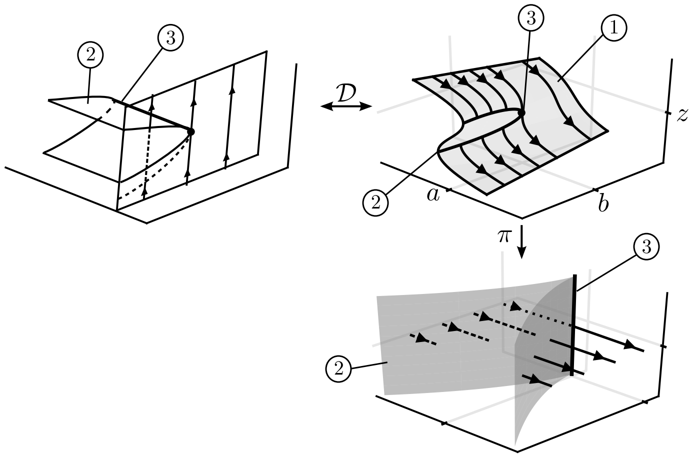


Figure 2.16: Phase portrait of the flow box case. 1 A tomography (the variable c is fixed and suppressed) of the 3-dimensional manifold S_V. 2 The 2-dimensional fold manifold. 3 The 1-dimensional cusp manifold. Compare with [39] figure 3 and note the resemblance with these projections.

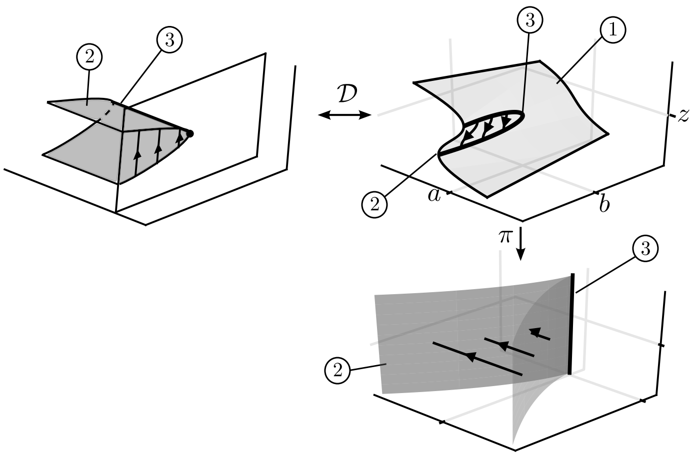


Figure 2.17: Phase portrait of the dual flow box case. 1 A tomography (the variable c is fixed and suppressed) of the 3-dimensional manifold S_V. 2 The 2-dimensional fold manifold. 3 The 1-dimensional cusp manifold. Compare with [39] figure 3 and note the resemblance with these projections.

The corresponding constraint manifold reads as

$$S_V = \left\{(a,b,c,z) \in \mathbb{R}^4 \,|\, z^4 + cz^2 + bz + a = 0\right\},$$

while the attracting region of S_V is given by

$$S_{V,min} = \left\{(b,c,z) \in S_V \,|\, 4z^3 + 2cz + b = 0\right\}.$$

Locally, the vector field is a flow-box and is depicted in figure 2.3.3. Note also that the integral curves of the desingularized vector field $\overline{X}$ go in opposite direction with respect to the solution curves of the CDE. It is straightforward to see that if we consider a potential function $-V$, the topology of the solutions does not change. Observe the jumping feature in the case $c < 0$, see section 2.3.4 for more details on such phenomenon.

Hyperbolic Umbilic

The total space is $\mathbb{R}^5$. The constraint manifold and the bifurcation set are detailed in figure 2.2. The origin of the desingularized vector field is a non-hyperbolic equilibrium point (see the proof of theorem 2.3.1) . We show in figures 2.19 and 2.20 the phase portraits of the center-saddle and center-center cases respectively. We take advantage on the fact that $\{a = 0\}$ is an invariant set. This means that the integral curves are arranged by those in the subspace $\{(0, b, c, z_1, z_2)\}$. Note that both phase portraits satisfy the geometric description given in section 2.3.1. That is, the integral curves are transversal to the singular sets. We have decided to show only the solution curves within $S_{V,min}$ as those are the ones we are interested in.

Elliptic Umbilic

The constraint manifold and the bifurcation set are described in figure 2.2. We show in figure 2.21 the phase portrait of the center-saddle. It is easy to check that $S_{V,min}|a = 0$ is just a point, so unlike in the hyperbolic umbilic case, there are no solutions curves of the corresponding CDE at $\{a = 0\}$. Therefore, we show projections into $S_{V,min}|a > 0$ with the value of a fixed, of some integral curves.

2.3.4 Jumps in CDEs with three parameters

Constrained differential equations and slow-fast systems are closely related. CDEs may represent an approximation of some generic dynamical systems with two or more different time scales. One interesting behavior of the latter type of systems is formed by jumps. Roughly speaking a jump is a rapid transition from one stable part of S_V to another. One common example of such a behavior

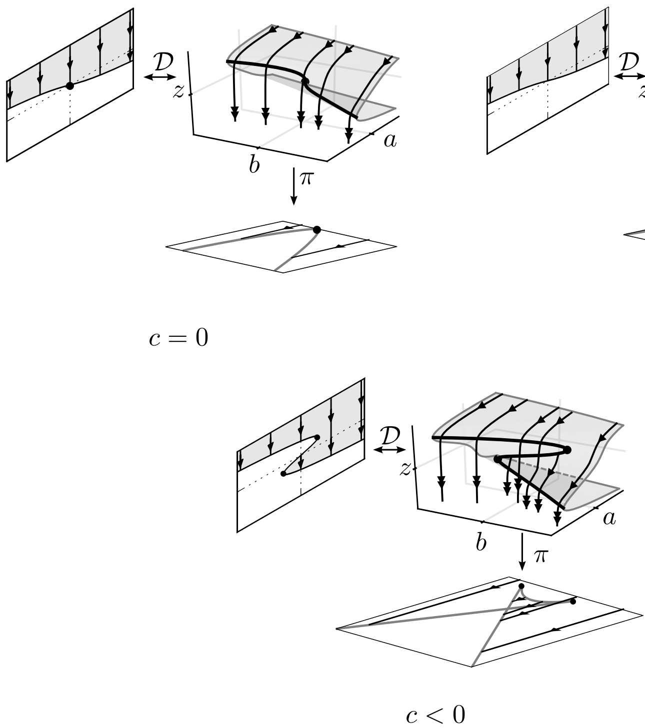


Figure 2.18: Tomographies for different values of the parameter a of the phase portraits of the swallowtail case. The catastrophe is stratified in the sets shown in figure 2.1. Note the particular behavior of the solutions when $c < 0$. In such case, there exists a region near the origin where jumps may occur. Observe that the shown solutions are in accordance with our description in section 2.3.1, that is X is transverse to the projection of the singular set.

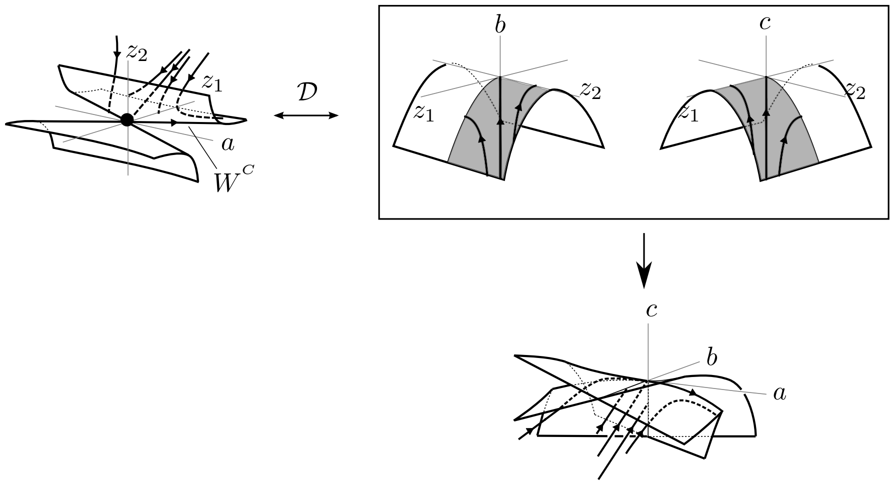


Figure 2.19: Phase portraits of the center-saddle case of the hyperbolic umbilic. Top left: the desingularized vector field. The origin is a semihyperbolic equilibrium point. Two directions correspond to a saddle, and one to a center manifold. Locally, such manifold is tangent to the singularity cone depicted. The center manifold changes direction depending on the $\pm$ sign of the normal form. The trajectories shown are within the projection of $S_{V,min}$. Top right: Trajectories of the CDE (V, X) restricted to $S_{V,min}$. The latter set is shown as a shaded region. Bottom: the projection of the solution curves into the parameter space. Note that the phase portrait shown satisfy the conjecture given in section 2.3.1.

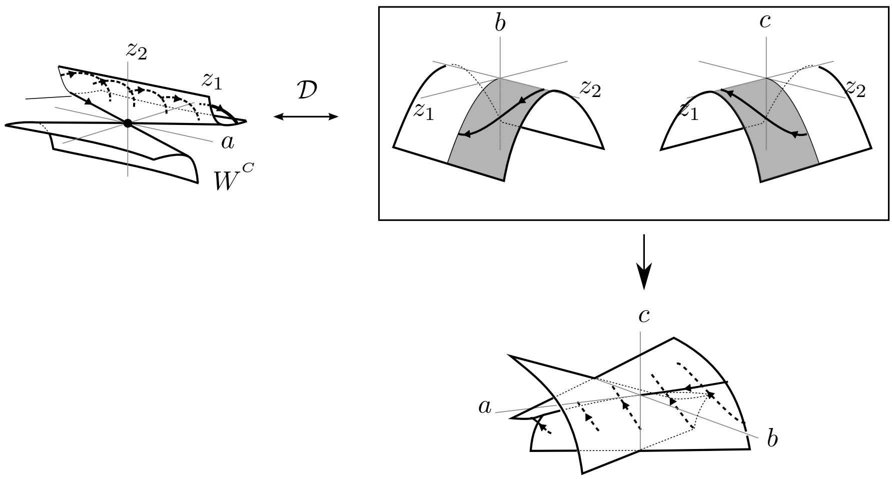


Figure 2.20: Phase portraits of the center case of the hyperbolic umbilic singularity. Top left: the desingularized vector field. Such vector field has an equilibrium at the origin and a 3-dimensional center manifold. The direction of the 1-dimensional center manifold depicted changes according to the $\pm$ sign of the normal form. Top right: Solutions curves in the invariant space $S_{V,min}|a = 0$. The latter set is shown as a shaded region. Bottom: the projection of the solution curves into the parameter space. Note that the phase portrait shown satisfy the conjecture given in section 2.3.1.

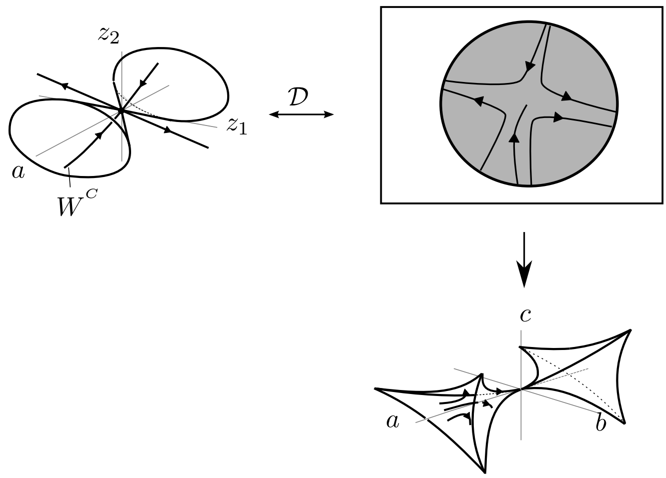


Figure 2.21: Phase portraits of the center-saddle case of the elliptic umbilic. Top left: the desingularized vector field. The origin is a semi-hyperbolic equilibrium point with two hyperbolic and one center directions. The center manifold is locally tangent to the singularity cone depicted. The hyperbolic directions shown (corresponding to a saddle) together with the center manifold arrange all the integral curves sufficiently close to the origin. Top right: Projection of some solutions curves into a tomography (a fixed) of $S_{V,min}$. Observe that $S_{V,min}$ is the inside region of a cone (refer to figure 2.3 and section 2.3.1). Bottom: the projection of the solution curves into the parameter space.

are relaxation oscillations. See also the examples in section chapter 1, where the characteristic property of jumps is described.

In this section we discuss the possibility of encountering such jumping behavior in generic CDEs with a swallowtail, hyperbolic, or elliptic umbilical singularity.

Definition 2.3.1 (FINITE JUMP)**.** *Let γ be a solution curve of a CDE (V, X). Let $q \in B$. We say that γ has a finite jump at q if the following conditions are satisfied:*

1. *there exists a point $p \in S_{V,min}$ such that $\pi(p) = \pi(q)$,*
2. *there exists a curve from p to q along which V is monotonically decreasing.*

In the case of the fold singularity, there are no finite jumps. In the case of the cusp singularity, a solution curve γ has the jump [39]

$$(a, b, z) \to (a, b, -2z).$$

For the existence (or nonexistence) of finite jumps in the generic CDEs with three parameters, we have the following proposition.

Proposition 2.3.1 (Jumps in the generic CDEs with 3 parameters)**.** *Let (V, X) be a generic CDE with potential function V one of the codimension 3 catastrophes. Let γ be a solution curve of (V, X). Then*

1. *If V is the swallowtail catastrophe, then there are finite jumps as follows. Let (a, b, c, z) be coordinates of $\gamma \cap B$, then the finite jump is given by*

$$(a, b, c, z) \mapsto (-z - \sqrt{-2z^2 - c}, a, b, c),$$

where it is readily seen that

$$z \in \left(-\sqrt{-\frac{c}{2}}, \sqrt{-\frac{c}{2}}\right), \ c < 0.$$

2. *If V is the hyperbolic or the elliptic umbilic catastrophe, then there are no finite jumps.*

Proof. See section 2.5.3. □

2.4 Normal form of A_k-CDEs

In this section, a topological classification of constrained differential equations with only one fast variable is presented. Such a classification is motivated by the A_k singularities [4]. Briefly speaking, we consider the potential function V to be the universal unfolding of a singularity of the type z^{k+1} with $k \geq 2$. In chapter 3 we embed the classification obtained below into the theory of slow fast systems.

Definition 2.4.1. *Let (V, X) be a generic constrained differential equation (CDE). We say that (V, X) is an A_k-CDE if and only if V is the universal unfolding of a function V_0 with a singularity of type A_k [4]. That is, an A_k-CDE has the local form*

$$\begin{aligned} \dot{x} &= f(x, z) \\ 0 &= z^k + \sum_{i=1}^{k-1} x_i z^{i-1}. \end{aligned}$$

The motivation behind A_k-CDEs is that we will use them in the next section to study slow fast systems with one fast variable and $k-1$ slow variables.

Remark 2.4.1. *For $k = 2, 3, 4$ we have the fold, the cusp, and the swallowtail cases respectively, which have been studied above.*

Now, we provide a classification of A_k-CDEs.

Proposition 2.4.1. *Let (V, X) be an A_k-CDE, and assume that $f(0,0) \neq 0$. Then (V, X) is topologically equivalent to $\left(V, \dfrac{\partial}{\partial x_1}\right)$.*

Proof. See section 2.5.4

∎

2.5 Proofs

2.5.1 Proof of lemma 2.2.1

Note that we can write each elementary catastrophe in the form

$$V(x, z) = V(0, z) + \sum_{i=1}^{m} x_i \frac{\partial V(x, z)}{\partial x_i},$$

where $z \in \mathbb{R}^n,\ x \in \mathbb{R}^m$, and with $n \leq m$. The constraint manifold is given by $\dfrac{\partial}{\partial z} V(x, z) = 0$, which means

$$\frac{\partial V(0, z)}{\partial z_j} + \sum_{i=1}^{m} x_i \frac{\partial^2 V(x, z)}{\partial z_j \partial x_i} = 0, \qquad \forall j \in [1, n].$$

Next, note that we can always solve the previous equation for n of the x_j's, obtaining

$$x_j = -\frac{\partial V(0, z)}{\partial z_j} - \sum_{i=j+1}^{m} x_i \frac{\partial^2 V(x, z)}{\partial z_j \partial x_i}.$$

This expresses that x_j is the coefficient of the linear term z_j in the potential function $V(x,z)$. Now, we can choose coordinates in S_V as

$$\begin{aligned}&(z_1,\ldots,z_n,\\&\left.-\frac{\partial V(0,z)}{\partial z_1}-\sum_{i=j+1}^{m}x_i\frac{\partial^2 V(x,z)}{\partial z_1\partial x_i},\ldots,-\frac{\partial V(0,z)}{\partial z_n}-\sum_{i=j+1}^{m}x_i\frac{\partial^2 V(x,z)}{\partial z_n\partial x_i},x_{n+1},\ldots,x_m\right).\end{aligned}$$

Next, we define the projection $\tilde{\pi}$ by $\tilde{\pi}=\pi|S_V$, this is

$$\tilde{\pi}(p)=\left(-\frac{\partial V(0,z)}{\partial z_1}-\sum_{i=j+1}^{m}x_i\frac{\partial^2 V(x,z)}{\partial z_1\partial x_i},\ldots,-\frac{\partial V(0,z)}{\partial z_n}-\sum_{i=j+1}^{m}x_i\frac{\partial^2 V(x,z)}{\partial z_n\partial x_i},x_{n+1},\ldots,x_m\right),$$

where p is a point in S_V. In the original coordinates, X has the general form

$$X=\sum_{i=1}^{m}f_i(x,z)\frac{\partial}{\partial x_i}.$$

The set S_V is the phase space of a constrained differential equation. So, for a point in S_V with coordinates $(z_1,\ldots,z_n,x_{n+1},\ldots,x_m)$, $\tilde{X}$ is given by

$$\tilde{X}=(d\tilde{\pi})^{-1}X(z,\tilde{\pi}(x,z)).$$

It is clear that $\tilde{X}$ is defined only for points where the projection in non-singular. Next, recall that the map $A\mapsto\det(A)A^{-1}$ can be extended to a $\mathcal{C}^\infty$ map on the space of square matrices. This means that we can define a smooth vector field by

$$\overline{X}=\det(d\tilde{\pi})(d\tilde{\pi})^{-1}X(z,\tilde{\pi}(x,z)).$$

Note that for all points where $\det(d\tilde{\pi})\neq 0$, the solutions of (V,X) are obtained from the integral curves of $\overline{X}$. First by a reparametrization due to the smooth projection $\tilde{\pi}$, and in cases where $\det(d\tilde{\pi})<0$, by next reversing the direction of the solutions.

2.5.2 Proof of theorem 2.3.1

We only detail the hyperbolic umbilic case as it is the most interesting one. All the other cases follow similar arguments and steps. The procedure of the proof is summarized as follows.

1. Desingularization of (V,X). With this we obtain the desingularized vector field $\overline{X}$. Then we are able to use standard techniques of dynamical systems theory to obtain a polynomial normal form of $\overline{X}$ following the next two steps.

2. Reduction to a center manifold, see appendix A.3. This reduction greatly simplifies the expressions of the normal forms.

3. Apply Takens's normal form theorem, see appendix A.3.

4. At this stage, we have a polynomial local normal form of the vector field $\overline{X}$. Now, recall that the form of $\overline{X}$ is obtained by following the desingularization process described in section 2.2. So, the last step in order to write the local normal forms of a constrained differential equation (V,X) is to carry out the inverse coordinate transformation performed when obtaining $\overline{X}$.

The Hyperbolic Umbilic

Following table A.1, we deal with the constrained differential equation

$$V(a,b,c,z_1,z_2) = z_1^3 + z_2^3 + az_1z_2 + bz_1 + cz_2$$
$$X(a,b,c,z_1,z_2) = f_a\frac{\partial}{\partial a} + f_b\frac{\partial}{\partial b} + f_c\frac{\partial}{\partial c},$$

The functions $f_i(a,b,c,z_1,z_2) : \mathbb{R}^5 \to \mathbb{R}$, for $i = a,b,c$, are considered to be $\mathcal{C}^\infty$ with the generic condition $f_i(0) \neq 0$.

Remark 2.5.1. *To simplify notation let* $(z_1,z_2) = (x,y)$.

The constraint manifold is the critical set of the potential function V

$$S_V = \left\{(a,b,c,x,y) \in \mathbb{R}^5 \mid b = -3x^2 - ay,\ c = -3y^2 + ax\right\}.$$

The attracting region of S_V is

$$S_{V,min} = \left\{(a,b,c,x,y) \in S_V \mid \begin{bmatrix} 6x & a \\ a & 6y \end{bmatrix} \geq 0\right\},$$

which is equivalent to the conditions $36xy - a^2 \geq 0$ and $x + y \geq 0$. Consequently, the catastrophe set reads

$$B = \left\{(a,b,c,x,y) \in S_V \mid \det \begin{bmatrix} 6x & a \\ a & 6y \end{bmatrix} = 0\right\}.$$

The pictures of S_V and B are given by 2.2. Following the desingularization process, we choose coordinates in S_V. The projection into the parameter space restricted to S_V is

$$\tilde{\pi} = (a, -3x^2 - ay, -3y^2 - ax).$$

Observe that $\det(D\tilde{\pi}) \geq 0$ for points in $S_{V,min}$. By following corollary 2.2.1, the corresponding desingularized vector field is

$$\overline{X} = (36xy - a^2)f_a\frac{\partial}{\partial a} + \left(\left(-6y^2 + ax\right)f_a - 6yf_b + af_c\right)\frac{\partial}{\partial x} +$$
$$\left(\left(-6x^2 + ay\right)f_a + af_b - 6xf_c\right)\frac{\partial}{\partial y}.$$

The vector field $\overline{X}$ has an equilibrium point at the origin. The corresponding linearization shows the spectrum

$$\left\{0, +6\sqrt{f_b(0)f_c(0)}, -6\sqrt{f_b(0)f_c(0)}\right\}.$$

Considering the generic conditions on f_b and f_c, and by referring to the center manifold theorem A.3.1, we study the cases where $\overline{X}$ is topologically equivalent to

1. $\overline{X}'(u,v,w) = f_u(u)\frac{\partial}{\partial u} + v\frac{\partial}{\partial v} - w\frac{\partial}{\partial w}$, or
2. $\overline{X}'(u,v,w) = f_u(u,v,w)\frac{\partial}{\partial u} + (v + f_w(u,v,w))\frac{\partial}{\partial w} + (-w + f_v(u,v,w))\frac{\partial}{\partial v}$,

where $f_i(0) = Df_i(0) = 0$ for $i = u,v,w$. We study each case separately.

1. Here we consider that the spectrum of $\overline{X}$ is of the form $\{0, \lambda_1, \lambda_2\}$, $\lambda_1 > 0 > \lambda_2$, so we call it *the center-saddle case*. There exists a 1-dimensional center manifold passing through the origin. Following theorem A.3.2 and noting that

$$\left[u^2 \frac{\partial}{\partial u}, u^{k-1} \frac{\partial}{\partial u}\right] = (k-3)u^k,$$

we have that the $k-$jet of $\overline{X}'$ is smoothly equivalent to

$$\left(\delta_1 u^2 + \delta_2 u^3\right) \frac{\partial}{\partial u} + v \frac{\partial}{\partial v} - w \frac{\partial}{\partial w}$$

for all $k \geq 3$, where $\delta_1 \in \mathbb{R} \setminus \{0\}$, and $\delta_2 \in \mathbb{R}$. With this we can further say that $\overline{X}$ is topologically equivalent to

$$\overline{X}' = \left(\pm u^2 + \delta u^3\right) \frac{\partial}{\partial u} + v \frac{\partial}{\partial v} - w \frac{\partial}{\partial w}, \qquad \delta \in \mathbb{R}. \tag{2.4}$$

Observe that u is the center direction and v, w are the hyperbolic (saddle) directions. Locally, the direction of the center manifold depends on the $\pm$ sign in front of the u^2 term of the normal form (2.4).

2. Now we deal with a 3-dimensional center manifold. The vector field $\overline{X}'$ has spectrum $\{0, \lambda\imath, -\lambda\imath\}$, $\lambda \in \mathbb{R}$, so we call it *the center case*. It is convenient to introduce complex coordinates

$$\begin{aligned} z &= u + \imath w, \\ \overline{z} &= u - \imath w. \end{aligned}$$

In these coordinates we have that the $1-jet$ of $\overline{X}'$ is

$$\overline{X}'_1(u, z, \overline{z}) = \imath \left(z \frac{\partial}{\partial z} - \overline{z} \frac{\partial}{\partial \overline{z}}\right).$$

Following the normal form theorem A.3.2, we write the elements of $\mathcal{H}^k \otimes \mathbb{C}$ as a combination of the monomials $u^{m_1} z^{m_2} \overline{z}^{m_3}$, where $m_1 + m_2 + m_3 = k$, having the relations

$$\begin{aligned}
\left[\overline{X}'_1,\ u^{m_1} z^{m_2} \overline{z}^{m_3} \frac{\partial}{\partial u}\right] &= \imath u^{m_1} z^{m_2} \overline{z}^{m_3} (m_2 - m_3) \frac{\partial}{\partial u}, \\
\left[\overline{X}'_1,\ u^{m_1} z^{m_2} \overline{z}^{m_3} \frac{\partial}{\partial z}\right] &= \imath u^{m_1} z^{m_2} \overline{z}^{m_3} (m_2 - m_3 - 1) \frac{\partial}{\partial z}, \\
\left[\overline{X}'_1,\ u^{m_1} z^{m_2} \overline{z}^{m_3} \frac{\partial}{\partial \overline{z}}\right] &= \imath u^{m_1} z^{m_2} \overline{z}^{m_3} (m_2 - m_3 + 1) \frac{\partial}{\partial \overline{z}}.
\end{aligned}$$

We can choose as a complement of the image of $\left[\overline{X}'_1, -\right]_k$ the space spanned by

$$\left\{u^{k-2m} z^m \overline{z}^m \frac{\partial}{\partial u}\right\}_{m=0}^{m=k/2} \bigcup$$
$$\left\{u^{k-1-2m} z^m \overline{z}^m \left(z \frac{\partial}{\partial z} + \overline{z} \frac{\partial}{\partial \overline{z}}\right),\ \imath u^{k-1-2m} z^m \overline{z}^m \left(z \frac{\partial}{\partial z} - \overline{z} \frac{\partial}{\partial \overline{z}}\right)\right\}_{m=0}^{m=\frac{k-1}{2}}.$$

This base is chosen so that we can easily write the normal form in the original coordinates by identifying $\left(z\frac{\partial}{\partial z}+\overline{z}\frac{\partial}{\partial \overline{z}}\right)$, and $\imath\left(z\frac{\partial}{\partial z}-\overline{z}\frac{\partial}{\partial \overline{z}}\right)$ with $\left(v\frac{\partial}{\partial v}+w\frac{\partial}{\partial w}\right)$, and $\left(v\frac{\partial}{\partial w}-w\frac{\partial}{\partial v}\right)$ respectively. Then, we have that the $k-$th order polynomial normal form of $\overline{X}'$ reads

$$\begin{aligned}\overline{X}' = \overline{X}'_1+\sum_{\ell=2}^{k}&\left(\sum_{j=0}^{2j=\ell}\rho_{\ell j}u^{\ell-2j}(v^2+w^2)^j\frac{\partial}{\partial u}+\right.\\&\left.\sum_{j=0}^{2j+1=l}u^{\ell-1-2j}(v^2+w^2)^j\left(\eta_{\ell j}\left(v\frac{\partial}{\partial v}+w\frac{\partial}{\partial w}\right)+\sigma_{\ell j}\left(v\frac{\partial}{\partial w}-w\frac{\partial}{\partial v}\right)\right)\right),\end{aligned} \tag{2.5}$$

where $\rho_{\ell j}$, $\eta_{\ell j}$, and $\sigma_{\ell j}$ are some nonzero constants. Compare with [38], where the case of a vector field having eigenvalues of its Jacobian equal to $\{\alpha,\pm\imath\}$, $\alpha\neq 0$ is studied.

At this point then, we have two normal forms of the vector field $\overline{X}'$ depending on the eigenvalues of $D_0\overline{X}$. Recall that the solutions of (V,X) are related to the integral curves of $\overline{X}$ and therefore also to the integral curves of $\overline{X}'$. In order to locally identify the coordinates in which we expressed $\overline{X}'$ with the original coordinates (a,b,c,x,y), we perform a linear change of coordinates such that $D_0\overline{X} = D_0\overline{X}'$. This linear transformation is given by

$$\begin{bmatrix}a\\x\\y\end{bmatrix}=\begin{bmatrix}6&0&0\\1&-1&1\\1&1&1\end{bmatrix}\begin{bmatrix}u\\v\\w\end{bmatrix}$$

in the case of the center-saddle vector field (2.4), and

$$\begin{bmatrix}a\\x\\y\end{bmatrix}=\begin{bmatrix}6&0&0\\-1&0&1\\-1&1&0\end{bmatrix}\begin{bmatrix}u\\v\\w\end{bmatrix}$$

in the case of the vector field (2.5). By carrying out the computations, $\overline{X}$ has respectively the $k-$th order local normal form

1. Center-saddle case

$$\begin{aligned}\overline{X} = &\left(\pm a^2+\delta a^3\right)\frac{\partial}{\partial a}+\frac{1}{6}\left(\left(\pm a^2+\delta a^3\right)+a-6y\right)\frac{\partial}{\partial x}+\\&\frac{1}{6}\left(\left(\pm a^2+\delta a^3\right)+a-6x\right)\frac{\partial}{\partial y},\end{aligned} \tag{2.6}$$

where $\delta\in\mathbb{R}$.

2. Center case

$$\begin{aligned}\overline{X} = &\left(\frac{1}{6}a+y\right)\frac{\partial}{\partial x}-\left(\frac{1}{6}a+x\right)\frac{\partial}{\partial y}+6\sum_{\ell=2}^{k}\sum_{j=0}^{2j=\ell}\rho_{\ell j}\left(\frac{a}{6}\right)^{\ell-j}\Delta^j\frac{\partial}{\partial a}+\\&\left(-\sum_{\ell=2}^{k}\sum_{j=0}^{2j=\ell}\rho_{\ell j}\left(\frac{a}{6}\right)^{\ell-j}\Delta^j+\sum_{\ell=2}^{k}\sum_{j=0}^{2j+1=\ell}\left(\frac{a}{6}\right)^{\ell-1-j}\Delta^jA_{\ell,j}\right)\frac{\partial}{\partial x}+\\&\left(-\sum_{\ell=2}^{k}\sum_{j=0}^{2j=\ell}\rho_{\ell j}\left(\frac{a}{6}\right)^{\ell-j}\Delta^j+\sum_{\ell=2}^{k}\sum_{j=0}^{2j+1=\ell}\left(\frac{a}{6}\right)^{\ell-1-j}\Delta^j\overline{A}_{\ell,j}\right)\frac{\partial}{\partial y}\end{aligned} \tag{2.7}$$

where

$$\Delta = \frac{a}{108}\left(a^2 + 6ax + 6ay + 18x^2 + 18y^2\right)$$
$$A_{\ell,j} = \eta_{\ell,j}\left(\frac{a}{6}+x\right) + \sigma_{\ell,j}\left(\frac{a}{6}+y\right)$$
$$\overline{A}_{\ell,j} = \eta_{\ell,j}\left(\frac{a}{6}+y\right) - \sigma_{\ell,j}\left(\frac{a}{6}+x\right), \qquad \eta_{\ell,j}, \sigma_{\ell,j} \in \mathbb{R}$$

The phase portraits of (2.6) and (2.7) are shown in figures 2.19 and 2.20 respectively.

Finally, by following lemma 2.2.1 we can obtain the form of (V, X). Recall that the desingularized vector field is defined by $\overline{X} = \det(D\tilde{\pi})(D\tilde{\pi})^{-1}X$. This means that in principle, once we know $\overline{X}$, X is obtained as $X = \frac{1}{\det(D\tilde{\pi})}D\tilde{\pi}\overline{X}$. Clearly, the map X is not defined at points of the bifurcation set. Away from such a set, X is smoothly equivalent to the smooth map $\pm D\tilde{\pi}\overline{X}$, where the sign $\pm$ depends on the sign of $\det(D\tilde{\pi})$. So, since in this case we have that $\det(D\tilde{\pi}) > 0$ in $S_{V,min}$, the solution curves of (V, X) are obtained from the integral curves of $\overline{X}$ and by the reparametrization

$$b = -3x^2 - ay, \qquad c = -3y^2 - ax.$$

Straightforward computations show that the CDE $(V, X = D\tilde{\pi}\overline{X})$ with a hyperbolic umbilic singularity has the local normal forms as stated in theorem 2.3.1.

2.5.3 Proof of proposition 2.3.1

We detail the proof of the hyperbolic umbilic case. The other cases follow the same methodology. To simplify notation, let $(z_1, z_2, a, b, c) = (x, y, a, b, c,)$.

Recall that for the hyperbolic umbilic

$$S_V = \left\{(a, b, c, x, y) \in \mathbb{R}^5 \mid b = -3x^2 - ay,\ c = -3y^2 + ax\right\},$$

$$S_{V,min} = \left\{(a, b, c, x, y) \in S_V \mid 36xy - a^2 \geq 0,\ x + y > 0\right\},$$

and

$$B = \left\{(a, b, c, x, y) \in S_V \mid 36xy - a^2 = 0\right\}.$$

Let $p = (x_1, y_1, a_1, b_1, c_1) \in S_V$ and $q = (x_2, y_2, a_2, b_2, c_2) \in B$. So we have that the projections $\pi(p)$ and $\pi(q)$ read as

$$\pi(p) = (a_1, -3x_1^2 - a_1y_1, -3y_1^2 - a_1x_1)$$
$$\pi(q) = (a_2, -3x_2^2 - a_2y_2, -3y_2^2 - a_2x_2), \quad a_2^2 = 36x_2y_2.$$

The point $q = \gamma \cap B$ is known. The point p is unknown, it corresponds to a possible arriving point when a finite jump occurs. If such a point p exists, then it is a nontrivial solution of $\pi(p) = \pi(q)$. The easiest case is when $a_2 = 0$. We have

$$\pi(p) = (0, -3x_1^2, -3y_1^2)$$
$$\pi(q) = (0, -3x_2^2, -3y_2^2), \quad 0 = x_2y_2.$$

Here we have two cases: 1) $0 = x_2y_2 \implies x_2 = 0,$ and $y_2 \neq 0$, or 2) $0 = x_2y_2 \implies x_2 \neq 0,$ and $y_2 = 0$.

1. $a_2 = 0,\ x_2 = 0,\ y_2 \neq 0$. We have

$$\begin{aligned} -3x_1^2 &= 0 \\ -3y_1^2 &= -3y_2 \end{aligned}$$

The non trivial solution is $(x_1, y_1) = (0, -y_2)$. So, there is a possible finite jump of the form

$$q_1 = (0, y_2, 0, b_2, c_2) \mapsto p_1 = (0, -y_2, 0, b_2, c_2)$$

2. $a_2 = 0,\ x_2 \neq 0,\ y_2 = 0$. Similarly we have the possible jump

$$q_2 = (x_2, 0, 0, b_2, c_2) \mapsto p_2 = (-x_2, 0, 0, b_2, c_2).$$

Now we check if any of such arriving points are in $S_{V,min}$. The conditions for a point $p = (a, b, c, x, y)$ to be in $S_{V,min}$ are

$$\begin{aligned} -3x^2 - ay - b &= 0 \\ -3y^2 - ax - c &= 0 \\ 36xy - a^2 &\geq 0 \\ x + y &\geq 0. \end{aligned}$$

It is readily seen then that for $a = 0$, p_1 and p_2 are not points in $S_{V,min}$ as the last inequality is not satisfied. Now, we study the case $a_2 \neq 0$. The problem $\pi(p) = \pi(q)$ can be rewritten as the nonlinear simultaneous equation

$$\begin{aligned} -3x_1^2 - a_2y_1 + 3x_2^2 + a_2y_2 &= 0 \\ -3y_1^2 - a_2x_1 + 3y_2^2 + a_2x_2 &= 0. \end{aligned}$$

Since $a_2 \neq 0$ we can write from the first equation

$$y_1 = \frac{-3x_1^2 + 3x_2^2 + a_2y_2}{a_2},$$

and substituting in the second equation we get

$$27x_1^4 - (54x_2^2 + 18a_2y_1)x_2^2 + a_2^3x_1 + 18x_2^2a_2y_2 - a_2^3x_2 + 27x_2^4 = 0.$$

It is not difficult to see that $x_1 = x_2$ is a double root, so we have the factorization

$$(x_1 - x_2)^2(3x_1^2 + 6x_2x_1 + 3x_2^2 - 2a_2y_2) = 0.$$

The roots of $3x_1^2 + 6x_2x_1 + 3x_2^2 - 2a_2y_2 = 0$ are

$$X_\pm = -x_2 \pm \frac{2}{\sqrt{6}}\sqrt{a_2y_2}.$$

The corresponding y_1 solutions are

$$Y_\pm = -y_2 \pm 2\sqrt{6}x_2\sqrt{\frac{y_2}{a_2}}.$$

This is, for a trajectory γ such that $\gamma|B = (x_2, y_2, a_2, b_2, c_2)$, there are possible jumps towards

$$p_1 = \left(-x_2 + \frac{2}{\sqrt{6}}\sqrt{a_2 y_2}, -y_2 + \frac{2\sqrt{6}\sqrt{y_2}x_2}{\sqrt{a_2}}, a_2, b_2, c_2 \right)$$
$$p_2 = \left(-x_2 - \frac{2}{\sqrt{6}}\sqrt{a_2 y_2}, -y_2 - \frac{2\sqrt{6}\sqrt{y_2}x_2}{\sqrt{a_2}}, a_2, b_2, c_2 \right).$$

Just as in the previous case, we shall check if the points $(X_+, Y_+, a_2, b_2, c_2)$, $(X_-, Y_-, a_2, b_2, c_2)$ are contained in $S_{V,min}$. This is, we have to check if the following inequalities are satisfied.

$$\begin{aligned} X_+ + Y_+ &\geq 0, \\ 36X_+Y_+ - a_2^2 &\geq 0, \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} X_- + Y_- &\geq 0, \\ 36X_-Y_- - a_2^2 &\geq 0. \end{aligned} \tag{2.9}$$

In both cases we have the further properties $36x_2y_2 - a_2^2 = 0$ and $x_2 + y_2 \geq 0$ since $(x_2, y_2, a_2, b_2, c_2) \in B$. By substituting the value $y_2 = \frac{a^2}{36x_2}$ in $X_\pm$ and $Y_\pm$ we have

$$\begin{aligned} X_\pm &= -x_2 \pm \frac{a_2^{3/2}}{3\sqrt{6}x_2^{1/2}}, \\ Y_\pm &= -\frac{a_2^2}{36x_2} \pm \frac{2}{\sqrt{6}} x_2^{1/2} a_2^{1/2}. \end{aligned}$$

Now, (2.8) and (2.9) read as

$$\begin{aligned} -x_2 + \frac{a_2^{3/2}}{3\sqrt{6}x_2^{1/2}} - \frac{a_2^2}{36x_2} + \frac{2}{\sqrt{6}} x_2^{1/2} a_2^{1/2} &\geq 0, \\ \left(-x_2 + \frac{a_2^{3/2}}{3\sqrt{6}x_2^{1/2}} \right) \left(-\frac{a_2^2}{36x_2} + \frac{2}{\sqrt{6}} x_2^{1/2} a_2^{1/2} \right) - a_2^2 &\geq 0, \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} -x_2 - \frac{a_2^{3/2}}{3\sqrt{6}x_2^{1/2}} - \frac{a_2^2}{36x_2} - \frac{2}{\sqrt{6}} x_2^{1/2} a_2^{1/2} &\geq 0, \\ \left(-x_2 - \frac{a_2^{3/2}}{3\sqrt{6}x_2^{1/2}} \right) \left(-\frac{a_2^2}{36x_2} - \frac{2}{\sqrt{6}} x_2^{1/2} a_2^{1/2} \right) - a_2^2 &\geq 0, \end{aligned} \tag{2.11}$$

respectively. It is readily seen that (2.11) is not satisfied. Now we focus on (2.10). First we check the conditions for $X_+ \geq 0$ and $Y_+ \geq 0$. We have

$$\begin{aligned} X_+ \geq 0 &\implies x_2^{3/2} \leq \frac{a^{3/2}}{3\sqrt{6}}, \\ Y_+ \geq 0 &\implies \frac{1}{12\sqrt{6}} a^{3/2} \leq x_2^{3/2}. \end{aligned}$$

This is $\frac{1}{12\sqrt{6}}a_2^{3/2} \leq x_2^{3/2} \leq \frac{1}{3\sqrt{6}}a_2^{3/2}$. Of course this would imply that $X_+ + Y_+ \geq 0$. Now we have to check if for such interval $36X_+Y_+ - a^2 \geq 0$. So we have

$$36X_+Y_+ - a^2 = 12\sqrt{6}x_2^{3/2}a_2^{1/2} - \frac{a_2^{7/2}}{3\sqrt{6}x_2^{3/2}} + 4a^2,$$

so we check if

$$-12\sqrt{6}x_2^{3/2}a_2^{1/2} - \frac{a_2^{7/2}}{3\sqrt{6}x_2^{3/2}} + 4a^2 \geq 0$$

in the interval

$$\frac{1}{12\sqrt{6}}a_2^{3/2} \leq x_2^{3/2} \leq \frac{1}{3\sqrt{6}}a_2^{3/2}. \tag{2.12}$$

We have that

$$12\sqrt{6}x_2^{3/2}a_2^{1/2} + \frac{a_2^{7/2}}{3\sqrt{6}x_2^{3/2}} = \frac{216x_2^3a_2^{1/2} + a_2^{7/2}}{3\sqrt{6}x_2^{3/2}},$$

but note that from (2.12) we obtain

$$\frac{1}{4}a_2^{3/2} \leq 3\sqrt{6}x_2^{3/2},$$

so we have

$$\frac{216x_2^3a_2^{1/2} + a_2^{7/2}}{3\sqrt{6}x_2^{3/2}} \geq 864x_2^{3/2}a_2^{-3/2}x_2^{3/2}a_2^{1/2} + 4a_2^2 \geq \frac{72}{\sqrt{6}}a_2^{1/2}x_2^{3/2} + 4a_2^2 \geq 5a_2^2.$$

This means that the inequality

$$-12\sqrt{6}x_2^{3/2}a_2^{1/2} - \frac{a_2^{7/2}}{3\sqrt{6}x_2^{3/2}} + 4a^2 \geq 0$$

can not be satisfied, which implies that $\pi(p) = \pi(q)$ does not have nontrivial solutions in $S_{V,min}$. Therefore, it is not possible to have finite jumps.

