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ON THE CHARACTERIZATION OF HAMILTONIAN SYSTEMS VIA DIFFERENTIAL INPUT-OUTPUT EQUATIONS

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Abstract

This work extends a result of Crouch and Lamnabhi which characterized Hamiltonian systems described by second order SISO input-output differential equations. A general formalism is given which defines the adjoint variational equations from the input-output differential equation representation. This is then used to characterize Hamiltonian systems by employing the general results of Crouch and van der Schaft: Hamiltonian systems are those systems for which the variational and adjoint variational systems coincide.

1 The adjoint system

We consider state-space systems which may be written in the form

$$\Sigma_s \quad \begin{aligned} \dot{x} &= f(x, u), & y &= h(x), \\ u &\in \mathbf{R}^m, & y &\in \mathbf{R}^m, \end{aligned} \quad (1)$$

and corresponding input-output representations

$$\Sigma_{i/o} \quad F(y, \dot{y}, \dots, y^{(N)}, u, \dot{u}, \dots, u^{(N-1)}) = 0. \quad (2)$$

If the system (1) is minimal in the sense of Crouch and van der Schaft [2], the corresponding representation (2) will also be called a minimal representation. Note that we do not insist here on the relationship between state-space representations (1) and input-output differential equations representations (2). We assume that all conditions are met for obtaining one representation from the other representation. See Sontag

and Wang [9, 10] for a complete solution to this problem.

A variational system Σ_s^v , about a given trajectory of Σ_s , is defined in the usual way, see [2]. We write the input-output map of the variational system in the form

$$y_v(t) = \int_0^t W(t, \sigma, u(\sigma)) u_v(\sigma) d\sigma$$

where $y_v \in \mathbf{R}^m$ is the variation in y due to a variation $u_v \in \mathbf{R}^m$ in u . (We assume $y_v(0) = 0$ and $u_v(0) = 0$)

The input-output variational systems $\Sigma_{i/o}^v$ are defined by the system of equations

$$\begin{aligned} \Sigma_{i/o}^v : \quad & \frac{\partial F}{\partial y} y_v + \frac{\partial F}{\partial \dot{y}} \dot{y}_v + \dots + \frac{\partial F}{\partial y^{(N)}} y_v^{(N)} + \\ & \frac{\partial F}{\partial u} u_v + \frac{\partial F}{\partial \dot{u}} \dot{u}_v + \dots + \frac{\partial F}{\partial u^{(N-1)}} u_v^{(N-1)} = 0, \end{aligned} \quad (3)$$

along solutions of (2). Note that (3) results from differentiation of an one-parameter family of solutions to (2). In [11] conditions are derived under which the linearization of the system (2) is, indeed, (3).

Definition(See also [8]): Consider a variational system (3). The adjoint variational system $\Sigma_{i/o}^a$ consists of the following set of input-output pairs (u_a, y_a) : there exists a function

$$Q(y, \dot{y}, \dots, u, \dot{u}, \dots, y_a, \dot{y}_a, \dots, u_a, \dot{u}_a, \dots, y_v, \dot{y}_v, \dots, u_v, \dot{u}_v, \dots)$$

such that for all $t \in \mathbf{R}$,

$$y_a^T u_v - u_a^T y_v = \frac{d}{dt} Q$$

for all input output pairs (u_v, y_v) which are solutions of $\Sigma_{i/o}^v$.

An explicit expression for Q is provided by the state space representation of Σ_s . Indeed, define the variational and adjoint variational systems as in [2, 3]. Then

$$y_a^T u_v - u_a^T y_v = \frac{d}{dt} p^T v,$$

where p is the state of the adjoint variational system and v is the state of the variational system. Denote the variational and adjoint variational system along a particular trajectory of Σ_s by the equations:

$$\Sigma_s^v : \begin{cases} \dot{v} &= F(t)v + G(t)u_v \\ y_v &= H(t)v \end{cases} \quad (4)$$

$$\Sigma_s^a : \begin{cases} \dot{p} &= -F^T(t)p + H^T(t)u_a \\ y_a &= G^T(t)p \end{cases}$$