2.5.4 Proof of proposition 2.4.1

The constraint differential equation (V, X) is defined by

$$V = \frac{1}{k+1}z^{k+1} + \sum_{i=1}^{k-1}\frac{1}{i}x_iz^i,$$

and $X = \sum_{i=1}^{k-1} X_i\dfrac{\partial}{\partial x_i}$. Then the desingularized vector field reads as

$$\overline{X} = \left(X_1 - \sum_{i=2}^{k-1} z^{i-1}X_i\right)\frac{\partial}{\partial z} + Z\sum_{i=2}^{k-1} X_i\frac{\partial}{\partial x_i},$$

where $Z = kz^{k-1} + \sum_{i=2}^{k-1}(i-1)x_i z^{i-2}$. The constraint manifold reads as

$$S_V = \left\{ (x,z) \in \mathbb{R}^k \mid z^k + \sum_{i=1}^{k-1} x_i z^{i-1} = 0 \right\},$$

while the singularity set is a submanifold of S_V given by

$$B = \{(x,z) \in S_V \mid Z = 0\}.$$

Generically $\overline{X}(0) = X_1(0) \neq 0$, therefore, by the rectification theorem [1], there exists a diffeomorphism $\overline{\Phi}$ such that in a sufficiently small neighborhood of the origin we have

$$\overline{\Phi}_* \overline{X} = \frac{\partial}{\partial z}.$$

This implies that the equivalence in the CDE is given by $\Phi = \tilde{\pi}^{-1} \circ \overline{\Phi} \circ \tilde{\pi}$. Naturally Φ is just continuous, but note that $\Phi|_{S_V \setminus B}$ is a local diffeomorphism.

Chapter 3

Slow fast systems

This chapter presents several techniques which are useful to study the local dynamics of SFSs, and in particular of what we call A_k-SFSs. These slow fast systems have *one fast direction*, and their principal characteristic is that their corresponding slow manifold is given by the critical set of an A_k catastrophe [3, 4].

This chapter is arranged as follows. First, the definition, and a qualitative description of A_k slow fast system is given. Next we briefly recall results on normal forms and transition of slow fast systems along Normally Hyperbolic slow manifolds. After this, we discuss a class of maps called "exponential type maps". These appear frequently in our forthcoming analysis. Following these preliminary results, we provide a formal normal form of A_k-SFSs. Afterwards, we shall briefly recall the Geometric Desingularization method. All these techniques will be used in chapter 4, where we study A_2 (fold), A_3 (cusp), and A_4 (swallowtail) Slow Fast Systems.

3.1 Introduction

We consider, as a slow fast system, a parameter family of vector fields X_ε which in coordinates has the form

$$X_\varepsilon : \begin{cases} x' &= \varepsilon f(x,z,\varepsilon) \\ z' &= g(x,z,\varepsilon), \end{cases} \tag{3.1}$$

where ε is a small positive parameter, this is $0 < \varepsilon \ll 1$, and where f and g are considered of class $\mathcal{C}^\infty$. Throughout this chapter, we study SFSs with one fast variable, that is $z \in \mathbb{R}$. Rescaling the time by $t = \varepsilon\tau$ we can also write (3.1) as

$$\begin{aligned} \dot{x} &= f(x,z,\varepsilon) \\ \varepsilon\dot{z} &= g(x,z,\varepsilon). \end{aligned} \tag{3.2}$$

For $\varepsilon \neq 0$, the systems (3.1) and (3.2) are equivalent. Recall that the limit $\varepsilon \to 0$ of (3.2) is the CDE

$$\begin{aligned} \dot{x} &= f(x,z,0) \\ 0 &= g(x,z,0). \end{aligned}$$

Definition 3.1.1 (Slow manifold)**.** *The set*

$$S = \left\{(x,z) \in \mathbb{R}^{k-1} \times \mathbb{R} \,|\, g(x,z,0) = 0\right\}$$

is called the slow manifold *of* (3.1).

It is now worth recalling the definition of Normally Hyperbolic Invariant Manifolds in the present context.

Definition 3.1.2 (Normally Hyperbolic Invariant Manifold)**.** *Consider a slow fast system given by* (3.1)*. The associated slow (invariant) manifold $S = \{g(x,z,0) = 0\}$ is said to be normally hyperbolic if each point of S is a hyperbolic equilibrium point of X_0.*

NHIMs are relevant in the context of the geometric study of slow fast systems, see for example [16]. It turns out that compact NHIMs persist under $\mathcal{C}^1$ small perturbation of X_0, see appendix A.4. In the particular context presented above, a normally hyperbolic compact subset of the slow manifold S persists as an invariant manifold of the slow fast system X_ε.

A point of S at which S is tangent to the one dimensional fast foliation is called singular, and the set of all singular points is denoted by B, that is

$$B = \left\{(x,z) \in \mathbb{R}^k \times \mathbb{R} \,|\, \frac{\partial g}{\partial z} = 0\right\}.$$

Let us now define the particular slow fast system studied here. Briefly speaking, we shall define a class of slow fast systems by a generic perturbation of the A_k-CDEs of chapter 2.

Definition 3.1.3. *Let $k \geq 1$. An A_k slow fast system (for brevity A_k-SFS) is a smooth vector field obtained as a generic perturbation of an A_k-CDE. More specifically, an A_k-SFS is a singularly perturbed ordinary differential equation given as*

$$\begin{aligned} \dot{x}_1 &= 1 + \varepsilon\tilde{f}_1 \\ \dot{x}_j &= \varepsilon\tilde{f}_j \\ \varepsilon\dot{z} &= g + \varepsilon\tilde{f}_k, \end{aligned} \tag{3.3}$$

where

$$g = g(x,z) = -\left(z^k + \sum_{i=1}^{k-1} x_i z^{i-1}\right),$$

and where $\tilde{f}_i = \tilde{f}_i(x,z,\varepsilon)$, *for all* $i = 1,\dots,k$, *are smooth functions. In this context, generic means that* $\tilde{f}_i(0,0,0) \neq 0$. *Observe that the corresponding limit* $\varepsilon \to 0$ *of* (3.3) *corresponds to a* A_k*-CDE in topological normal form, see section 2.4.*

Equivalently, in the fast time regime, and A_k-SFS can be written as

$$\begin{aligned} x_1' &= \varepsilon(1+\varepsilon\tilde{f}_1) \\ x_j' &= \varepsilon^2\tilde{f}_j \\ z' &= g + \varepsilon\tilde{f}_k. \end{aligned}$$

Our main goal in this chapter is to provide a systematic method to study the flow of an A_k-SFS near the origin. For this task, we use the techniques presented below. Before going any further, a qualitative description of A_k-SFSs is given in the following section.

3.1.1 Qualitative description of A_k slow fast systems

The slow manifold of (3.3) is of dimension $k-1$ and is defined by

$$S = \left\{(x,z) \in \mathbb{R}^{k-1}\times\mathbb{R} \,|\, \varepsilon = 0,\ \left(z^k + \sum_{i=1}^{k-1} x_i z^{i-1}\right) = 0\right\}.$$

For $k \geq 2$, the projection $\pi : S \to \mathbb{R}^{k-1}$, from S to the parameter space (x), has singularities. This is the case we further study, so from now on, let $k \geq 2$. From the expression of S we distinguish two cases, namely

- k is even. The slow manifold S is of "U-shape", for example like in the A_2 (fold) case.
- k is odd. The slow manifold S has an "S-shape", for example, like in the A_3 (cusp) case.

The related CDE of (3.3) has the simple form

$$\begin{aligned} \dot{x}_1 &= 1 \\ \dot{x}_j &= 0 \\ 0 &= g, \end{aligned} \tag{3.4}$$

where we recall that $g = \left(z^k + \sum_{i=1}^{k-1} x_i z^{i-1} \right)$. Then, we expect that the flow of (3.3) is close to the flow of (3.4). Thus, to study the orbits of X within a small neighborhood of the origin, we define the so called "entry and exit sections" as follows

- If k is even then

$$\begin{aligned}\Sigma^{\mathrm{en}} &= \left\{ (x,z) \in \mathbb{R}^k \,|\, x_1 = -x_{1,\mathrm{en}},\, z \in J \right\}\\ \Sigma^{\mathrm{ex}} &= \left\{ (x,z) \in \mathbb{R}^k \,|\, z = -z_{\mathrm{ex}},\, x_1 \in J' \right\},\end{aligned}$$

  where $x_{1,\mathrm{en}}$ and z_{ex} are small positive constants, and where $J \subset \mathbb{R}$ and $J' \subset \mathbb{R}$ are suitable intervals.

- If k is odd then

$$\begin{aligned}\Sigma^{\mathrm{en}} &= \left\{ (x,z) \in \mathbb{R}^k \,|\, x_1 = -x_{1,\mathrm{en}},\, z \in J \right\}\\ \Sigma^{\mathrm{ex}} &= \left\{ (x,z) \in \mathbb{R}^k \,|\, x_1 = -x_{1,\mathrm{ex}},\, z \in J' \right\},\end{aligned}$$

  where $x_{1,\mathrm{en}}$ and $x_{1,\mathrm{ex}}$ are small positive constants, and where $J \subset \mathbb{R}$ and $J' \subset \mathbb{R}$ are suitable intervals.

From these sections, a transition Π is defined as follows.

Definition 3.1.4. *Let X be an A_k-SFS and let Σ^{en}, Σ^{ex} be sections as above. A transition Π is a map $\Pi : \Sigma^{\mathrm{en}} \to \Sigma^{\mathrm{ex}}$ induced by the flow of X.*

See fig. 3.1 for a description of the sections and the transition defined above.

Observe in fig. 3.1 that the trajectories of X spend a long time along Normally Hyperbolic regions of S. Accordingly, before studying the A_k-SFS near the non-hyperbolic singularity at the origin, let us discuss its flow near regular parts of the manifold S.

3.1.2 Regular slow fast systems

In this section, the dynamics of a regular slow fast system is studied. We remark that only SFS with one fast variable are considered in this document. To be more precise we have the following.

Definition 3.1.5. *Let X_ε denote a slow fast system and S its slow manifold. Then X_ε is said to be (locally) regular around a point $p_0 \in S$, if S is normally hyperbolic in a neighborhood of p_0.*

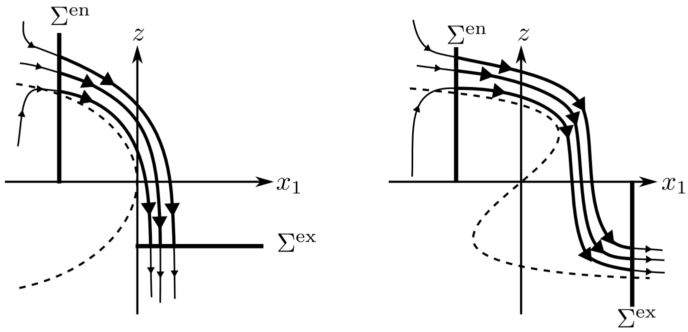


Figure 3.1: Qualitative picture of the sections Σ^{en}, Σ^{ex} and of a transition Π. Left: for k even. Right: for k odd. A transition is a map from the entry section Σ^{en} to the section Σ^{ex} induced by the flow of the vector field X. Note that the trajectories depicted spend a long time along normally hyperbolic regions of the slow manifold.

The slow vector field

For simplicity, let us consider a slow fast system written as

$$X_\varepsilon = \sum_{i=1}^{m} \varepsilon f_i(x,z,\varepsilon)\frac{\partial}{\partial x_i} + H(x,z,\varepsilon)\frac{\partial}{\partial z}, \tag{3.5}$$

where $x \in \mathbb{R}^m$, $z \in \mathbb{R}$, and as usual $0 < \varepsilon \ll 1$. Furthermore, assume that $f = (f_1, \ldots, f_m)$ is such that $f(0,0,0) \neq 0$; and $H(0,0,0) = 0$, with $\frac{\partial H}{\partial z}(0,0,0) < 0$. Thus X_ε is regular around $0 \in \mathbb{R}^{m+2}$. From the defining assumptions of (3.5), the slow manifold S is normally hyperbolic in a sufficiently small neighborhood of the origin. By looking at the Jacobian of X_ε it follows that there exists an $m+1$ dimensional a center manifold. Since X_ε is smooth, we can choose a $\mathcal{C}^\ell$ center manifold $\mathcal{W}^C$ for any $\ell < \infty$. This choice of center manifold is given as a graph $z = \phi(x,\varepsilon)$ where ϕ is a $\mathcal{C}^\ell$ function.

Remark 3.1.1. *Along the rest of the document we frequently make use of a finite class of differentiability. As it is customary in the present context, when we say that a manifold (or a map) is $\mathcal{C}^\ell$, we mean that such a manifold (or map) is ℓ-differentiable for ℓ as large as necessary.*

The slow manifold S is naturally given by the restriction $\mathcal{W}^C|_{\varepsilon=0} = S$. Next, let us consider the vector field $\frac{1}{\varepsilon}X_\varepsilon(x,\phi,\varepsilon)$. Since $\mathcal{W}^C$ is locally invariant, it follows that $\frac{1}{\varepsilon}X_\varepsilon$ is tangent to $\mathcal{W}^C$. Therefore the vector field

$$X^{slow} = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}X_\varepsilon(x,\phi,\varepsilon),$$

is tangent to S at each point of S, and we call it *the slow vector field*. We remark that the slow vector field X^{slow} is only well defined whenever ϕ is invertible.

The slow divergence integral

Associated to a regular slow fast system and the corresponding slow vector field, the *slow divergence integral* is defined here. For this, let Σ^- and Σ^+ be two sections which are transversal to the flow of X_ε given by (3.5). For $\varepsilon \neq 0$ but sufficiently small, these sections are also transversal to the slow manifold S. Let γ_ε be a solution curve of X_ε chosen along a center manifold $\mathcal{W}^C$, thus γ_ε is transversal to the sections Σ^- and Σ^+. In the limit $\varepsilon = 0$, the curve γ_0 is a curve along the slow manifold S. The idea now is to borrow the well-known divergence theorem [33] to get some sense on how the trajectories of X_ε are attracted to S (recall that we made the assumption $\frac{\partial H}{\partial z} < 0$). The divergence of X_ε (given by (3.5)) reads as

$$\operatorname{div} X_\varepsilon = \frac{\partial H(x,z,\varepsilon)}{\partial z} + O(\varepsilon).$$

We can now take the integral of $\operatorname{div} X_\varepsilon$ along the orbit γ_ε of X_ε parametrized by the fast time τ, we have

$$\int_{\gamma_\varepsilon} \operatorname{div} X_\varepsilon \, d\tau = \int_{\gamma_\varepsilon} \left(\frac{\partial H(x,z,\varepsilon)}{\partial z} + O(\varepsilon) \right) d\tau. \tag{3.6}$$

Let us now define the *slow divergence integral* by

$$I(t) = \int_{\gamma_0} \operatorname{div} X_0 \, dt,$$

where t is the slow time defined by the slow vector field X^{slow}. Our goal then is to relate the divergence integral (3.6) with I.

Proposition 3.1.1. *Under the assumptions made in this section, we have that*

$$\int_{\gamma_\varepsilon} \operatorname{div} X_\varepsilon \, d\tau = \frac{1}{\varepsilon} \left(I(t) + o(1) \right),$$

where $I(t)$ is the slow divergence integral.

Proof. Recall that the expression of the slow vector field reads as $X^{slow} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} X_\varepsilon(x, \phi, \varepsilon)$, where $\phi = \phi(x, \varepsilon)$ is a $\mathcal{C}^\ell$ function. By our assumptions, the curve γ_ε is transversal to the sections Σ^- and Σ^+ for ε small enough. Without loss of generality we can assume that γ_ε is parametrized by x_1. Then let x_1^- and x_1^+ be defined by $\gamma_\varepsilon(x_1^-) = \gamma_\varepsilon \cap \Sigma^-$ and $\gamma_\varepsilon(x_1^+) = \gamma_\varepsilon \cap \Sigma^+$. Next, the integral of the divergence of X_ε along γ_ε from Σ^- to Σ^+ reads as

$$\begin{aligned}\int_{\gamma_\varepsilon} \operatorname{div} X_\varepsilon \, d\tau &= \frac{1}{\varepsilon} \int_{x_1^-}^{x_1^+} \left(\frac{\partial H(x,z,0)}{\partial z} + O(\varepsilon) \right) \frac{dx_1}{f_1(x,z,0) + o(1)} \\ &= \frac{1}{\varepsilon} \left(\int_{x_1^-}^{x_1^+} \frac{\partial H(x,z,0)}{\partial z} \frac{dx_1}{f_1(x,z,0)} + o(1) \right) \\ &= \frac{1}{\varepsilon} \left(\int_{\gamma_0} \operatorname{div} X_0 dt + o(1) \right),\end{aligned}$$

where t is the slow time induced by X^{slow}, which in coordinates means that $\frac{dx_1}{dt} = f_1$. □

Observe that the slow divergence integral is a first order approximation of the divergence along orbits of X_ε. This will useful when presenting our main result in section 3.4.

Normal form and transition of a regular slow fast system

Now we consider the problem of finding a suitable normal form of a regular SFS.

Proposition 3.1.2. *Consider a regular slow fast system on $\mathbb{R}^{m+2}$ given by*

$$X_\varepsilon = \varepsilon(1 + f_1)\frac{\partial}{\partial u} + \sum_{j=1}^{m} \varepsilon g_j \frac{\partial}{\partial v_j} + H\frac{\partial}{\partial z},$$

where $(u, v_1, \ldots, v_m, z, \varepsilon) \in \mathbb{R}^{m+3}$; where the functions $f_1 = f_1(u, v, z, \varepsilon)$ and $g_j = g_j(u, v, z, \varepsilon)$, for $1 \leq j \leq m$, are smooth and where the function $H = H(u, v, z, \varepsilon)$ is smooth with $H(0,0,0,0) = 0$ and $\frac{\partial H}{\partial z}(0,0,0,0) < 0$. Then, the vector field X is $\mathcal{C}^\ell$-equivalent to a normal form given by

$$X_\varepsilon^N = \varepsilon\frac{\partial}{\partial U} + \sum_{j=1}^{m} 0\frac{\partial}{\partial V_j} - Z\frac{\partial}{\partial Z}, \tag{3.7}$$

where $\{Z = 0\}$ corresponds to a choice of the center manifold $\mathcal{W}^C$ of X_ε.

Proof. See section 3.5.1 □

Motivated by proposition 3.1.2, the dynamics of (3.7) are now discussed. The slow manifold S, corresponding to the normal form (3.7), is given by

$$S = \{\varepsilon = 0, \; Z = 0\}.$$

Furthermore, we can parametrize the solution of (3.7) by U. Let us define the sections

$$\Sigma^{-} = \left\{(U, V, Z, \varepsilon) \in \mathbb{R} \times \mathbb{R}^{m} \times \mathbb{R} \times \mathbb{R} \,|\, U = U^{-}\right\}$$
$$\Sigma^{+} = \left\{(U, V, Z, \varepsilon) \in \mathbb{R} \times \mathbb{R}^{m} \times \mathbb{R} \times \mathbb{R} \,|\, U = U^{+}\right\},$$

where $U^{-} < U^{+}$. The sections Σ^{-} and Σ^{+} are transversal to the manifold S and therefore, for $\varepsilon \neq 0$, are also transversal to the flow of (3.7). Associated to these sections, we define the transition

$$\begin{aligned} \Pi : \Sigma^{-} &\to \Sigma^{+} \\ (V, Z, \varepsilon) &\mapsto (\tilde{V}, \tilde{Z}, \tilde{\varepsilon}). \end{aligned}$$

To compute the component $\tilde{Z}$ we only need to integrate $\frac{\partial dZ}{\partial dU} = -\frac{1}{\varepsilon} Z$. Then it follows that $\tilde{Z} = Z(T)$, where T is the time to go from Σ^{-} to Σ^{+}, which is $T = U_f - U_i$. Since (3.7) is very simple to integrate we get

$$\begin{aligned} \tilde{V} &= V \\ \tilde{Z} &= Z \exp\left(-\frac{1}{\varepsilon}(U_f - U_i)\right) \\ \tilde{\varepsilon} &= \varepsilon. \end{aligned}$$

Observe the particular format of the transition Π. The Z component is an *exponential* contraction towards the center manifold $\{Z = 0\}$. Maps with this characteristic will appear frequently in our text. Thus, we proceed in section 3.2 by discussing in a rather general way, the properties of such maps.

3.2 Exponential type maps

In this section, we discuss a particular type of function which will be found and used frequently throughout the main text. First, however, let us give two preliminary definitions.

Definition 3.2.1 ($\mathcal{C}^{\ell}$-admissible function)**.** *Let $U \in \mathbb{R}^{n}$. A function $f : \mathbb{R}^{n} \to \mathbb{R}$ is said to be a $\mathcal{C}^{\ell}$-*admissible function* if f is $\mathcal{C}^{\ell}$-smooth away form the origin (for any $\ell > 0$), $\mathcal{C}^{0}$ at the origin and if for all $n_i \in \mathbb{N}$ and $n_i < \ell$, there exists an $N(n_i) \in \mathbb{N}$ such that*

$$\frac{\partial^{n_i} f}{\partial U_i^{n_i}} \in O\left(U_i^{-N(n_i)}\right), \qquad \text{as } U_i \to 0.$$

Now, we define a particular type of differentiability. For this we need to extend the common concept of monomial. In our context, a monomial, e.g. in two variables, $\omega(u,v)$ is any expression of the form $u^\alpha v^\beta$ or of the form $u^\alpha(\ln v)^\beta$, with $\alpha, \beta \in \mathbb{R}$. In general, if we let $u \in \mathbb{R}^m$ and $v \in \mathbb{R}^n$, we allow a monomial to be any expression of the type $u^p(\ln v)^q$, where $u^p = u_1^{p_1} \cdots u_m^{p_m}$ and $(\ln v)^q = (\ln v_1)^{q_1} \cdots (\ln v_n)^{q_n}$. We note that these monomials are admissible functions.

Definition 3.2.2 ($\mathcal{C}^\ell$ function with respect to monomials)**.** *Let $(U,V) \in \mathbb{R}^m \times \mathbb{R}^n$. We say that a function $f(U,V)$ is $\mathcal{C}^\ell$-function with respect to a monomial $\omega(U)$, if f is $\mathcal{C}^\ell$ w.r.t. V in a neighborhood of $0 \in \mathbb{R}^n$, and if there is a quadrant $\mathcal{U} = [0,u_1) \times \cdots \times [0,u_n) \subset \mathbb{R}^m$ where the monomial $\omega(U)$ is defined and such that the function $\tilde{f}(\omega,U,V) = f(U,V)$ is $\mathcal{C}^\ell$ with respect to ω in $\mathcal{U}$. Similarly, the function f is said to be a $\mathcal{C}^\ell$-function with respect to monomials $\omega_1,\ldots,\omega_s$ if there is a quadrant $\mathcal{U}$ where the monomials are defined and such that the function $\tilde{f}(\omega_1,\ldots,\omega_s,U,V) = f(U,V)$ is $\mathcal{C}^\ell$ with respect to $\omega_1,\ldots,\omega_s$ in $\mathcal{U}$.*

Observe that a function f which is differentiable w.r.t monomials is an admissible function. As an example, consider $f(U) = U_1 \ln U_1 \phi(U)$ where $\phi(U)$ is smooth. This function is smooth away from $U = 0$ and C^0 at the origin. However, it is not differentiable w.r.t. U_1 at $U_1 = 0$ but it is differentiable with respect to $\omega = U_1 \ln U_1$ at $\omega = 0$.

Let $V \in \mathbb{R}^m$, $Z \in \mathbb{R}$, and as usual ε denotes a small parameter.

Definition 3.2.3 (Exponential type function)**.** *A function $D(V,Z,\varepsilon)$ is called of exponential type if it has the following form*

$$D(V,Z,\varepsilon) = \mathcal{B}(V,\varepsilon) + Z \exp\left(-\frac{\mathcal{A}(V,\varepsilon) + \Phi(V,\varepsilon,Z)}{\varepsilon}\right), \tag{3.8}$$

where $\mathcal{A}$ and $\mathcal{B}$, are $\mathcal{C}^\ell$ admissible functions with $\mathcal{A} > 0$, and $\mathcal{B}(V,0) = 0$; and where Φ is $\mathcal{C}^\ell$ in z and $\mathcal{C}^\ell$ w.r.t. monomials of (V,ε) with $\Phi(V,0,Z) = 0$. We distinguish two particular cases

1. *the exponential type function D is* without shift *if $\mathcal{B} \equiv 0$.*

2. *the exponential type function D is* linear *if $\Phi(V,Z,\varepsilon) \equiv \Phi(V,\varepsilon)$*

Remark 3.2.1. *Given a function D and if it is of exponential type, the representation of D is unique in the sense that all the functions in r.h.s of* (3.8) *are computable from D. In fact*

$$\begin{aligned}
\mathcal{B} &= D(V,0,\varepsilon) \\
\mathcal{A} &= \lim_{Z\to 0}\left(-\varepsilon \ln\left(\frac{D(V,Z,\varepsilon)-D(V,0,\varepsilon)}{Z}\right)\right) \\
\Phi &= -\varepsilon \ln\left(\frac{D(V,Z,\varepsilon)-D(V,0,\varepsilon)}{Z}\right) - \mathcal{A}.
\end{aligned}$$

We want to study the scenario where we have to compose D with some other functions and want to keep the exponential type structure. To be more precise, we consider D as an $(V,\varepsilon)-$parameter family of functions (in Z) and compose it with a $(V,\varepsilon)-$parameter family of diffeomorphisms $\Psi_{(V,\varepsilon)}$ on $\mathbb{R}$.

Proposition 3.2.1 (Composition on the left). *Let $\Psi_{(V,\varepsilon)} : \mathbb{R} \to \mathbb{R}$ be a family of diffeomorphisms, and let D be an exponential type function. Then, the composition $\Psi_{(V,\varepsilon)} \circ D$ is also of exponential function of the form*

$$\tilde{D} = \tilde{\mathcal{B}}(V,\varepsilon) + Z\exp\left(-\frac{\mathcal{A}(V,\varepsilon)+\tilde{\Phi}(V,Z,\varepsilon)}{\varepsilon}\right).$$

Proof. Let us simplify the notation by writing $\Psi = \Psi_{(V,\varepsilon)}$. Since Ψ is a diffeomorphism we can write $\Psi(a+b) = \Psi(a) + C(1+\psi(a,b))b$, near $b=0$, with ψ a $\mathcal{C}^\ell$ function such that $\psi(a,0)=0$ and with $C>0$. Then we have

$$\begin{aligned}
\Psi \circ D(z) &= \Psi\left(\mathcal{B} + Z\exp\left(-\frac{\mathcal{A}+\Phi}{\varepsilon}\right)\right) \\
&= \Psi(\mathcal{B}) + C(1+\psi(V,Z,\varepsilon))Z\exp\left(-\frac{\mathcal{A}+\Phi}{\varepsilon}\right).
\end{aligned}$$

Since $C>0$ we can take the logarithm of $C(1+\psi(V,Z,\varepsilon))$ and then we have

$$\begin{aligned}
\Psi \circ D(z) &= \Psi(\mathcal{B}) + \exp(\ln(C(1+\psi))Z\exp\left(-\frac{\mathcal{A}+\Phi}{\varepsilon}\right) \\
&= \Psi(\mathcal{B}) + Z\exp\left(-\frac{\mathcal{A}+\Phi+\varepsilon\ln(C(1+\psi)}{\varepsilon}\right).
\end{aligned}$$

The result is obtained by setting $\tilde{\mathcal{B}} = \Psi(\mathcal{B})$ and $\tilde{\Phi} = \Phi + \varepsilon\ln(C(1+\psi)$. □

Proposition 3.2.2 (Composition on the right). *Let $\Psi_{(V,\varepsilon)} : \mathbb{R} \to \mathbb{R}$ be a family of diffeomorphisms with no shift, that is $\Psi_{(V,\varepsilon)}(0) = 0$ for all (V,ε), and let D be an exponential type function. Then, the composition $D \circ \Psi_{(V,\varepsilon)}$ is also of exponential function of the form*

$$\tilde{D} = \tilde{\mathcal{B}}(V,\varepsilon) + Z\exp\left(-\frac{\mathcal{A}(V,\varepsilon)+\tilde{\Phi}(V,Z,\varepsilon)}{\varepsilon}\right).$$

Proof. Let us simplify the notation by writing $\Psi = \Psi_{(V,\varepsilon)}$. Since $\Psi(0) = 0$ we can write $\Psi(z) = C(1 + O(z))z$ with $C > 0$. Then we have

$$\begin{aligned} D \circ \Psi(z) &= D(C(1+O(z))z) = \mathcal{B}(V,\varepsilon) + C(1+O(z))z \exp\left(-\frac{\mathcal{A}(V,\varepsilon) + \Phi(V,\varepsilon,\Psi)}{\varepsilon}\right) \\ &= \mathcal{B}(V,\varepsilon) + z \exp\left(-\frac{\mathcal{A}(V,\varepsilon) + \Phi(V,\varepsilon,\Psi) + \varepsilon \ln(C(1+O(z)))}{\varepsilon}\right). \end{aligned}$$

The result then is obtained by setting $\tilde{\Phi} = \Phi(V,\varepsilon,\Psi) + \varepsilon \ln(C(1+O(z)))$. $\square$

Remark 3.2.2. *If we want the composition $\Pi \circ \Psi_{(V,\varepsilon)}$ to be of exponential type, the family $\Psi_{(V,\varepsilon)}$ cannot be arbitrary. In order to preserve the structure, $\Psi_{(V,\varepsilon)}$ should satisfy the hypothesis of proposition 3.2.2. In corollary 3.2.2 we show a particular case in which the diffeomorphism Ψ can have a shift and yet preserve the structure of the exponential type function.*

Let us proceed by presenting a couple of useful corollaries.

Corollary 3.2.1. *Let D_1 and D_2 be two exponential type functions of the form*

$$\begin{aligned} D_1(V,Z,\varepsilon) &= Z \exp\left(-\frac{\mathcal{A}_1(V,\varepsilon) + \Phi_1(V,Z,\varepsilon)}{\varepsilon}\right) \\ D_2(V,Z,\varepsilon) &= \mathcal{B}_2(V,\varepsilon) + Z \exp\left(-\frac{\mathcal{A}_2(V,\varepsilon) + \Phi_2(V,Z,\varepsilon)}{\varepsilon}\right), \end{aligned}$$

that is, D_1 is an exponential type function with no shift. Then $D_2 \circ D_1$ is an exponential type function.

Corollary 3.2.2. *Let D_1 and D_2 be two exponential type functions with D_2 linear, this is*

$$\begin{aligned} D_1(V,Z,\varepsilon) &= \mathcal{B}_1(V,\varepsilon) + Z \exp\left(-\frac{\mathcal{A}_1(V,\varepsilon) + \Phi_1(V,Z,\varepsilon)}{\varepsilon}\right) \\ D_2(V,Z,\varepsilon) &= \mathcal{B}_2(V,\varepsilon) + Z \exp\left(-\frac{\mathcal{A}_2(V,\varepsilon)}{\varepsilon}\right). \end{aligned}$$

Then the composition $D_2 \circ D_1$ is of exponential type.

Later in this document, we will find transitions in which a component is given by an exponential type transition. To be more specific, let $X(V,Z,\varepsilon)$ be a given vector field on $\mathbb{R}^{m+2}$, and let Σ_0 and Σ_1 be codimension one subsets of $\mathbb{R}^{m+2}$ which are transversal to the flow of X. For the moment it is sufficient to think of a section Σ_i given by $\{V_j = v_0\}$ or by $\{\varepsilon = \varepsilon_0\}$ with v_0 and ε_0 fixed constants. Induced from definition 3.2.3 we then have the following.

Definition 3.2.4 (Exponential type transition). *A transition* $\Pi : \Sigma_0 \to \Sigma_1$ *is called of exponential type if and only if its* Z*-component is an exponential type function. This is, an exponential type transition is of the form*

$$\begin{aligned}\Pi(V, Z, \varepsilon) &= (G, D, H)\\ &= \left(G(V, \varepsilon),\, \mathcal{B}(V, \varepsilon) + Z \exp\left(-\frac{\mathcal{A}(V, \varepsilon) + \Phi(V, Z, \varepsilon)}{\varepsilon} \right),\, H(V, \varepsilon) \right),\end{aligned}$$

where $G : \mathbb{R}^{m+1} \to \mathbb{R}^m$ *and* $H : \mathbb{R}^{m+1} \to \mathbb{R}$ *are* $\mathcal{C}^\ell$ *with* $G(V, 0) = V$ *and* $H(V, 0) = 0$*; where* A*,* B *and* Φ *are* $\mathcal{C}^\ell$*-admissible functions. The names exponential type transition with no shift and linear are inherited as well from the type of* D*.*

Let us now discuss a particular situation which will be useful later. Suppose X is a given vector field on $\mathbb{R}^{m+2}$, as above, and let Σ_i with $i = 0, 1, 2, 3, 4, 5$ be sections which are all transversal to the flow of X. Assume that X induces exponential type transitions $\Pi_i : \Sigma_{i-1} \to \Sigma_i$ with $i = 1, 2, 3, 4, 5$ of the following form

1. Π_1 is with no shift and linear
2. Π_2 is with no shift
3. Π_3 is a general diffeomorphism
4. Π_4 is with no shift
5. Π_5 is with no shift and linear.

We need to show that the composition of all these five maps is an exponential type transition.

Proposition 3.2.3. *Let* $\Pi_i : \Sigma_{i-1} \to \Sigma_i$ *as described above. Then the composition* $\Pi = \Pi_5 \circ \Pi_4 \circ \Pi_3 \circ \Pi_2 \circ \Pi_1$ *is an exponential type map of the form*

$$\Pi = \left(\tilde{G}(V, \varepsilon),\, \tilde{\mathcal{B}}(V, \varepsilon) + Z \exp\left(-\frac{\tilde{\mathcal{A}}(V, \varepsilon) + \tilde{\Phi}(V, Z, \varepsilon)}{\varepsilon} \right),\, \tilde{H}(V, \varepsilon) \right),$$

where $\tilde{\mathcal{A}} = \mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_4 + \mathcal{A}_5$*.*

Proof. Let us write each of the transitions as follows.

1. $\Pi_1(V, Z, \varepsilon) = (G_1, D_1, H_1) = \left(G_1, Z \exp\left(-\frac{\mathcal{A}_1(V, \varepsilon)}{\varepsilon} \right), H_1 \right)$
2. $\Pi_2(V, Z, \varepsilon) = (G_2, D_2, H_2) = \left(G_2, Z \exp\left(-\frac{\mathcal{A}_2(V, \varepsilon) + \Phi_2(V, Z, \varepsilon)}{\varepsilon} \right), H_2 \right)$

3. $\Pi_3(V,Z,\varepsilon) = (G_3, D_3, H_3)$
4. $\Pi_4(V,Z,\varepsilon) = (G_4, D_4, H_4) = \left(G_4, Z\exp\left(-\frac{\mathcal{A}_4(V,\varepsilon)+\Phi_4(V,Z,\varepsilon)}{\varepsilon}\right), H_4\right)$
5. $\Pi_5(V,Z,\varepsilon) = (G_5, D_5, H_5) = \left(G_5, Z\exp\left(-\frac{\mathcal{A}_5(V,\varepsilon)}{\varepsilon}\right), H_5\right)$

For brevity let $\Pi_2 \circ \Pi_1 = (\tilde{G}_2, \tilde{D}_2, \tilde{H}_2)$. Then we have

$$(\tilde{G}_2, \tilde{D}_2, \tilde{H}_2) =$$
$$\left(G_2(G_1,H_1), D_1\exp\left(-\frac{\mathcal{A}_2(G_1,H_1)+\Phi_2(G_1,D_1,H_1)}{H_1}\right), H_2(G_1,H_1)\right).$$

Now, we take care only of the Z-component of the composition $\Pi_2 \circ \Pi_1$. From the hypothesis on G_1 and H_1 we can write $G_1 = V + O(\varepsilon)$ and $H_1 = \alpha\varepsilon(1+O(\varepsilon))$ with $\alpha > 0$, then

$$\tilde{D}_2 = Z\exp\left(-\frac{\mathcal{A}_1(V,\varepsilon)+\mathcal{A}_2(V,\varepsilon)+\bar{\Phi}_2(V,Z,\varepsilon)}{\varepsilon}\right),$$

where we have gathered in $\bar{\Phi}_2$ the function Φ_1 and the terms resulting from taking $G_1 = V + O(\varepsilon)$ and $H_1 = \alpha\varepsilon(1+O(\varepsilon))$. In a similar way, letting $\Pi_5 \circ \Pi_4 = (\tilde{G}_5, \tilde{D}_5, \tilde{H}_5)$ we get

$$\tilde{D}_5 = Z\exp\left(-\frac{\mathcal{A}_4(\varepsilon)+\mathcal{A}_5(\varepsilon)+\bar{\Phi}_5(V,Z,\varepsilon)}{\varepsilon}\right)$$

Next, and following similar arguments as above, we know from proposition 3.2.1 that the composition $\Pi_{321} = \Pi_3 \circ \Pi_2 \circ \Pi_1$ is of exponential type *with shift*. Finally since the transition $\Pi_{54} = \Pi_5 \circ \Pi_4$ is of exponential type with no shift, and using proposition 3.2.1, we have that $\Pi_{54} \circ \Pi_{321}$ is an exponential type transition as claimed in the proposition.

Remark 3.2.3. *In the case where Π_3 is an exponential type map, we get a similar result with $\tilde{\mathcal{A}} = \mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3 + \mathcal{A}_4 + \mathcal{A}_5$.*

□

3.3 Formal normal form of A_k slow fast systems

In several works dealing with the geometric desingularization of slow fast systems, e.g. [8, 13, 27, 26, 28, 36], the blow-up technique is directly applied to the vector field being studied. Here, however, we propose an intermediate step that simplifies the analysis. Namely, before applying the blow-up we normalize the A_k-SFS (given in definition 3.1.3). Briefly speaking, if we consider a small neighborhood of the origin then we may regard an A_k-SFS as a vector field X written as

$$X : \left\{ \begin{array}{rl} x_1' = & \\ x_j' = & \\ z' = & \\ \varepsilon' = & \end{array} \right. \underbrace{\boxed{\begin{array}{c} \varepsilon \\ 0 \\ g \\ 0 \end{array}}}_{F} + \underbrace{\boxed{\begin{array}{l} \varepsilon^2 \tilde{f}_1 \\ \varepsilon^2 \tilde{f}_j \\ \varepsilon \tilde{g} \\ 0 \end{array}}}_{P}$$

where F is "the principal part" of X and P is a "perturbation term". The vector field F contains the essential information of X, in particular it contains the relevant expression of the slow manifold S. Thus, we want to find a transformation $\Phi : \mathbb{R}^{k+1} \to \mathbb{R}^{k+1}$ which conjugates X to its principal part F.

In fact, we can do better by considering X to be given as $X = F + P$, where

$$\begin{aligned} F &= \varepsilon \frac{\partial}{\partial x_1} + \sum_{i=2}^{k-1} 0 \frac{\partial}{\partial x_i} + g \frac{\partial}{\partial z} + 0 \frac{\partial}{\partial \varepsilon}, \\ P &= \sum_{i=1}^{k-1} \varepsilon \tilde{f}_i + \varepsilon \tilde{g} \frac{\partial}{\partial z} + 0 \frac{\partial}{\partial \varepsilon}, \end{aligned} \tag{3.9}$$

and where

$$g = -\left(z^k + \sum_{j=1}^{k-1} x_j z^{j-1} \right).$$

Remark 3.3.1. *We shall show that F is (formally) stable in the following sense: any ε-perturbation $X = F + P$ (see* (3.9)*) of F is formally conjugate to F.*

The procedure of normalizing the vector field X is motivated by [29], where normal forms of quasihomogeneous vector fields are investigated. The formulation of this section is based on the concepts of quasihomogeneous polynomials and vector fields [4, 29]; refer also to appendix A.5.

In brief terms (see more details in appendix A.5), let $y \in \mathbb{R}^n$ and let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function. The function f is said to be *quasihomogeneous of quasidegree δ and type $r = (r_1, \ldots, r_n)$* if and only if for every $\lambda \in \mathbb{R}$ we have

$$f(\lambda^{r_1} y_1, \ldots, \lambda^{r_n} y_n) = \lambda^{\delta} f(y_1, \ldots, y_n).$$

A vector field $Y = \sum_{i=1}^{n} f_i \frac{\partial}{\partial y_i}$ is said to be quasihomogeneous of quasidegree γ and type $r = (r_1, \ldots, r_n)$ if each of the functions f_i is quasihomogeneous of quasidegree $\gamma + r_i$ and type r. An r-quasihomogeneous vector field Y is said to be of *quasiorder* γ if all its components are functions of quasidegree $\sigma + r_i$ or higher.

Following the previous description, the vector field F is quasihomogeneous of quasidegree $k-1$ and type $r = (k, k-1, \ldots, 1, 2k-1)$. From now on, we fix the type of quasihomogeneity r. For shortness, we shall write that a polynomial (or a vector field) is r-quasihomogeneous if its type of quasihomogeneity is r.

Definition 3.3.1 (Good perturbation). *Let F be an r-quasihomogeneous vector field of quasidegree $k-1$. A good perturbation X of F is a smooth vector field $X = F + P$, where $P = P(x_1, \ldots, x_{k-1}, z, \varepsilon)$ has the following properties*

1. $P = \sum_{i=1}^{k-1} P_i \dfrac{\partial}{\partial x_i} + P_k \dfrac{\partial}{\partial z} + 0 \dfrac{\partial}{\partial \varepsilon}$ *is a smooth vector field of quasiorder k. That is, each function P_j is r-quasihomogeneous of quasidegree $k + r_j$ or higher.*

2. $P(x_1, \ldots, x_{k-1}, z, 0) = 0$.

Remark 3.3.2. *A straightforward computation shows that a good perturbation $X = F + P$ of F satisfies*

$$P = \sum_{i=1}^{k-1} \varepsilon \bar{P}_i \frac{\partial}{\partial x_i} + \varepsilon \bar{P}_k \frac{\partial}{\partial z} + 0 \frac{\partial}{\partial \varepsilon},$$

where $\bar{P}_1(0) = 0$ and $\bar{P}_j$ is any r-quasihomogeneous polynomial of quasiorder 0.

Now, we need the operators d, d^* and $\Box$ as introduced in [29]. Let us denote by $\mathcal{H}_\gamma$ the space of $k+1$ dimensional r-quasihomogeneous vector fields of degree γ whose elements are of the form

$$U = \sum_{i=1}^{k+1} U_i \frac{\partial}{\partial y_i},$$

where $U_{k+1} = 0$ and $U_i(y_1, \ldots, y_k, 0) = 0$ for all i. We identify then y with (x, z, ε).

Definition 3.3.2 (The operators d, d^* and $\Box$). *The operator $d : \mathcal{H}_\gamma \to \mathcal{H}_{\gamma+k-1}$ is defined by $d(U) = [F, U]$ for any $U \in \mathcal{H}_\gamma$ and where $[\cdot, \cdot]$ denotes the Lie bracket. The operator d^* is the adjoint operator of d with respect to the inner product of definition A.5.7. That is, given $U \in \mathcal{H}_\gamma$, $V \in \mathcal{H}_{\gamma+k-1}$ we have*

$$\langle d(U), V \rangle_{r,\gamma+k-1} = \langle U, d^*(V) \rangle_{r,\gamma}$$

For any quasidegree $\beta > k-1$, the self adjoint operator $\Box_\beta : \mathcal{H}_\beta \to \mathcal{H}_\beta$ is defined by $\Box_\beta(U) = dd^*(U)$ for $U \in \mathcal{H}_\beta$.

Definition 3.3.3.

- *A vector field $U \in \mathcal{H}_\beta$ is called resonant if $U \in \ker \Box_\beta$.*
- *A formal vector field is called resonant if all its quasihomogeneous components are resonant.*
- *A good perturbation $X = F + R$ of F is a* normal form with respect to *F if R is resonant.*

Now, recall a result of [29] (Proposition 4.4), only adapted for the present context.

Theorem 3.3.1 (Formal normal form [29]). *Let $X = F + P$ be a good perturbation of F as in definition 3.3.1. Then there exists a formal diffeomorphism $\hat{\Phi}$ that conjugates $\hat{X}$ to a vector field $F + R$, where R is a resonant formal vector field in the sense of definition 3.3.3.*

Using theorem 3.3.1, we next show that a good perturbation X of F is in fact formally conjugate to F.

Theorem 3.3.2. *Let $X = F + P$ be a good perturbation of the vector field*

$$F = \varepsilon \frac{\partial}{\partial x_1} + \sum_{i=2}^{k-1} 0 \frac{\partial}{\partial x_i} - \left(z^k + \sum_{j=1}^{k-1} x_j z^{j-1} \right) \frac{\partial}{\partial z} + 0 \frac{\partial}{\partial \varepsilon}.$$

Then, there exists a formal diffeomorphism $\hat{\Phi}$ such that $\hat{\Phi}_* X = F$.

Proof. See section 3.5.2 □

Theorem 3.3.2 shows that any good perturbation $X = F + P$ is (formally) conjugate to F. Next, by Borel's lemma [10], the vector field F can be realized as a smooth vector field $X^N = F + \tilde{P}$ where $\tilde{P}$ is of the form

$$\tilde{P} = \sum_{i=1}^{k-1} \tilde{P}_i \frac{\partial}{\partial x_i} + \tilde{P}_k \frac{\partial}{\partial z} + 0 \frac{\partial}{\partial \varepsilon},$$

where each $\tilde{P}_j$ is flat at the origin and $\tilde{P}_j|_{\varepsilon=0} = 0$, $j = 1, \ldots, k$. More specifically, in the rest of the document we assume that an A_k-SFS has a smooth normal form given by

$$X^N : \begin{cases} x_1' &= \varepsilon(1+\varepsilon\tilde{f}_1) \\ x_j' &= \varepsilon^2\tilde{f}_j \\ z' &= -\left(z^k + \sum_{i=1}^{k-1} x_i z^{i-1}\right) + \varepsilon\tilde{f}_k \\ \varepsilon' &= 0 \end{cases} \tag{3.10}$$

where all the functions $\tilde{f}_i = \tilde{f}_i(x,z,\varepsilon)$ are flat at $(x,z,\varepsilon) = (0,0,0) \in \mathbb{R}^{k+1}$. This normal form X^N is of course not unique and it is given up to flat terms. The smooth (and maybe analytic) version of theorem 3.3.2 is an open problem. However, the fact that the perturbations $\tilde{f}_i$ are flat greatly simplifies the local analysis of (3.10) as shown in chapter 4.

3.4 Geometric desingularization

In this section, the geometric desingularization technique (also known as blow-up) is briefly recalled. For brevity purposes, we restrict ourselves to a vector field X given by (3.10). We remark however, that this is only one of several methods to study singular perturbation problems, see e.g. [30, 31, 44] for a recollection of asymptotic methods.

Observe that the origin is a non-hyperbolic singularity of X. Since the vector field X is quasihomogeneous (see appendix A.5), we make use of the so called *quasihomogeneous blow-up*. This technique consists on performing the change of coordinates

$$x_1 = r^k\bar{x}_1,\ x_2 = r^{k-1}\bar{x}_2, \ldots,\ x_{k-1} = r^2\bar{x}_{k-1},\ z = r\bar{z},\ \varepsilon = r^{2k-1}\bar{\varepsilon}, \tag{3.11}$$

where $\bar{x}_1^2 + \cdots + \bar{x}_{k-1}^2 + \bar{z}^2 + \bar{\varepsilon}^2 = 1$ and $r \in [0,+\infty)$. That is $(\bar{x},\bar{z},\bar{\varepsilon},r) \in S^k \times \mathbb{R}^+$. Since $\varepsilon \geq 0$, we can restrict the coordinates to $\bar{\varepsilon} \geq 0$. Note that $S^k \times \{0\}$ is mapped, via the blow-up map (3.11), to the origin of $\mathbb{R}^{k+1}$. The powers or weights of the blow-up map (3.11) are obtained from the type of quasihomogeneity of X, see section 3.3 and appendix A.5.

Let us denote by $\Phi(\bar{x},\bar{z},\bar{\varepsilon})$ the blow-up map (3.11). This map induces a smooth vector field $\tilde{X}$ on $S^{k-1} \times \mathbb{R}^+$ defined by $\Phi_*\tilde{X} = X$. It is often the case in which the vector field $\tilde{X}$ is degenerate along $S^{k-1} \times \{0\}$. Then one defines $\bar{X} = \frac{1}{r^m}\tilde{X}$ for a well chosen positive integer m. Since $r \in \mathbb{R}^+$, the phase portraits of $\tilde{X}$ and $\bar{X}$ are equivalent outside $S^k \times \{0\}$, and therefore it is equally useful to study $\bar{X}$ instead of $\tilde{X}$. It turns out that the vector field $\bar{X}$ is simpler that X in the sense that the singularities of $\bar{X}$ are semi-hyperbolic or even hyperbolic, in contrast to the non-hyperbolic singularity of X. One obtains a complete description of the flow of X near the origin by studying the flow of $\bar{X}$ for $(\bar{x}, \bar{z}, \bar{\varepsilon}, r) \in S^k \times [0, r_0)$ for $r_0 > 0$.

For problems of dimension greater than 2, performing computations in spherical coordinates becomes tedious. Therefore, it is more convenient to consider *charts*. A chart is a parametrization of a hemisphere of the ball $S^k \times [0, r_0)$. More specifically, the charts are given by

$$K_{\pm\bar{x}_i} = \{\bar{x}_i = \pm 1\}\,,\; K_{\pm\bar{z}} = \{\bar{z} = \pm 1\}\,,\; K_{\bar{\varepsilon}} = \{\bar{\varepsilon} = 1\}\,, \tag{3.12}$$

and we always keep $r \in [0, r_0)$. Naturally, an analysis on all charts provides a complete picture near $S^k \times [0, r_0)$. Finally, we remark that since ε is a parameter, the function $r^{2k-1}\bar{\varepsilon}$ is constant along the orbits of $\bar{X}$. This fact is useful in the analysis of $\bar{X}$. See fig. 3.2 for a schematic description of the blow-up map and the charts.

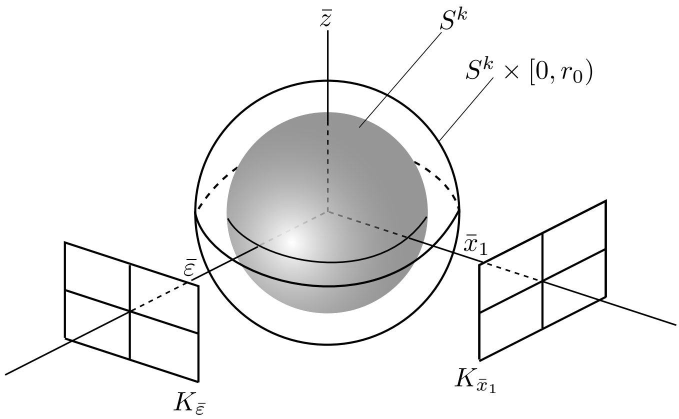


Figure 3.2: The blow-up space and the charts. Each chart K_ℓ parametrizes a region of the ball $S^k \times [0, r_0)$. A local analysis in the charts provides a full picture of the dynamics of the vector field $\bar{X}$.

In order to be systematic with our exposition, first we present some general properties of the vector fields obtained on each chart.

3.4.1 The local vector fields on the blow-up charts

In this section we provide the list of vector fields which are obtained via blow-up on each chart. We remark that to obtain such "blown up vector fields", one performs the directional blow-up define on each chart. For example, to obtain the blown up vector field in the chart $K_{-\bar{x}_1} = \{\bar{x}_1 = -1\}$ one performs the change of coordinates

$$x_1 = -r^k,\, x_2 = r^{k-1}\bar{x}_2,\, \ldots,\, x_{k-1} = r^2\bar{x}_{k-1},\, z = r\bar{z},\, \varepsilon = r^{2k-1}\bar{\varepsilon}.$$

Carrying out standard computations on finds the following.

Proposition 3.4.1. *Let X be an A_k-SFS given in normal form*

$$X : \begin{cases} x_1' &= \varepsilon(1+\varepsilon\tilde{f}_1) \\ x_j' &= \varepsilon^2\tilde{f}_j \\ z' &= -\left(z^k + \sum_{i=1}^{k-1} x_i z^{i-1}\right) + \varepsilon\tilde{f}_k \\ \varepsilon' &= 0 \end{cases},$$

where all the functions $\tilde{f}_i = \tilde{f}_i(x, z, \varepsilon)$ are flat at $(x, z, \varepsilon) = (0, 0, 0) \in \mathbb{R}^{k+1}$. Let $\Phi : S^k \times \mathbb{R}_+ \to \mathbb{R}^{k+1}$ be the blow-up map (3.11)*. Consider the charts defined by* (3.12)*. Then, up to smooth equivalence, the blown up vector fields on each of the charts are given as follows.*

- *In the charts $K_{\pm\bar{x}_1}$ the corresponding local vector field reads as*

$$\overline{X}_{\pm\bar{x}_1} : \begin{cases} r' &= \pm r\bar{\varepsilon} \\ \bar{x}_j' &= \mp(k-j+1)\bar{\varepsilon}\bar{x}_j + \bar{\varepsilon}^2\bar{f}_j \\ \bar{\varepsilon}' &= \mp(2k-1)\bar{\varepsilon}^2 \\ \bar{z}' &= -k\left(\bar{z}^k + \sum_{i=2}^{k-1} \bar{x}_i\bar{z}^{i-1} \pm 1 \pm \frac{1}{k}\bar{\varepsilon}\bar{z}\right) + \bar{\varepsilon}\bar{g} \end{cases}$$

  where the functions $\bar{f}_j$ and $\bar{g}$ are flat along $\{r = 0\}$.

- *In the charts $K_{\pm\bar{x}_j}$, for $j \in \{2, \ldots, k-1\}$, the corresponding local vector field reads as*

$$\overline{X}_{\pm\bar{x}_j} : \begin{cases} r' &= \bar{\varepsilon}\bar{f}_0 \\ \bar{x}_1 &= \bar{\varepsilon}(1+\bar{f}_1) \\ \bar{x}_p' &= \bar{\varepsilon}\bar{f}_p \\ \bar{\varepsilon} &= \bar{\varepsilon}\bar{f}_{k+1} \\ \bar{z}' &= -\left(\bar{z}^k + \sum_{i=1, i\neq j}^{k-1} \bar{x}_i\bar{z}^{i-1} \pm \bar{z}^{j-1}\right) + \bar{\varepsilon}\bar{f}_k, \end{cases}$$

  where the functions $\bar{f}_p$, for $p = 0, 1, 2, \ldots, j-1, j+1, \ldots, k-1$, are flat along $\{r = 0\}$.

- *In the chart $K_{\pm \bar{z}}$ the corresponding local vector field reads*

$$\overline{X}_{\pm \bar{z}} : \begin{cases} r' &= \mp r + \bar{\varepsilon} \bar{g} \\ \bar{x}_1' &= \pm k \bar{x}_1 + \bar{\varepsilon} L(\bar{x}) + \bar{\varepsilon} \bar{f}_1 \\ \bar{x}_j' &= \pm (k-j+1) \bar{x}_j + \bar{\varepsilon} \bar{f}_j \\ \bar{\varepsilon}' &= \pm (2k-1) \bar{\varepsilon} + \bar{\varepsilon}^2 \bar{h}, \end{cases}$$

where

$$L(\bar{x}) = \frac{1}{(\pm 1)^k + \sum_{i=1}^{k-1} (\pm 1)^{i+1} x_i},$$

and where the functions $\bar{g}$, $\bar{f}_i$ and $\bar{h}$ are all flat along $\{r = 0\}$.

- *In the chart $K_{\bar{\varepsilon}}$ the corresponding local vector field reads*

$$\overline{X}_{\bar{\varepsilon}} : \begin{cases} r' &= 0 \\ \bar{x}_1' &= 1 + \bar{f}_1 \\ \bar{x}_j' &= \bar{f}_j \\ \bar{z}' &= -\left(\bar{z}^k + \sum_{i=1}^{k-1} x_i z^{i-1} \right) + \bar{g}, \end{cases}$$

where the functions $\bar{f}_i$ and $\bar{g}$ are all flat along $\{r = 0\}$.

Remark 3.4.1. *It becomes important to relate the local coordinates of the charts. This is done by means of a "matching map" $M_i^j : K_i \to K_j$. For more details see chapter 4.*

Remark 3.4.2. *Whenever suitable, normal form theory is used to normalize the blown up vector fields of proposition 3.4.1. In this way, the transitions are expressed in a simpler format.*

Proof of proposition 3.4.1. The proof is carried out by standard computations following the coordinate transformation of each chart. The coordinate transformation of the chart K_m (m being $\pm \bar{x}_i$, $\pm \bar{z}$ or $\bar{\varepsilon}$) induces a vector field $\tilde{X}_m$ on each chart. Such a vector field is degenerate and thus we must define $\overline{X}_m = \frac{1}{r^{k-1}} \tilde{X}_m$. Finally one multiplies $\overline{X}_m$ by a suitable non-zero smooth function to obtain the expressions of the proposition. □

3.5 Proofs

3.5.1 Proof of proposition 3.1.2

This result is an application of Takens's normalization theorem [37]. However, we remark that in some situations, this normalizing change of coordinates must respect some constraints, for example, we could ask that the change of coordinates preserves a certain invariant manifold or more generally, a certain invariant foliation. This requires the adaptations due to Bonckaert [6, 7].

As a preliminary step, and instead of working with X_ε, we consider the equivalent vector field X defined on $\mathbb{R} \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}$ given as

$$X = \varepsilon(1+f_1)\frac{\partial}{\partial u} + \sum_{j=1}^{m} \varepsilon g_j \frac{\partial}{\partial v_j} + H\frac{\partial}{\partial z} + 0\frac{\partial}{\partial \varepsilon}.$$

The first step is to divide the vector field X by $1+f_1$. In a sufficiently small neighborhood of the origin this is a smooth equivalence relation. That is $Y = \frac{1}{1+f_1}X$ reads as

$$Y = \varepsilon\frac{\partial}{\partial u} + \sum_{j=1}^{m} \varepsilon^2 \tilde{g}_j \frac{\partial}{\partial v_j} + \tilde{H}\frac{\partial}{\partial z},$$

where $\tilde{g}_j$, for $2 \geq j \geq k-1$, and $\tilde{H}$ are smooth with $\tilde{H}(0) = 0$ and $\frac{\partial \tilde{H}}{\partial z}(0) < 0$. Now we note that the origin of $\mathbb{R}^{m+3}$ is a semyhyperbolic equilibrium point with (u, v, ε) being center coordinates and z being the hyperbolic coordinate. We can now use Takens-Bonckaert results on the normal form of partially hyperbolic vector fields [6, 7, 37]. This result tells us that there exists a $\mathcal{C}^\ell$ change of coordinates (maybe respecting some constraints if required) under which Y is conjugated to

$$\bar{Y} = \varepsilon\frac{\partial}{\partial U} + \sum_{j=1}^{m} \varepsilon^2 \bar{G}_j \frac{\partial}{\partial V_j} + \bar{H}Z\frac{\partial}{\partial Z},$$

where $\bar{G}_j = \bar{G}_j(U, V, \varepsilon)$, for $2 \geq j \geq k-1$, and $\bar{H} = \bar{H}(U, V, \varepsilon)$ are $\mathcal{C}^\ell$ functions, and where $\{Z = 0\}$ corresponds to a chosen center manifold $\mathcal{W}^C$. We remark that in the vector field $\bar{Y}$, the functions $\bar{G}_j$ and $\bar{H}$ are independent of Z. Furthermore we have

$$\bar{H}(0,0,0) = \frac{\partial \tilde{H}}{\partial z}(0,0,0,0) < 0.$$

This means that in a small neighborhood of the origin $\bar{Y}$ can be divided by $|\bar{H}|$. In other words, $\bar{Y}$ is $\mathcal{C}^\ell$-equivalent to

$$\mathcal{Y} = \varepsilon\mathcal{G}\frac{\partial}{\partial U} + \sum_{j=1}^{m} \varepsilon^2 \bar{K}_j \frac{\partial}{\partial V_j} - Z\frac{\partial}{\partial Z},$$

where $\mathcal{G}(0,0,0) \neq 0$ and $\bar{K}_j = \bar{K}_j(U, V, \varepsilon)$, for $2 \geq j \geq k-1$, are $\mathcal{C}^\ell$. Next, since $\mathcal{W}^C = \{Z = 0\}$ is invariant under the flow of $\mathcal{Y}$, we can study the restriction $\mathcal{Y}|_{Z=0}$. This is

$$\mathcal{Y}|_{Z=0} = \varepsilon\mathcal{G}\frac{\partial}{\partial U} + \sum_{j=1}^{m} \varepsilon^2 \bar{K}_j \frac{\partial}{\partial V_j}.$$

For $\varepsilon \neq 0$, the vector field $\mathcal{Y}|_{Z=0}$ is regular because $\mathcal{G}(0,0,0) \neq 0$. Thus, by the flow-box theorem, there exists a change of coordinates, depending in a $\mathcal{C}^\ell$ way on ε, under which $\mathcal{Y}|_{Z=0}$ can be written as

$$\varepsilon\frac{\partial}{\partial U} + \sum_{j=1}^{m} 0\frac{\partial}{\partial V_j}.$$

This implies that $\mathcal{Y}$ is $\mathcal{C}^\ell$-equivalent to

$$X_{reg}^N = \varepsilon \frac{\partial}{\partial U} + \sum_{j=1}^{m} 0 \frac{\partial}{\partial V_j} - Z \frac{\partial}{\partial Z},$$

as stated in the proposition.

3.5.2 Proof of theorem 3.3.2

First, it is important to note the following.

Lemma 3.5.1. $\ker \Box_\beta = \ker d^*|_{\mathcal{H}_\beta}$.

Proof. Let $\alpha = k - 1$, then $d : \mathcal{H}_\gamma \to \mathcal{H}_{\gamma+\alpha}$ and $d^* : \mathcal{H}_{\gamma+\alpha} \to \mathcal{H}_\gamma$. Due to the fact that d^* is the adjoint of d, we have the decomposition $\mathcal{H}_\gamma = \operatorname{Im} d^*|_{\mathcal{H}_{\gamma+\alpha}} \oplus \ker d|_{\mathcal{H}_\gamma}$. Now let $U \in \mathcal{H}_{\gamma+\alpha} = \mathcal{H}_\beta$, then $\Box_\beta(U) = dd^*(U) = 0$ if and only if $d^*U \in \ker d$. Furthermore, $d^*U \in \operatorname{Im} d^*$. That is $d^*U \in \operatorname{Im} d^* \cap \ker d$. However $\operatorname{Im} d^*$ and $\ker d$ are orthogonal. Then $\Box_\beta(U) = 0$ if and only if $d^*U = 0$.

□

From theorem 3.3.1 and lemma 3.5.1 we will show that if $P \in \ker d^*|_{\mathcal{H}_{\geq k}}$ then $P = 0$. Let us start by rewriting $d^*(P)$ in a more workable format.

Let $\alpha \geq k$ and $P \in \mathcal{H}_\alpha$. Let also $\beta = \alpha - k + 1$ and $Q \in \mathcal{H}_\beta$. We proceed from the definition

$$\langle d(Q), P\rangle_{r,\alpha} = \langle Q, d^*(P)\rangle_{r,\beta}.$$

For simplicity of notation let $x = (x_1, \ldots, x_{k-1}, z, \varepsilon) = (x_1, \ldots, x_{k-1}, x_k, x_{k+1})$. Note that we can write $d(Q) = \sum_{i=1}^{k+1} F(Q_i) - Q(F_i)$, where $F(Q_i) = \sum_{j=1}^{k+1} F_j \dfrac{\partial Q_i}{\partial x_j}$ and similarly for $Q(F_i)$, then

$$\begin{aligned}
\langle d_0(Q), P\rangle_{r,\alpha} &= \sum_{i=1}^{k+1} \langle F(Q_i) - Q(F_i), P_i\rangle_{r,\beta} \\
&= \sum_{i=1}^{k+1} \langle F(Q_i), P_i\rangle_{r,\alpha+r_i} - \langle Q(F_i), P_i\rangle_{r,\alpha+r_i} \\
&= \sum_{i=1}^{k+1} \langle Q_i, F^*(P_i)\rangle_{r,\beta+r_i} - \langle Q(F_i), P_i\rangle_{\alpha+r_i} \\
&= \sum_{i=1}^{k+1} \langle Q_i, F^*(P_i)\rangle_{r,\beta+r_i} - \sum_{j=1}^{k+1} \langle Q_j, \left(\frac{\partial F_i}{\partial x_j}\right)^* (P_i)\rangle_{\beta+r_j} \\
&= \sum_{i=1}^{k+1} \langle Q_i, F^*(P_i) - \sum_{j=1}^{k+1} \left(\frac{\partial F_j}{\partial x_i}\right)^* (P_j)\rangle_{\beta+r_i}
\end{aligned}$$

Comparing with $\langle Q, d^*(P)\rangle_{r,\beta}$, we can write

$$d^*(P) = \begin{bmatrix} F^* - \left(\dfrac{\partial F_1}{\partial x_1}\right)^* & -\left(\dfrac{\partial F_2}{\partial x_1}\right)^* & \cdots & -\left(\dfrac{\partial F_{k+1}}{\partial x_1}\right)^* \\ -\left(\dfrac{\partial F_1}{\partial x_2}\right)^* & F^* - \left(\dfrac{\partial F_2}{\partial x_2}\right)^* & \cdots & -\left(\dfrac{\partial F_{k+1}}{\partial x_2}\right)^* \\ \vdots & \vdots & \ddots & \vdots \\ -\left(\dfrac{\partial F_1}{\partial x_{k+1}}\right)^* & -\left(\dfrac{\partial F_2}{\partial x_{k+1}}\right)^* & \cdots & F^* - \left(\dfrac{\partial F_{k+1}}{\partial x_{k+1}}\right)^* \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_{k+1} \end{bmatrix}. \tag{3.13}$$

Recall that P (in our context) is of the form

$$P = \sum_{i=1}^{k-1} P_i \frac{\partial}{\partial x_i} + P_k \frac{\partial}{\partial z} + 0\frac{\partial}{\partial \varepsilon},$$

then, plugging in F and P into (3.13), we are now concerned with

$$d^*(P) = \begin{bmatrix} F^* & 0 & \cdots & 0 & 1 & 0 \\ 0 & F^* & \cdots & 0 & z^* & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \\ 0 & 0 & \cdots & F^* & \left(z^{k-1}\right)^* & 0 \\ 0 & 0 & \cdots & 0 & F^* + Z^* & 0 \\ -1 & 0 & \cdots & 0 & 0 & F^* \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_{k-1} \\ P_k \\ 0 \end{bmatrix} = 0. \tag{3.14}$$

where $Z^* = \left(kz^{k-1} + \sum_{i=2}^{k-1}(i-1)x_i z^{i-2}\right)^*$. Now note that (3.14) implies $F^*(P_j) = 0$ for all $j = 2, \ldots, k-1$ and $P_1 = P_k = 0$.

Remark 3.5.1. *For $k = 2$, the fold case, the result is trivial. The vector field F is*

$$F = \varepsilon\frac{\partial}{\partial x_1} - (z^2 + x_1)\frac{\partial}{\partial z} + 0\frac{\partial}{\partial \varepsilon},$$

and $d^*(P) = 0$ is written as

$$d^*(P) = \begin{bmatrix} F^* & 1 & 0 \\ 0 & F^* + 2z^* & 0 \\ -1 & 0 & F^* \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ 0 \end{bmatrix} = 0,$$

which immediately implies $P_1 = P_2 = 0$.

So, we study the problem $F^*(P_j) = 0$. Recall that $P = P(x_1, \ldots, x_{k-1}, z, \varepsilon)$ is not any vector field, but it has the property that $P(x_1, \ldots, x_{k-1}, z, 0) = 0$. That is, we can write

$$P = \sum_{i=1}^{k-1} \varepsilon\bar{P}_i \frac{\partial}{\partial x_i} + \varepsilon\bar{P}_k \frac{\partial}{\partial z} + 0\frac{\partial}{\partial \varepsilon},$$

where $\bar{P}_j \in \mathcal{P}_{\alpha+r_i-2k+1}$. That is because the weight of ε is $2k-1$. Now, since it is complicated to work with the adjoint, we first rewrite the problem $F^*(\varepsilon\bar{P}_j) = 0$. We then prove that $F^*(\varepsilon\bar{P}_j) = 0$ implies that $\bar{P}_j = 0$.

Note that $F^*(\varepsilon\bar{P}_j) = 0$ is equivalent to $\langle Q, F^*(\varepsilon\bar{P}_j)\rangle_{\alpha+r_j-k+1} = 0$ for all $Q \in \mathcal{P}_{\alpha+r_j-k+1}$. Next, we use the definition of F^* that is

$$\langle Q, F^*(\varepsilon\bar{P}_j)\rangle_{\alpha+r_j-k+1} = \langle F(Q), \varepsilon\bar{P}_j\rangle_{r,\alpha+r_j} = 0.$$

Now, we look for conditions on $\varepsilon \bar{P}_j$ given that the latter equality is satisfied.

Assume $\langle F(Q), \varepsilon \bar{P}_j \rangle_{\alpha+r_j} = 0$ holds for all $Q \in \mathcal{P}_{\alpha+r_j-k+1}$. Then we choose an element x^q of the basis of $\mathcal{P}_{\alpha+r_j-k+1}$ given as

$$x^q = x_1^{q_1} \cdots x_{k-1}^{q_{k-1}} z^{q_k} \varepsilon^{q_{k+1}}, \qquad (r,q) = \alpha + r_j - k + 1.$$

Next we have

$$F(x^q) = q_1 x_1^{q_1-1} \cdots x_{k-1}^{q_{k-1}} z^{q_k} \varepsilon^{q_{k+1}+1} - \left(z^k + \sum_{i=1}^{k-1} x_i z^{i-1} \right) q_k x_1^{q_1} \cdots x_{k-1}^{q_{k-1}} z^{q_k-1} \varepsilon^{q_{k+1}}.$$

Let us write $\varepsilon \bar{P}_j$ as

$$\varepsilon \bar{P}_j = \varepsilon \sum_{(r,p)=\alpha+r_j} a_p x_1^{p_1} \cdots x_{k-1}^{p_{k-1}} z^{p_k} \varepsilon^{p_{k+1}},$$

where $a_p \in \mathbb{R}$. We now proceed by recursion on the exponent of ε.

Let $q_{k+1} = 0$, then the inner product $\langle F(Q), \varepsilon \bar{P}_j \rangle_{\alpha+r_j}$ has only one term since $F(Q)$ has only one monomial containing ε. That is

$$\langle F(Q), \varepsilon \bar{P}_j \rangle_{\alpha+r_j} |_{q_{k+1}=0} = \langle q_1 x_1^{q_1-1} \cdots x_{k-1}^{q_{k-1}} z^{q_k} \varepsilon, \varepsilon a_p x_1^{p_1} \cdots x_{k-1}^{p_{k-1}} z^{p_k} \rangle_{r,\alpha+r_j} = 0. \tag{3.15}$$

We naturally consider $q_1 > 0$. If $q_1 = 0$, then the equality is automatically satisfied. Recalling the definition A.5.7 of the inner product, the equality (3.15) means that

$$\langle q_1 x_1^{q_1-1} \cdots x_{k-1}^{q_{k-1}} z^{q_k} \varepsilon, \varepsilon a_p x_1^{p_1} \cdots x_{k-1}^{p_{k-1}} z^{p_k} \rangle_{r,\alpha+r_j} = q_1 a_p \frac{(q!)^r}{(\alpha + r_j)!} = 0,$$

and therefore

$$a_p = a_{q_1-1,p_2,\ldots,p_k,1} = 0, \tag{3.16}$$

for all $q_1 > 0,\ p_2, \ldots, p_k \geq 0$ (naturally, also satisfying the degree condition $(r,p) = \alpha + r_j$).

Next, let $q_{k+1} = 1$. Then

$$F(x^q) = q_1 x_1^{q_1-1} \cdots x_{k-1}^{q_{k-1}} z^{q_k} \varepsilon^2 - \left(z^k + \sum_{i=1}^{k-1} x_i z^{i-1} \right) q_k x_1^{q_1} \cdots x_{k-1}^{q_{k-1}} z^{q_k-1} \varepsilon.$$

Once again, the inner product $\langle F(Q), \varepsilon \bar{P}_j \rangle_{r,\alpha+r_j}$ has only one term, now this is due to the fact that all coefficients a_p of monomials containing ε are zero due to (3.16). Then

$$\langle F(Q), \varepsilon \bar{P}_j \rangle_{\alpha+r_j} |_{q_{k+1}=1} = \langle q_1 x_1^{q_1-1} \cdots x_{k-1}^{q_{k-1}} z^{q_k} \varepsilon^2, \varepsilon a_p x_1^{p_1} \cdots x_{k-1}^{p_{k-1}} z^{p_k} \varepsilon \rangle_{r,\alpha+r_j} = 0.$$

Therefore, similarly as above, we have the condition

$$a_p = a_{q_1-1,p_2,\ldots,p_k,2} = 0,$$

for all $q_1 > 0,\ p_2, \ldots, p_k \geq 0$ (naturally, also satisfying the degree condition $(r,p) = \alpha + r_j$).

By recursion, assume $q_{k+1} = n$ and that all the coefficients

$$a_p = a_{p_1,p_2,\dots,p_k,m} = 0, \qquad \forall m \leq n.$$

Then again the inner product $\langle F(Q), \varepsilon \bar{P}_j \rangle_{r,\alpha+r_j}$ has only one term, namely

$$\langle F(Q), \varepsilon \bar{P}_j \rangle_{\alpha+r_j} |_{q_{k+1}=n} = \langle q_1 x_1^{q_1-1} \cdots x_{k-1}^{q_{k-1}} z^{q_k} \varepsilon^{n+1}, \varepsilon a_p x_1^{p_1} \cdots x_{k-1}^{p_{k-1}} z^{p_k} \varepsilon^n \rangle_{r,\alpha+r_j} = 0,$$

which implies

$$a_p = a_{q_1-1,p_2,\dots,p_k,n+1} = 0.$$

This finishes the proof of $\langle F(Q), \varepsilon \bar{P}_j \rangle_{r,\alpha+r_j} = 0$ implies $\bar{P}_j = 0$.

Chapter 4

Applications

In this chapter, we investigate the local dynamics of the A_2 (fold), A_3 (cusp) and A_4 (swallowtail) slow fast systems. Our analysis is based on the several techniques described in chapter 3. We also use the technical results (regarding normal forms and transition maps) described in appendix A.6.

Remark 4.0.2.

- *Slow fast systems near a fold, that is A_2-SFSs, have been already studied in e.g. [27]. However, here we use our proposed normal form of section 3.3 and compare the relevant results.*
- *Similarly, the A_3-SFSs have been studied in [8]. In section section 4.2, however, we provide a refinement of the the results presented in [8].*
- *A similar methodology as the presented here can be used to study all A_k-SFS. In this way, our approach is systematic.*

To simplify the notation we let the parameters be denoted as $(x_1, x_2, x_3) = (a, b, c)$. This matches our exposition of chapter 2 as well as the list of catastrophes provided in table A.1.

4.1 Analysis of the A_2-SFS

In this section we study an A_2 slow fast system, which is given as

$$X : \begin{cases} a' &= \varepsilon(1+f) \\ z' &= -\left(z^2+a\right)+\varepsilon g \\ \varepsilon' &= 0, \end{cases} \tag{4.1}$$

where, thanks to theorem 3.3.2, $f = f(a, z, \varepsilon)$ and $g = g(a, z, \varepsilon)$ are flat functions at $(a, z, \varepsilon) = (0, 0, 0)$. The corresponding contrained differential equation and the corresponding layer equation have phase portraits as shown in fig. 4.1.

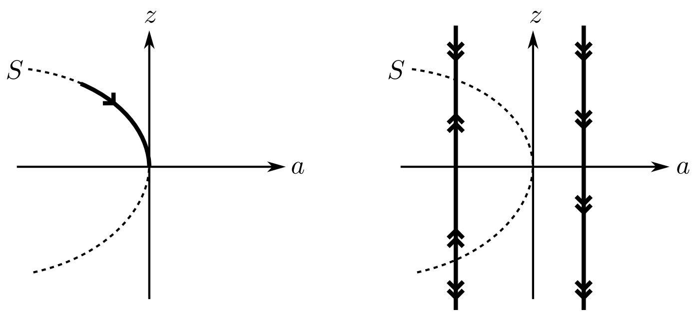


Figure 4.1: Left: the phase portrait of the corresponding CDE of (4.1). Right: the phase portrait of the corresponding layer equation of (4.1). The set S corresponds to the slow (or critical) manifold.

We are interested in computing the transition map $\Pi : \Sigma_{\text{en}} \to \Sigma_{\text{ex}}$ defined by the flow of X and where

$$\begin{aligned}\Sigma_{\text{en}} &= \left\{(a, z, \varepsilon) \in \mathbb{R}^3 \,|\, a = -a_0, z > 0\right\}\\ \Sigma_{\text{ex}} &= \left\{(a, z, \varepsilon) \in \mathbb{R}^3 \,|\, z = -z_0, a > 0\right\},\end{aligned}$$

where $a_0 > 0$ and $z_0 > 0$ are small constants. Using the techniques of chapter 3 we have the following.