It is well known that Σ_s^v is *observable* if

$$\text{rank} \begin{bmatrix} H(t) \\ H(t)F(t) + \dot{H}(t) \\ \vdots \end{bmatrix} = n \text{ and that } \Sigma_s^a \text{ is}$$

$$\text{observable if rank} \begin{bmatrix} G^T(t) \\ -G^T(t)F^T(t) + \dot{G}^T(t) \\ \vdots \end{bmatrix} =$$

n which is equivalent to the fact that Σ_s^v is *controllable*. If this holds we will see that p and v can be expressed as functions of (u_a, y_a, \dots) , respectively (u_v, y_v, \dots) , and thus

$$p^T v = Q(y, \dot{y}, \dots, u, \dot{u}, \dots, y_a, \dot{y}_a, \dots, u_a, \dot{u}_a, \dots, y_v, \dot{y}_v, \dots, u_v, \dot{u}_v, \dots).$$

On the other hand, direct calculation yields:

$$y_a^T(t)u_v(t) - u_a^T(t)y_v(t) = \frac{d}{dt} \left[- \int_{-\infty}^t \int_{-\infty}^t u_a^T(\tau)H(\tau)\Phi(\tau,\sigma)G(\sigma)u_v(\sigma)d\sigma d\tau \right]$$

where $H(\tau)\Phi(\tau,\sigma)G(\sigma)$ is the impulse response of the variational system. This suggests that Q is equal to

$$- \int_{-\infty}^t \int_{-\infty}^t u_a^T(\tau)H(\tau)\Phi(\tau,\sigma)G(\sigma)u_v(\sigma)d\sigma d\tau. \quad (5)$$

In the following we are aiming to express (5) directly as function of

$$(y, \dot{y}, \dots, u, \dot{u}, \dots, y_a, \dot{y}_a, \dots, u_a, \dot{u}_a, \dots, y_v, \dot{y}_v, \dots, u_v, \dot{u}_v, \dots).$$

The most direct way of obtaining Q and an explicit representation of the adjoint variational system, in input-output differential equation form, is given by the following method based on partial integration. For simplicity we will only consider the case $N = 2$ in the rest of the paper but no intrinsic difficulty is presented by the general case. Thus we have:

$$\Sigma_{i/o} : F(y, \dot{y}, \ddot{y}, u, \dot{u}) = 0, \quad u, y \in \mathbf{R}^m \quad (6)$$

and

$$\Sigma_{i/o}^v : Ay_v + B\dot{y}_v + C\ddot{y}_v + Du_v + E\dot{u}_v = 0, \quad (7)$$

where

$y_v = (y_v^1, y_v^2, \dots, y_v^m)^T$, $u_v = (u_v^1, u_v^2, \dots, u_v^m)^T$, and the components of the $m \times m$ matrices A, B, C, D and E are given by

$$A_{ik} := \frac{\partial F_i}{\partial y_k}, \quad B_{ik} := \frac{\partial F_i}{\partial \dot{y}_k}, \quad C_{ik} := \frac{\partial F_i}{\partial \ddot{y}_k}, \\ D_{ik} := \frac{\partial F_i}{\partial u_k}, \quad E_{ik} := \frac{\partial F_i}{\partial \dot{u}_k}.$$

We shall always assume that the matrix C is invertible. Using summation convention, two consecutive partial integrations yield,

$$0 = \int_{t_1}^{t_2} \xi^i [A_{ik} \dot{y}_v^k + B_{ik} \ddot{y}_v^k + C_{ik} \dddot{y}_v^k + D_{ik} u_v^k + E_{ik} \dot{u}_v^k] dt \\ = \int_{t_1}^{t_2} [\xi^i A_{ik} \dot{y}_v^k - \dot{\xi}^i B_{ik} \dot{y}_v^k + \ddot{\xi}^i C_{ik} \dot{y}_v^k + \xi^i D_{ik} u_v^k - \dot{\xi}^i E_{ik} u_v^k] dt + [\xi^i B_{ik} \dot{y}_v^k + \xi^i C_{ik} \dot{y}_v^k + \xi^i E_{ik} u_v^k - \dot{\xi}^i C_{ik} \dot{y}_v^k]_{t_1}^{t_2}$$

Thus the adjoint variational systems are

$$\Sigma_{i/o}^a : \begin{cases} \dot{u}_a^k &= \xi^i A_{ik} - \dot{\xi}^i B_{ik} + \ddot{\xi}^i C_{ik} \\ y_a^k &= -\xi^i D_{ik} + \dot{\xi}^i E_{ik}, \end{cases} \quad (8)$$

$k = 1, \dots, m$ while Q is given by the term between the brackets $[\cdot]$.