Proposition 4.1.1. *Consider X, Σ_{en} and Σ_{ex} given as above. Then the transition map $\Pi : \Sigma_{\text{en}} \to \Sigma_{\text{ex}}$ is given by*

$$\begin{aligned}\Pi :& \Sigma_{\text{en}} \to \Sigma_{\text{ex}}\\ &(z, \varepsilon) \mapsto (\tilde{a}, \tilde{\varepsilon}),\end{aligned}$$

where

$$\begin{aligned}\tilde{\varepsilon} &= \varepsilon\\ \tilde{a} &= \varepsilon^{2/3} + O(\varepsilon).\end{aligned}$$

See fig. 4.2 for a qualitative description of this result.

Remark 4.1.1. *Compare proposition 4.1.1 with e.g. theorem 2.2 of [27] where it is shown that*

$$\tilde{a} = c_1\varepsilon^{2/3} + c_2\varepsilon \ln \varepsilon + c_3\varepsilon + O(\varepsilon^{4/3} \ln(\varepsilon)), \tag{4.2}$$

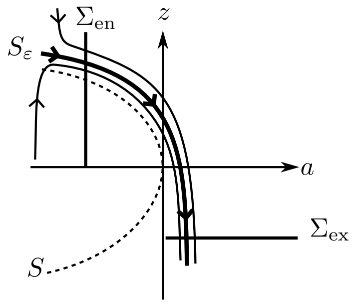


Figure 4.2: Qualitative picture of the flow of (4.1) near the origin. We study trajectories passing through $\Sigma_{\rm en}$ and that arrive at $\Sigma_{\rm ex}$ transversally. Before $\Sigma_{\rm en}$, the manifold S_ε is given by Fenichel's theory. Afterwards we compute a continuation of S_ε passing through the fold point.

for some constants c_1, c_2 and c_3. We remark that our result does not have the logarithmic terms of (4.2) *thanks to the formal normal form step that we have performed.*

Through the following sections, we prove proposition 4.1.1.

Blow up and charts

Following our exposition of section 3.4 we shall use the blow-up technique to study the flow of X near the origin. The blow-up map $\Phi : S^2 \times \mathbb{R}^+ \to \mathbb{R}^3$ is given by

$$a = r^2\bar{a},\ z = r\bar{z},\ \varepsilon = r^3\bar{\varepsilon},$$

where $r \in \mathbb{R}^+$ and $(\bar{a}, \bar{z}, \bar{\varepsilon}) \in S^2$. The charts are given by

$$K_{\rm en} = \{\bar{a} = -1\}, \qquad K_\varepsilon = \{\bar{\varepsilon} = 1\}, \qquad K_{\rm ex} = \{\bar{z} = -1\}.$$

Qualitatively speaking, in chart $K_{\rm en}$ we study the trajectories of X before they arrive at the fold point; in $K_{\bar{\varepsilon}}$ we study the trajectories of X passing through the fold point; and in $K_{\rm ex}$ we study the trajectories of X as they leave the fold point. We provide the local analysis in each of the above charts. Later we compose the local transitions in each chart in order to compute the map $\Pi : \Sigma_{\rm en} \to \Sigma_{\rm ex}$.

Remark 4.1.2 (On notation). *Since we have to compose the maps obtained in each chart $K_{\rm en}$, K_ε and $K_{\rm ex}$, and to avoid confusion, we shall use the following notation. Any object O in chart $K_{\rm en}$ (resp K_ε and $K_{\rm ex}$) shall be denoted by O_1 (resp O_2 and O_3).*

Analysis in the chart K_{en}

The local coordinates in this chart are given by

$$a = -r_1^2,\ z = r_1 z_1,\ \varepsilon = r_1^3 \varepsilon_1. \tag{4.3}$$

The induced vector filed reads as

$$\begin{aligned} r_1' &= -\varepsilon_1 r_1 (1 + \bar{f}_1) \\ \varepsilon_1' &= 3\varepsilon_1^2 (1 + \bar{f}_1) \\ z_1' &= -2(z_1^2 - 1 - \frac{1}{2}\varepsilon_1 z_1) + r_1 \varepsilon_1 \bar{g}_1, \end{aligned} \tag{4.4}$$

where $\bar{f}_1$ and $\bar{g}_1$ are flat functions along $\{r_1 = 0\}$. Observe that the equilibrium set of (4.4) is

$$\Gamma_1 = \left\{ (r_1, \varepsilon_1, z_1) \in \mathbb{R}^3 \,|\, \varepsilon_1 = 0,\ z_1 = \pm 1 \right\}.$$

Let us define the sections

$$\begin{aligned} \Delta_1^{\text{en}} &= \left\{ (r_1, \varepsilon_1, z_1) \in \mathbb{R}^3 \,|\, r_1 = r_0,\ z_1 > 0 \right\} \\ \Delta_1^{\text{ex}} &= \left\{ (r_1, \varepsilon_1, z_1) \in \mathbb{R}^3 \,|\, \varepsilon_1 = \delta \right\}, \end{aligned}$$

where $r_0 = a_0^{1/3}$. Observe that the section Δ_1^{en} corresponds to Σ_{en} written in the blow-up coordinates. That is $\Sigma_{\text{en}} = \Phi(\Delta_1^{\text{en}})$ with Φ given by (4.3). We are interested in computing the transition

$$\begin{aligned} \Pi_1 :& \Delta_1^{\text{en}} \to \Delta_1^{\text{ex}} \\ & (z_1, \varepsilon_1) \mapsto (\tilde{r}_1, \tilde{z}_1). \end{aligned}$$

Let $\zeta_1 = z_1 - 1$. This shift puts the origin at the equilibrium point $(0, 0, 1)$. With this new coordinate we can write (4.4) as

$$\begin{aligned} r_1' &= -\varepsilon_1 r_1 (1 + \bar{f}_1) \\ \varepsilon_1' &= 3\varepsilon_1^2 (1 + \bar{f}_1) \\ \zeta_1' &= -2\zeta(2 + \zeta - \frac{1}{2}\varepsilon_1) + r_1 \varepsilon_1 \bar{g}_1. \end{aligned} \tag{4.5}$$

The line $\ell_1 = (r_1, \varepsilon_1, \zeta_1) = (r_1, 0, 0)$ is an equilibrium set of (4.5) and it is equivalent, under the change of coordinates we performed, to Γ_1. The linearization of (4.5) along ℓ_1 has eigenvalues eigenvalues $(0, 0, -2)$. This means that there exists a 2-dimensional center manifold $\mathcal{W}^C$ which is tangent to $\{\zeta_1 = 0\}$ along ℓ_1. Observe that (4.5) satisfies the conditions of proposition A.6.2. Therefore there exist a near identity $\mathcal{C}^\ell$ transformation $\phi_1 : (r_1, \varepsilon_1, \zeta_1) \mapsto (r_1, \varepsilon_1, Z_1)$ such that (4.5) is $\mathcal{C}^\ell$ equivalent to

$$X^N : \begin{cases} r_1' &= -\varepsilon_1 r_1 \\ \varepsilon_1' &= 3\varepsilon_1^2 \\ Z_1' &= -2(2 + G(r_1, \varepsilon_1))Z_1, \end{cases} \tag{4.6}$$

where $G(0,0) = 0$. With the transformation performed, the center manifold is now given as $\mathcal{W}^C = \{Z_1 = 0\}$. Note that the flow of X^N restricted to the center manifold $\mathcal{W}^C$ is, up to multiplication by ε_1 linear. More specifically we have

$$\frac{1}{\varepsilon_1} X^N|_{\mathcal{W}^C} : \begin{cases} r_1' &= -r_1 \\ \varepsilon_1' &= 3\varepsilon_1. \end{cases} \tag{4.7}$$

From the format of (4.7) we see that the origin $(r_1, \varepsilon_1) = (0,0)$ is a is a saddle equilibrium point of (4.7). This allows us to understand the flow of X^N in the neighborhood of the origin and for $\varepsilon_1 > 0$ but sufficiently small. See fig. 4.3 for a qualitative description of the flows of (4.4), (4.4), and of (4.6).

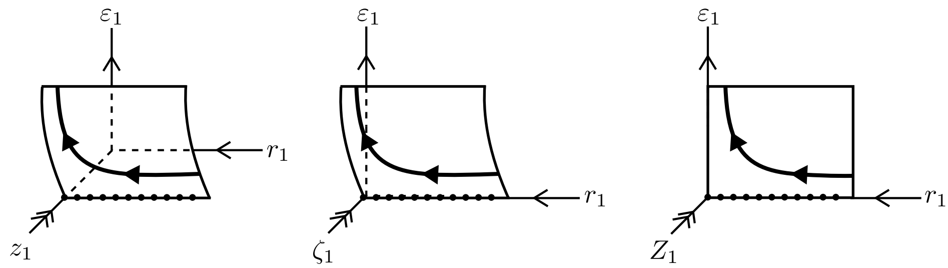


Figure 4.3: From left to right we show the phase portraits of (4.4), (4.4), and of (4.6) respectively. We show only the flow restricted to $z_1 > 0$ as it is the one which interests us the most. The shown surface is an invariant center manifold $\mathcal{W}^C$. This invariant manifold is stable.

The manifold $\mathcal{W}^C$ is very important in our analysis. From a geometric point of view, $\mathcal{W}^C$ is a perturbation of the critical manifold S written in the blow-up coordinates. Therefore, we will keep track of $\mathcal{W}^C$ as in passes through the other charts.

Since X^N is in normal form, we can now compute the transition $\Pi_1 : (\varepsilon_1, Z_1) \mapsto (\tilde{r_1}, \tilde{Z_1})$ by following proposition A.6.5. The time of integration is $T = \ln\left(\frac{\delta}{\varepsilon_1}\right)$

(see the details in proposition A.6.5). We then have

$$\begin{aligned}
\tilde{r_1} &= r_0 \left(\frac{\varepsilon_1}{\delta}\right)^{1/3} \\
\tilde{Z}_1 &= Z_1 \exp\left(-\frac{4}{3\varepsilon_1}(1 + \tilde{\alpha}_1\varepsilon_1 \ln(\varepsilon_1) + \varepsilon_1 \tilde{G}_1)\right),
\end{aligned}$$

where $\tilde{\alpha}_1 = \tilde{\alpha}_1(r_0\varepsilon_1^{1/3})$ and $\tilde{G}_1 = \tilde{G}_1(r_0\varepsilon_1^{1/3})$ are $\mathcal{C}^\ell$ functions (recall that r_0 is a positive constant). As expected, the trajectories are attracted to the center manifold $\mathcal{W}^C$ (see the expression of $\tilde{Z}_1$).

Now we return to the coordinate ζ_1. We can write $\zeta_1 = \psi(r_1, \varepsilon_1, Z_1)$ where ψ is a $\mathcal{C}^\ell$ near identity transformation. This means that the relationship between ζ_1 and Z_1 can be written as follows

$$\begin{aligned}
\zeta_1 &= Z_1(1 + \psi_0(r_1, \varepsilon_1, Z_1)) \\
Z_1 &= \zeta_1(1 + \phi_0(r_1, \varepsilon_1, \zeta_1))
\end{aligned}$$

where ψ_0 and ϕ_0 are unknown $\mathcal{C}^\ell$ functions. Then we obtain the expression for $\tilde{\zeta}_1$ as follows.

$$\begin{aligned}
\tilde{\zeta}_1 &= \tilde{Z}_1(1 + \psi_0) \\
&= Z_1 \exp\left(-\frac{4}{3\varepsilon_1}(1 + \tilde{\alpha}_1\varepsilon_1 \ln(\varepsilon_1) + \varepsilon_1 \tilde{G}_1)\right)(1 + \psi_0) \\
&= \zeta_1(1 + \phi_0) \exp\left(-\frac{4}{3\varepsilon_1}(1 + \tilde{\alpha}_1\varepsilon_1 \ln(\varepsilon_1) + \varepsilon_1 \tilde{G}_1)\right)(1 + \psi_0) \\
&= \zeta_1 \exp(\ln(1 + \phi_0)) \exp(\ln(1 + \psi_0)) \exp\left(-\frac{4}{3\varepsilon_1}(1 + \tilde{\alpha}_1\varepsilon_1 \ln(\varepsilon_1) + \varepsilon_1 \tilde{G}_1)\right) \\
&= \zeta_1 \exp\left(-\frac{4}{3\varepsilon_1}(1 + \tilde{\alpha}_1\varepsilon_1 \ln(\varepsilon_1) + \varepsilon_1 \tilde{H}_1)\right),
\end{aligned}$$

where we have absorbed the unknown function $\ln(1 + \psi_0)$ and $\ln(1 + \phi_0)$ into the unknown function $\tilde{H}_1$. Finally recall that z_1 is just a translation of ζ_1 ($\zeta_1 = z_1 - 1$), and therefore, in the coordinate z_1 we have a transition that reads as

$$\tilde{z}_1 = (z_1 - 1) \exp\left(-\frac{4}{3\varepsilon_1}(1 + \tilde{\alpha}_1\varepsilon_1 \ln(\varepsilon_1) + \varepsilon_1 \tilde{H}_1)\right) + 1.$$

Analysis in the chart $K_{\bar{\varepsilon}}$

In this chart the blow-up map is given by

$$a = r_2^2 a_2,\ z = r_2 z_2,\ \varepsilon = r_2^3.$$

Therefore, the blown up vector field reads as

$$\begin{aligned} r_2' &= 0 \\ a_2' &= 1 + \bar{f}_2 \\ z_2' &= -(z_2^2 + a_2) + \bar{g}_2, \end{aligned} \tag{4.8}$$

where $\bar{f}_2$ and $\bar{g}_2$ are flat functions along $r_2 = 0$. Since $r_2' = 0$, the trajectories of (4.4) are invariant on each plane given by $\{r_2 = R\}$, with $R \in \mathbb{R}$. We are interested in a transition $\Pi_2 : \Delta_2^{\text{en}} \to \Delta_2^{\text{ex}}$ where

$$\begin{aligned} \Delta_2^{\text{en}} &= \{(r_2, a_2, z_2) \,|\, a_2 = -a_{2,in}\} \\ \Delta_2^{\text{ex}} &= \{(r_2, a_2, z_2) \,|\, z_2 = -z_{2,out}\}, \end{aligned}$$

where $a_{2,in}$ and $z_{2,out}$ are sufficiently small positive constants.

Let S_2 be the set

$$S_2 = \left\{(r_2, a_2, z_2) \in \mathbb{R}^3 | r_2 = 0, a_2 = -z_2^2\right\}.$$

See fig. 4.4 for a qualitative description of the flow of (4.8).

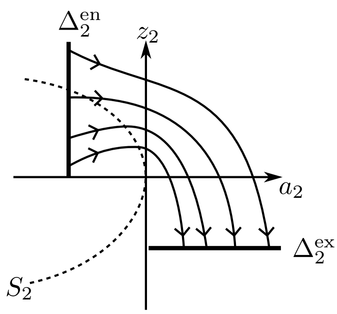


Figure 4.4: Qualitative picture of the flow of (4.8) with initial conditions at Δ_2^{en}. Observe that the trajectories cross horizontally the set S_2. This guarantees that all the trajectories starting at Δ_2^{en} arrive in finite time to Δ_2^{ex}.

Note that since the (4.8) is regular, the map

$$\begin{aligned} \Pi_2 : &\Delta_2^{\text{en}} \to \Delta_2^{\text{ex}} \\ &(r_2, z_2) \mapsto (r_2, \tilde{a_2}) \end{aligned}$$

is a diffeomorphism. Therefore, there exists a smooth function $\phi_2 = \phi_2(r_2, z_2)$ such that $\tilde{a_2} = \phi_2$. However, since r_2 is a constant, we shall write $\tilde{a_2} = \phi_{2,r_2}(z_2)$, where ϕ_{2,r_2} is an r_2-parameter family of diffeomorphisms.

Analysis in the chart K_{ex}

In this chart the blow-up map reads as

$$a = r_3^2 a_3,\; z = -r_3,\; \varepsilon = br^3\varepsilon_3.$$

Accordingly, the blown up vector field is given by

$$\begin{aligned} r_3' &= r_3(1+a_3) + \bar{f}_3 \\ a_3' &= -2a_3(1+a_3) + \varepsilon_3 + \bar{g}_3 \\ \varepsilon_3' &= -3\varepsilon_3(1+a_3) + \bar{h}_3, \end{aligned} \tag{4.9}$$

where the functions $\bar{f}_3$, $\bar{g}_3$ and $\bar{h}_3$ are flat along $r_3 = 0$. Note that the origin is a hyperbolic equilibrium point of (4.9). This hyperbolic point has two stable manifolds which are tangent to the a_3 and the ε_3 axes, and one unstable manifold that is tangent to the r_3 axis. Implied by this behavior, we are interested in a transition of the form

$$\begin{aligned} \Pi_3 :&\Sigma_3^{\text{en}} \to \Sigma_3^{\text{ex}} \\ &(r_3, a_3) \mapsto (\tilde{a_3}, \tilde{\varepsilon_3}), \end{aligned}$$

where

$$\begin{aligned} \Sigma_3^{\text{en}} &= \left\{(r_3, a_3, \varepsilon_3) \in \mathbb{R}^3 \,|\, \varepsilon_3 = \delta\right\} \\ \Sigma_3^{\text{ex}} &= \left\{(r_3, a_3, \varepsilon_3) \in \mathbb{R}^3 \,|\, r_3 = r_{3,out}\right\}. \end{aligned}$$

Normal forms of vector fields of the form of (4.9) are obtained in appendix A.6.1. Using proposition A.6.1 we have that there exists a $\mathcal{C}^\infty$ transformation that conjugates (4.9) to

$$\begin{aligned} r_3' &= r_3 \\ a_3' &= -2a_3 + \varepsilon_3\left(1 - \frac{a_3}{1+a_3}\right) \\ \varepsilon_3' &= -3\varepsilon_3. \end{aligned} \tag{4.10}$$

The transition map of vector fields of the form of (4.10) are computed in appendix A.6.4. Accordingly, we use proposition A.6.4 to compute the transition

Π_3. We then have

$$\tilde{a}_3 = \left(\frac{r_3}{r_{3,out}}\right)^2 \left(a_3 + \varepsilon_3 O\left(\frac{r_3}{r_{3,out}}\right)\right)$$
$$\tilde{\varepsilon}_3 = \varepsilon_3 \left(\frac{r_3}{r_{3,out}}\right)^3$$

Composition of the chart transitions (proof of proposition 4.1.1)

We now compose all the chart transitions we computed above. To recall what we have obtained see fig. 4.5.

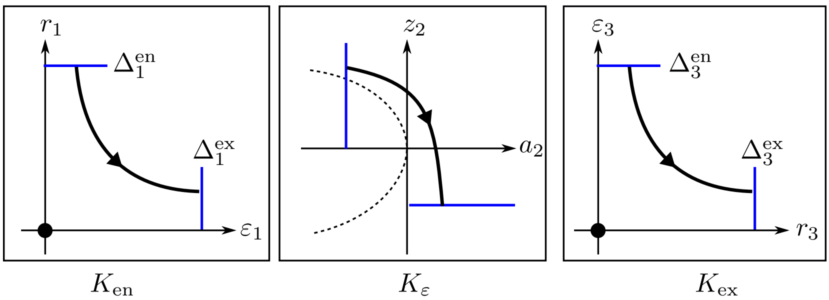


Figure 4.5: Flow in each of the charts. In the chart K_{en} we have a transition near a semihyperbolic point. In the chart K_ε we have a regular flow, and therefore the transition is a diffeomorphism. In the chart K_{ex} we have a transition near a hyperbolic point.

In order to compose the transitions we need also the so called matching maps. These maps relate the coordinates in each chart and they are given as

$$\begin{aligned} M_{\mathrm{en}}^{\varepsilon} :& K_{\mathrm{en}} \to K_\varepsilon \\ & (r_1, z_1, \varepsilon_1) \mapsto (r_2, a_2, z_2) \\ M_{\varepsilon}^{\mathrm{ex}} :& K_\varepsilon \to K_{\mathrm{ex}} \\ & (r_2, a_2, z_2) \mapsto (r_3, a_3, \varepsilon_3) \end{aligned}$$

where

$$r_2 = r_1 \varepsilon_1^{1/3}, \qquad a_2 = -\varepsilon_1^{-2/3}, \qquad z_2 = z_1 \varepsilon_1^{-1/3}, \qquad \text{for } \varepsilon_1 > 0$$
$$r_3 = -r_2 z_2, \qquad a_3 = z_2^{-2} a_2, \qquad \varepsilon_3 = -z_2^{-3}, \qquad \text{for } z_2 < 0.$$

We now need to compute the transition $\tilde{\Pi} : \Delta_1^{\text{en}} \to \Delta_3^{\text{ex}}$ given as

$$\tilde{\Pi} = \Pi_3 \circ M_{\varepsilon}^{\text{ex}} \circ \Pi_2 \circ M_{\text{en}}^{\varepsilon} \circ \Pi_1$$

Let us proceed step by step.

1. We have in the chart K_{en} a transition of the form

$$\Pi_1 : (\zeta_1, \varepsilon_1) \mapsto (\tilde{r}_1, \tilde{z}_1),$$

   where

$$\begin{aligned}
\tilde{r}_1 &= r_0 \left(\frac{\varepsilon_1}{\delta}\right)^{1/3}, \\
\tilde{z}_1 &= 1 + (z_1 - 1) \exp\left(-\frac{4}{3\varepsilon_1}(1 + \tilde{\alpha}_1 \varepsilon_1 \ln(\varepsilon_1) + \varepsilon_1 \tilde{H})\right).
\end{aligned}$$

2. Next, we relate the coordinates of the chart K_{en} with the coordinates of the chart $K_{\bar{\varepsilon}}$ by the matching map $M_{\text{en}}^{\varepsilon}$. Thus we have

$$(r_2, a_2, z_2)|_{\Delta_2^{\text{en}}} = M_{\text{en}}^{\varepsilon}\left((r_1, \varepsilon_1, z_1)|_{\Delta_1^{\text{ex}}}\right) = M_{\text{en}}^{\varepsilon}(\tilde{r}_1, \delta, \tilde{z}_1)$$

   and then the coordinates of the image of the map Π_1, in the chart $K_{\bar{\varepsilon}}$, read as

$$\begin{aligned}
r_2 &= r_0 \varepsilon_1^{1/3} \\
a_2 &= -\delta^{-2/3} \\
z_2 &= \delta^{-1/3} \tilde{z}_1.
\end{aligned}$$

3. Next we have that the transition in the chart $K_{\bar{\varepsilon}}$

$$\begin{aligned}
\Pi_2 : &\Delta_2^{\text{en}} \to \Delta_2^{\text{ex}} \\
&(r_2, z_2) \mapsto (r_2, \tilde{a_2})
\end{aligned}$$

   is a diffeomorphism with $\tilde{a}_2 = \phi_2(z_2) = \phi_2(\tilde{z}_1)$ where ϕ_2 is a smooth function.

4. Next we apply the matching map from $K_{\bar{\varepsilon}}$ to K_{ex}. This is

$$(r_3, \varepsilon_3, a_3)|_{\Delta_3^{\text{en}}} = M_{\varepsilon}^{\text{ex}}(r_2, a_2, z_2)|_{\Delta_2^{\text{ex}}} = M_{\varepsilon}^{\text{ex}}(r_2, \tilde{a}_2, -z_{2,out}).$$

Since $z_{2,out}$ is an arbitrary positive constant we choose $z_{2,out} = \delta^{-1/3}$. This allows us to match the exit section Δ_2^{ex} with the entry section Δ_3^{en}. That is, $\Delta_3^{\text{en}} = M_\varepsilon^{\text{ex}}(\Delta_2^{\text{ex}})$. So we have

$$\begin{aligned} r_3 &= r_0 \delta^{-1/3} \varepsilon_1^{1/3} \\ \varepsilon_3 &= \delta \\ a_3 &= \delta^{2/3} \phi_2(\tilde{z}_1). \end{aligned} \tag{4.11}$$

5. The transition $\Pi_3 : (r_3, a_3) \mapsto (\tilde{\varepsilon}_3, \tilde{a}_3)$ is given by

$$\begin{aligned} \tilde{\varepsilon}_3 &= \varepsilon_3 \left(\frac{r_3}{r_{3,out}} \right)^3 \\ \tilde{a}_3 &= \left(\frac{r_3}{r_{3,out}} \right)^2 \left(a_3 + \varepsilon_3 O \left(\frac{r_3}{r_{3,out}} \right) \right). \end{aligned} \tag{4.12}$$

The constant $r_{3,out}$ is chosen as $r_{3,out} = r_0$ and therefore, substituting (4.11) into (4.12) we get

$$\begin{aligned} \tilde{\varepsilon}_3 &= \varepsilon_1 \\ \tilde{a}_3 &= \phi_2(\tilde{z}_1) \varepsilon_1^{2/3} \left(1 + O(\varepsilon_1^{1/3}) \right) \end{aligned} \tag{4.13}$$

We now have to express $\phi_2(\tilde{z}_1)$ in the coordinates (ε_1, z_1). We can expand $\phi_2(\tilde{z}_1)$ at $\tilde{z}_1 = 1$ as follows

$$\phi_2(\tilde{z}_1) = 1 + (\tilde{z}_1 - 1)(A + B(\tilde{z}_1)),$$

for some positive constant A and a smooth function B. Then we have

$$\begin{aligned} \phi_2(\tilde{z}_1) &= 1 + (z_1 - 1) \exp \left(-\frac{4}{3\varepsilon_1} (1 + \tilde{\alpha}_1 \varepsilon_1 \ln(\varepsilon_1) + \varepsilon_1 \tilde{H}) \right) (A + B(\tilde{z}_1)) \\ &= 1 + (z_1 - 1) \exp \left(-\frac{4}{3\varepsilon_1} (1 + \tilde{\alpha}_1 \varepsilon_1 \ln(\varepsilon_1) + \varepsilon_1 \tilde{H}) \right), \end{aligned} \tag{4.14}$$

where we have absorbed the unknown function $A + B(\tilde{z}_1) = A + B(\varepsilon_1, z_1)$ into the also unknown function $\tilde{H}$. Now we substitute (4.14) into (4.13) to obtain

$$\tilde{a}_3 = \left(1 + (z_1 - 1) \exp \left(-\frac{4}{3\varepsilon_1} (1 + \tilde{\alpha}_1 \varepsilon_1 \ln(\varepsilon_1) + \varepsilon_1 \tilde{H}) \right) \right) \varepsilon_1^{2/3} \left(1 + O(\varepsilon_1^{1/3}) \right). \tag{4.15}$$

We can write $(1+O(\varepsilon^{1/3}))$ as $\exp(\ln(1+O(\varepsilon^{1/3})))$ and thus we reduce (4.15) to

$$\tilde{a}_3 = \varepsilon_1^{2/3}\left(1+O(\varepsilon_1^{1/3})\right) + \varepsilon_1^{2/3}(z_1-1)\exp\left(-\frac{4}{3\varepsilon_1}(1+\tilde{\alpha}_1\varepsilon_1\ln(\varepsilon_1)+\varepsilon_1\tilde{H})\right).$$

Now observe that the exponential $\exp\left(-\frac{4}{3\varepsilon_1}(1+\tilde{\alpha}_1\varepsilon_1\ln(\varepsilon_1)+\varepsilon_1\tilde{H})\right)$ is flat along $\varepsilon_1 = 0$. Therefore, we can write

$$\tilde{a}_3 = \varepsilon_1^{2/3} + O(\varepsilon_1) + \Psi(z_1, \varepsilon_1),$$

where Ψ is a flat function at $\varepsilon_1 = 0$. However, a function which is flat at $\varepsilon_1 = 0$ is also of order $O(\varepsilon_1)$ and thus we arrive to our final expression

$$\tilde{a}_3 = \varepsilon_1^{2/3} + O(\varepsilon_1).$$

We have then computed $\tilde{\Pi}$, where the relevant data is $\tilde{a}_3 = \varepsilon_1^{2/3} + O(\varepsilon_1)$. To obtain the final result we return to the the original coordinates (a, z) via blow-down. It follows that

$$\tilde{a} = \varepsilon^{2/3} + O(\varepsilon).$$

4.2 Analysis of the A_3-SFS

In this section we study an A_3 slow fast system based on techniques described in chapter 3. To simplify the notation, let us write the A_3-SFS as

$$X = \varepsilon(1+f_1)\frac{\partial}{\partial a} + \varepsilon^2 f_2\frac{\partial}{\partial b} - (z^3+bz+a+\varepsilon f_3)\frac{\partial}{\partial z} + 0\frac{\partial}{\partial \varepsilon}, \tag{4.16}$$

where thanks the functions $f_i = f_i(a, b, z, \varepsilon)$ are smooth. We investigate the transition associated to (4.16) between the sections

$$\begin{aligned}\Sigma^- &= \left\{(a,b,z,\varepsilon)\in\mathbb{R}^4 \,|\, a=-a^-,\ z>0\right\}\\ \Sigma^+ &= \left\{(a,b,z,\varepsilon)\in\mathbb{R}^4 \,|\, a=a^+,\ z<0\right\},\end{aligned}$$

where $a^- > 0$ and $a^+ > 0$ are arbitrarily large constants. However, since the trajectories of X spend a long time along regular parts of S, it will be useful to define the "entry" and "exit" sections

$$\begin{aligned}\Sigma^{\text{en}} &= \left\{(a,b,z,\varepsilon)\in\mathbb{R}^4 \,|\, a=-a_0,\ z>0\right\}\\ \Sigma^{\text{ex}} &= \left\{(a,b,z,\varepsilon)\in\mathbb{R}^4 \,|\, a=a_0,\ z<0\right\},\end{aligned}$$

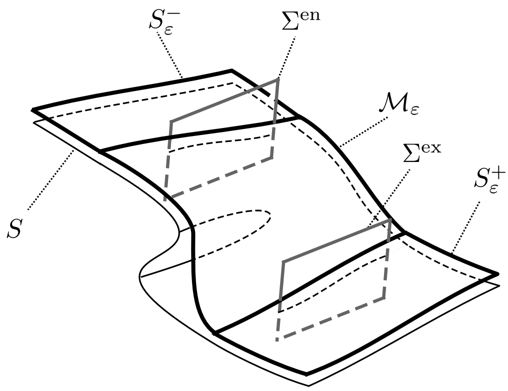


Figure 4.6: Qualitative representation of the investigation performed in this section. The sections Σ^{en} and Σ^{ex} are arbitrarily close to the cusp point. On the other hand the sections Σ^- and Σ^+ (not shown) are parallel to Σ^{en} and Σ^{ex} but far away from the cusp point. In a qualitative sense, we will construct an invariant manifold $\mathcal{M}_\varepsilon$ and then extend it all the the way up to the sections Σ^- and Σ^+. Our analysis aims for simplicity and thus depends extensively on the usage of normal forms. This, of course, makes our results depend on the choice of coordinates.

where a_0 is a positive but sufficiently small constant, for reference see fig. 4.6.

It will be clear from our analysis in the blow-up space, see section 4.2.1, that the section Σ^- needs to be partitioned as follows.

Definition 4.2.1 (The inner layer and the lateral regions)**.** *Let $0 < L < M < \infty$ be constants. The inner layer $\Sigma^{\mathrm{inner}} \subset \Sigma^-$ is defined as*

$$\Sigma^- \supset \Sigma^{\mathrm{inner}} = \left\{(b, z, \varepsilon) \in \Sigma^- \mid |b| < M\varepsilon^{2/5}\right\}.$$

On the other hand, the lateral regions are defined as

$$\Sigma^- \supset \Sigma^{+b} = \left\{(b, z, \varepsilon) \in \Sigma^- \mid b > L\varepsilon^{2/5}\right\}$$
$$\Sigma^- \supset \Sigma^{-b} = \left\{(b, z, \varepsilon) \in \Sigma^- \mid -b > L\varepsilon^{2/5}\right\}.$$

Note that the set $\left\{\Sigma^{\mathrm{inner}}, \Sigma^{+b}, \Sigma^{-b}\right\}$ is an open cover of Σ^-, see fig. 4.7.

We are now in position to present our main result. In the following theorem, we characterize the transition $\Pi : \Sigma^- \to \Sigma^+$ under a suitable choice of coordinates at the section Σ^- and Σ^+. Furthermore, we give details on the differentiability of this map according to the cover of Σ^-, see definition 4.2.1.

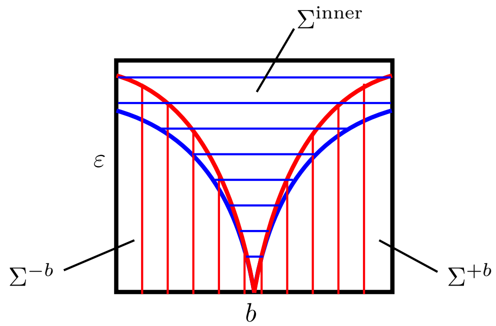


Figure 4.7: The section Σ^- needs to be partitioned into three subsections: the inner layer Σ^{inner} and the lateral regions Σ^{+b}, Σ^{-b}. From a qualitative point of view, these three layers correspond to three different types of trajectories: 1. Trajectories starting at Σ^{inner} pass close to the cusp point. Observe that $\lim_{\varepsilon\to 0}(\Sigma^{\text{inner}}) = \{b = 0\}$ and then corresponds to a solution of the associated CDE passing exactly through the cusp point. 2. Trajectories starting at Σ^{+b} pass sufficiently away from the cusp point along the regular side of the manifold S. 3. Trajectories starting at Σ^{-b} pass sufficiently away from the cusp point along the folded side of the manifold S.

Theorem 4.2.1 (Transition map of an A_3-SFS). *Let X be an A_3 slow fast system. This is, X is a vector field defined by*

$$X = \varepsilon(1+\varepsilon f_1)\frac{\partial}{\partial a} + \varepsilon^2 f_2 \frac{\partial}{\partial b} - \left(z^3 + bz + a + \varepsilon f_3\right)\frac{\partial}{\partial z} + 0\frac{\partial}{\partial \varepsilon}, \tag{4.17}$$

where each $f_i = f_i(a,b,z,\varepsilon)$, $i = 1,2,3$, is smooth. Let the sections Σ^-, Σ^+ be defined as above. Then we can choose suitable $\mathcal{C}^\ell$-coordinates (B,Z,ε) in Σ^- and $\mathcal{C}^\ell$-coordinates $(\tilde{B},\tilde{Z},\tilde{\varepsilon})$ in Σ^+ such that the transition $\Pi : (B,Z,\varepsilon) \mapsto (\tilde{B},\tilde{Z},\tilde{\varepsilon})$ is an exponential type map of the form

$$\Pi(B,Z,\varepsilon) = \left(B + h,\ \phi(B,\varepsilon) + Z\exp\left(-\frac{A(B,\varepsilon)+\Psi(B,Z,\varepsilon)}{\varepsilon}\right), \varepsilon\right), \tag{4.18}$$

where h is flat at the origin, $A > 0$ is $\mathcal{C}^\ell$, ϕ is $\mathcal{C}^\ell$-admissible with $\phi(B,0) = 0$, and Ψ is $\mathcal{C}^\ell$-admissible with $\Psi(B,Z,0) = 0$, see section 3.2 for the definition of $\mathcal{C}^\ell$-admissible. Moreover, we have the following properties of the function A, ϕ and Ψ.

1. *$-A(B,0) = I(B)$ where I is the slow divergence integral associated to* (4.17).
2. *Restricted to $(B,Z,\varepsilon) \in \Sigma^{\text{inner}}$, there are functions $\tilde{\phi}$ and $\tilde{\Psi}$ such that*

$$\phi(B,\varepsilon) = \tilde{\phi}\left(\mu, \varepsilon^{1/5}\right)$$
$$\Psi(B,Z,\varepsilon) = \tilde{\Psi}\left(|B|^{1/2}, \varepsilon^{1/5}, \varepsilon\ln\varepsilon, \mu, Z\right),$$

where $\tilde{\phi}$ and $\tilde{\Psi}$ are $\mathcal{C}^\ell$-functions with respect to monomials (see definition 3.2.2) with $\mu = B\varepsilon^{-2/5}$. Note that in this domain, μ is well defined in the sense that μ is bounded by a constant as $\varepsilon \to 0$.

3. *Restricted to $(B, Z, \varepsilon) \in \Sigma^{+b}$, there is a function $\tilde{\Psi}$ such that*

$$\begin{aligned}\phi(B,\varepsilon) &= 0\\ \Psi(B,Z,\varepsilon) &= \tilde{\Psi}\left(|B|^{1/2}, \varepsilon^{1/5}, \varepsilon\ln(|B|), \sigma, Z\right),\end{aligned}$$

where $\tilde{\Psi}$ is a $\mathcal{C}^\ell$-function with respect to monomials (see definition 3.2.2) with $\sigma = \varepsilon|B|^{-5/2}$. Note that in this domain, σ is well defined since $|B| > 0$.

4. *Restricted to $(B, Z, \varepsilon) \in \Sigma^{-b}$, there are functions $\tilde{\phi}$ and $\tilde{\Psi}$ such that*

$$\begin{aligned}\phi(B,\varepsilon) &= \tilde{\phi}\left(|B|^{1/2}, \sigma\right)\\ \Psi(B,Z,\varepsilon) &= \tilde{\Psi}\left(|B|^{1/2}, \varepsilon^{1/5}, \varepsilon\ln(|B|), \sigma\right),\end{aligned}$$

where $\tilde{\phi}$ and $\tilde{\Psi}$ are $\mathcal{C}^\ell$-functions with respect to monomials (see definition 3.2.2) with $\sigma = \varepsilon|B|^{-5/2}$. Note that in this domain, σ is well defined since $|B| > 0$.

Sketch of the proof. The first step is to recall theorem 3.3.2, which shows that X is formally conjugate to

$$F = \varepsilon\frac{\partial}{\partial a} + 0\frac{\partial}{\partial b} - \left(z^3 + bz + a\right)\frac{\partial}{\partial z} + 0\frac{\partial}{\partial \varepsilon}.$$

Next, by means of the Borel's lemma [10], the vector field F can be realized as a smooth vector field $X^N = F + \varepsilon H$ where H is flat at $(a, b, z, \varepsilon) = (0, 0, 0, 0)$. Thus, from now on, we only treat an A_3-SFS given as

$$X = \varepsilon(1 + \varepsilon\tilde{f}_1)\frac{\partial}{\partial a} + \varepsilon^2\tilde{f}_2\frac{\partial}{\partial b} - \left(z^3 + bz + a + \varepsilon\tilde{f}_3\right)\frac{\partial}{\partial z} + 0\frac{\partial}{\partial \varepsilon},$$

where each $\tilde{f}_i = \tilde{f}_i(a, b, z, \varepsilon)$ is flat at $(a, b, z, \varepsilon) = (0, 0, 0, 0)$.

Another important ingredient of the proof is the blow-up technique, described in section 3.4. This method provides several local vector fields whose corresponding transitions are of exponential type, refer to section 3.2. Later all these local transitions are composed to produce an exponential type transition between the sections Σ^- and Σ^+. Along the analysis of the local vector fields (in the blow-up space) we will take advantage of the flatness of the higher order terms of X. The complete proof follows sections 4.2.1 to 4.2.4 and is given in section 4.2.5.

Now, assuming that the transition Π is of the form (4.18), we can show that $A(B, 0)$ is given by the slow divergence integral of X. For this, let us recall the Poincaré-Leontovich-Sotomayor formula [12], which in general is given as follows.

Proposition 4.2.1. *Let X be a vector field on a manifold M^n with a volume form Ω. Let Σ^- and Σ^+ be two open sections of M and transverse to the flow of X. Let γ_ε be an orbit of X along a center manifold $\mathcal{W}^C$ of X, starting at $p = \gamma_\varepsilon \cap \Sigma^-$ and reaching $q = \gamma_\varepsilon \cap \Sigma^+$ in finite time. Let $\Pi : \Sigma^- \to \Sigma^+$ be the transition map defined in a neighborhood of p. If $\psi^- : U \to \Sigma^-$ and $\psi^+ : V \to \Sigma^+$, with $U \subset \mathbb{R}^{n-1}$ and $V \subset \mathbb{R}^{n-1}$, are coordinates in Σ^- and in Σ^+ respectively, then*

$$\det \left(D \left((\psi^+)^{-1} \circ \Pi \circ \psi^- \right) \right) (s^-) = \frac{\langle \Omega(p), D\psi^-(s^-) \times X(p) \rangle}{\langle \Omega(q), D\psi^+(s^+) \times X(p) \rangle} \exp \left(\int_{\gamma_\varepsilon} \operatorname{div}_\Omega X \, d\tau \right), \tag{4.19}$$

where $s^- = (\psi^-)^{-1}(p)$ and $s^+ = (\psi^+)^{-1}(q)$. The integral is taken along the orbit γ_ε from p to q parametrized by the fast time τ.

So we have the following.

Proposition 4.2.2. *Consider an A_3-SFS and assume that the transition $\Pi : \Sigma^- \to \Sigma^+$ is given by* (4.18)*. Then $-A(B, 0) = I(B)$, where $I(B)$ is the slow divergence integral associated to the A_3-SFS.*

Proof. The only relevant component is Z, so denote by Π_Z the Z-component of Π. The factor multiplying the exponential in (4.19) can be taken as a constant $C > 0$. Then we have that (4.19) for the vector field of theorem 4.2.1 reads as

$$\frac{\partial \Pi_Z}{\partial Z} = C \exp \left(\int_{\gamma_\varepsilon} \operatorname{div}_\Omega X \, d\tau \right).$$

Using the properties of the slow divergence integral described in section 3.1.2, and since $C \neq 0$, we have

$$\begin{aligned} \frac{\partial \Pi_Z}{\partial Z} &= C \exp \left(\int_{\gamma_\varepsilon} \operatorname{div}_\Omega X \, d\tau \right) \\ &= \exp \left(\frac{1}{\varepsilon} \left(\int_{\gamma_0} \operatorname{div} X_0 \, dt + \varepsilon \ln C + o(1) \right) \right) \\ &= \exp \left(\frac{1}{\varepsilon} \left(I + O(\varepsilon) \right) \right), \end{aligned} \tag{4.20}$$

where I is the slow divergence integral of X along a curve in the slow manifold S from Σ^- to Σ^+. In principle, the limit $\varepsilon \to 0$ of (4.20) is not well defined.

However, according to our theorem 4.2.1, we have by differentiating (4.18) w.r.t. Z

$$\frac{\partial \Pi_Z}{\partial Z} = \exp\left(-\frac{A(B,\varepsilon)+\varepsilon\Psi(B,Z,\varepsilon)}{\varepsilon}\right). \tag{4.21}$$

Identifying (4.20) with (4.21) and taking the limit $\varepsilon \to 0$ we have indeed that

$$\lim_{\varepsilon\to 0}(I+O(\varepsilon)) = \lim_{\varepsilon\to 0}(-A(B,\varepsilon)+\varepsilon\Psi(B,Z,\varepsilon)),$$

which shows the claim.

Note that the slow divergence integral in the coordinates (a,b,z) reads as

$$I(b) = \tilde{I}(b,\zeta^+) - \tilde{I}(b,\zeta^-),$$

where straightforward computations show that

$$\tilde{I}(b,\zeta) = \frac{9}{5}\zeta^5 + 2\zeta^3 b + b^2\zeta,$$

and where $\zeta^\pm$ is a constant defined by $(a^\pm, b, \zeta^\pm) \in \Sigma^\pm \cap S$.

On the other hand, in normal coordinates and along regular parts of the slow manifold, the A_3-SFS can be written as (see section 3.1.2)

$$X(A,B,Z,\varepsilon) = \varepsilon\frac{\partial}{\partial A} + 0\frac{\partial}{\partial B} - Z\frac{\partial}{\partial Z} + 0\frac{\partial}{\partial \varepsilon}.$$

In these coordinates the slow divergence integral reads as

$$I = A^+ - A^-,$$

where A^+ and A^- are the corresponding parametrizations of Σ^+ and Σ^- (respectively) in the coordinates (A,B,Z,ε). □

4.2.1 Analysis in the chart K_{en}

Taking into account our notation convention, the blow-up map in this chart is given by

$$a = -r_1^3,\ b = r_1^2 b_1,\ z = r_1^3 z_1,\ \varepsilon = r_1^5\varepsilon_1. \tag{4.22}$$

The corresponding vector field in this chart (after multiplication by 3) has the form

$$X_{\text{en}}:\begin{cases} r_1' &= -\varepsilon_1 r_1\left(1+\tilde{f}_1\right)\\ b_1' &= 2\varepsilon_1 b_1\left(1+\tilde{f}_1\right)+r_1^6\varepsilon_1^2\tilde{f}_2\\ z_1' &= -3\left(z_1^3+b_1z_1-1-\frac{1}{3}\varepsilon_1 z_1\right)+r_1^2\varepsilon_1\tilde{f}_3\\ \varepsilon_1' &= 5\varepsilon_1^2\left(1+\tilde{f}_1\right)\end{cases}$$

where the functions $\tilde{f}_i = f_i(r_1, b_1, z_1, \varepsilon_1)$ are flat along $r_1 = 0$, recall that $S^3 \times \{r = 0\} \mapsto 0 \in \mathbb{R}^4$ via the blow-up map. We study a transition $\Pi_1 : \Delta_1^{\text{en}} \to \Delta_1^{\text{ex}}$ where

$$\begin{aligned}\Delta_1^{\text{en}} &= \left\{(r_1, b_1, z_1, \varepsilon_1) \in \mathbb{R}^4 \,|\, r_1 = r_0, \varepsilon_1 < \delta,\, z_1 > 0\right\}\\ \Delta_1^{\text{ex}} &= \left\{(r_1, b_1, z_1, \varepsilon_1) \in \mathbb{R}^4 \,|\, \varepsilon_1 = \delta, r_1 < r_0\right\},\end{aligned}$$

where r_0 and δ are sufficiently small positive constants. In fact, we choose $r_0 = a_0^{1/3}$.

Remark 4.2.1. *The section Δ_1^{en} corresponds to Σ^{en} in the blow-up space, that is $\Sigma^{\text{en}} = \Phi(\Delta_1^{\text{en}})$, where Φ is the blow-up map* (4.22)*. This implies that trajectories of X crossing Σ^{en} correspond to trajectories of X_{en} crossing Δ_1^{en}.*

Before going any further, let us provide a qualitative description of X_{en} based on [8]. This process can be repeated, following similar arguments, in all the local charts; however, for brevity we only detail it for the current one.

Qualitative description of the flow of X_{en} The subspaces $\{r_1 = 0\}$, $\{\varepsilon_1 = 0\}$ and $\{r_1 = 0\} \cap \{\varepsilon_1 = 0\}$ are invariant. Therefore, it is useful to study the flow of X_{en} restricted to the aforementioned subspaces.

Restriction to $\{r_1 = 0\} \cap \{\varepsilon_1 = 0\}$. In this space X_{en} is reduced to

$$\begin{aligned} b_1' &= 0\\ z_1' &= -3\left(z_1^3 + b_1 z_1 - 1\right).\end{aligned} \tag{4.23}$$

The set

$$\gamma_1 = \left\{(b_1, z_1) \,|\, z_1^3 + b_1 z_1 - 1 = 0\right\}$$

is a curve of equilibrium points. The phase portrait of (4.23) is shown in figure 4.8.

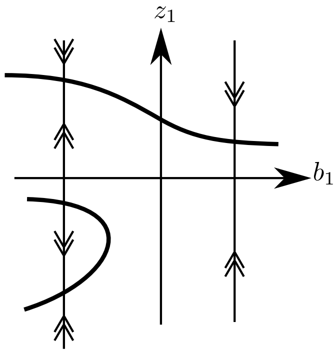


Figure 4.8: The phase portrait of X_{en} restricted to the invariant space $\{r_1 = 0\} \cap \{\varepsilon_1 = 0\}$. The shown curve is γ_1 and it comprises a set of equilibrium points. Note that locally, all trajectories with initial condition $z_1(0) > 0$ are attracted to $\gamma_1|_{\{z_1>0\}}$.

Remark 4.2.2. *All the trajectories of* (4.23) *restricted to an initial condition* $z_0 > 0$ *are attracted to the curve* $\gamma_1|_{z_1>0}$. *Furthermore, due to our definition of* Δ_1^{en}, *we are interested* only *in trajectories satisfying this initial condition. Thus, from now on, we restrict our analysis to the subspace* $\{z_1 > 0\}$.

Restriction to $\{\varepsilon_1 = 0\}$. In this space X_{en} is reduced to

$$\begin{aligned} r_1' &= 0 \\ b_1' &= 0 \\ z_1' &= -3\left(z_1^3 + b_1 z_1 - 1\right). \end{aligned} \tag{4.24}$$

The set $\Gamma_1 = \left\{(r_1, b_1, z_1) \,|\, z_1^3 + b_1 z_1 - 1 = 0\right\}$ is a surface of equilibrium points given by $\Gamma_1 = (r_1, \gamma_1)$. Since $r_1' = 0$, the phase space of (4.24) is foliated by two dimensional leaves in which the flow looks like fig. 4.8.

Restriction to $\{r_1 = 0\}$. In this space X_{en} is reduced to

$$\begin{aligned} b_1' &= 2\varepsilon_1 b_1 \\ z_1' &= -3\left(z_1^3 + b_1 z_1 - 1 - \frac{1}{3}\varepsilon_1 z_1\right) \\ \varepsilon_1' &= 5\varepsilon_1^2, \end{aligned} \tag{4.25}$$

Once again, the set $\gamma_1 = \left\{(b_1, z_1, \varepsilon_1) \,|\, \varepsilon_1 = 0,\ z > 0,\ z_1^3 + b_1 z_1 - 1 = 0\right\}$ is a curve of equilibrium points. The Jacobian of (4.25) evaluated along γ_1 shows that, for small enough ε_1, there exists an invariant center manifold that passes through γ_1. Furthermore, the non-zero eigenvalue corresponding to the z-direction is negative along γ_1. The phase portrait of (4.25) is shown in figure 4.9.

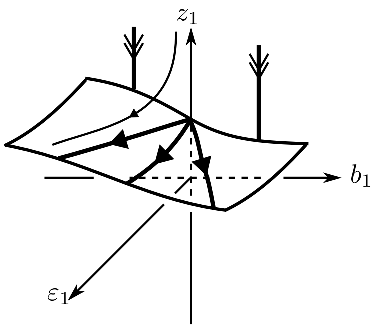


Figure 4.9: Phase portrait of (4.25) restricted to $z_1 > 0$. The shown surface is an invariant center manifold, which is attracting in the z_1-direction.

Observe that the b_1 and the ε_1 directions are expanding. It is important to know the relation between such two expanding variables. We have

$$\frac{db_1}{d\varepsilon_1} = \frac{2}{5}\frac{b_1}{\varepsilon_1},$$

which has the solution

$$b_1 = b_1^* \left(\frac{\varepsilon_1}{\varepsilon_1^*}\right)^{2/5},$$

where $b_1^* \leq b_1$ and $\varepsilon_1^* \leq \varepsilon_1$ are the initial conditions, that is $(b_1^*, \varepsilon_1^*) = (b_1, \varepsilon_1)|_{\Delta_1^{\text{en}}}$. It is important to look at the ratio of initial conditions $\frac{b_1^*}{(\varepsilon_1^*)^{2/5}}$. This ratio tells us that b_1 is bounded as $\varepsilon_1 \to 0$ (and therefore as $\varepsilon_1^* \to 0$) if and only if $b_1^* \in O\left((\varepsilon_1^*)^{2/5}\right)$. In other words, if the initial condition b_1^* is not of order $O((\varepsilon_1^*)^{2/5})$ then the value of b_1 at Δ_1^{ex} blows up as $\varepsilon_1^* \to 0$. This leads us to partition the section Δ_1^{en} into three open regions as follows.

$$\begin{aligned}
\Delta_1^{\text{en},\varepsilon_1} &= \Delta_1^{\text{en}}|_{|b_1|<M\varepsilon_1^{2/5}}\\
\Delta_1^{\text{en},b_1} &= \Delta_1^{\text{en}}|_{b_1>K\varepsilon_1^{2/5}}\\
\Delta_1^{\text{en},-b_1} &= \Delta_1^{\text{en}}|_{-b_1>K\varepsilon_1^{2/5}},
\end{aligned}$$

where $0 < K < M < \infty$. Observe that the open sets $\Delta_1^{\text{en},\varepsilon_1}$, Δ_1^{en,b_1} and $\Delta_1^{\text{en},-b_1}$ form an open cover of Δ_1^{en}. Accordingly, these sets induce an open cover of the entry section Σ^{en} via the blow-up map (4.22). See fig. 4.10 for a representation of the aforementioned partition.

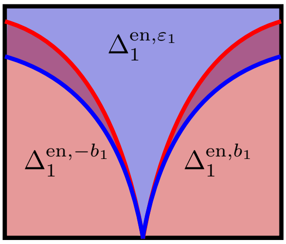


Figure 4.10: Partition of Δ_1^{en}. Trajectories crossing through $\Delta_1^{\text{en},\varepsilon_1}$ corresponding to the inner wedge area, have a continuation on the chart $K_{\bar{\varepsilon}}$. On the other hand, outside $\Delta_1^{\text{en},\varepsilon_1}$ we must consider the lateral regions Δ_1^{en,b_1} and $\Delta_1^{\text{en},-b_1}$.

Based on the partition of the entry section Δ_1^{en}, we define three transitions as follows

$$\begin{aligned} \Pi_1^{\varepsilon_1} &: \Delta_1^{\text{en},\varepsilon_1} \to \Delta_1^{\text{ex}} \\ \Pi_1^{+b_1} &: \Delta_1^{\text{en},+b_1} \to \Delta_1^{\text{ex},+b_1} \\ \Pi_1^{-b_1} &: \Delta_1^{\text{en},-b_1} \to \Delta_1^{\text{ex},-b_1}, \end{aligned} \tag{4.26}$$

where

$$\begin{aligned} \Delta_1^{\text{ex}} &= \left\{ (r_1, b_1, z_1, \varepsilon_1) \in \mathbb{R}^4 \,|\, \varepsilon_1 = \delta, r_1 < r_0 \right\}, \\ \Delta_1^{\text{ex},\pm b_1} &= \left\{ (r_1, b_1, z_1, \varepsilon_1) \in \mathbb{R}^4 \,|\, b_1 = \pm\eta, r_1 < r_0 \right\}. \end{aligned}$$

To finish with the qualitative description, note that there exists a (non-unique) 3-dimensional center manifold $\mathcal{W}_1^C$. This is shown by evaluating the Jacobian of X_{en} all along the surface

$$\Gamma_1 = \left\{ (r_1, b_1, z_1, \varepsilon_1) \,|\, \varepsilon_1 = 0,\ z_1 > 0\, z_1^3 + b_1 z_1 - 1 = 0 \right\}.$$

Moreover, by the analysis provided above, the center manifold $\mathcal{W}_1^C|_{z_1>0}$ is attracting for ε_1 small enough. Note that $\mathcal{W}_1^C|_{\varepsilon_1=0} = \Gamma_1$. This means that $\mathcal{W}_1^C$ can be interpreted as a perturbation of the slow manifold S, written in the coordinates of the current chart. See fig. 4.11 for a representation of the previous exposition.

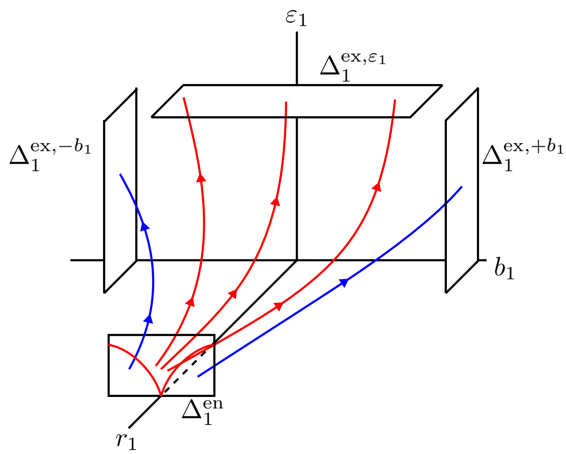


Figure 4.11: Phase portrait of the trajectories of X_{en} depending on their initial condition. If the trajectories satisfy the estimate $y \in O(\varepsilon^{2/5})$, then they arrive to $\Delta_1^{\mathrm{ex},\varepsilon_1}$ in finite time. If the estimate $y \in O(\varepsilon^{2/5})$ is not satisfied, then we must choose one of the outgoing sections $\Delta_1^{\mathrm{ex},\pm b}$ in order to have a well defined transition map.

Let us recall that the vector field X_{en} is of the form

$$X_{\mathrm{en}}: \begin{cases} r_1' &= -\varepsilon_1 r_1 \left(1+\tilde{f}_1\right) \\ b_1' &= 2\varepsilon_1 b_1 \left(1+\tilde{f}_1\right) + r_1^6\varepsilon_1^2\tilde{f}_2 \\ z_1' &= -3\left(z_1^3 + b_1 z_1 - 1 - \frac{1}{3}\varepsilon_1 z_1\right) + r_1^2\varepsilon_1\tilde{f}_3 \\ \varepsilon_1' &= 5\varepsilon_1^2\left(1+\tilde{f}_1\right) \end{cases} \tag{4.27}$$

We now proceed to describe the transitions Π_1 given by (4.26). For this, first we write (4.27) in a suitable normal form. Next, based on this normal form, we compute the corresponding transition.

First of all, let us move the origin to the point $(r_1, b, z_1, \varepsilon_1) = (0, 0, 1, 0)$. This is done by defining a new variable ζ_1 by $\zeta_1 = z_1 - 1$. With this variable we have a new local vector field Y_{en} which is defined by

$$Y_{\mathrm{en}}: \begin{cases} r_1' &= -\varepsilon_1 r_1 \left(1+\tilde{f}_1\right) \\ b_1' &= 2\varepsilon_1 b_1 \left(1+\tilde{f}_1\right) + r_1^6\varepsilon_1^2\tilde{f}_2 \\ \varepsilon_1' &= 5\varepsilon_1^2\left(1+\tilde{f}_1\right) \\ \zeta_1' &= -3G(b_1,\varepsilon_1,\zeta_1) + \varepsilon_1\tilde{h}, \end{cases}$$

where $G(0,0,0)=0$ and $\frac{\partial G}{\partial \zeta_1}(0,0,0)=3$. Now, we want to write Y_{en} in a suitable normal form. From proposition A.6.2, we know that Y_{en} is $\mathcal{C}^\ell$ equivalent to

$$X_{\mathrm{en}}^N : \begin{cases} r_1' &= -\varepsilon_1 r_1 \\ B_1' &= 2\varepsilon_1 B_1 \\ \varepsilon_1' &= 5\varepsilon_1^2 \\ Z_1' &= -9(1+H_1(r_1,B_1,\varepsilon_1))Z_1, \end{cases}$$

where H_1 is a C^ℓ-function vanishing at the origin. This normal form X_{en}^N is convenient since the chosen center manifold $\mathcal{W}_1^C$ is now simply given by $\mathcal{W}_1^C = \{Z_1 = 0\}$. Furthermore, from the format of X_{en}^N, it is evident the "hyperbolic nature" of the flow restricted to the center manifold: the restriction of X_{en}^N to the center manifold $\mathcal{W}_1^C$ has a simple structure, namely

$$X_{\mathrm{en}}^N|_{\mathcal{W}_1^C} : \begin{cases} r_1' &= -\varepsilon_1 r_1 \\ B_1' &= 2\varepsilon_1 B_1 \\ \varepsilon_1' &= 5\varepsilon_1^2. \end{cases}$$

The vector field X_{en}^N is of the form studied in proposition A.6.5, therefore we have that the transition

$$\Pi_1^{\varepsilon_1} : (B_1, \varepsilon_1, z_1) \mapsto (\tilde{r}_1, \tilde{B}_1, \tilde{Z}_1)$$

is of the form

$$\begin{aligned} \tilde{r}_1 &= r_0 \left(\frac{\varepsilon_1}{\delta}\right)^{1/5} \\ \tilde{B}_1 &= B_1 \left(\frac{\delta}{\varepsilon_1}\right)^{2/5} \\ \tilde{Z}_1 &= Z_1 \exp\left(-\frac{9}{5\varepsilon_1}(1+\alpha_1\varepsilon_1 \ln\varepsilon_1 + \varepsilon_1 G_1)\right), \end{aligned}$$

where $\alpha_1 = \alpha_1(r_0|B_1|^{1/2}, r_0\varepsilon_1^{1/5})$ and $G_1 = G_1(r_0|B_1|^{1/2}, r_0\varepsilon_1^{1/5}, \mu)$ where $\mu = B_1\varepsilon_1^{-2/5}$. Recall that for this transition we have the condition $B_1 \in O(\varepsilon_1^{2/5})$ so μ is well defined.

On the other hand, the transition

$$\Pi_1^{\pm B_1} : (B_1, \varepsilon_1, Z_1) \mapsto (\tilde{r}_1, \tilde{\varepsilon}_1, \tilde{Z}_1)$$

is (see proposition A.6.5) of the form

$$\begin{aligned}\tilde{r}_1 &= r_0\left(\frac{B_1}{\eta}\right)^{1/2}\\ \tilde{\varepsilon}_1 &= \varepsilon_1\left(\frac{\eta}{B_1}\right)^{5/2}\\ \tilde{Z}_1 &= Z_1\exp\left(-\frac{9}{5\varepsilon_1}(1+\beta_1\varepsilon_1\ln(|B_1|)+\varepsilon_1 H_1)\right),\end{aligned}$$

where $\beta_1 = \beta_1(r_0|B_1|^{1/2}, r_0\varepsilon_1^{1/5})$ and $H_1 = H_1(r_0|B_1|^{1/2}, r_0\varepsilon_1^{1/5}, \sigma)$, where $\sigma = \varepsilon_1|B_1|^{-5/2}$. Note that since $B_1 \notin O(\varepsilon_1^{2/5})$, σ is well defined. We observe that the transitions $\Pi_1^{\varepsilon_1}$ and $\Pi_1^{\pm B_1}$ are exponential type maps.

4.2.2 Analysis in the chart $K_{\bar{\varepsilon}}$

Taking into account our notation convention, the blow-up map in this chart is given by

$$a = r_2^3 a_2,\ b = r_2^2 b_2,\ z = r_2^3 z_2,\ \varepsilon = r_2^5.$$

Then, the blown up vector field reads as

$$X_{\bar{\varepsilon}}: \begin{cases} r_2' &= 0\\ a_2' &= 1+\tilde{g}_1\\ b_2' &= \tilde{g}_2\\ z_2' &= -\left(z_2^3+b_2z_2+a_2\right)+\tilde{g}_3,\end{cases} \tag{4.28}$$

where the function $\tilde{g}_i = \tilde{g}_i(r_2, a_2, b_2, z_2)$ are flat along $r_2 = 0$. Note that in this chart r_2 acts as a parameter and that the flow of $X_{\bar{\varepsilon}}$ is regular. From the equation $a_2' = 1 + \tilde{g}_1$, we define the following "entry" and "exit" sections.

$$\begin{aligned}\Delta_2^{\mathrm{en},\bar{\varepsilon}} &= \{(r_2, a_2, b_2, z_2)\,|\, a_2 = -A_0,\ z_2 \geq 0\},\\ \Delta_2^{\mathrm{ex},\bar{\varepsilon}} &= \{(r_2, a_2, b_2, z_2)\,|\, a_2 = A_0,\ z_2 \leq 0\}.\end{aligned}$$

Therefore, we define a transition $\Pi_2^{\bar{\varepsilon}}$ as

$$\begin{aligned}\Pi_2^{\bar{\varepsilon}} :&\Delta_2^{\mathrm{en},\bar{\varepsilon}} \to \Delta_2^{\mathrm{en},\bar{\varepsilon}}\\ &(r_2, b_2, z_2) \mapsto (\tilde{r}_2, \tilde{b}_2, \tilde{z}_2).\end{aligned}$$

Since (4.28) is regular, by the flow box theorem all trajectories starting at $\Delta_2^{\mathrm{en},\bar{\varepsilon}}$ arrive at $\Delta_2^{\mathrm{ex},\bar{\varepsilon}}$ in finite time. Moreover, the transition $\Pi_2^{\bar{\varepsilon}}$ is a diffeomorphism and then, from (4.28) we have that $\Pi_2^{\bar{\varepsilon}}$ reads as

$$\begin{aligned}\Pi_2\bar{\varepsilon}(r_2, b_2, z_2) &= (\tilde{r}_2, \tilde{b}_2, \tilde{z}_2)\\ &= (r_2, b_2 + h_{b_2}, \phi_1(r_2, b_2) + \phi_2(r_2, b_2)(1 + \phi_3(r_2, b_2, z_2))z_2),\end{aligned}$$

where the ϕ_i's are smooth functions. Observe that in this chart, the transition is not an exponential type map.

4.2.3 Analysis in the chart K_{ex}

Taking into account our notation convention, the blow-up map in this chart is given by

$$a = r_3^3,\ b = r_3^2 b_3,\ z = r_3^3 z_3,\ \varepsilon = r_3^5 \varepsilon_3.$$

Then, the corresponding blown up vector field reads as

$$X_{\mathrm{ex}} : \begin{cases} r_3' &= \varepsilon_3 r_3 \left(1 + \tilde{f}_1\right) \\ b_3' &= -2\varepsilon_3 b_3 \left(1 + \tilde{f}_1\right) + r_3^6 \varepsilon_3^2 \tilde{f}_2 \\ z_3' &= -3\left(z_3^3 + b_3 z_3 + 1 + \frac{1}{3}\varepsilon_3 z_3\right) + r_3^2 \varepsilon_3 \tilde{f}_3 \\ \varepsilon_3' &= -5\varepsilon_3^2 \left(1 + \tilde{f}_1\right) \end{cases}$$

where the function $\tilde{f}_i = \tilde{f}_i(r_3, b_3, \varepsilon_3, z_3)$ are flat along $r_3 = 0$. Observe that the vector field X_{ex} resembles the vector field X_{en}. Therefore, we have a similar behavior of the trajectories, the main difference is that in the case of X_{ex}, there is one expanding (r_3) and three contracting (b_3, ε_3 and z_3) directions. The flow of X_{ex} is obtained following similar arguments as for the flow of X_{en}.

From the fact that X_{ex} has three contracting and one expanding direction, we define the entry sections

$$\begin{aligned}\Delta_3^{\mathrm{en},\bar{\varepsilon}} &= \{(r_3, b_3, \varepsilon_3, z_3) \,:\, \varepsilon_3 = \delta,\, z_3 < 0,\, r_3 < r_0\}\\ \Delta_3^{\mathrm{en},+b_3} &= \{(r_3, b_3, \varepsilon_3, z_3) \,:\, b_3 = \eta,\, z_3 < 0,\, r_3 < r_0\}\\ \Delta_3^{\mathrm{en},-b_3} &= \{(r_3, b_3, \varepsilon_3, z_3) \,:\, b_3 = -\eta,\, z_3 < 0,\, r_3 < r_0\},\end{aligned}$$

where all the constants are positive and sufficiently small, and the exit section

$$\Delta_3^{\mathrm{ex}} = \{(r_3, b_3, \varepsilon_3, z_3) \,:\, r_3 = r_0,\, z_3 < 0,\, \varepsilon_3 < \delta,\, |b_3| < \eta\}.$$

Then, accordingly, we define three transition maps as follows

$$\begin{aligned}\Pi_3^{\varepsilon_3} &: \Delta_3^{\mathrm{en},\bar{\varepsilon}} \to \Delta_3^{\mathrm{ex}}\\ &: (r_3, b_3, z_3) \mapsto (\tilde{b}_3, \tilde{\varepsilon}_3, \tilde{z}_3)\\[1em] \Pi_3^{+b_3} &: \Delta_3^{\mathrm{en},+b_3} \to \Delta_3^{\mathrm{ex}}\\ &: (r_3, \varepsilon_3, z_3) \mapsto (\tilde{b}_3, \tilde{\varepsilon}_3, \tilde{z}_3)\\[1em] \Pi_3^{-b_3} &: \Delta_3^{\mathrm{en},-b_3} \to \Delta_3^{\mathrm{ex}}\\ &: (r_3, \varepsilon_3, z_3) \mapsto (\tilde{b}_3, \tilde{\varepsilon}_3, \tilde{z}_3).\end{aligned}$$

Now we proceed to write X_{ex} in a normal form just as we did with X_{en} in section 4.2.1. Following proposition A.6.2 we have that X_{ex} is $\mathcal{C}^\ell$ equivalent to

$$X_{\mathrm{ex}}^N : \begin{cases} r_3' &= \varepsilon_3 r_3\\ B_3' &= -2\varepsilon_3 B_3\\ \varepsilon_3' &= -5\varepsilon_3^2\\ Z_3' &= -9(1 + H_3) Z_3, \end{cases}$$

where $H_3 = H_3(r_3, B_3, \varepsilon_3)$ is a C^ℓ function vanishing at the origin. Just as in the chart K_{en}, there exists a three dimensional center manifold $\mathcal{W}_3^C$ associated to X_{ex}^N and which has been chosen such that $\mathcal{W}_3^C = \{Z_3 = 0\}$. Since r_3 is the only expanding direction, we take as transition time $T_3 = \ln\left(\frac{r_0}{r_3}\right)$. This transition time is computed from the dynamics restricted to $\mathcal{W}_3^C$, that is, from the equation $r_3' = r_3$. In contrast to what happened in the chart K_{en}, the time T_3 is well defined for all the three transitions $\Pi_3^{\varepsilon_3}$, $\Pi_3^{+B_3}$ and $\Pi_3^{-B_3}$. Following proposition A.6.5 we have

$$\begin{aligned}\tilde{B}_3 &= B_3\left(\frac{r_3}{r_0}\right)^2\\ \tilde{\varepsilon}_3 &= \varepsilon_3\left(\frac{r_3}{r_0}\right)^5\\ \tilde{Z}_3 &= Z_3 \exp\left(-\frac{9}{5\varepsilon_3}\left(\left(\frac{r_0}{r_3}\right)^5 - 1 + \alpha_3\varepsilon_3 \ln r_3 + \varepsilon_3 H_3\right)\right),\end{aligned}$$

where $\alpha_3 = \alpha_3(r_3|B_3|^{1/2}, r_3\varepsilon_3^{1/5})$ and $H_3 = H_3(r_3|B_3|^{1/2}, r_3\varepsilon_3^{1/5}, r_3)$. Therefore, by taking into account the definitions of the entry sections we have

$$\Pi_3^{\varepsilon 3}(r_3, B_3, Z_3) = \left(B_3\left(\frac{r_3}{r_0}\right)^2, \delta\left(\frac{r_3}{r_0}\right)^5, Z_3 \exp\left(-\frac{9}{5\delta}\left(\left(\frac{r_0}{r_3}\right)^5 - 1 + \alpha_3\delta\ln r_3 + \delta H_3\right)\right)\right)$$
$$\Pi_3^{\pm b_3}(r_3, \varepsilon_3, Z_3) = \left(\pm\eta\left(\frac{r_3}{r_0}\right)^2, \varepsilon_3\left(\frac{r_3}{r_0}\right)^5, Z_3 \exp\left(-\frac{9}{5\varepsilon_3}\left(\left(\frac{r_0}{r_3}\right)^5 - 1 + \alpha_3\varepsilon_3\ln r_3 + \varepsilon_3 H_3\right)\right)\right).$$

Observe that these transitions are of exponential type.

4.2.4 Analysis in the charts $K_{\pm\bar{b}}$

In this section we study the local flow at the charts $K_{+\bar{b}}$ and $K_{-\bar{b}}$. In a qualitative sense, these charts come into play when the initial condition $b_0 = b|_{\Sigma^{\mathrm{en}}}$ does not satisfy the estimate $b_0 \in O(\varepsilon^{2/5})$. This implies that the corresponding trajectory passes away from the cusp point. The chart $K_{+\bar{b}}$ "sees" trajectories with initial condition $b|_{\Sigma^{\mathrm{en}}} > 0$ while $K_{-\bar{b}}$ "sees" trajectories with initial condition $b|_{\Sigma^{\mathrm{en}}} < 0$.

Analysis in the chart $K_{+\bar{b}}$

In this chart the blow-up maps reads

$$a = r_2^3 a_2,\ b = r_2^2,\ z = r_2 z_2,\ \varepsilon = r_2^5\varepsilon_2.$$

Then we have that the blow-up vector field is given by

$$X_{+\bar{b}} : \begin{cases} r_2' &= \varepsilon_2 \bar{f}_r \\ a_2' &= \varepsilon_2(1 + \bar{f}_{a_2}) + \varepsilon_2\bar{g}_{a_2} \\ \varepsilon_2' &= -\varepsilon_2\bar{f}_{\varepsilon_2} \\ z_2' &= -(z_2^3 + z_2 + a_2) + \varepsilon_2\bar{f}_{z_2} \end{cases}$$

where all the functions $\bar{f}_\ell$ are flat along $\{r_2 = 0\}$. Observe that the set

$$\Gamma_2 = \left\{(r_2, a_2, \varepsilon_2, z_2)\,|\,\varepsilon_2 = 0,\ z_2^3 + z_2 + a_2 = 0\right\}$$

is a NHIM of $X_{+\bar{b}}$. However, $X_{+\bar{b}}$ is not exactly a slow fast system since $\varepsilon_2' \neq 0$, but the restriction of $X_{+\bar{b}}$ to $\{r_2 = 0\}$ is indeed a slow fast system. This restriction reads as

$$X_{+\bar{b}}|_{\{r_2=0\}} : \begin{cases} a_2' &= \varepsilon_2 \\ \varepsilon_2' &= 0 \\ z_2' &= -(z_2^3 + z_2 + a_2). \end{cases}$$

Since the subspace $\{r_2 = 0\}$ is invariant and $X_{+\bar{b}}$ is a flat perturbation of $X_{+\bar{b}}|_{\{r_2=0\}}$, it is equally useful to study the restriction $X_{+\bar{b}}|_{\{r_2=0\}}$ than the whole system $X_{+\bar{b}}$. After all, the flow of $X_{+\bar{b}}|_{r_2>0}$ is equivalent to the flow of $X_{+\bar{b}}|_{\{r_2=0\}}$. The slow manifold of $X_{+\bar{b}}|_{\{r_2=0\}}$ is defined by $\Gamma_2|_{r_2=0}$ and it is normally hyperbolic. Let us define the sections

$$\begin{aligned} \Delta_2^{\mathrm{en},+b_2} &= \left\{(r_2, a_2, \varepsilon_2, z_2) \in \mathbb{R}^4 \mid a_2 = -A_0\right\} \\ \Delta_2^{\mathrm{ex},+b_2} &= \left\{(r_2, a_2, \varepsilon_2, z_2) \in \mathbb{R}^4 \mid a_2 = A_0\right\}. \end{aligned} \tag{4.29}$$

Accordingly, we study the transition

$$\begin{aligned} \Pi_2^{+b_2} :& \Delta_2^{\mathrm{en},+b_2} \to \Delta_2^{\mathrm{ex},+b_2} \\ & (r_2, \varepsilon_2, z_2) \mapsto (\tilde{r}_2, \tilde{\varepsilon}_2, \tilde{z}_2). \end{aligned}$$

For a qualitative description of $X_{+\bar{b}}|_{\{r_2=0\}}$ and the objects defined above see fig. 4.12.

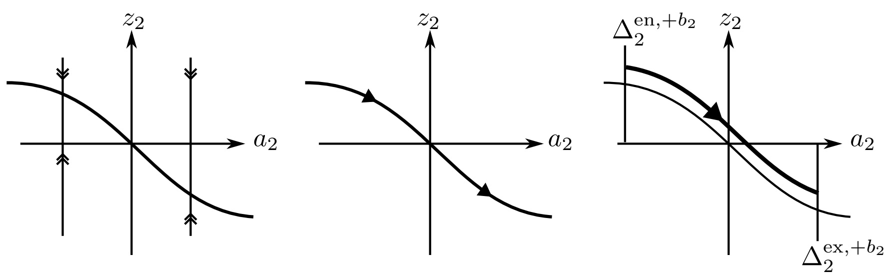


Figure 4.12: Left: phase portrait of the corresponding layer equation of $X_{+\bar{b}}|_{\{r_2=0\}}$. Center: phase portrait of the corresponding CDE of $X_{+\bar{b}}|_{\{r_2=0\}}$. Right: Since the critical manifold is regular, by Fenichel theory we know that the manifold Γ_2 is perturbed to an invariant manifold Γ_{2,ε_2} which is at distance of order $O(\varepsilon_2)$ from Γ_2.

For sufficiently small ε_2, there exists a C^ℓ change of coordinates (see our exposition of section 3.1.2) that transforms $X_{+\bar{b}}|_{\{r_2=0\}}$ into the vector field

$$Y^N : \begin{cases} a_2' &= \varepsilon_2 \\ \varepsilon_2' &= 0 \\ Z_2' &= -Z_2, \end{cases}$$

From the definition of the entry and exit sections (4.29), we choose the time of integration as $T = 2A_0$, which is obtained from the equation $\dot{a_2} = 1$. To obtain the component Z_2 of the transition $\Pi_2^{+b_2}|_{\{r_2=0\}}$ we need to integrate

$$Z_2' = -\frac{1}{\varepsilon_2} Z_2,$$

and then $\tilde{Z}_2 = Z_2(T)$. Therefore the transition $\Pi_2^{+b_2}$ reads as

$$\Pi_2^{+b_2}(0, \varepsilon_2, Z_2) = \left(0, \varepsilon_2, Z_2 \exp\left(-\frac{2A_0}{\varepsilon_2}\right)\right)$$

Remark 4.2.3. *By regular perturbation theory, the flow of $X_{+\bar{b}}|_{\{r_2=0\}}$ is equivalent to that of $X_{+\bar{b}}|_{\{r_2>0\}}$. In other words, the image of $\Pi_2^{+b_2}|(\Delta_2^{\mathrm{en},+b_2}|_{r=0})$ is diffeomorphic to the image of $\Pi_2^{+b_2}|(\Delta_2^{\mathrm{en},+b_2})$.*

Analysis in the chart $K_{-\bar{b}}$

In this chart the blow-up maps reads

$$a = r_2^3 a_2,\ b = -r_2^2,\ z = r_2 z_2,\ \varepsilon = r^5 \varepsilon_2.$$

Then we have that the blow-up vector field is given by

$$X_{-\bar{b}} : \begin{cases} r_2' &= -\varepsilon_2 \bar{f}_r \\ a_2' &= \varepsilon_2(1 + \bar{f}_{a_2}) + \varepsilon_2 \bar{g}_{a_2} \\ \varepsilon_2' &= \varepsilon_2 \bar{f}_{\varepsilon_2} \\ z_2' &= -(z_2^3 - z_2 + a_2) + \varepsilon_2 \bar{f}_{z_2} \end{cases}$$

where all the functions $\bar{f}_\ell$ and $\bar{g}_{a_2}$ are flat along $\{r_2 = 0\}$. Observe that the subspace $\{r_2 = 0\}$ is invariant. The restriction of $X_{-\bar{b}}$ to this subspace reads as

$$X_{-\bar{b}}|_{\{r_2=0\}} : \begin{cases} a_2' &= \varepsilon_2 \\ \varepsilon_2' &= 0 \\ z_2' &= -(z_2^3 - z_2 + a_2). \end{cases}$$

The flow of $X_{-\bar{b}}$ is a flat perturbation of the flow of $X_{-\bar{b}}|_{\{r_2=0\}}$. Therefore, let us continue our analysis restricted to the invariant space $\{r_2 = 0\}$.

The manifold Γ_2, which is defined by

$$\Gamma_2 = \left\{(r_2, a_2, \varepsilon_2, z_2) \,|\, r_2 = 0,\ \varepsilon_2 = 0,\ z_2^3 - z_2 + a_2 = 0\right\}$$

is normally hyperbolic except at the two points $p_\pm = \pm\left(\frac{2}{3\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$. Let us define the sections

$$\begin{aligned}\Delta_2^{\mathrm{en},-b_2} &= \left\{(r_2, a_2, \varepsilon_2, z_2) \in \mathbb{R}^4 \mid a_2 = -A_0\right\}\\ \Delta_2^{\mathrm{ex},-b_2} &= \left\{(r_2, a_2, \varepsilon_2, z_2) \in \mathbb{R}^4 \mid a_2 = A_0\right\},\end{aligned}$$

where $A_0 > 0$ is a sufficiently large constant. We are interested in the transition

$$\begin{aligned}\Pi_2^{-b_2} : &\Delta_2^{\mathrm{en},-b_2} \to \Delta_2^{\mathrm{ex},-b_2}\\ &(r_2, \varepsilon_2, z_2) \mapsto (\tilde{r}_2, \tilde{\varepsilon}_2, \tilde{z}_2).\end{aligned}$$

For a qualitative description of $X_{-\bar{b}}|_{\{r_2=0\}}$ and the objects defined above see fig. 4.13.