In order to express Q as a function of y_a, u_a, y_v and u_v , we must solve (8) for ξ . The assumed invertibility of C ensures that we may write

$$\xi^j = P_{jk}u_a^k + T_{jk}\dot{y}_a^k + R_{jk}y_a^k, \quad j = 1, \dots, m$$

which upon substituting from (8) yields the defining equation for ξ

$$\begin{aligned} \xi^j = & \sum_{k=1}^m P_{jk}(\xi^i A_{ik} - \overline{\xi^i B_{ik}} + \overline{\xi^i C_{ik}}) + \\ & \overline{T_{jk}(-\xi^i D_{ik} + \overline{\xi^i E_{ik}})} + \\ & \overline{R_{jk}(-\xi^i D_{ik} + \overline{\xi^i E_{ik}})} \end{aligned} \quad (9)$$

which is reminiscent of a Bezout identity. We go even further by relating the solution of the equations to controllability and observability conditions.

Lemma 1: *Solvability* of the Bezout identity (9) is equivalent to *observability* of the adjoint variational system (8), and thus to controllability of the variational system.

Proof: Let us consider again (7) where without loss of generality we may assume $C = I_m$. This equation can also be written

$$\ddot{y}_v + \overline{B} \dot{y}_v + (A - \dot{B})y_v + \overline{E}u_v + (D - \dot{E})u_v = 0 \quad (10)$$

The associated variational system is then

$$\Sigma_s^v : \begin{cases} \dot{v} &= F(t)v + G(t)u_v \\ y_v &= H(t)v \end{cases} \quad (11)$$

where v is a $2m$ -dimensional vector

$$(v_1, v_2)^T = (y_v, \dot{y}_v + By_v + Eu_v)^T$$

and where

$$\begin{aligned} F(t) &= \begin{bmatrix} -B & I_m \\ \dot{B} - A & 0 \end{bmatrix} \\ G(t) &= \begin{bmatrix} -E \\ \dot{E} - D \end{bmatrix} \quad \text{and} \quad H(t) = (I_m, 0). \end{aligned}$$

Moreover, we have

$$\begin{aligned} p^T v &= (p_1, p_2)(y_v, \dot{y}_v + By_v + Eu_v)^T \\ &= \xi^T By_v + \xi^T \dot{y}_v + \xi^T Eu_v - \dot{\xi}^T y_v \\ &\quad \text{and} \quad \xi = Pu_a + T\dot{y}_a + Ry_a. \end{aligned}$$

Therefore, if we can solve equations (9) for the matrices P, T and R then $p = (p_1, p_2)^T = (-\dot{\xi}, \xi)^T$ can be expressed as function of (u_a, y_a) .

Now, by equating coefficients of derivative of ξ , solvability of the Bezout identity (9) is equivalent to the existence of $m \times m$ -matrices P, T and R such that

$$\begin{aligned} P(A^T - \dot{B}^T) + T(-\dot{D}^T + \ddot{E}^T) \\ + R(-D^T + \dot{E}^T) = I_m \end{aligned}$$

$$\begin{aligned} P(-B^T) + T(-D^T + 2\dot{E}^T) + RE^T = 0 \\ P + TE^T = 0 \end{aligned}$$

Using the third equality gives

$$\begin{aligned} T(-E^T(A^T - \dot{B}^T) + (-\dot{D}^T + \ddot{E}^T)) \\ + R(-D^T + \dot{E}^T) = I_m \end{aligned}$$

and

$$T(E^T B^T + (-D^T + 2\dot{E}^T)) + RE^T = 0.$$

The matrices T and R exist if and only if

$$\text{rank} \begin{bmatrix} -E^T(A^T - \dot{B}^T) + & E^T B^T + \\ (-\dot{D}^T + \ddot{E}^T) & (-D^T + 2\dot{E}^T) \\ -D^T + \dot{E}^T & E^T \end{bmatrix} = 2m,$$

but this condition is nothing else than the controllability condition of the variational system (11).