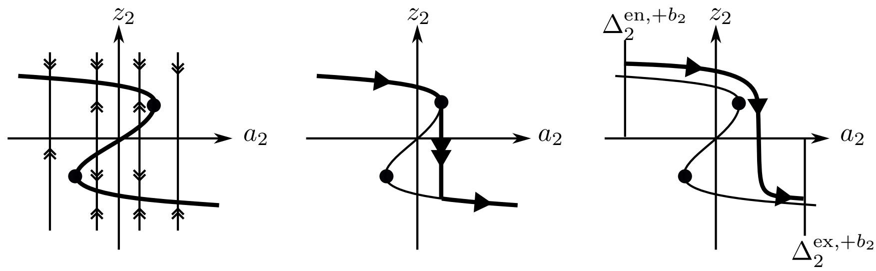


Figure 4.13: Left: phase portrait of the corresponding layer equation of $X_{-\bar{b}}|_{\{r_2=0\}}$. Center: phase portrait of the corresponding CDE of $X_{-\bar{b}}|_{\{r_2=0\}}$. Right: The expected perturbed invariant manifold obtained from the flow of the corresponding CDE and layer equation.

Away from the fold points $p_\pm$, the manifold Γ_2 is regular and thus, Fenichel's theory applies. However, we need to take care of the transition near the fold point p_+. The local transition of a slow fast system near a fold point is investigated in e.g. [27]. However, in our current problem this transition is not essential. By this we mean that the passage through the fold point is seen as a flat perturbation of the trajectory along the stable branch of Γ_2. In a qualitative sense, this is due to the fact that the transition $\Pi_2^{-b_2}$ goes along a large NHIM, which fails to be normally hyperbolic only at one point.

Proposition 4.2.3. *We can choose appropriate coordinates (Z_2, ε_2) in $\Delta_2^{\mathrm{en},-b_2}$ and $(\tilde{Z}_2, \tilde{\varepsilon_2})$ in $\Delta_2^{\mathrm{ex},-b_2}$ such that the transition $\Pi_2^{-b_2} : \Delta_2^{\mathrm{en},-b_2} \to \Delta_2^{\mathrm{ex},-b_2}$ associated to $X_{-\bar{b}}$ is an exponential type map of the form*

$$\Pi_2^{-b_2}(0,\varepsilon_2,Z_2)=\left(0,\varepsilon_2,\phi_2+Z_2\exp\left(-\frac{1}{\varepsilon_2}(A_0+\varepsilon_2\psi_2)\right)\right),$$

where $\phi_2=\phi_2(\varepsilon_2)$ and $\psi_2=\psi_2(Z_2,\varepsilon_2)$ are $\mathcal{C}^\ell$-admissible functions, and where A_0 is given by the slow divergence integral of $X_{-\bar{b}}|_{\{r_2=0\}}$.

Remark 4.2.4. *By regular perturbation theory, the flow of $X_{-\bar{b}}|_{\{r_2=0\}}$ is equivalent to that of $X_{-\bar{b}}|_{\{r_2>0\}}$. In other words, the image of $\Pi_2^{-b_2}|(\Delta_2^{\mathrm{en},-b_2}|_{r=0})$ is diffeomorphic to the image of $\Pi_2^{-b_2}|(\Delta_2^{\mathrm{en},-b_2})$.*

Proof. To prove that A_0 is given by the slow divergence integral we proceed along the same lines as in proposition 4.2.2, so we do not repeat it here. In figure fig. 4.14 we show the three transitions that we must consider.

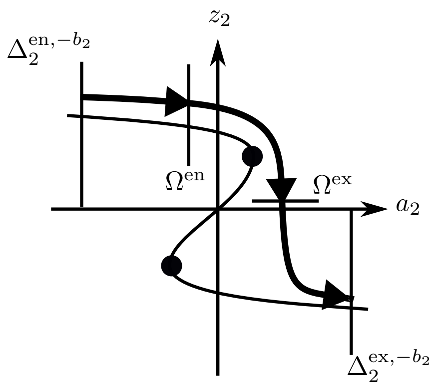


Figure 4.14: The three different transitions in which $\Pi_2^{-b_2}$ is decomposed. The central transitions is locally an A_2 problem. The other two transitions at the sides are regular.

The three transitions are defined as

$$\begin{aligned}\Pi_2^{reg_1} &: \Delta_2^{\mathrm{en},-b_2}\to\Omega^{\mathrm{en}}\\ \Pi_2^{fold} &: \Omega^{\mathrm{en}}\to\Omega^{\mathrm{ex}}\\ \Pi_2^{reg_2} &: \Omega^{\mathrm{ex}}\to\Delta_2^{\mathrm{ex},-b_2},\end{aligned}$$

where Ω^{en} and Ω^{en} are given as

$$\begin{aligned}\Omega^{\mathrm{en}} &= \left\{(r_2,a_2,\varepsilon_2,Z_2)\in\mathbb{R}^4 \mid a_2=-a_{2,\mathrm{en}}\right\}\\ \Omega^{\mathrm{ex}} &= \left\{(r_2,a_2,\varepsilon_2,Z_2)\in\mathbb{R}^4 \mid Z_2=Z_{2,\mathrm{ex}}\right\},\end{aligned}$$

where $a_{2,\mathrm{en}}$ and $Z_{2,\mathrm{ex}}$ are well chosen constants.

The total transition $\Pi_2^{+b_2}$ is given by $\Pi_2^{b_2} = \Pi_2^{reg_2} \circ \Pi_2^{fold} \circ \Pi_2^{reg_1}$. Recall from section 3.2 that if we want to write the transition $\Pi_2^{+b_2}$ as an exponential type map, we require that $\Pi_2^{reg_1}$ is expressed as an exponential type map with no shift. The transition Π_2^{fold} is studied in section 4.1. In there, it is shown that there are local coordinates $(\bar{Z}_2, \varepsilon_2)$ in Ω^{en}, and $(\tilde{a}_2, \tilde{\varepsilon_2})$ in Ω^{ex}, such that the transition Π_2^{fold} is given by

$$\begin{aligned}\Pi_2^{fold}(\bar{Z}_2, \varepsilon_2) &= (\tilde{a}_2, \tilde{\varepsilon_2}) \\ &= \left(\varepsilon_2^{2/3} + O(\varepsilon_2), \varepsilon_2\right).\end{aligned}$$

Assume now that we have characterized an invariant manifold $\mathcal{M}_{\varepsilon_2}^{fold}$ from Ω^{en} to Ω^{ex} via the map Π_2^{fold}. Now we want to "extend" $\mathcal{M}_{\varepsilon_2}^{fold}$ all the way up to the sections $\Delta_2^{\mathrm{en},-b_2}$ and $\Delta_2^{\mathrm{ex},-b_2}$ via transitions along regular regions of Γ_2. For this, it is more convenient to regard $\mathcal{M}_{\varepsilon_2}^{fold}$ as a graph $\zeta_2 = \phi_{2,\varepsilon_2}(A_2)$ where for $\varepsilon_2 > 0$, ϕ_{2,ε_2} is an ε_2-family of diffeomorphisms. In this way we have that we can equivalently express the map Π_2^{fold} as

$$\Pi_2^{fold}(\bar{Z}_2, \varepsilon_2) = (\phi_{2,\varepsilon_2}(\bar{Z}_2), \varepsilon_2).$$

Next, following section 3.1.2 we can find coordinates (Z_2, ε_2) in $\Delta_2^{\mathrm{en},-b_2}$, and coordinates $(\tilde{Z}_2, \varepsilon_2)$ in $\Delta_2^{\mathrm{ex},-b_2}$ in such a way that the transitions $\Pi_2^{reg_1}$ and $\Pi_2^{reg_2}$ are given as

$$\begin{aligned}\Pi_2^{reg_1}(Z_2, \varepsilon_2) &= \left(Z_2 \exp\left(-\frac{1}{\varepsilon_2}(A_0 - a_{2,\mathrm{en}})\right)\right) = (\bar{Z}_2, \varepsilon_2) \\ \Pi_2^{reg_2}(Z_{2,\mathrm{ex}}, \varepsilon_2) &= \left(Z_{2,\mathrm{ex}} \exp\left(-\frac{1}{\varepsilon_2}(A_0 - \tilde{a}_2)\right)\right) = (\tilde{Z}_2, \varepsilon_2).\end{aligned}$$

Remark 4.2.5. *Recall that along normally hyperbolic slow manifolds, it is possible to make a normal form transformation in such a way that this transformation respects certain constraint or structure of the vector field, [6, 7]. In this particular case, we respect the choice of the invariant manifold* $\mathcal{M}_{\varepsilon_2}^{fold}$.

Next, we can compute the composition $\Pi_2^{-b_2} = \Pi_2^{reg_2} \circ \Pi_2^{fold} \circ \Pi_2^{reg_1}$ by following section 3.2 and it thus follows that

$$\Pi_2^{-b_2}(0, Z_2, \varepsilon_2) = \left(0, \bar{\phi}_{\varepsilon_2} + Z_2 \exp\left(-\frac{1}{\varepsilon_2}(A_1 + A_3 + \varepsilon_2 \psi_2)\right), \varepsilon_2\right),$$

where $\bar{\phi}_{\varepsilon_2} = \phi_{\varepsilon_2}(0)\exp\left(-\frac{A_3}{\varepsilon_2}\right)$ and where $\psi_2 = \psi_2(Z, \varepsilon_2)$ is a $\mathcal{C}^\ell$-admissible function. The result is finally obtained by regular perturbation theory.

□

4.2.5 Proof of theorem 4.2.1

In this section we prove theorem 4.2.1. For this, we need to compose all the maps that we have computed in sections 4.2.1 to 4.2.4. In words, we have to construct a transition from the entry section Δ_1^{en} to the exit section Δ_3^{ex}. Recall that Δ_1^{en} and Δ_3^{ex} are the blow-up versions of Σ^{en} and Σ^{ex} respectively, and therefore, a transition $\Delta_1^{\mathrm{en}} \to \Delta_3^{\mathrm{ex}}$ is equivalent to a transition $\Sigma^{\mathrm{en}} \to \Sigma^{\mathrm{ex}}$. Once we obtain the latter transition, we extend it to a transition $\Sigma^- \to \Sigma^+$ given as an exponential type map. We have three possibilities given by the partition of the entry section Δ_1^{en}.

- If $b_1|_{\Delta_1^{\mathrm{en}}} \in O(\varepsilon_1^{2/5})$ then we construct a transition passing through the charts $K_{\mathrm{en}} \to K_{\bar{\varepsilon}} \to K_{\mathrm{ex}}$.
- If $b_1|_{\Delta_1^{\mathrm{en}}} \notin O(\varepsilon_1^{2/5})$ and $b_1|_{\Delta_1^{\mathrm{en}}} > 0$ then we construct a transition passing through the charts $K_{\mathrm{en}} \to K_{+\bar{b}} \to K_{\mathrm{ex}}$.
- If $b_1|_{\Delta_1^{\mathrm{en}}} \notin O(\varepsilon_1^{2/5})$ and $b_1|_{\Delta_1^{\mathrm{en}}} < 0$ then we construct a transition passing through the charts $K_{\mathrm{en}} \to K_{-\bar{b}} \to K_{\mathrm{ex}}$.

In fig. 4.15 we give a qualitative diagram of the local transitions obtained and their relationship.

As we have seen in our previous analysis, we consider three transitions which are given as

$$\begin{aligned}
\Pi^{\varepsilon} &= \Pi_3^{\varepsilon} \circ M_{\varepsilon}^{\mathrm{ex}} \circ \Pi_2^{\varepsilon} \circ M_{\mathrm{en}}^{\varepsilon} \circ \Pi_1^{\varepsilon} \\
\Pi^{+b} &= \Pi_3^{+b_1} \circ M_{+b_1}^{\mathrm{ex}} \circ \Pi_2^{+b_1} \circ M_{\mathrm{en}}^{+b_1} \circ \Pi_1^{+b_1} \\
\Pi^{-b} &= \Pi_3^{-b_1} \circ M_{-b_1}^{\mathrm{ex}} \circ \Pi_2^{-b_1} \circ M_{\mathrm{en}}^{-b_1} \circ \Pi_1^{-b_1},
\end{aligned}$$

where the matching maps are obtained from the blow-up map. For example, to obtain the matching map from the chart K_{en} to the chart $K_{\bar{\varepsilon}}$ we relate the two directional blow-up maps

$$a = -r_1^3,\ b = r_1^2 b_1,\ z = r_1 z_1,\ \varepsilon = r_1^5 \varepsilon_1$$

and

$$a = r_2^3 a_2,\ b = r_2^2 b_2,\ z = r_2 z_2,\ \varepsilon = r_2^5.$$

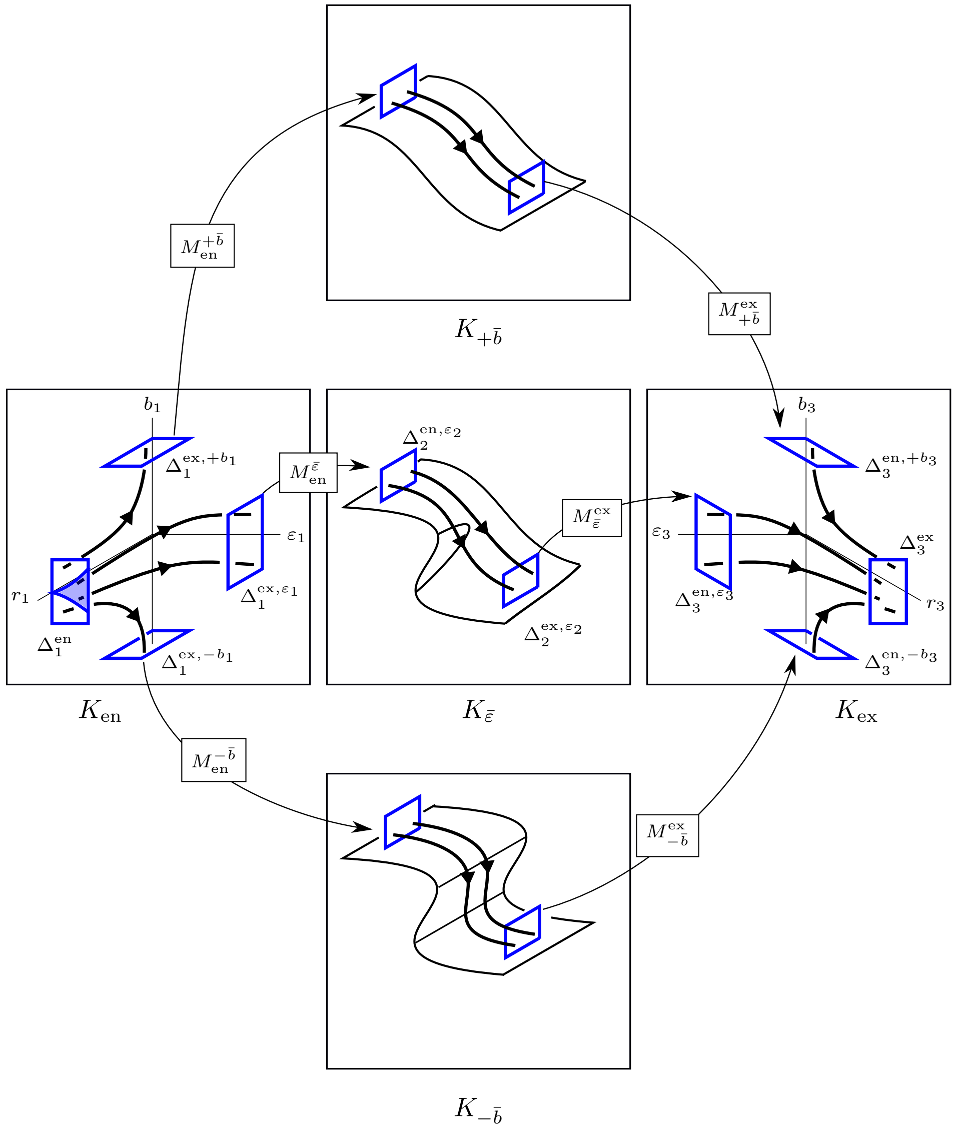


Figure 4.15: All the transitions obtained in the charts. We have to compose all such transitions through the matching maps M_i^j. A matching map M_i^j relates the coordinates between the charts K_i and K_j.

Let us detail how to obtain the map Π^ε. The other two cases are obtained by following similar arguments and thus the details are omitted.

The transition Π^ε

Here we compute first the transition

$$\Pi^\varepsilon = \Pi_3^{\varepsilon_3} \circ M_{\bar{\varepsilon}}^{\mathrm{ex}} \circ \Pi_2^{\varepsilon_2} \circ M_{\mathrm{en}}^{\bar{\varepsilon}} \circ \Pi_1^{\varepsilon_1} : \Delta_1^{\mathrm{en},\varepsilon_1} \to \Delta_3^{\mathrm{ex}}.$$

A matching map M_i^j relates the exit section in the chart K_i with the entry section of the chart K_j. For example we have $\Delta_2^{\mathrm{en},\varepsilon_2} = M_{\mathrm{en}}^{\bar{\varepsilon}}(\Delta_1^{\mathrm{ex},\varepsilon_1})$ and so on.

Let us work out only with the z-component of the transitions as it is the only relevant one. Recall from section 4.2.1 that $\Pi_1^{\varepsilon_1}$ is an exponential type map with no shift. Next, the composition $\Pi^{central} = M_{\bar{\varepsilon}}^{\mathrm{ex}} \circ \Pi_2^{\varepsilon_2} \circ M_{\mathrm{en}}^{\bar{\varepsilon}}$ can be seen as a general diffeomorphism as $\Pi_2^{\varepsilon_2}$ is a general diffeomorphism, and the matching maps are local diffeomorphisms on their domain of definition. Next, the last transition $\Pi_2^{\varepsilon_3}$ is an exponential type map with no shift. Finally, following section 3.2 we have that $\Pi_2^{\varepsilon_3} \circ \Pi^{central} \circ \Pi_1^{\varepsilon_1}$ is an exponential type map of the form

$$\Pi_{Z_1}^\varepsilon = \bar{\phi}(B_1, \varepsilon_1) + Z_1 \exp\left(-\frac{1}{\varepsilon_1}\left(\bar{\mathcal{A}}(B_1, \varepsilon_1) + \varepsilon_1 \bar{\Psi}(B_1, \varepsilon_1, Z_1)\right)\right),$$

where $\bar{\mathcal{A}} > 0$ and ϕ and Ψ are $\mathcal{C}^\ell$ admissible functions. The differentiability of ϕ and Ψ with respect to monomials is obtained by performing the straightforward computation $\Pi_2^{\varepsilon_3} \circ \Pi^{central} \circ \Pi_1^{\varepsilon_1}$ and it is evident from the results of section 4.2.1. By blowing down we obtain that the transition $\Pi^{\mathrm{inner}} : \Sigma^{\mathrm{en}} \to \Sigma^{\mathrm{ex}}$ (in a small neighborhood of the cusp point and within the inner layer as domain) reads as

$$\Pi_Z^{inner} = \phi(B, \varepsilon) + Z \exp\left(-\frac{1}{\varepsilon}\left(\mathcal{A}(B, \varepsilon) + \varepsilon \bar{\Psi}(B, \varepsilon, Z)\right)\right).$$

To obtain the transition $\Pi : \Sigma^- \to \Sigma^+$ we now need to compose Π_Z^{inner} with exponential type maps on the left and on the right corresponding to

$$\begin{aligned} \Pi^- &: \Sigma^- \to \Sigma^{\mathrm{en}} \\ \Pi^+ &: \Sigma^{\mathrm{ex}} \to \Sigma^+. \end{aligned}$$

However, we must proceed carefully. As shown in section 3.2, in order to express the transition Π as an exponential type map, we need that at least the transition Π^- is an exponential type map with no shift. In other words, we need to choose appropriate coordinates on Σ^- and on Σ^+ in such a way that we respect the already chosen coordinates in Σ^{en} and in Σ^{ex}, and the already chose invariant manifold between Σ^{en} and Σ^{ex}. Fortunately, this is possible with the extensions of Bonckaert [6, 7] to the normalization results of Takens [37].

For sake of clarity, let $(B_{\text{en}}, Z_{\text{en}})$ be coordinates in Σ^{en} and $(B_{\text{ex}}, Z_{\text{ex}})$ be coordinates in Σ^{ex}. We have shown that these coordinates can be chosen in such a way that the "vertical" component of the transition map $\Pi^{\text{inner}} : \Sigma^{\text{en}} \to \Sigma^{\text{ex}}$ reads as

$$\begin{aligned}\Pi_{Z_{\text{en}}}(B_{\text{en}}, Z_{\text{en}}, \varepsilon) &= Z_{\text{ex}} \\ &= \phi(B_{\text{en}}, \varepsilon) + Z_{\text{en}} \exp\left(-\frac{1}{\varepsilon}\left(\mathcal{A}(B_{\text{en}}, \varepsilon) + \varepsilon\bar{\Psi}(B_{\text{en}}, \varepsilon, Z_{\text{en}})\right)\right).\end{aligned}$$

In this case the invariant manifold, say $\mathcal{M}_\varepsilon$, is given by $Z_{\text{en}} = 0$. Using [6, 7] we can find suitable coordinates (B_-, Z_-) in Σ^- in such a way that

$$\Pi^-_{Z_-}(B_-, Z_-, \varepsilon) = Z_- \exp\left(-\frac{1}{\varepsilon}(A_0)\right) = Z_{\text{en}}.$$

In other words, there is a change of coordinates respecting the invariant manifold $\mathcal{M}_\varepsilon$ under which the transition Π^- is an exponential type map with no shift and linear. Similar arguments hold for the choice of coordinates in Σ^+. Finally performing the composition $\Pi^+ \circ \Pi^{\text{inner}} \circ \Pi^-$ we prove our claim.

4.3 Analysis of the A_4-SFS

In this section we study the transition of an A_4 slow fast system. As we have seen in the previous examples (the fold and the cusp) the most important transition is through a small strip around the central singularity.

To simplify notation, we let $x = (a, b, c) \in \mathbb{R}^3$ and $z \in \mathbb{R}$. We then consider a vector field X given by

$$X = \varepsilon(1 + f_1)\frac{\partial}{\partial a} + \varepsilon^2 f_2 \frac{\partial}{\partial b} + \varepsilon^2 f_3 \frac{\partial}{\partial c} - \left(z^4 + cz^2 + bz + a + f_4\right)\frac{\partial}{\partial z} + 0\frac{\partial}{\partial \varepsilon}, \tag{4.30}$$

where, thanks to theorem 3.3.2, all f_i's are flat at the origin. The critical manifold is locally defined by

$$S = \left\{(a, b, c, z) \,|\, z^4 + cz^2 + bz + a = 0\right\}.$$

The singularity set is then defined as

$$B = \left\{(a, b, c, z) \,|\, 4z^3 + 2cz + b = 0\right\}.$$

Refer to section 2.3.1 for the description of S and B. The flow of the corresponding CDE is found in section 2.3.3. The flow of the CDE leads us to define the following sections

$$\Sigma^{\text{en}} = \left\{(a,b,c,z,\varepsilon) \in \mathbb{R}^5 \,|\, a = -a_0,\ z > 0\right\}$$
$$\Sigma^{\text{ex}} = \left\{(a,b,c,z,\varepsilon) \in \mathbb{R}^5 \,|\, a > -0,\ z = -z_0\right\},$$

where a_0 and z_0 are positive but sufficiently small constants. Then we study the transition

$$\Pi : \Sigma^{\text{en}} \to \Sigma^{\text{ex}}$$

defined by the flow of X. In this section we only focus on a transition along a small strip around the swallowtail point. Sufficiently away from the swallowtail point, the constraint manifold S has only cusp and fold singularities, an therefore, the transitions near those points are already known. Following the same type of analysis as in sections 4.1 and 4.2 we can obtain the following.

Proposition 4.3.1. *Consider an A_4-SFS given by* (4.30) *and the transition*

$$\begin{aligned}\Pi :&\Sigma^{\text{en}} \to \Sigma^{\text{ex}}\\ &(b,c,z,\varepsilon) \mapsto (\tilde{a},\tilde{b},\tilde{c},\tilde{\varepsilon}).\end{aligned}$$

Let $\Sigma^{\text{inner}} \subset \Sigma^{\text{en}}$ be defined as

$$\Sigma^{\text{inner}} = \left\{(b,c,z,\varepsilon) \in \Sigma^{\text{en}} \,|\, b \in O(\varepsilon^{3/7}),\ c \in O(\varepsilon^{2/7})\right\}.$$

Then, the transition $\Pi|_{\Sigma^{\text{inner}}}$ is given by

$$\begin{aligned}\tilde{a} &= \varepsilon^{4/7} + O(\varepsilon^{5/7})\\ \tilde{b} &= b + f_b\\ \tilde{c} &= c + f_c\\ \tilde{\varepsilon} &= \varepsilon,\end{aligned}$$

where each function $f_i = f_i(a,b,c,z,\varepsilon)$, $i = b, c$, is flat at $(a,b,c,z,\varepsilon) = (0,0,0,0,0)$.

For a qualitative picture of the result stated in proposition 4.3.1 see fig. 4.16.

Proof of proposition 4.3.1. This proof follows similar arguments as in sections 4.1 and 4.2, thus we shall be brief and point out only the key ingredients.

The blow-up map reads as

$$a = r^4\bar{a},\ b = r^3\bar{b},\ c = r^2\bar{c},\ z = r\bar{z},\ \varepsilon = r^7\bar{\varepsilon}.$$

According to this blow-up map, the induced vector fields on the charts $K_{\text{en}} = \{\bar{a} = -1\}$, $K_{\bar{\varepsilon}} = \{\bar{\varepsilon} = 1\}$, and $K_{\text{ex}} = \{\bar{z} = -1\}$ are studied.

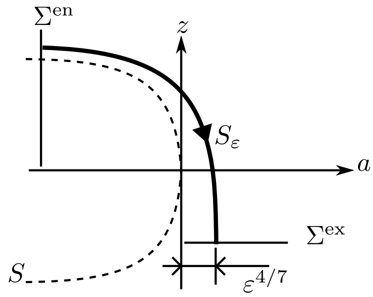


Figure 4.16: Qualitative description of the result stated in proposition 4.3.1. In words, after passing through the swallowtail point, the distance from S_ε to the fast fiber passing through the origin is of order $O(\varepsilon^{4/7})$.

Analysis in the chart K_{en}

The induced vector field in this chart reads as

$$\begin{aligned} r_1' &= -r_1\varepsilon_1 \\ b_1' &= 3\varepsilon_1 b_1 + \varepsilon_1 f_{b_1} \\ c_1' &= 2\varepsilon_1 c_1 + \ f_{c_1} \\ \varepsilon_1' &= 7\varepsilon_1^2 \\ z_1' &= -4\left(-z_1^4 + c_1 z_1^2 + b_1 z_1 - 1 - \tfrac{1}{4}\varepsilon_1 z_1\right) + \varepsilon_1 f_{z_1}, \end{aligned} \tag{4.31}$$

where the functions $f_i = f_i(r_1, b_1, c_1, \varepsilon_1, z_1)$, for $i = b_1, c_1, z_1$, are flat along $\{r_1 = 0\}$. According to appendix A.6.2 we have that in a small neighborhood of $(r_1, b_1, c_1, \varepsilon_1, z_1) = (0, 0, 0, 0, 1)$, the system (4.31) is $\mathcal{C}^\ell$ equivalent to

$$\begin{aligned} r_1' &= -r_1\varepsilon_1 \\ B_1' &= 3\varepsilon_1 B_1 \\ C_1' &= 2\varepsilon_1 C_1 \\ \varepsilon_1' &= 7\varepsilon_1^2 \\ Z_1' &= -\Lambda(r_1, B_1, C_1, \varepsilon_1) Z_1, \end{aligned}$$

where $\Lambda(0, 0, 0, 0) = 16$. The sections of interest are defined as

$$\begin{aligned} \Delta_1^{\mathrm{inner}} &= \left\{(r_1, B_1, C_1, \varepsilon_1, Z_1) \in \mathbb{R}^5 \,|\, r_1 = -r_1^0\right\} \\ \Delta_1^{\mathrm{ex}} &= \left\{(r_1, B_1, C_1, \varepsilon_1, Z_1) \in \mathbb{R}^5 \,|\, \varepsilon_1 = \delta\right\}, \end{aligned}$$

where $r_1^0 = a_0^{1/4}$ and δ are small constants. Then, in normal coordinates and according to appendix A.6.5, we have that the transition $\Pi^{\mathrm{inner}} : \Delta_1^{\mathrm{inner}} \to \Delta_1^{\mathrm{ex}}$

reads as

$$\Pi_1(B_1, C_1, \varepsilon_1, Z_1) = (\tilde{r_1}, \tilde{B_1}, \tilde{C_1}, \tilde{Z_1}),$$

with

$$\begin{aligned}
\tilde{r_1} &= r_1^0 \left(\frac{\varepsilon_1}{\delta}\right)^{1/7} \\
\tilde{B_1} &= B_1 \left(\frac{\delta}{\varepsilon_1}\right)^{3/7} \\
\tilde{C_1} &= C_1 \left(\frac{\delta}{\varepsilon_1}\right)^{2/7} \\
\tilde{Z_1} &= Z_1 \exp\left(-\frac{16}{7\varepsilon_1}(1 + \alpha_1 \varepsilon_1 \ln \varepsilon_1 + \varepsilon_1 G_1)\right),
\end{aligned}$$

where $\alpha = \alpha(r_1, B_1, C_1)$ and $B_1 = G_1(r_1, B_1, C_1, Z_1)$ are admissible functions.

Analysis in the chart $K_{\bar{\varepsilon}}$

In this chart the induced blown up vector field reads as

$$\begin{aligned}
r_2' &= 0 \\
a_2' &= 1 + f_{a_2} \\
b_2' &= f_{b_2} \\
c_2' &= f_{c_2} \\
z_2' &= -(z_2^4 + c_2 z_2^2 + b_2 z_2 + a_2) + f_{z_2},
\end{aligned} \tag{4.32}$$

where each $f_i = f_i(r_2, a_2, b_2, c_2, z_2)$, for $i = a_2, b_2, c_2, z_2$, is flat along $\{r_2 = 0\}$. The sections of interest read as

$$\begin{aligned}
\Delta_2^{\text{en}} &= \left\{(r_2, a_2, b_2, c_2 z_2) \in \mathbb{R}^5 \mid a_2 = -a_2^0\right\} \\
\Delta_2^{\text{ex}} &= \left\{(r_2, a_2, b_2, c_2, z_2) \in \mathbb{R}^5 \mid z_2 = -z_2^0\right\}.
\end{aligned}$$

The transition is then $\Pi_2 : \Delta_2^{\text{en}} \to \Delta_2^{\text{ex}}$. Since the vector field (4.32) is regular, then Π is a local diffeomorphism. Moreover if we write $\Pi_2(r_2, b_2, c_2, z_2) = (\tilde{r_2}, \tilde{a_2}, \tilde{b_2}, \tilde{c_2})$ we have

$$\begin{aligned}
\tilde{r_2} &= r_2 \\
\tilde{a_2} &= \phi(r_2, b_2, c_2, z_2) \\
\tilde{b_2} &= b_2 + \bar{f}_{b_2} \\
\tilde{c_2} &= c_2 + \bar{f}_{c_2},
\end{aligned}$$

where $\phi(r_2, b_2, c_2, z_2) = \phi_{(r_2, b_2, c_2)}(z_2)$, where $\phi_{(r_2, b_2, c_2)} : \mathbb{R} \to \mathbb{R}$ is a family of local diffeomorphisms.

Analysis in the chart K_{ex}

The corresponding blown up vector field is smoothly equivalent to

$$\begin{aligned} r_3' &= r_3 + \varepsilon_3 f_{r_3} \\ a_3' &= -4a_3 + \varepsilon_3 L + \varepsilon_3 f_{a_3} \\ b_3' &= -3b_3 + \varepsilon_3 f_{b_3} \\ c_3' &= -2c_3 + \varepsilon_3 f_{c_3} \\ \varepsilon_3' &= -7\varepsilon_3 + \varepsilon_3 f_{\varepsilon_3}, \end{aligned} \tag{4.33}$$

where $L = L(a_3, b_3, c_3) = \frac{1}{1+a_3-b_3+c_3}$ and where the functions $f_i = f_i(r_3, a_3, b_3, c_3, \varepsilon_3)$ for $i = r_3, a_3, b_3, c_3, \varepsilon_3$ are flat along $\{r_3 = 0\}$. The sections of interest are given by

$$\begin{aligned} \Delta_3^{\mathrm{en}} &= \left\{(r_3, a_3, b_3, c_3, \varepsilon_3) \in \mathbb{R}^5 \,|\, \varepsilon_3 = \delta\right\} \\ \Delta_3^{\mathrm{ex}} &= \left\{(r_3, a_3, b_3, c_3, \varepsilon_3) \in \mathbb{R}^5 \,|\, r_3 = r_3^0\right\}, \end{aligned}$$

where δ and r_3^0 are small positive constants. Thus, we describe the transition $\Pi_3 : \Delta_3^{\mathrm{en}} \to \Delta_3^{\mathrm{ex}}$. According to appendix A.6.1 we have that (4.33) is smoothly conjugate to

$$\begin{aligned} r_3' &= r_3 \\ a_3' &= -4a_3 + \varepsilon_3 L \\ b_3' &= -3b_3 \\ c_3' &= -2c_3 \\ \varepsilon_3' &= -7\varepsilon_3. \end{aligned}$$

Then, following appendix A.6.4 we have that Π_3 is given by $\Pi_3(r_3, a_3, b_3, c_3) = (\tilde{a_3}, \tilde{b_3}, \tilde{c_3}, \tilde{\varepsilon_3})$ where

$$\begin{aligned} \tilde{a_3} &= a_3 \left(\frac{r_3}{r_3^0}\right)^4 \left(1 + O\left(\frac{r_3}{r_3^0}\right)\right) \\ \tilde{b_3} &= b_3 \left(\frac{r_3}{r_3^0}\right)^3 \\ \tilde{c_3} &= c_3 \left(\frac{r_3}{r_3^0}\right)^2 \\ \tilde{\varepsilon_3} &= \delta \left(\frac{r_3}{r_3^0}\right)^7 \end{aligned}$$

Composition of the transitions

The last step of the proof is to compose the local transitions taking into account the matching maps $M_{\text{en}}^{\bar{\varepsilon}}$ and $M_{\bar{\varepsilon}}^{\text{ex}}$. That is, one obtains the map $\Pi : \Delta_1^{\text{en}} \to \Delta_1^{\text{ex}}$. The matching maps are given by

$$M_{\text{en}}^{\bar{\varepsilon}} : \qquad r_2 = r_1 \varepsilon_1^{1/7},\ a_2 = -\varepsilon_1^{-4/7},\ b_2 = \varepsilon_1^{-3/7} b_1,\ c_2 = \varepsilon_1^{-2/7} c_1,\ z_2 = \varepsilon_1^{-1/7} z_1,$$

and

$$M_{\bar{\varepsilon}}^{\text{ex}} : \qquad r_3 = -r_2 z_2,\ a_3 = z_2^{-4} a_2,\ b_3 = -z_2^{-3} b_2,\ c_3 = z_2^{-2} c_2,\ \varepsilon_3 = -z_2^{-7}.$$

The composition $\Pi = \Pi_3 \circ M_{\bar{\varepsilon}}^{\text{ex}} \circ \Pi_2 \circ M_{\text{en}}^{\bar{\varepsilon}} \circ \Pi_1$ is obtained by following the procedure detailed in section 4.1. By doing this one proves the result.

□

4.4 Digression on the A_k-SFSs

In this section, we briefly discuss a way to study the local transition of A_k-SFS based on the exposition of this thesis. Recall that, in normal form, an A_k-SFS reads as

$$X = \varepsilon(1+\varepsilon f_1)\frac{\partial}{\partial x_1} + \sum_{i=2}^{k-1} \varepsilon^2 f_i \frac{\partial}{\partial x_i} - \left(z^k + \sum_{i=1}^{k-1} x_i z^{i-1} - \varepsilon f_k \right) \frac{\partial}{\partial z} + 0\frac{\partial}{\partial \varepsilon},$$

where all the functions $f_i = f_i(x_1, \ldots, x_{k-1}, z, \varepsilon)$ are smooth. The cases $k = 2, 3, 4$ have been studied in sections 4.1 to 4.3. However, similar steps can be followed to study any A_k SFS. The procedure is as follows.

1. The formal normal form result, given in theorem 3.3.2, shows that there exists a formal diffeomorphism $\hat{\Phi} : \mathbb{R}^{k+1} \to \mathbb{R}^{k+1}$ such that $\hat{\Phi}_* \hat{X} = F$, where F is known as the principal part and is given as

$$F = \varepsilon \frac{\partial}{\partial x_1} + \sum_{i=2}^{k-1} 0 \frac{\partial}{\partial x_i} - \left(z^k + \sum_{i=1}^{k-1} x_i z^{i-1} \right) \frac{\partial}{\partial z} + 0 \frac{\partial}{\partial \varepsilon}.$$

   By Borel's lemma [10], this principal part F can be realized as a smooth vector field $X^N = F + \varepsilon H$, where H is smooth and flat at $0 \in \mathbb{R}^{k+1}$.

   Remark 4.4.1. *Suppose this formal normal form result can be extended to a smooth and/or analytic version. That is, there exists a smooth/analytic diffeomorphism*

$\Phi : \mathbb{R}^{k+1} \to \mathbb{R}^{k+1}$ *conjugating X to F. This implies that F is a smooth/analytic normal form of X. However, even in such a case, F cannot be integrated. This means that even with such a smooth/analytic normal form, some technique must be used to study the local dynamics of F. We propose to use the Geometric Desingularization method as in chapters 3 and 4.*

2. Depending on k, define the entry and the exist sections.

   If k is even then

$$\Sigma^{\text{en}} = \left\{(x, z, \varepsilon) \in \mathbb{R}^{k-1} \times \mathbb{R} \times \mathbb{R} \,|\, x_1 = -x_1^0,\, z > 0\right\}$$
$$\Sigma^{\text{ex}} = \left\{(x, z, \varepsilon) \in \mathbb{R}^{k-1} \times \mathbb{R} \times \mathbb{R} \,|\, z = -z^0,\, x_1 > 0\right\},$$

   where x_1^0 and z^0 are sufficiently small small positive constants.

   If k is odd then

$$\Sigma^{\text{en}} = \left\{(x, z, \varepsilon) \in \mathbb{R}^{k-1} \times \mathbb{R} \times \mathbb{R} \,|\, x_1 = -x_1^0,\, z > 0\right\}$$
$$\Sigma^{\text{ex}} = \left\{(x, z, \varepsilon) \in \mathbb{R}^{k-1} \times \mathbb{R} \times \mathbb{R} \,|\, x_1 = x_1^0,\, z < 0\right\},$$

   where x_1^0 is a sufficiently small small positive constant.

3. The blow-up map is given by

$$x_1 = r^k \bar{x}_1,\ x_2 = r^{k-1} \bar{x}_2,\ \ldots,\ z = r\bar{z},\ \varepsilon = r^{2k-1}\bar{\varepsilon}.$$

4. To study the transition through a small neighborhood of the origin, one considers the following three charts

$$K_{\text{en}} = \{\bar{x}_1 = -1\},\ K_{\varepsilon} = \{\bar{\varepsilon} = -1\},$$

   and if k is even

$$K_{\text{ex}} = \{\bar{z} = -1\},$$

   while if k is odd

$$K_{\text{ex}} = \{\bar{x}_1 = 1\}.$$

5. A study in the chart K_{en} shows that a transition through the charts K_{en}, K_ε and K_{ex} is well defined only if the coordinates $(x_2, \dots, x_{k-1}) \in \Sigma_{\text{en}}$ satisfy the estimate

$$x_i \in O\left(\varepsilon^{\frac{k-i+1}{2k-1}}\right).$$

   Therefore, a subset of Σ^{en}, called the inner layer, is defined as

$$\Sigma^{\text{inner}} = \left\{(x,z) \in \Sigma^{\text{en}} \,|\, x_i < K_i \varepsilon^{\frac{k-i+1}{2k-1}},\, 0 < K_i < \infty,\, \forall\, i = 2, 3, \dots, k-1\right\}.$$

6. A local study through the charts provides several local transitions, which have to be composed.

7. Finally, by composing the local transitions obtained in each chart, the map $\Pi : \Sigma^{\text{en}} \to \Sigma^{\text{ex}}$ can be computed. In a broad sense, we conjecture that the type of results to be obtained are as follows.

   If k is odd. There exist local coordinates $(X_2, \dots, X_{k-1}, Z, \varepsilon) \in \Sigma^{\text{en}}$ and $(\tilde{X}_2, \dots, \tilde{X}_{k-1}, \tilde{Z}, \tilde{\varepsilon}) \in \Sigma^{\text{en}}$ such that the transition map

$$\Pi(X_2, \dots, X_{k-1}, Z, \varepsilon) \mapsto (\tilde{X}_2, \dots, \tilde{X}_{k-1}, \tilde{Z}, \tilde{\varepsilon})$$

   is given by

$$\Pi(X_2, \dots, X_{k-1}, Z, \varepsilon) = (X_2 + h_2, \dots, X_{k-1} + h_{k-1}, D, \varepsilon),$$

   where each $h_i = h_i(X_2, \dots, X_{k-1}, Z, \varepsilon)$, $i = 2, \dots, k-1$, is a flat function and where $D = D(X_2, \dots, X_{k-1}, Z, \varepsilon)$ is an exponential type function in Z, that is

$$D = \Phi + Z \exp\left(-\frac{1}{\varepsilon}(A + \varepsilon\Psi)\right),$$

   where $\Phi = \Phi(X_2, \dots, X_{k-1}, \varepsilon)$, $\Psi = \Psi(X_2, \dots, X_{k-1}, Z, \varepsilon)$ and $A = A(X_2, \dots, X_{k-1}, \varepsilon)$ are admissible functions with $A|_{\varepsilon=0} > 0$ and given by the slow divergence integral.

   If k is even. There exist local coordinates $(X_2, \dots, X_{k-1}, Z, \varepsilon) \in \Sigma^{\text{en}}$ and $(\tilde{X}_1, \tilde{X}_2, \dots, \tilde{X}_{k-1}, \tilde{\varepsilon}) \in \Sigma^{\text{en}}$ such that the transition map

$$\Pi(X_2, \dots, X_{k-1}, Z, \varepsilon) \mapsto (\tilde{X}_1, \tilde{X}_2, \dots, \tilde{X}_{k-1}, \tilde{\varepsilon})$$

is given by

$$\Pi(X_2, \ldots, X_{k-1}, Z, \varepsilon) = \left(\varepsilon^{\frac{k}{2k-1}} + O\left(\varepsilon^{\frac{k+1}{2k-1}}\right), X_2 + h_2 \ldots, X_{k-1} + h_{k-1}, \varepsilon\right),$$

where each $h_i = h_i(X_2, \ldots, X_{k-1}, Z, \varepsilon)$, $i = 2, \ldots, k-1$, is a flat function at the origin of $\mathbb{R}^{k+1}$.

Chapter 5

Conclusions and future research

This thesis deals with

1. Constrained Differential Equations (CDEs),
2. Slow Fast Systems (SFSs).

Studying CDEs is important since these are zeroth-order approximations of SFSs. Thus, in chapter 2 we have studied the classification problem of CDEs. In this context, Takens classified CDEs related to the A_2 (fold) and A_3 (cusp) catastrophes [39] . As a contribution, we have extended Takens' results by classifying CDEs related to the A_k, the D^4 (hyperbolic umbilic), and the D^{-4} (elliptic umbilic) catastrophes. The class of CDEs related to the A_k catastrophes have further relevance in our research, and therefore we have called them A_k-CDEs.

Our next step, in chapter 3, has been to embed the A_k-CDEs into the theory of slow fast systems. By doing this we have studied what we called A_k-SFSs, which are ε-perturbations of the A_k-CDEs of chapter 2. The slow manifold corresponding to the A_k-SFSs is not Normally Hyperbolic and thus, the method called Geometric Desingularization (GD) becomes useful. With the aim of refining the results obtained by the GD-process, we have proposed a normal form of A_k-SFSs. Later, in chapter 4, we have shown that this normal form greatly simplifies the local analysis of the A_k-SFSs via the GD-method. Our final contribution is the description of the flow of A_k-SFSs for $k = 2, 3, 4$ near the fold, the cusp, and the swallowtail points respectively. However, we remark that the methodology developed here is systematic and is applicable to all the A_k-SFSs.

Motivated by our research, a series of open problems arise.

- On Constrained Differential Equations

  1. Complete the topological classification of CDEs (V, X) for all potential functions V given by Thom's catastrophes. That consists on extending our results by considering potential functions given by the Butterfly and the Parabolic Umbilic catastrophes.
  2. It is also interesting to investigate the topological classification of generic CDEs for other known stable potential functions. That is, one may consider potential functions V given as the universal unfolding of singularities of type D_n, E_6, E_7, E_8, etcetera [4].
  3. The research reported in this thesis relies on the smooth (up to diffeomorphism) classification of singularities of smooth maps. In contrast, it is known that the topologically stable maps form a dense set in the space of smooth maps [3]. Thus, it is also interesting to investigate if the classification of CDEs is possible in the case where the potential function V is given just up to homeomorphism.

- On Slow fast systems

  1. The first natural question is to investigate the corresponding SFSs obtained as ε-perturbations of the CDEs described above. For example: to consider SFSs with two (or more) fast variables.
  2. Regarding A_k-SFSs, it is interesting to investigate whether the formal normal form provided in section 3.3 can be made smooth and/or analytic. If the answer is affirmative, then many of the arguments of this thesis get simpler. However, since the normal forms are still not integrable, we cannot avoid using, e.g., the geometric desingularization method.
  3. Related to normal forms of SFSs, it is interesting to investigate if our proposed normal form can be extended to more complicated slow fast systems. For example, SFSs with two or more fast variables.
  4. Another interesting application of normal forms of SFSs is to obtain a "true" normal form of the canard phenomenon.
  5. Global problems are also interesting, in particular because they involve the intersection of invariant manifolds. Thus, the investigation of slow fast systems with some "global return mechanism" may lead to interesting results. In this setting, the class of differentiability of the transition maps involved becomes important.

Appendix A

Auxiliary results

A.1 Elementary catastrophe theory

Catastrophe theory has its origins in the 1960's with the work of René Thom [40, 41, 42]. One of its goals was to qualitatively study the sudden (or catastrophic) way in which solutions of biological systems change upon a small variation of parameters. The most basic setting of this theory is called *elementary* catastrophe theory [18, 32, 35]. It is concerned with gradient dynamical systems

$$\dot{z} = -\frac{\partial}{\partial z} V(x, z). \tag{A.1}$$

The variables $z \in \mathbb{R}^n$ represent the *states* or the measurable quantities of a certain process, and $x \in \mathbb{R}^m$ represent *control parameters*. One concern is to find the equilibrium points of (A.1), this is, to solve

$$\frac{\partial}{\partial z} V(x, z) = 0. \tag{A.2}$$

In mathematical terminology, one is interested in the qualitative behavior of the solutions z of (A.2) as the parameters x change. It is also interesting to know to what extent different functions V may show the same local behavior. These ideas led to the classification of families of degenerate functions $V(x, z) : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$ for $m \leq 4$, which is known as the "seven elementary catastrophes", see table A.1.

Theorem A.1.1 (Thom's classification theorem [10]). *Let $V_x(z) = V(x, z) : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$ be an $m-$parameter family of smooth functions with $m \leq 4$. If $V(x, z)$ is generic then it is right-equivalent (up to multiplication by ± 1, up to addition of Morse functions and up to addition of functions on the parameters) to one of the forms shown in table A.1.*

A.2 Thom-Boardman symbols

Let N^n, M^m be smooth manifolds, and consider that $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_m)$ are some local coordinates in N and M respectively. Let a smooth map $f : N^n \to M^m$ be given by $y_i = f_i(x)$. Let i_1 be a nonnegative integer. The set $\Sigma^{i_1}(f)$ consists of all points at which the kernel of Df has dimension i_1. Given a finite sequence $I = (i_1, i_2, \ldots, i_k)$ of non-increasing nonnegative numbers, $\Sigma^I(f)$ is defined inductively as follows.

Definition A.2.1 (Thom-symbol). *Assume that $\Sigma^I(f) = \Sigma^{i_1, i_2, \ldots, i_k}(f) \subset N$ is a smooth manifold. Then*

$$\Sigma^{i_1, i_2, \ldots, i_k, i_{k+1}} = \Sigma^{i_{k+1}} \left(f | \Sigma^I(f) \right)$$

Name	$V(x,z)$	Codimension
Non-critical	z	
Non-degenerate (Morse)	z^2	0
Fold	$\frac{1}{3}z^3 + x_1 z$	1
Cusp	$\frac{1}{4}z^4 + \frac{1}{2}x_2 z^2 + x_1 z$	2
Swallowtail	$\frac{1}{5}z^5 + \frac{1}{3}x_3 z^3 + \frac{1}{2}x_2 z^2 + x_1 z$	3
Elliptic Umbilic	$z_1^3 - 3z_1 z_2^2 + x_3(z_1^2 + z_2^2) + x_2 z_1 + x_1 z_2$	3
Hyperbollic Umbilic	$z_1^3 + z_2^3 + x_3 z_1 z_2 + x_2 z_1 + x_1 z_2$	3
Butterfly	$\frac{1}{6}z^6 + \frac{1}{4}x_4 z^4 + \frac{1}{3}x_3 z^3 + \frac{1}{2}x_2 z^2 + x_1 z$	4
Parabolic Umbilic	$z_1^2 z_2 + z_2^4 + x_4 z_1^2 + x_3 z_2^2 + x_2 z_1 + x_1 z_2$	4

Table A.1: Thom's classification of families of functions for $m \leq 4$. Each elementary catastrophe is a structurally stable m-parameter unfolding of the germ $V(z,0)$. Loosely speaking, the codimension of a singularity is the minimal number of parameters m for which a singularity persistently occurs in an $m-$parameter family of functions.

is the set of all points at which the kernel of $D(f|\Sigma^I(f))$ has dimension i_{k+1}.

Naturally, we have the inclusions

$$N \supset \Sigma^{i_1}(f) \supset \Sigma^{i_1,i_2}(f) \supset \cdots$$

Denote by $\mathscr{E}_n$ the ring of germs of $\mathcal{C}^\infty$ functions on $\mathbb{R}^n$ at 0. Let $\mathscr{I}$ be an ideal of $\mathscr{E}_n$.

Definition A.2.2 (JACOBIAN EXTENSION). *The* Jacobian extension $\Delta_k(\mathscr{I})$ *of $\mathscr{I}$ is the ideal generated by $\mathscr{I}$ and all the Jacobians* $\det\left(\frac{\partial \phi_i}{\partial x_j}\right)$ *of order k, and where ϕ_i are functions in $\mathscr{I}$.*

Remark A.2.1.

- *The ideal $\Delta_k(\mathscr{I})$ is independent on the choice of coordinates.*
- $\Delta_{k+1}(\mathscr{I}) \subseteq \Delta_k(\mathscr{I})$.

Definition A.2.3 (CRITICAL JACOBIAN EXTENSION). *A Jacobian extension $\Delta_k(\mathscr{I})$ is said to be* critical *if $\Delta_k \neq \mathscr{E}_n$ but $\mathscr{E}_n = \Delta_{k-1}(\mathscr{I})$. This is, the order k of the Jacobians is the smallest for which the extension does not coincide with $\mathscr{E}_n$.*

Now, we change lower indices to upper indices as follows.

Definition A.2.4. $\Delta^k = \Delta_{n-k+1}$.

By using the upper indices as in definition A.2.4, we have

$$i_1 = \operatorname{corank}(\mathscr{I}),\ i_2 = \operatorname{corank}(\Delta^{i_1}\mathscr{I}),\ \ldots,\ i_k = \operatorname{corank}(\Delta^{i_{k-1}} \cdots \Delta^{i_k}\mathscr{I}).$$

Definition A.2.5 (THOM-BOARDMAN SYMBOL). *Let $I = (i_1, i_2, \ldots, i_k)$ be a non-increasing sequence of non-negative integer numbers. The ideal $\mathscr{I}$ is said to have* Thom-Boardman symbol *I if its successive critical extensions are*

$$\Delta^{i_1}\mathscr{I},\ \Delta^{i_2}\Delta^{i_1}\mathscr{I}, \ldots, \Delta^{i_k}\Delta^{i_{k-1}} \cdots \Delta^{i_2}\Delta^{i_1}\mathscr{I}.$$

Definition A.2.6 (SYMBOL OF A SINGULARITY). *Let the map $f : N^n \to M^m$ be such that $f(0) = 0$. We say that* f has a singularity of Thom-Boardman symbol Σ^I at 0 *if the ideal generated by the m coordinate functions f_i has Thom-Boardman symbol I.*

Definition A.2.7 (NICE MAP). *A map f is said to be* nice *if its k-jet extension is transverse to the manifolds Σ^I.*

The importance of a nice map is contained in the following result.

Theorem A.2.1 (ON NICE MAPS). *[5]*

1. *If $f : N^n \to M^m$ is a nice map, then $\Sigma^I(f) = (j^k f)^{-1}(\Sigma^I)$. This is $\Sigma^I(f)$ is a submanifold of N and $x \in \Sigma^I(f)$ if and only if $j^k f(x) \in \Sigma^I$.*

2. *Any smooth map $f : N^n \to M^m$ can be arbitrarily well approximated by a nice map.*

A.3 Some classical normal forms

Let a $\mathcal{C}^\infty$ vector field $Y(x)$ be given as $Y = \sum_{i=1}^n f_i \frac{\partial}{\partial x_i}$. Assume that the origin is an isolated equilibrium point, this is $Y(0) = 0$. Assume also that the Jacobian of Y has c eigenvalues in the imaginary axis, and let ℓ be a positive integer. We have the following result.

Theorem A.3.1 (TOPOLOGICAL REDUCTION TO THE CENTER MANIFOLD). *There exists a $\mathcal{C}^\ell$, $c-$dimensional manifold W^c containing the origin, and a neighborhood U of $0 \in \mathbb{R}^n$, such that for any point $x \in W^c \cap U$, $Y(x)$ is tangent to W^c at x. Moreover, there exists an integer r, with $0 \leq r \leq n - c$ such that Y is* topologically equivalent *to the vector field*

$$\overline{Y} = \sum_{i=1}^{c} \tilde{f}_i(y_1, \ldots, y_c) \frac{\partial}{\partial y_i} + \sum_{i=c+1}^{c+r} y_i \frac{\partial}{\partial y_i} - \sum_{i=c+r+1}^{n} y_i \frac{\partial}{\partial y_i},$$

where $(y_1, \cdots, y_c)$ are coordinates in the center manifold W^c, and all eigenvalues of $D_0 \tilde{f}$ are on the imaginary axis.

It is important to note that the center manifold W^c in theorem A.3.1 is not unique. However, different choices of W^c lead to topologically equivalent phase portraits [2, 38].

Define by $Y_1(x)$ the vector field which has the same $1-$jet at $x = 0$ as Y, and whose coefficients are linear in x. Denote by $\mathcal{H}^k$ the space of vector fields whose coefficients are homogeneous polynomials of degree k.

The linear map $[Y_1, -]_k : \mathcal{H}^k \to \mathcal{H}^k$ assigns to each $H \in \mathcal{H}^k$ the Lie product $[Y_1, H]$. Observe that for a fixed Y_1 there is a splitting $\mathcal{H}^k = B^k + G^k$, where $B^k = \mathrm{Im}\,([Y_1, -]_k)$, and G^k is some complementary space.

Theorem A.3.2 (NORMAL FORM THEOREM [38]). *Let $Y,\ Y_1,\ B^k,\ G^k$ be as above. Then, for $\ell \leq k$, there exists a $\mathcal{C}^\infty-$diffeomorphism $\phi : \mathbb{R}^n \to \mathbb{R}^n$, which fixes the origin, such that $\phi_*(Y) = Y'$ is of the form*

$$Y' = Y_1 + g_2 + \cdots + g_\ell + R_\ell$$

where $g_j \in G^j$, $j = 2, \ldots, \ell$ and R_ℓ is a vector field with vanishing $\ell - jet$ at the origin, $\ell = k = \infty$ is not excluded.

Remark A.3.1. *In case the $1 - jet$ of Y is identically 0, one proceeds as follows. Let s be the smallest integer such that the $s - jet$ of Y does not vanish at 0, denote by Y_s the vector field whose component functions are homogeneous polynomials of degree s, and such that the $s-$jets of Y and Y_s are the same. As in the normal form theorem, define the map*

$$[Y_s, -]_k : \mathcal{H}^k \to \mathcal{H}^{k+s-1}.$$

For $k > s$, the splitting of the space $\mathcal{H}^k$ is $\mathcal{H}^k = B^k + G^k$, where $B^k = \mathrm{im}\,([Y_s, H])$, with now $H \in \mathcal{H}^{k-s+1}$. In this way, the conclusion of the normal form theorem remains valid by replacing the Y' from above by

$$Y' = Y_s + g_{s+1} + \ldots + g_\ell + R_\ell.$$

Theorem A.3.3 (TAKENS NORMAL FORM THEOREM ON SEMI-HYPERBOLIC SINGULARITIES [37]). *Let X be a vector field on $\mathbb{R}^n$ which is zero at the origin. If X satisfies the non-resonant condition of Stenberg, then there are C^k coordinates $(x_1, \ldots, x_c)$, $(y_1, \ldots, y_z)$ and $(z_1, \ldots, z_u)$ on $\mathbb{R}^n$ such that X is locally of the form*

$$X = \sum_{i=1}^{c} X_i(x_1, \ldots, x_c)\frac{\partial}{\partial x_1} + \sum_{i=1}^{s} A(x_1, \ldots, x_c)y_i\frac{\partial}{\partial y_i} + \sum_{i=1}^{s} B(x_1, \ldots, x_c)z_i\frac{\partial}{\partial z_i},$$

where

- *All the eigenvalues of $D_xX(0)$ have zero real part,*
- $A(0) < 0$
- $B(0) > 0$

Theorem A.3.4 (Sternberg-Chen [11, 34]). *Let X and Y be two hyperbolic vector fields. Then X and Y are $\mathcal{C}^\infty$ equivalent if and only if they are formally equivalent.*

A.4 Fenichel's theory

Let X_ε be a slow fast system whose slow manifold S is regular, say at the origin, that is

$$X_\varepsilon : \begin{cases} x' & = \varepsilon f(x, z, \varepsilon) \\ z' & = g(x, z, \varepsilon), \end{cases} \tag{A.3}$$

where $\dfrac{\partial g}{\partial z}(0)$ is non-singular. As usual, we denote by S the slow manifold of (A.3), and by definition this manifold S is Normally Hyperbolic in a neighborhood of the origin. Let S_0 be a compact subset of S.

Theorem A.4.1 (Fenichel [16, 46]). *If $\varepsilon > 0$ but sufficiently small, there exists a manifold S_ε which lies within distance of order $O(\varepsilon)$ from S_0. Moreover, the manifold S_ε is invariant under the flow of X_ε and is C^ℓ for any $0 < \ell < \infty$.*

This theorem tells us that the flow of the corresponding CDE gives an approximation of the flow along the invariant manifold S_ε of (A.3). We remark that the compactness property guarantees the existence and uniqueness of S_ε [45, 46].

A.5 Quasihomogeneous functions and vector fields

In this section we briefly recall the principal properties of quasihomogeneous functions, and vector fields. This section is mainly based in [3, 29].

Let $x \in \mathbb{R}^n$. Germs of functions are denoted by $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$. Similarly, vector fields on $\mathbb{R}^n$ are denote by X. We shall use the usual multi-index notation as follows: let $q = (r_1, \ldots, q_n) \in \mathbb{N}^n$, then x^q expresses the monomial $x_1^{q_1} \cdots x_n^{q_n}$, $|q| = q_1 + \cdots + q_n$ is the length of q. Let $r \in \mathbb{N}^n$, then $(r, q) = r_1q_1 + \cdots + r_nq_n$.

Definition A.5.1 (Quasihomogeneous function). *Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be a (germ of a) smooth function. The function f is called* quasihomogeneous of degree δ and type $r = (r_1, \ldots, r_n)$ *if for any $\lambda > 0$ the following holds*

$$f(\lambda^{r_1}x_1, \lambda^{r_2}x_2 \ldots, \lambda^{r_n}x_n) = \lambda^\delta f(x_1, x_2, \ldots, x_n).$$

In the rest we are interested only in quasihomogeneous polynomials.

Definition A.5.2 (Degree of a monomial). *We say that the r-monomial $x^q = x_1^{q_1} \cdots x_n^{q_n}$ has quasidegree δ if $(r, q) = \delta$.*

Let us, from now on, fix the type $r = (r_1, \dots, r_n)$. Observe then that any r-quasihomogeneous polynomial of degree δ can be written in the form

$$\sum_{(r,Q)=\delta} a_Q x^Q,$$

where $a_Q \in \mathbb{R}$. Since r is fixed, whenever no confusion is possible, we will omit to mention the type of a quasihomogeneous polynomial.

Definition A.5.3 (Quasidegree). *Let f be a quasihomogeneous polynomial. If all the monomials of f are of degree δ we call δ the* quasidegree *of f. By convention, the degree of 0 is $+\infty$.*

Definition A.5.4 (Order). *A polynomial f* has order $\operatorname{ord}(f) = \delta$ *if all of its monomials have degree δ or higher.*

We denote by $\mathcal{P}_\delta$ the linear space formed by polynomials of order δ. Observe that $\mathcal{P}_\delta \subset \mathcal{P}_{\delta'}$ for $\delta' < \delta$.

Definition A.5.5 (Quasihomogeneous vector field). *A (formal) vector field $X = \sum X_i \dfrac{\partial}{\partial x_i}$ is called* quasihomogeneous of degree δ and type $r = (r_1, \dots)$ *if each non-zero X_i is a quasihomogeneous (polynomial) function of degree $\delta + r_i$ and of type $r = (r_1, \dots)$.*

Definition A.5.6 (Order of a vector field). *A formal vector field $X = \sum X_i \frac{\partial}{\partial x_i}$ is said to have* order δ *all its quasihomogeneous components X_i are of degree higher than $\delta + r_i$.*

The set of all quasihomogeneous vector fields of order δ is denoted by $\mathcal{H}_\delta$. The following lemma summarizes some of the properties of quasihomogeneous objects that are useful in the present context.

Lemma A.5.1 (Properties of quasihomogeneous objects [4, 29]). *Let $\delta, \sigma \in \mathbb{N}$ be nonnegative, and $f \in \mathcal{P}_\delta$, $X \in \mathcal{H}_\sigma$. The following facts hold*

- *if $g \in \mathcal{P}_\sigma$, then $fg \in \mathcal{P}_{\delta+\sigma}$.*
- $L_X f \in \mathcal{P}_{\delta+\sigma}$
- $fX \in \mathcal{H}_{\delta+\sigma}$
- *if $Y \in \mathcal{H}_\delta$, then $DX \cdot Y \in \mathcal{H}_{\sigma+\delta}$ and $[X, Y] \in \mathcal{H}_{\sigma+\delta}$.*
- *Let $X = \sum_{i=1}^n X_i \dfrac{\partial}{\partial x_i}$, then $\operatorname{ord}(X) = \delta$ if and only if $\operatorname{ord}(X_i) = \delta + r_i$.*

We shall define a convenient inner product in the space of quasihomogeneous polynomials and vector fields.

Definition A.5.7 (The inner product $\langle \cdot, \cdot \rangle_{r,\delta}$). *Let $x = (x_1, \dots, x_n)$, and $s, q \in \mathbb{N}^n$. Let $f, g \in \mathcal{P}_\delta$, that is*

$$f = \sum_{(r,s)=\delta} f_s x^s,$$

and similarly for g. Then the inner product $\langle \cdot, \cdot \rangle_{r,\delta}$ is defined as

$$\langle f, g \rangle_{r,\delta} = \sum_{(r,s)=\delta} f_s g_s \frac{(s!)^r}{\delta!},$$

where $(s!)^r = (s_1!)^{r_1} \cdots (s_n!)^{r_n}$. So for monomials one has

$$\langle x^s, x^q\rangle_{r,\delta} = \begin{cases} \frac{(s_1!)^{r_1}\cdots(s_n!)^{r_n}}{\delta!} & \textit{if} \quad s=q \quad \textit{with} \quad (s,r)=\delta, \\ 0 & \textit{otherwise.} \end{cases}$$

For vector fields we have the following. Let X, Y be quasihomogeneous vector fields such that $X = \sum_{i=1}^n X_i \frac{\partial}{\partial x_i} \in \mathcal{H}_\delta$, and $Y = \sum_{i=1}^n Y_i \frac{\partial}{\partial x_i} \in \mathcal{H}_\delta$. Then

$$\langle X, Y\rangle_{r,\delta} = \sum_{i=1}^n \langle X_i, Y_i\rangle_{r,\delta+r_i}.$$

Examples of quasihomogeneous polynomials and vector fields

The following examples are used in the main text.

Example A.5.1. *The function $f(x_1, z) = z^3 + x_1 z$ is quasihomogeneous of degree $\delta = 3$ and type $r = (2,1)$.*

Example A.5.2. *Consider the quasihomogeneous polynomial $f^k = z^k + \sum_{i=1}^{k-1} x_i z^{i-1}$. All its monomials are of degree k. Therefore, we say that f has quasidegree k.*

Example A.5.3 (Order)**.**

- *The quasihomogeneous polynomial $f^k = z^k + \sum_{i=1}^{k-1} x_i z^{i-1}$ has order k.*
- *Consider the quasihomogeneous polynomial $f^k + P$, where $\deg(P) \geq k$. Then $f^k + P$ has order k.*
- *Consider the quasihomogeneous polynomial $f^k + O$, where $\mathrm{ord}(O) \geq k$. Then $f^k + O$ has order k.*

Example A.5.4. *The n-dimensional linear vector field $X = \sum_{i=1}^n a_i x_i \dfrac{\partial}{\partial x_i}$ is a quasihomogeneous vector field of degree $\delta = 0$ and type $r = (\underbrace{1, 1, \ldots, 1}_{n})$.*

Example A.5.5. *The 3-dimensional vector field $X = \varepsilon \dfrac{\partial}{\partial x_1} - (z^2 + x_1)\dfrac{\partial}{\partial z} + 0\dfrac{\partial}{\partial \varepsilon}$ is a quasihomogeneous of degree $\delta = 1$ and type $r = (2, 1, 3)$.*

Example A.5.6. *The $k+1$ dimensional vector field*

$$X = \varepsilon\frac{\partial}{\partial x_1} - \left(z^k + \sum_{i=1}^{k-1} x_i z^{i-1}\right)\frac{\partial}{\partial z} + 0\frac{\partial}{\partial \varepsilon}$$

is quasihomogeneous of degree $\delta = k-1$ and type $r = (k, k-1, \ldots, 1, 2k-1)$.

A.6 Normal forms and transitions

Here we present a procedure to obtain C^ℓ normal forms of hyperbolic and semi-hyperbolic vector fields. We do this because those are the type of blown up vector fields that we encounter when studying A_k-SFS.

A.6.1 Normal form near a hyperbolic point

Proposition A.6.1. *Let X be the $m+2$ dimensional vector field*

$$X : \begin{cases} u' &= uL(v) + f_u \\ v_1' &= \alpha_1 v_1 L(v) + w + f_{v_1} \\ v_j' &= \alpha_j v_j L(v) + f_{v_j} \\ w' &= \gamma w L(v) + f_w, \end{cases}$$

where $j = 2, \ldots, m$, $L(v) = 1 + \sum_{i=1}^{m} v_i$ and all the f_ℓ's are flat along $u = 0$. Assume that the coefficients α_i and γ are non-zero. Then, there exists a $\mathcal{C}^\infty$ transformation that brings X intro the normal form

$$X_{\mathrm{h}}^N : \begin{cases} U' &= U \\ V_1' &= -\alpha_1 V_1 + W \frac{1}{L(V)} \\ V_j' &= -\alpha_j V_j \\ W' &= -\gamma W. \end{cases}$$

Proof of proposition A.6.1. The proof consists of two steps. First divide (re-parametrize time) X by $L(v)$. Finally apply Sternberg-Chen theorem appendix A.3. □

A.6.2 Normal form near a semi-hyperbolic point

Proposition A.6.2. *Let $(u, v, w, z) \in \mathbb{R} \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}$ and consider the vector field X given as*

$$X : \begin{cases} u' &= \alpha w u(1+f) \\ v_j' &= \beta_j w v_j (1+f) + w g_j \\ w' &= \gamma w^2 (1+f), \\ z' &= -\Lambda + h \end{cases}$$

where $j = 1, \ldots, m$; where the coefficients α, β_i, γ are all nonzero, the functions f, g_i and h are all smooth functions flat along $u = 0$ and $\Lambda = \Lambda(u, v, w, z)$ is a smooth function such that $\Lambda(0) = 0$ and $\frac{\partial \Lambda}{\partial z}(0) > 0$. Then there exist C^ℓ coordinates around the origin such that X is C^ℓ equivalent to

$$X_{sh}^N : \begin{cases} U' &= \alpha W U \\ V_j' &= \beta_j W V_j \\ W' &= \gamma W^2 \\ Z' &= -GZ, \end{cases}$$

where $G = G(U, V, W)$ is a C^ℓ function such that $G(0) > 0$.

Proof of proposition A.6.2. From the definition of the vector field X we note that the origin is a semi-hyperbolic singular point. The hyperbolic eigenspace is 1-dimensional while the center eigenspace is $(m+1)$-dimensional. In the following lines we obtain a C^ℓ normal form X_{sh}^N for X. We detail all the steps that take X into the normal form X^N.

Divide by $1+f$. We define a new vector field Y by $Y = \frac{1}{1+f} X$, which reads

$$Y : \begin{cases} u' &= \alpha w u \\ v_j' &= \beta_j w v_j + w \tilde{g}_j \\ w' &= \gamma w^2, \\ z' &= -\Lambda + \tilde{h}, \end{cases}$$

where the functions $\tilde{g}_j$, $j = 1, \ldots, m$, and $\tilde{h}$ are flat along $u = 0$. Then, in a small neighborhood of the origin, the vector fields X and Y are smoothly conjugate.

Characterize a center manifold. Observe that linearization of Y at the origin shows the existence of an $(m+1)$-dimensional center manifold $\mathcal{W}^C$ [19]. In this step we simplify the expression of the center manifold. Let M_0 be the set of critical points of Y, that is

$$M_0 = \{(u,v,w,z) \mid \Lambda(u,v,0,z) = 0\}.$$

By definition, the manifold M_0 is invariant and normally hyperbolic. Now, assume $|w| \ll 1$. Such a condition appears naturally in our applications. By Fenichel's theory [16] the manifold M_0 will persist as an invariant normally hyperbolic manifold M_w, for sufficiently small $w \neq 0$. We identify such manifold M_w with $\mathcal{W}^C$. In other words, there exists a C^ℓ function $m = m(u,v,w)$ such that the center manifold $\mathcal{W}^C$ is given as a graph

$$\mathcal{W}^C = \mathrm{Graph}(u,v,w,m).$$

Let $\zeta = z - m$, then $\zeta' = z' - m'$. But we know, due to invariance of $\mathcal{W}^C$ under the flow of Y, that $\zeta'|_{\zeta=0} = 0$. This is, there exists a C^ℓ function $g = g(u,v,w,\zeta)$ such that $\zeta' = -g\zeta$. With $g(0) = 0$ and $\frac{\partial g}{\partial \zeta}(0) > 0$.

In conclusion of this step, there exists a C^ℓ transformation $\psi : (u,v,w,z) \mapsto (u,v,w,\zeta)$ that transforms the vector field Y into

$$\tilde{Y} : \begin{cases} u' &= \alpha w u \\ v_j' &= \beta_j w v_j + w\tilde{g}_j \\ w' &= \gamma w^2, \\ \zeta' &= -g\zeta, \end{cases}$$

where $g = g(u,v,w,\zeta)$ is a C^ℓ function such that $g(0) = 0$ and $\dfrac{\partial g}{\partial z}(0) = \dfrac{\partial \Lambda}{\partial z}(0)$.

Eliminate the flat terms in the center manifold $\mathcal{W}^C$. Observe that thanks to the previous step, the center manifold $\mathcal{W}^C$ has the simple expression $\mathcal{W}^C = \{\zeta = 0\}$. The flow of $\tilde{Y}$ restricted to $\mathcal{W}^C$ reads

$$\tilde{Y}^c = \tilde{Y}|_{\mathcal{W}^C} : \begin{cases} u' &= \alpha w u \\ v_j' &= \beta_j w v_j + w\tilde{g}_j \\ w' &= \gamma w^2, \end{cases}$$

where we recycle the notation of the flat perturbation. Note that the vector field $\tilde{Y}^c$ is, up to multiplication by w and up to flat perturbation, linear. That is

$$j^\infty\left(\frac{1}{w}\tilde{Y}^c\right) : \begin{cases} u' &= \alpha u \\ v_j' &= \beta_j v_j \\ w' &= \gamma w. \end{cases}$$

The goal now is to conjugate $\frac{1}{w}\tilde{Y}^c$ with its infinite jet, which is linear. However we must do this fixing the coordinate w. Note that this is not possible by only using Sternberg-Chen. But it is possible via a result of Roussarie [15]. Thus, for $w \neq 0$, we have that there is a smooth change of coordinates $(u,v,w) \mapsto (\tilde{u},\tilde{v},w)$ conjugating $\tilde{Y}^c$ to

$$\mathcal{Y}^c : \begin{cases} \tilde{u}' &= \alpha \tilde{w}\tilde{u} \\ \tilde{v}_j' &= \beta_j \tilde{w}\tilde{v}_j \\ w' &= \gamma w^2. \end{cases}$$

Finally, since the center dynamics are independent of ζ, we have that the vector $\tilde{Y}$ is smoothly conjugate to

$$\mathcal{Y}: \begin{cases} \tilde{u}' &= \alpha w \tilde{u} \\ \tilde{v}_j' &= \beta_j w \tilde{v}_j \\ w' &= \gamma w^2 \\ \zeta' &= -\tilde{g}\zeta, \end{cases}$$

where $\tilde{g} = g(\tilde{u}, \tilde{v}, w, \zeta)$.

Linearize the hyperbolic direction. Roughly speaking, the last step in order to obtain the normal form X^N is to make the equation of ζ' linear in ζ. We do so by invoking the Bonckaert extensions to the theorem of Takens on semi-hyperbolic fixed points [37, 6, 7]. Such result guarantees the existence of C^ℓ change of coordinates coordinates $(\tilde{u}, \tilde{v}, w, \zeta) \mapsto (\tilde{u}, \tilde{v}, w, Z)$ bringing $\mathcal{Y}$ into the normal form

$$X_{\text{sh}}^N: \begin{cases} \tilde{u}' &= \alpha w \tilde{u} \\ \tilde{v}_j' &= \beta_j w \tilde{v}_j \\ w' &= \gamma w^2 \\ Z' &= -GZ, \end{cases}$$

where $G = G(\tilde{u}, \tilde{v}, w)$ is a C^ℓ function such that $G(0) > 0$.

□

A.6.3 Partition of a smooth function

In this section we investigate the problem of partitioning a smooth function. The result presented below is important since it is used to simplify the computation of transition maps. To be more specific, let us give a brief example. Consider the three dimensional differential equation

$$\begin{aligned} x' &= x \\ y' &= -y \\ z' &= g(x, y)z, \end{aligned}$$

where g is a smooth function. We want to take advantage from the fact that xy is a first integral. We show below that the function g can be partitioned as $g(x, y) = g_1(xy, x) + g_2(xy, y)$. This makes the integration of z' simpler.

Lemma A.6.1. *Let $u \in \mathbb{R}$ and $v \in \mathbb{R}^m$. Let $f = f(u, v)$ be a smooth function such that $f(0, 0) = 0$. Then there exist smooth functions $f_0 = f_0(uv, u)$ and $f_1(uv, v)$ such that the function f can be written as*

$$f = f_0 + f_1,$$

where $f_0(0, 0) = 0$ and $f_1(0, 0) = 0$.

Proof of lemma A.6.1. We proceed in two steps. The first consists in proving the formal version of the statement. The second step is to extend the formal result to the smooth case.

Formal step

Let $\hat{f}$ denote the formal expansion of the smooth function f. Let $p \in \mathbb{N}$ and $q \in \mathbb{N}^m$. We use the following notation:

- By $q \geq 0$ we mean $q_i \geq 0$ for all $i \in [1, m]$.
- For a vector $v \in \mathbb{R}^m$ we write $v^q = v_1^{q_1} \cdots v_m^{q_m}$.
- The L_1 norm of q is denote by $|q|$, and thus for $q > 0$ we have $|q| = \sum_{j=1}^m q_j$.
- We denote by $\tilde{q}_i$ the vector

$$\tilde{q}_i = (q_1, \ldots, q_{i-1}, q_{i+1}, \ldots, q_m)$$

and therefore we have that $v^{\tilde{q}_i}$ reads as

$$v^{\tilde{q}_i} = \frac{v^q}{v_i^{q_i}} = v_1^{q_1} \cdots v_{i-1}^{q_{i-1}} v_{q_{i+1}}^{q_{i+1}} \cdots v_m^{q_m}.$$

Besides, we have that the L_1 norm of $\tilde{q}_i$ is given by $|\tilde{q}_i| = |q| - q_i = \sum_{j=1, j\neq i}^m q_j$.

The formal series expansion of f reads as

$$\hat{f} = \sum_{p\geq 0, q\geq 0} a_{pq} u^p v^q,$$

where $a_{00} = 0$. With the notation introduced above, we can partition $\hat{f}$ as follows

$$\hat{f} = \sum_{p\geq |q|} a'_{pq} (uv)^q u^{p-|q|} + \sum_{i=1}^m \sum_{q_i \geq p + |\tilde{q}_i|} a'_{pq} (uv_i)^p (v_i v)^{\tilde{q}_i} v_i^{q_i - p - |\tilde{q}_i|},$$

where

$$(uv)^q = (uv_1)^{q_1} \cdots (uv_m)^{q_m}$$
$$(v_i v)^{\tilde{q}_i} = \frac{(v_i v)^q}{v_i^{2q_i}},$$

and where $a'_{pq} \in \mathbb{R}$ are suitable chosen coefficients. Let $r \in \mathbb{N}^m,\ s \in \mathbb{N}$. Define the following formal polynomials

$$\hat{h}(uv, u) = \sum_{r,s\geq 0} \alpha_{rs} (uv)^r u^s = \sum_{p\geq |q|} a'_{pq} (uv)^q u^{p-|q|},$$

where $\alpha_{rs} \in \mathbb{R}$, and

$$\begin{aligned} \hat{g}_i(uv_i, v) &= \sum_{r,s,t\geq 0} \beta_{irs} (uv_i)^s v^r \\ &= \sum_{q_i \geq p + |\tilde{q}_i|} a'_{pq} (v_i v)^{\tilde{q}_i} (uv_i)^p v_i^{q_i - p - |\tilde{q}_i|}, \end{aligned}$$

where $\beta_{irs} \in \mathbb{R}$. The coefficients α_{rs} and β_{irs} are conveniently chosen to make the definitions hold. Let $uv = (uv_1, \ldots, uv_m)$. Define $\hat{g} = \hat{g}(uv, v)$ by $\hat{g}(uv, v) = \sum_{i=1}^m \hat{g}_i(uv_i, v)$, then we can write $\hat{f}$ as

$$\hat{f}(u, v) = \hat{h}(uv, u) + \hat{g}(uv, v).$$

This shows that the proposition holds for formal series.

Smooth step

By Borel's lemma [10], there exist smooth functions $h = h(uv, u)$ and $g = g(uv, v)$ (whose formal series expansions are $\hat{h}$ and $\hat{g}$ respectively) such that

$$f = h + g + R,$$

where R (reminder) is a flat function. We now show the following.