We can summarize the previous results in the following theorem

Theorem 1: Consider a control system (2). If the variational system given by (4) is controllable then the variables (u_a, y_a) of the adjoint variational system are given by (8) after solving the Bezout identity (9) for ξ . Moreover the state variables p and v can be expressed as functions of (u_a, y_a) , respectively (u_v, y_v) and their derivatives.

2 Hamiltonian systems

In the next section we will use the self-adjointness property of the variational system for Hamiltonian systems, i.e. the fact that the adjoint variational systems and the variational systems are identical. This will allow us to characterize Hamiltonian systems from a set of input-output differential equations.

Note that for a system described by a set of equations (6) with $m = 1$ the conditions under which this system represents a Hamiltonian system have been found already in an earlier paper by Crouch and Lamnabhi-Lagarrigue [1] i.e.,

$$\frac{\partial F}{\partial \dot{u}} = 0, \\ \frac{\partial F}{\partial u} \left[\frac{d}{dt} \frac{\partial F}{\partial \dot{y}} - \frac{\partial F}{\partial y} \right] = \frac{\partial F}{\partial \dot{y}} \frac{d}{dt} \frac{\partial F}{\partial u}.$$

for every point $y, \dot{y}, \dots, u, \dot{u}, \dots$ such that $F(y, \dot{y}, \dots, u, \dot{u}, \dots) = 0$.

Self-adjointness in the second order case (6) follows if the equation (7) for (u_v, y_v) is satisfied by the expressions (8) for (u_a, y_a) :

$$\sum_{k=1}^m [-A_{jk} D_{ik} \xi^i + A_{jk} \overline{E_{ik} \xi^i} - B_{jk} \overline{D_{ik} \xi^i} \\ + B_{jk} \overline{E_{ik} \xi^i} - C_{jk} \overline{D_{ik} \xi^i} + C_{jk} (E_{ik} \xi^i)^{(3)} \\ + D_{jk} A_{ik} \xi^i - D_{jk} \overline{B_{ik} \xi^i} + D_{jk} \overline{C_{ik} \xi^i} \\ + E_{jk} \overline{A_{ik} \xi^i} - E_{jk} \overline{B_{ik} \xi^i} + E_{jk} \overline{C_{ik} \xi^i}] = 0$$

Hence we have the following theorem,

Theorem 2: Consider a minimal system (6). Then it is Hamiltonian if and only if the following conditions hold

- (i). $CE^T + EC^T = 0$,
- (ii). $BE^T - CD^T + 3C\dot{E}^T + DC^T - EB^T + 3E\dot{C}^T = 0$,
- (iii). $AE^T - BD^T + 2B\dot{E}^T - 2C\dot{D}^T + 3C\ddot{E}^T - DB^T + 2D\dot{C}^T + EA^T - 2E\dot{B}^T + 3E\ddot{C}^T = 0$,
- (iv). $-AD^T + A\dot{E}^T - B\dot{D}^T + B\ddot{E}^T - C\ddot{D}^T + CE^{(3)T} + DA^T - D\dot{B}^T + D\dot{C}^T + E\dot{A}^T - E\ddot{B}^T + EC^{(3)T} = 0$.

at every point $y, \dot{y}, \dots, u, \dot{u}, \dots$ such that $F(y, \dot{y}, \dots, u, \dot{u}, \dots) = 0$.

For a general problem (2) of order N in the time derivative of y , it is not difficult to see that the number of conditions is $2N$.

This result assumes controllability of the variational system (or observability of the adjoint variational system) and also the existence of a state-space representation. We believe that these

conditions are not necessary. Indeed, the equation we have to solve for ξ in (9) using the above method is of the form

$$\left[D(t, \frac{d}{dt}) \quad N(t, \frac{d}{dt}) \right] \begin{bmatrix} w(t) \\ \xi(t) \end{bmatrix} = 0. \quad (12)$$

But we know that for a simpler structure of it:

$$\left[D(s) \quad N(s) \right] \begin{bmatrix} w \\ \xi \end{bmatrix} = 0$$

there exists (see for instance [5, 6]) a unimodular matrix $U(s)$ such that

$$U(s) \left[D(s) \quad N(s) \right] \begin{bmatrix} w \\ \xi \end{bmatrix} = \\ \begin{bmatrix} \tilde{D}_1(s) & \tilde{N}_1(s) \\ \tilde{D}_2(s) & 0 \end{bmatrix} \begin{bmatrix} w \\ \xi \end{bmatrix}$$

where \tilde{N}_1 is of full row rank. Therefore there is no more condition for the elimination of ξ . The solution is given by

$$\tilde{D}_2 w = 0.$$

However no such result exists for the general form (12). This will be a theme of a future investigation, perhaps using results from [4].