Proposition A.6.3. *Let $u \in \mathbb{R}$, $v \in \mathbb{R}^m$, and $R(u, v)$ be a smooth flat function at $(0, 0) \in \mathbb{R} \times \mathbb{R}^m$. There exist flat functions $r_0 = r_0(uv, u)$ and $r_1 = r_1(uv, v)$ such that*

$$R = r_0 + r_1.$$

Remark A.6.1. *Proposition A.6.3 together with the formal step $\hat{f} = \hat{h} + \hat{g}$ imply our result.*

Proof of proposition A.6.3. For this proof we shall use the blow-up technique. Let $\Phi : S^m \times \mathbb{R}^+ \to \mathbb{R}^{m+1}$ be a blow-up map. The map Φ maps $S^m \times \{0\}$ to the origin in $\mathbb{R}^{m+1}$. Let $\tilde{R}$ be a function defined by $\tilde{R} = R \circ \Phi$. Since R is flat at the origin, the function $\tilde{R}$ is flat along the sphere S^m. We assume that the function $R = R(u, v)$ is defined on a small neighborhood $\mathcal{R}$ of the origin in $\mathbb{R} \times \mathbb{R}^m$; this neighborhood is defined as

$$\mathcal{R} = \{|u| \leq A,\ |v_i| \leq B_i\},$$

for some A, B_i positive scalars. Let $0 < \delta < 1$. The sphere S^m can be partitioned into $m + 1$ regions as follows:

$$\begin{aligned} \mathcal{U} &= S^m \setminus \{|\bar{u}| \leq \delta\} \\ \mathcal{V}_i &= S^m \setminus \{|\bar{v}_i| \leq \delta\}, \end{aligned}$$

where $(\bar{u}, \bar{v}) = (\bar{u}, \bar{v}_1, \ldots, \bar{v}_m) \in S^m$. We can then take a partition of unity to split $\tilde{R}$ as

$$\tilde{R}(\bar{u}, \bar{v}) = \tilde{R}_0(\bar{u}, \bar{v}) + \sum_{i=1}^{m} \tilde{R}_i(\bar{u}, \bar{v}), \tag{A.4}$$

where $\mathrm{Supp}(\tilde{R}_0) \subset \mathcal{U}$ and $\mathrm{Supp}(\tilde{R}_i) \subset \mathcal{V}_i$ for $i \in [1, m]$. We define as R_0 and R_i the corresponding functions on $\mathbb{R}^{m+1}$ flat at the origin given by the blow-up map Φ, that is $\tilde{R}_j = R_j \circ \Phi$, for $j = 0, 1, \ldots, m$. Note that $R \to \tilde{R}$ is an isomorphism between the space of functions on $(u, v) \in \mathbb{R}^{m+1}$ flat at the origin, and the space of functions on $((\bar{u}, \bar{v}), \rho) \in S^m \times \mathbb{R}^+$ flat at $S^m \times \{0\}$. Therefore, the splitting (A.4) induces the splitting

$$R(u, v) = R_0(u, v) + \sum_{i=1}^{m} R_i(u, v)$$

of functions on $\mathbb{R}^{m+1}$. We will now prove that there exist flat functions r_0 and r_i such that

$$\begin{aligned} R_0(u, v) &= r_0(uv, u) \\ R_i(u, v) &= r_i(uv_i, v). \end{aligned}$$

Let us detail only the case of R_0. The other functions are obtained in a similar way.

The function $\tilde{R}_0$ has support in $\mathcal{U}$. We can parametrize $\mathcal{U}$ by the directional blow-up map Φ_u which reads as

$$(\bar{u}, \bar{v}_1, \ldots, \bar{v}_m) \mapsto (\bar{u}, u\bar{v}_1, \ldots, u\bar{v}_m) = (u, v_1, \ldots, v_m).$$

Now, suppose that there exists a flat function $\tilde{P}_0$ defined by

$$\tilde{R}_0(u,\bar{v}) = \tilde{P}_0(u, u^2\bar{v}).$$

This implies that there is a function $\tilde{r}_0 = \tilde{P}_0 \circ \Phi_u^{-1}$ such that

$$R_0(u,v) = \tilde{r}_0(u, uv),$$

which is precisely what we want to prove. So, now we only need to show that indeed a function $\tilde{P}_0$ as above exists. For this let us define coordinates $(U, V_1, \ldots, V_m)$ given by

$$U = u,\ V_1 = u^2\bar{v}_1,\ \ldots,\ V_m = u^2\bar{v}_m,$$

and let $\tilde{P}_0(u,V)$ be a function defined as

$$\tilde{P}_0(u,V) = \tilde{R}_0\left(\frac{V}{u^2}, u\right).$$

Note that $\tilde{P}_0$ is flat at $(u,V) = 0$. This is seen as follows. Since $\tilde{R}_0$ is flat along $\{u=0\}$, it follows that $\tilde{P}_0(0,0) = \tilde{R}_0|_{u=0} = 0$ and

$$\begin{aligned}
\frac{\partial \tilde{P}_0}{\partial u}(0) &= \frac{\partial \tilde{R}_0}{\partial u}|_{u=0} = 0\\
\frac{\partial \tilde{P}_0}{\partial V_i}(0) &= \frac{1}{u^2}\frac{\partial \tilde{R}_0}{\partial \bar{v}_i}|_{u=0} = 0,
\end{aligned}$$

and so on for the higher order derivatives.

Finally, for convenience of notation we define $r_0(uv,u) = \tilde{r}_0(u,uv)$, thus we can write $R_0(u,v) = r_0(uv,u)$ Following similar arguments as above we find the functions $r_i = r_i(uv_i, v)$ such that $R_i(u,v) = r_i(uv_i,v)$ for $i \in [1,m]$. Then we define $r_1(uv,v) = \sum_{i=1}^m r_i(uv_i,v)$. It follows that

$$R(u,v) = r_0(uv,u) + r_1(uv,v).$$

□

With this last proposition we can now write the function f as

$$\begin{aligned}
f &= h(uv,u) + g(uv,v) + R(u,v)\\
&= h(uv,u) + g(uv,v) + r_0(uv,u) + r_1(uv,v).
\end{aligned}$$

Finally, to show the lemma we define the smooth functions f_1, f_2 of the statement by

$$\begin{aligned}
f_1 &= h + r_0\\
f_2 &= g + r_1.
\end{aligned}$$

□

A.6.4 Transition near a hyperbolic point

In this section we study the flow near the hyperbolic equilibrium point of the vector field X_{h}^N obtained in appendix A.6.1.

We recall that X_{h}^N reads

$$X_{\mathrm{h}}^N : \begin{cases} u' &= u \\ v_1' &= -\alpha_1 v_1 + w\left(1 - \frac{L(v)-1}{L(v)}\right) \\ v_j' &= -\alpha_j v_j \\ w' &= -\gamma w, \end{cases}$$

where $j = 1, \ldots, m$. Observe that X_{h}^N has $m+1$ contracting directions and only one expanding direction. Therefore, it is convenient to study a transition map given as

$$\Pi : (u, v_1, \ldots, v_m, w) \mapsto (u_{out}, \tilde{v}_1, \ldots, \tilde{v}_m, \tilde{w}), .$$

where $u_{out} > u$. We then choose the time of integration to be $T = \ln\left(\frac{u_{out}}{u}\right)$. We have the following.

Proposition A.6.4. *Let X_{h}^N be as above and let*

$$\Pi : (u, v_1, \ldots, v_m, w) \mapsto (u_{out}, \tilde{v}_1, \ldots, \tilde{v}_m, \tilde{w}).$$

Then we have

$$\begin{aligned} \tilde{v}_1 &= v_1 \left(\frac{u}{u_{out}}\right)^{\alpha_1} \left(1 + O\left(\frac{u}{u_{out}}\right)\right) \\ \tilde{v}_j &= v_j \left(\frac{u}{u_{out}}\right)^{\alpha_j} \qquad \forall j \in [2, m] \\ \tilde{w} &= w \left(\frac{u}{u_{out}}\right)^{\gamma} \end{aligned}$$

Proof of proposition A.6.4. The solutions $\tilde{w}$ and $\tilde{v}_j$ for $j \in [2, m]$ are straightforward. So now the only problem we have is to integrate a one dimensional equation of the form

$$v_1' = -\alpha_1 v_1 + w(t) F(t, v_1), \tag{A.5}$$

where $F(t, v_1)$ reads

$$F(t, v_1) = \frac{1}{1 + v_1 + \sum_{j=2}^m v_j(0) e^{-\alpha_j t}}.$$

In our applications $w(t)$ is small in magnitude. Thus, observe that if $w \equiv 0$, we would have $v_1(t) = v_1(0) e^{-\alpha_1 t}$.

Let us define V_1 by $v_1 = e^{-\alpha_1 t} V_1$. Then substituting in (A.5) we get

$$V_1' = w e^{(-\gamma + \alpha_1)t} \tilde{F}(t, V_1), \tag{A.6}$$

where we have substituted $w(t) = w e^{-\gamma t}$ and

$$\tilde{F}(t, V_1) = \frac{1}{1 + e^{-\alpha_1 t} V_1 + \sum_{j=2}^m v_j(0) e^{-\alpha_j t}}.$$

The term $-\gamma+\alpha_1$ is always negative, and the term $we^{(-\gamma+\alpha_1)t}$ is dominant in forward time. This is because $\tilde{F} \to 1$ as $t \to \infty$. Then, even though we cannot explicitly integrate V_1', from the form of (A.6) we expect that

$$V_1 = Awe^{(-\gamma+\alpha_1)t}(1+E),$$

with some constant A and $E = E(t) \sim e^{-ct}$ with $c > 0$. Therefore we would have (adding the solution to the linear equation and the solution for V')

$$\begin{aligned} v_1(t) &= v_1(0)e^{-\alpha_1 t} + e^{-\alpha_1 t}V_1 \\ &= v_1(0)e^{-\alpha_1 t} + Awe^{-\gamma t}(1+E). \end{aligned}$$

Then taking into account the time of integration $T = \ln\left(\frac{u_{out}}{u}\right)$ we have

$$\tilde{v}_1 = v_1\left(\frac{u}{u_{out}}\right)^{\alpha_1} + Aw\left(\frac{u}{u_{out}}\right)^{\gamma}\left(1+O\left(\frac{u}{u_{out}}\right)\right).$$

Since $\gamma > \alpha_1$ we finally have

$$\tilde{v}_1 = v_1\left(\frac{u}{u_{out}}\right)^{\alpha_1}\left(1+O\left(\frac{u}{u_{out}}\right)\right).$$

□

A.6.5 Transition near a semi-hyperbolic point

In this section we investigate the transitions for the vector field X_{sh}^N computed in appendix A.6.2. Relabeling the coordinates we recall that X_{sh}^N reads

$$X_{\text{sh}}^N : \begin{cases} u' &= \alpha w u \\ v_j' &= \beta_j w v_j \\ w' &= \gamma w^2 \\ z' &= -gz, \end{cases}$$

where $j = 1, \ldots, m$, and where $g = g(u, v, w)$ is a C^ℓ function such that $g(0) = \Lambda > 0$. We assume that $w \in \mathbb{R}^+$. For our applications, we are interested in only two particular situations.

1. The saddle 1 case where $\alpha = -1$, $\beta_i > 0$ and $\gamma > 0$.
2. The saddle 2 case where $\alpha = 1$, $\beta_i < 0$ and $\gamma < 0$.

Saddle 1

In this case we investigate the transitions of a vector field of the form

$$Y : \begin{cases} u' &= -wu \\ v_j' &= \beta_j w v_j \\ w' &= \gamma w^2 \\ z' &= -gz, \end{cases} \tag{A.7}$$

where the coefficients β_j, for all $j = 1, \ldots, m$, and γ are positive. Observe that the flow in the direction of u and z is a contraction while it expands in all the other directions. Roughly speaking, this implies that a transition can go out at any expanding direction v_j of w.

We investigate two types of transitions that will be used later in our applications. For this, let us define the following sections

$$\begin{aligned}
\Sigma_{\rm en} &= \{(u,v,w,z) \mid u = u_i\} \\
\Sigma_{\rm ex}^{w} &= \{(u,v,w,z) \mid w = w_o\} \\
\Sigma_{\rm ex}^{\pm v_j} &= \{(u,v,w,z) \mid v_j = v_{j,o}\}.
\end{aligned}$$

In this section we compute the transitions

$$\begin{aligned}
\Pi^{w} : &\Sigma_{\rm en} \to \Sigma_{\rm ex}^{w} \\
&(v,w,z) \mapsto (\tilde{u},\tilde{v}_i,\tilde{z}),
\end{aligned}$$

for all $i \in [1,m]$, and

$$\begin{aligned}
\Pi^{\pm v_j} : &\Sigma_i \to \Sigma_{\rm ex}^{\pm v_j} \\
&(v,w,z) \mapsto (\tilde{u},\tilde{v}_i,\tilde{w},\tilde{z}),
\end{aligned}$$

for all $i \in [1,m]$ with $i \neq j$. The way each transition is used is made clear in section 3.4.

Proposition A.6.5. *Consider the vector field Y given by* (A.7) *and let $\Sigma_{\rm en}$, $\Sigma_{\rm ex}^{w}$, $\Sigma_{\rm ex}^{\pm v_j}$ and Π^{w}, $\Pi^{\pm v_j}$ be as above. Then*

- *The transition Π^{w} is given by*

$$\begin{aligned}
\tilde{u} &= u\left(\frac{w}{w_o}\right)^{1/\gamma}, \qquad \tilde{v}_i = v_i\left(\frac{w_o}{w}\right)^{\beta_i/\gamma} \\
\tilde{z} &= z\exp\left[-\frac{\Lambda}{\gamma w}\left(1+\tilde{\alpha}w\ln(w)+w\tilde{G}\right)\right]
\end{aligned}$$

  where $\tilde{\alpha} = \tilde{\alpha}(uv_i^{1/\beta_i}, uw^{1/\gamma})$ and $\tilde{G} = \tilde{G}(uv_i^{1/\beta_i}, uw^{1/\gamma}, w^{1/\gamma}, \mu_i)$ are C^{ℓ} functions with $\mu_i = v_i^{1/\beta_i}w^{-1/\gamma}$.

- *The transition $\Pi^{\pm v_j}$ is given by*

$$\begin{aligned}
\tilde{u} &= \left(\frac{v_j}{\eta_j}\right)^{1/\beta}, \qquad \tilde{v}_i = v_i\left(\frac{\eta_j}{v_j}\right)^{\beta_i/\beta_j}, \qquad \tilde{w} = w\left(\frac{\eta_j}{v_j}\right)^{\gamma/\beta_j} \\
\tilde{z} &= z\exp\left[-\frac{\Lambda}{\gamma w}\left(1+\tilde{\alpha}'w\ln(v_j)+w\tilde{G}'\right)\right],
\end{aligned}$$

  with $i \neq j$ and where

$$\begin{aligned}
\tilde{\alpha}' &= \tilde{\alpha}'(uv_i^{1/\beta_i}, uw^{1/\gamma}) \\
\tilde{G}' &= \tilde{G}'(uv_i^{1/\beta_i}, uw^{1/\gamma}, \mu_w, \mu_i)
\end{aligned}$$

  are C^{ℓ} functions with $\mu_w = w^{1/\gamma}v_j^{1/\beta_j}$ and $\mu_i = v_i^{1/\beta_i}v_j^{1/\beta_j}$.

Proof of proposition A.6.5. We detail first the computations for the transition Π^{w}. The transition $\Pi^{\pm v_j}$ is computed in a similar way so we only highlight the key parts of the computation.

The transition Π^w

In this case, the time of integration is $T = \ln\left(\frac{w_o}{w}\right)^{1/\gamma}$, where $w_o = w(t)|_{\Sigma^w_{\text{ex}}}$ and $w = w(t)|_{\Sigma_{\text{en}}}$. Such a time of integration is obtained form the equation $w' = \gamma w$. We also make the assumption that $v_i \in O(e^{\beta_i/\gamma})$. Such an assumption will appear in our applications. From the form of Y we evidently have

$$
\begin{aligned}
u(T) = \tilde{u} &= u\left(\frac{w}{w_o}\right)^{1/\gamma}\\
v_i(T) = \tilde{v}_i &= v_i\left(\frac{w_o}{w}\right)^{\beta_i/\gamma}.
\end{aligned}
$$

It only rests to compute the transition for the z coordinate. Let us rewrite Y as follows

$$
\begin{aligned}
u' &= -u\\
v_i &= \beta_i v_i\\
w &= \gamma w\\
z' &= -\frac{\Lambda + G(u, v, w)}{w} z,
\end{aligned}
$$

where G is a C^ℓ function vanishing at the origin. Observe that we have the first integrals $u^{b_i} v_i$ and $u^\gamma w$. We shall take advantage of such a fact. We define new coordinates (U, V, W) given by

$$
U = u,\ V_i^{\beta_i} = v_i,\ W^\gamma = w.
$$

In this new coordinates we have the system

$$
\begin{aligned}
U' &= -U\\
V_i' &= V_i\\
W' &= W\\
z' &= -\frac{\Lambda + G(U, V^{\beta_i}, W^\gamma)}{W^\gamma} z.
\end{aligned}
$$

In the new coordinates, the time of integration is given as $T = \ln\left(\frac{W_o}{W}\right)$. To have an idea of the expression of $\tilde{z}$, let us first study a simple example.

The case $G = 0$

Let us suppose $G = 0$. Therefore we have $z' = -\frac{\Lambda}{W^\gamma} z$, which has the solution

$$
z(t) = z(0)\exp\left(-\Lambda \int_0^t W(s)^{-\gamma} ds\right),
$$

where $W(s) = W(0)\exp(s)$. Substituting the time of integration T we have

$$
\begin{aligned}
z(T) = \tilde{z} &= z\exp\left(-\frac{\Lambda}{W^\gamma}\int_0^{\ln\left(\frac{W_o}{W}\right)} e^{-\gamma s} ds\right)\\
&= z\exp\left(-\frac{\Lambda}{\gamma W^\gamma}\left(1 - \left(\frac{W}{W_o}\right)^\gamma\right)\right).
\end{aligned}
$$

Observe that since $W < W_o$ we have that $\tilde{z} \to 0$ as $W \to 0$. Let us now study the general case.

The case $G \neq 0$

We now consider that $G \neq 0$, we have

$$z(T) = \tilde{z} = z \exp\left(I_0 + I_1\right),$$

where

$$I_0 = -\Lambda \int_0^T \frac{1}{W(s)} ds$$
$$I_1 = \int_0^T \frac{G(U(s), V(s)^{\beta_i}, W(s)^{\gamma})}{W(s)^{\gamma}} ds.$$

The integral I_0 has already been computed above. Let us write $F(U, V, W) = \frac{G(U(s),V(s)^{\beta_i},W(s)^{\gamma})}{W(s)^{\gamma}}$. We can do this because $G(U, 0, 0) = 0$ and $V^{\beta_i} \in O(W^{\gamma})$. Now we estimate the integral I_1. Using lemma A.6.1, we can write

$$I_1 = \int_0^T \left[F_1(s) + F_2(s)\right] ds,$$

where

$$F_1 = F_1(UV_1, \dots UV_m, UW, U)$$
$$F_2 = F_2(UV_1, \dots, UV_m, UW, V_1, \dots, V_m, W).$$

Observe that UW and all the UV_j's' are first integrals. Let $J_1 = \int F_1$ and $J_2 = \int F_2$. Then we have

$$J_1 = \int_0^T F_1(UV, UW, U(s))ds$$

$$= \int_0^{\ln\left(\frac{W_o}{W}\right)} F_1(UV, UW, Ue^{-s})ds.$$

Let us make the change of variables $y = e^{-s}$, we obtain

$$J_1 = -\int_1^{\frac{W}{W_o}} F_1(UV, UW, Uy)\frac{dy}{y}.$$

We expand the function F_1 in power of y that is

$$F_1(UV, UW, Uy) = F_1(UV, UW, 0) + O(y).$$

Then we have

$$J_1 = -\int_1^{\frac{W}{W_o}} \alpha_1 \frac{dy}{y} + \tilde{F}_1,$$

where $\alpha_1 = \alpha_1(UV, UW)$ and $\tilde{F}_1 = \tilde{F}_1(UV, UW, Uy(T))$ is some (unknown) C^ℓ function. Finally we get

$$J_1 = \alpha_1 \ln\left(\frac{W_0}{W}\right) + \tilde{F}_1\left(UV, UW, U\frac{W}{W_0}\right).$$

The function $\tilde{F}_1$ is C^ℓ but unknown, and W_0 is a fixed positive constant, then we can simplify the notation of $\tilde{F}_1$ as $\tilde{F}_1 = \tilde{F}_1(UV, UW, W)$.

Next we have

$$\begin{aligned} J_2 &= \int_0^T F_2(UV, UW, V(s), W(s))ds \\ &= \int_0^{\ln\left(\frac{W_o}{W}\right)} F_2(UV, UW, V_1 e^{\beta_1 s}, \dots, V_m e^{\beta_m s}, W e^{\gamma s})ds. \end{aligned}$$

Let us make the change of variables $y = e^s$. Then we obtain

$$J_2 = \int_1^{\frac{W_o}{W}} F_2(UV, UW, V_1 y^{\beta_1}, \dots, V_m y^{\beta_m}, W y^\gamma)\frac{dy}{y}.$$

As above, we expand in powers of y, that is

$$F_2 = \alpha_2 + O(y),$$

and then we have

$$J_2 = \alpha_2 \ln\left(\frac{W_0}{W}\right) + \tilde{F}_2,$$

where $\alpha_2 = \alpha_2(UV, UW)$, $F_2 = F_2(UV, UW, \mu_i)$ is a C^ℓ function with $\mu_i = V_i W^{-1}$ for all $i \in [1, m]$. Recall that since $v_i \in O(w^{\beta_i/\gamma})$ we also have that $V \in O(W)$.

Now we can write the integral I_1 as

$$\begin{aligned} I_1 &= J_1 + J_2 \\ &= \alpha_1 \ln\left(\frac{W_0}{W}\right) + \tilde{F}_1 + \alpha_2 \ln\left(\frac{W_0}{W}\right) + \tilde{F}_2 \\ &= \alpha \ln\left(\frac{W_0}{W}\right) + \tilde{F}, \end{aligned}$$

where $\alpha = \alpha(UV, UW)$ and $\tilde{F} = \tilde{F}(UV, UW, W, \mu_i)$ are C^ℓ functions. FInally we write $\tilde{z}$ in the original coordinates as follows

$$\begin{aligned} \tilde{z} &= z\exp(I_0 + I_1) \\ &= z\exp\left[-\frac{\Lambda}{\gamma w}\left(1 - \frac{w}{w_o}\right) + \frac{1}{\gamma}\alpha \ln\left(\frac{w_o}{w}\right) + \tilde{F}\right] \\ &= z\exp\left[-\frac{\Lambda}{\gamma w}\left(1 + \tilde{\alpha} w \ln(w) + w\tilde{G}\right)\right], \end{aligned}$$

where $\tilde{\alpha} = \tilde{\alpha}(uv_i^{1/\beta_i}, uw^{1/\gamma})$ and $\tilde{G} = \tilde{G}(uv_i^{1/\beta_i}, uw^{1/\gamma}, w^{1/\gamma}, \mu_i)$ are C^ℓ functions with $\mu_i = v_i w^{-\beta_i/\gamma}$.

The transition $\Pi^{\pm v_j}$

In this case the time of integration is given by $T = \ln\left(\frac{\eta_j}{v_j}\right)^{1/\beta_j}$. Such a time of integration is obtained from the equation $v_j' = \beta_j v_j$. The we have

$$\begin{aligned}
\tilde{u} &= u\left(\frac{v_j}{\eta_j}\right)^{1/\beta} \\
\tilde{v}_i &= v_i\left(\frac{\eta_j}{v_j}\right)^{\beta_i/\beta_j} \\
\tilde{w} &= w\left(\frac{\eta_j}{v_j}\right)^{\gamma/\beta_j}.
\end{aligned}$$

Then, only remains to compute $\tilde{z}$. Following similar arguments as for the transition Π^w we get in this case

$$\tilde{z} = z\exp\left[-\frac{\Lambda}{\gamma w}\left(1 + \tilde{\alpha}' w\ln(v_j) + w\tilde{G}'\right)\right],$$

where now

$$\begin{aligned}
\tilde{\alpha}' &= \tilde{\alpha}'(uv_i^{1/\beta_i}, uw^{1/\gamma}) \\
\tilde{G}' &= \tilde{G}'(uv_i^{1/\beta_i}, uw^{1/\gamma}, \mu_w, \mu_i)
\end{aligned}$$

are C^ℓ functions with $\mu_w = wv_j^{-\gamma/\beta_j}$ and $\mu_i = v_i v_j^{-\beta_i/\beta_j}$. □

Saddle 2

In this case we investigate the transitions of a vector field of the form

$$Y : \begin{cases} u' &= wu \\ v_j' &= -\beta_j w v_j \\ w' &= -\gamma w^2 \\ z' &= -gz, \end{cases} \tag{A.8}$$

where the coefficients β_j, for all $j = 1, \ldots, m$, and γ are positive. We assume that $u \in \mathbb{R}^+$. Observe that now, in contrast with case 1, we only have one expanding direction. This makes the study of the transition easier. Let $u_{\text{out}} > 0$ In this case, it is more convenient to study a transition

$$\Pi^u : \Sigma_{\text{en}} \to \Sigma_{\text{ex}},$$

where we let Σ_{en} be any codimension 1 subset of $\mathbb{R}^{m+3}$ given by setting one of the coordinates (v, w) to a constant and with $u < u_{\text{out}}$, and

$$\Sigma_{\text{ex}} = \{(u, \tilde{v}, \tilde{w}, \tilde{z}) \mid \tilde{u} = u_{\text{out}}\}.$$

Proposition A.6.6. *Consider the vector field Y given by* (A.8) *and let Σ_{en}, Σ_{ex} and Π^u be as above. Then*

$$\begin{aligned}
\tilde{v}_i &= v_i\left(\frac{u}{u_{out}}\right)^{\beta_i} \\
\tilde{w} &= w\left(\frac{u}{u_{out}}\right)^{\gamma} \\
\tilde{z} &= z\exp\left[-\frac{\Lambda}{\gamma w}\left(\left(\frac{u_{out}}{u}\right)^{\gamma} - 1 + \alpha w\ln(u) + w\tilde{F}\right)\right]
\end{aligned}$$

where $\alpha = \alpha(u^{\beta_i} v_i, u^\gamma w)$ and $\tilde{F} = \tilde{F}(u^{\beta_i} v_i, u^\gamma w, u)$ are C^ℓ functions.

Proof of proposition A.6.6. We have that the time of integration is $T = \ln\left(\frac{u_{\text{out}}}{u}\right)$. We then immediately have

$$\begin{aligned}\tilde{v}_i &= v_i \left(\frac{u}{u_{\text{out}}}\right)^{\beta_i} \\ \tilde{w} &= w \left(\frac{u}{u_{\text{out}}}\right)^{\gamma}.\end{aligned}$$

It only rests to compute $\tilde{z}$. Following very similar arguments as in case 1 we have

$$\tilde{z} = z \exp\left[-\frac{\Lambda}{\gamma w}\left(\left(\frac{u_{\text{out}}}{u}\right)^{\gamma} - 1 + \alpha w \ln(u) + w\tilde{F}\right)\right],$$

where $\alpha = \alpha(u^{\beta_i} v_i, u^{\gamma} w)$ and $\tilde{F} = \tilde{F}(u^{\beta_i} v_i, u^{\gamma} w, u)$ are C^{ℓ} admissible functions. □

Summary

In this thesis we study Constrained Differential Equations (CDEs) and Slow Fast Systems (SFSs). A SFS is a singularly perturbed ordinary differential equation given by

$$\begin{aligned} \dot{x} &= f(x, z, \varepsilon) \\ \varepsilon \dot{z} &= g(x, z, \varepsilon), \end{aligned} \tag{A.9}$$

while a CDE is obtained by taking the limit $\varepsilon \to 0$ of (A.9). A motivation to study SFSs is that several real life phenomena occur in two or more time scales, and thus can often be modeled by SFSs. For example, see in Chapter 1 the Zeeman's heartbeat and nerve impulse models. Our main goal is to contribute to the understanding of the local dynamics of SFSs and of CDEs, and to explore their mutual relationship. To accomplish this, we rely not only on classical techniques from dynamical systems, but new methods are being introduced.

Regarding CDEs, our primary interest is their topological classification, where elementary catastrophes (see Thom's list in appendix A.1) play an important role. This was shown by Takens [39] who classified those CDEs associated to the Fold (A_2) and the Cusp (A_3) catastrophes.

As our first contribution, in Chapter 2, we extend Takens's results by classifying CDEs related to the A_k ($k \geq 2$), and the $D^{\pm 4}$ (hyperbolic and elliptic umbilical) catastrophes, see [4, 10]. Further interest is given to the CDEs linked to the A_k catastrophes, which we call A_k-CDEs. Afterwards in Chapters 3 and 4, we embed these A_k-CDEs in the theory of slow fast systems. More precisely, we study A_k-SFSs which are ε-perturbations of the A_k-CDEs. The analysis performed is as follows: first, a formal normal form of A_k-SFSs is proposed. Next, the local flow of the A_k-SFSs is studied and the usefulness of this formal normal form is exploited. Such a study follows the Geometric Desingularization (or blow-up) method, see [8, 13, 14, 26]. In this way, we are able to generalize results dealing with SFSs near fold and cusp points like in [8, 26].

From the results obtained, many open problems arise, for example:

1. To study slow fast systems with more complicated dynamics, for example: 1) with two fast directions, and 2) with "canards".

2. To obtain a smooth and/or an analytic normal form of A_k-SFS.

An extensive digression of these and more open problems is given in Chapter 5.

Samenvatting

In dit proefschrift bestuderen we Constrained Differential Equations (CDEs) en Slow Fast Systems (SFSs). Een SFS is een singulier gestoorde gewone differentiaalvergelijking gegeven door

$$\begin{aligned} \dot{x} &= f(x, z, \varepsilon) \\ \varepsilon \dot{z} &= g(x, z, \varepsilon). \end{aligned}$$

Door het nemen van de limiet $\varepsilon \to 0$ ontstaat een CDE. Een van de motivaties om een SFS te bestuderen is het feit dat veel natuurlijke fenomenen plaatsvinden op verschillende tijdschalen. De modellen van Zeeman voor de hartslag en zenuwimpulsen in Hoofdstuk 1 vormen een voorbeeld hiervan. Het belangrijkste doel van dit proefschrift is het verkrijgen van inzicht in de lokale dynamica van zowel CDEs als SFSs en het verkennen van de relatie tussen deze twee. Hierbij maken we niet alleen gebruik van reeds bestaande technieken uit de theorie van Dynamische Systemen, maar ontwikkelen we ook nieuwe methoden.

Met betrekking tot CDEs is ons voornaamste doel de topologische classificatie van deze systemen, waarin elementaire catastrofen (zie de lijst van Thom in appendix A.1) een belangrijk rol vervullen. Dit is aangetoond door Takens, die de CDEs behorende bij de Vouw (Fold, A_2) en de Doorn (Cusp, A_3) catastrofen heeft geclassificeerd.

Als onze eerste bijdrage, in Hoofdstuk 2, breiden we de resultaten van Takens uit door middel van de catastrofen van de typen A_k ($k \geq 2$), en $D^{\pm 4}$ (hyperbolic and elliptic umbilical catastrophes), zie [4, 10]. De CDEs gerelateerd aan de A_k noemen we A_k-CDEs. In Hoofdstuk 3 en 4 bestuderen we de rol van deze normaalvormen in de context van een SFS. In het bijzonder beschouwen we A_k-SFSs gegeven door ε-storingen van de A_k-CDEs. De analyse geschiedt langs de volgende weg. Als eerste wordt een normaalvorm gegeven voor een A_k-SFS. Vervolgens wordt de lokale oplossingsstroming van deze normaalvorm bestudeerd. Hierbij maken we gebruik van een techniek genaamd "meetkundige desingularisatie", zie [8, 13, 14, 26]. Op deze manier zijn we in staat om resultaten voor een SFS nabij een Vouw en een Doorn te generaliseren zoals in [8, 26].

We noemen hier twee richtingen waarin het vervolgonderzoek zich kan ontwikkelen.

1. Een studie naar SFSs met meer gecompliceerde dynamica, bijvoorbeeld: met twee snelle richtingen, en met “canards”.

2. Het verkrijgen van een gladde en/of analytische normaalvorm voor een A_k-SFS.

Hoofdstuk 5 van dit proefschrift is gewijd aan een uitgebreide beschrijving van deze en andere open vragen.

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Bibliography

[1] V.I. Arnold. *Ordinary differential equations*, volume 17. The MIT Press, 1978.

[2] V.I. Arnold. *Geometrical Methods in the Theory of Ordinary Differential Equations*, volume 17. Springer, 1988.

[3] V.I. Arnold, V.V. Goryunov, O.V. Lyashko, A. Iacob, and V.A. Vasil'ev. *Singularity Theory: I*. Number VI in Encyclopaedia of mathematical sciences. Springer Berlin Heidelberg, 1998.

[4] V.I. Arnold, S.M. Gusein-Zade, and A.N. Varchenko. *Singularities of Differentiable Maps, Volume I*, volume 17. Birkhäuser, 1985.

[5] J.M. Boardman. Singularities of differentiable maps. *Publ. Math., Inst. Hautes Etud. Sci.*, 33:21–57, 1967.

[6] P. Bonckaert. Partially hyperbolic fixed points with contraints. *Transactions of the American mathematical society*, 33, 1996.

[7] P. Bonckaert. Conjugacy of vector fields respecting additional properties. *Journal of dynamics and control systems*, 3:419–432, 1997.

[8] H. W. Broer, T. J. Kaper, and M. Krupa. Geometric Desingularization of a Cusp Singularity in Slow–Fast Systems with Applications to Zeeman's Examples. *J. Dyn. Diff. Equat.*, 2013.

[9] Henk Broer and Floris Takens. *Dynamical Systems and Chaos*, volume 172 of *Applied Mathematical Sciences*. Springer, 2011.

[10] Th. Bröcker. *Differentiable Germs and Catastrophes*, volume 17 of *Lecture Note Series*. Cambridge University Press, 1975.

[11] K. Chen. Equivalence and decomposition of vector fields about an elementary critical point. *Amer. J. Math.*, 85, 1963.

[12] P. de Maesschalck and F. Dumortier. Time analysis and entry–exit relation near planar turning points. *Journal of Differential Equations*, 22(3–4):165 – 206, 2005.

[13] F. Dumortier and R. Roussarie. Geometric singular perturbation theory beyond normal hyperbolicity. In C.K.R.T. Jones and A. Khibnik, editors, *Multiple-Time-Scale Dynamical Systems*, volume 122, pages 29–63. Springer, 2001.

[14] Freddy Dumortier and Robert Roussarie. *Canard Cycles and Center Manifolds*, volume 121. American Mathematical Society, 1996.

[15] Freddy Dumortier, Robert Roussarie, and Peter de Maelsschack. *Canard cycles from the birth to the transition*. In preparation.

[16] N. Fenichel. Geometric singular perturbation theory. *JDE*, pages 53–98, 1979.

[17] M. Golubinsky and V. Guillemin. *Stable Mappings and their Singularities*. Springer-Verlag, 1973.

[18] M. Golubitsky. An introduction to catastrophe theory and its applications. *SIAM Review*, 20(2):352–387, 1978.

[19] John Guckenheimer and Philip Holmes. *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*. Springer-Verlag, 1983.

[20] A. L. Hodgkin and A. F. Huxley. A quantitative description of ion currents and its applications to conduction and excitation in nerve membranes. *J. Physiol. (Lond.)*, pages 500–544, 1952.

[21] H. Jardón-Kojakhmetov. Formal normal form of A_k slow fast systems. *Submitted*.

[22] H. Jardón-Kojakhmetov and Henk W. Broer. Polynomial normal forms of constrained differential equations with three parameters. *Journal of Differential Equations*, 257(4):1012–1055, 2014.

[23] H. Jardón-Kojakhmetov, Henk W. Broer, and R. Roussarie. Analysis of a slow fast system near a cusp singularity. *Submitted*.

[24] C. K. R. T. Jones. Geometric singular perturbation theory. In *Dynamical Systems*, LNM 1609, pages 44–120. Springer-Verlag, 1995.

[25] Tasso J. Kaper. An introduction to geometric methods and dynamical systems theory for singular peturbation problems. In *Symposia in Applied Mathematics*, volume 56, pages 85–131. AMS, 1999.

[26] M. Krupa and P. Szmolyan. Extending geometric singular perturbation theory to non hyperbolic points: fold and canard points in two dimensions. *SIAM J. Math. Anal.*, 33:286–314, 2001.

[27] M. Krupa and P. Szmolyan. Geometric analysis of the singularly perturbed planar fold. In *Multiple-Time.Scale Dynamical Systems*, LNM 1609, pages 89–116. Springer-Verlag, 2001.

[28] Martin Krupa and Martin Wechselberger. Local analysis near a folded saddle-node singularity. *Journal of Differential Equations*, 248(12):2841 – 2888, 2010.

[29] Eric Lombardi and Laurent Stolovitch. Normal forms of analytic perturbations of quasihomogeneous vector fields: rigidity, invariant analytic sets and exponentially small approximation. *Ann. Sci. Éc. Norm. Supér.*, 43(4), 2010.

[30] E.F. Mishchenko, Y.U. Kolesov, A. Yu. Kolesov, and N. Kh. Rozov. *Asymptotic methods in singularly perturbed systems*. Consultants Bureau, New York and London, 1994.

[31] E.F. Mishchenko and N. Kh. Rozov. *Differential equations with small parameters and relaxation oscillations*. Plenum, New York, 1980.

[32] T. Poston and I. Stewart. *Catastrophe Theory and its Applications*. Pitman, 1978.

[33] M. Spivak. *Calculos on manifolds*. Westview Press, 1965.

[34] S. Sternberg. On the structure of local homeomorphisms of Euclidean n-space. *Amer. J. Math*, 80, 1958.

[35] Ian Stewart. Elementary catastrophe theory. *IEEE Transactions on Circuits and Systems*, CAS-30(8):578–586, 1983.

[36] Peter Szmolyan and Martin Wechselberger. Canards in $\mathbb{R}^3$. *Journal of Differential Equations*, 177(2):419 – 453, 2001.

[37] F. Takens. Partially hyperbolic fixed points. *Topology*, 10:133–147, 1970.

[38] F. Takens. Singularities of vector fields. *Publications mathématiques de l'I.H.É.S*, pages 48–100, 1974.

[39] F. Takens. Constrained equations: a study of implicit differential equations and their discontinuous solutions. In *Structural Stability, the Theory of Catastrophes, and Applications in the Sciences*, LNM 525, pages 134–234. Springer-Verlag, 1976.

[40] R. Thom. L'evolution temporelle des catastrophes. *Applications of global analysis I*.

[41] R. Thom. Ensembles et morphismes stratifies. *B.A.M.S.*, 75:240–284, 1969.

[42] R. Thom. *Structural stability and morphogenesis. An outline of a general theory of models*. Addison-Wesley, second edition edition, 1986.

[43] B. van der Pol and J. van der Mark. The heartbeat considered as a relaxation oscillation, and an electrical model of the heart. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, Ser.7,6:763–775, 1928.

[44] Adelaida B. Vasil'eva, Valentin F. Butuzov, and Leonid V. Kalachev. *The Boundary Function Method for Singular Perturbation Problems*. Studies in Applied and Numerical Mathematics. SIAM, 1995.

[45] Ferdinand Verhulst. The dynamics of slow manifolds. *J. Indones. Math. Soc.*, 13:73–90.

[46] Ferdinand Verhulst. *Methods and Applications of Singular Perturbations*, volume 50 of *Texts in Applied Mathematics*. Springer, 2005.

[47] Ferdinand Verhulst. Singular perturbation methods for slow–fast dynamics. *Nonlinear Dynamics*, 50(4):747–753, 2007.

[48] E.C. Zeeman. Differential equations for the heart beat and nerve impulse. In *Towards a theoretical biology*, volume 4, pages 8–67. Edinburgh University Press.