Now let us sketch a second approach in order to find the conditions of theorem 2. Self-adjointness of the input-output map may be expressed as the following condition

$$H(t)\Phi(t, \sigma)G(\sigma) = -G^T(t)\Phi(\sigma, t)H(\sigma), \quad (13)$$

where $F(t), G(t)$ and $H(t)$ are defined in (11), and

$$\frac{d}{dt}\phi(t, \sigma) = F(t)\Phi(t, \sigma), \quad \text{with } \Phi(\sigma, \sigma) = I_m.$$

Now the conditions of theorem 2 can be also obtained by repeatedly differentiating with respect to t the equality (13) and setting $t = \sigma$. Indeed let us consider equation (10) corresponding to the input-output representation of the form (7) with $C = I_m$. Setting $t = \sigma$ leads to

$$HG = -G^T H^T,$$

which implies that

$$E = -E^T. \quad (14)$$

Now if $C \neq I_m$, and if $\det C \neq 0$, (7) may be written as equation (10) where A, B, D , and E have been replaced respectively by $C^{-1}A, C^{-1}B, C^{-1}D$, and $C^{-1}E$. Therefore equation (14) becomes

$$C^{-1}E = -E^T C^{-1T} \quad \text{or} \quad EC^T = -CE^T$$

which the first condition. We can repeat exactly the same computation using the derivative of equation (13) and by setting $t = \sigma$. Assuming first that $C = I_m$, we obtain

$$\begin{bmatrix} I_m & 0 \\ -E^T & \dot{E}^T - D^T \end{bmatrix} \begin{bmatrix} -B & I_m \\ \dot{B} - A & 0 \end{bmatrix} \begin{bmatrix} -E \\ \dot{E} - D \end{bmatrix} = \\ \begin{bmatrix} -B^T & \dot{B}^T - A^T \\ I_m & 0 \end{bmatrix} \begin{bmatrix} I_m \\ 0 \end{bmatrix} \\ - \begin{bmatrix} -\dot{E}^T & \ddot{E}^T - D^T \end{bmatrix} \begin{bmatrix} I_m \\ 0 \end{bmatrix}$$

or

$$BE + \dot{E} - D = E^T B^T + \dot{E}^T - D^T + \ddot{E}^T$$

Using $E = -E^T$ leads to condition ii). We would proceed in the same way to find the two other conditions.

3 Elaboration of the conditions in a special case

Let us assume that the input-output representation $F_i, i = 1, \dots, m$, have the following particular form

$$F_i(y, \dot{y}, \ddot{y}, u, \dot{u}) = S_i(y, \dot{y}, \ddot{y}) - u_i. \quad (15)$$

This is a classical case (see Santilli [7]). Conditions of theorem 2 reduce to:

- (i). void
- (ii). $\frac{\partial S}{\partial \ddot{y}} - \left(\frac{\partial S}{\partial \dot{y}}\right)^T = 0$
- (iii). $\frac{\partial S}{\partial \dot{y}} + \left(\frac{\partial S}{\partial y}\right)^T - 2 \frac{d}{dt} \left(\frac{\partial S}{\partial \dot{y}}\right)^T = 0$
- (iv). $\frac{\partial S}{\partial y} - \left(\frac{\partial S}{\partial \dot{y}}\right)^T + \frac{d}{dt} \left(\frac{\partial S}{\partial \dot{y}}\right)^T - \frac{d^2}{dt^2} \left(\frac{\partial S}{\partial \dot{y}}\right)^T = 0$

These conditions are precisely the condition (2.1.17) in [7]. It follows that (15) represents the input-output behavior of a Hamiltonian system if and only if

$$S_i = R_{ik}(y, \dot{y})\ddot{y}^k + T_i(y, \dot{y}), \quad i = 1, \dots, m$$

where R_{ik} and T_i satisfy (2.2.9) in [7].

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