On stable attraction and tauberian theorems

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CHAPTER I

INTRODUCTION

1.1 Summary

In this thesis we derive Abelian and Tauberian theorems by using probability theory methods. Let F be a non-degenerate distribution function with characteristic function φ. It is said that F or φ belongs to the domain of attraction of a characteristic function ψ with norming sequence $A_n > 0$ and centering sequence B_n if

$$(1.1) \qquad \lim_{n\to\infty} \varphi^n\left(\frac{u}{A_n}\right) e^{-iuB_n} = \psi(u), \quad -\infty < u < \infty .$$

It is known that all stable distributions, and only these, have non-empty domains of attraction. We want to find conditions for F and φ that are necessary or sufficient in order that a centering sequence B_n exists, such that (1.1) holds with prescribed norming sequence A_n and stable characteristic function ψ.
To solve this problem we make use of the fact that (1.1) turns out to be equivalent, unless $\int x^2 dF(x) < \infty$ and $\int x\, dF(x) \neq 0$, to the assertion that $e^{\varphi-1}$ belongs to the domain of attraction of ψ with norming sequence A_n and centering sequence B_n, i.e.

$$(1.2) \qquad \lim_{n\to\infty} \exp\left[n\left\{\varphi\left(\frac{u}{A_n}\right)-1\right\} - iuB_n\right] = \psi(u), \quad -\infty < u < \infty$$

See theorem 3.7 for the proof of this equivalence.
We prefer (1.2) to (1.1) since $\exp\left[n\left\{\varphi\left(\frac{u}{A_n}\right)-1\right\} - iuB_n\right]$

is the characteristic function of an infinitely divisible distribution and in consequence easier to handle by Lévy's or Khintchine's representation. So now we change our problem in trying to find necessary and sufficient conditions for (1.2) with prescribed A_n and ψ.
It will be shown in lemma 3.3 that if ψ is non-degenerate relation (1.2) implies the existence of a strictly increasing continuous function A defined on $(0,\infty)$ and α^{-1}- varying at ∞, where α is the exponent of ψ, such that $A_n \sim A(n)$ for $n \to \infty$. For $\alpha = 2$ we must have $x^{-1/2}A(x) \to c$, $c > 0$, for $x \to \infty$.

Now that we know this we start again. Let the function A defined on $(0,\infty)$ be continuous, strictly increasing and α^{-1}-varying at ∞, $\alpha \in (0,2]$, while for $\alpha = 2$ holds $x^{-1/2}A(x) \to c > 0$ for $x \to \infty$. Let ψ be a stable characteristic function with exponent α. We want to find conditions for F and φ that are necessary or sufficient in order that a sequence B_n exists, such that (1.2) holds with $A_n \sim A(n)$, $n \to \infty$, and prescribed ψ. These conditions are derived for F in terms of the behaviour of the tails of F in chapter IV, section 4.2. The conditions for the characteristic function φ of F are derived in terms of the behaviour of φ at the origin in chapter IV, section 4.3. In section 4.4 of chapter IV we derive necessary and sufficient conditions for the Laplace transform f of F, if this does exist, in order that a sequence B_n exists, such that (1.2) holds with $A_n \sim A(n)$, $n \to \infty$, and prescribed ψ.

These conditions are given in terms of the behaviour of f at the origin. So now we have conditions in terms of the tails of F and in terms of the behaviour of φ or f at the origin, which are equivalent. Combining these equivalent conditions we obtain a connection between the tail-behaviour of F and the behaviour of φ or f at the origin. This is respectively a Fourier or Laplace Abelian and Tauberian theorem.
In section 5.1 and 5.2 of chapter V we derive Fourier Abelian and Tauberian theorems for two-sided and one-sided measures, respectively. These theorems are new. A number of Laplace Abelian and Tauberian theorems are derived in section 5.3. of chapter V. These theorems are new in the case of integer-valued exponents. In chapter VI we derive Fourier Abelian and Tauberian theorems connected with attraction to a degenerate distribution.

1.2 Notations and conventions.

With $\mathcal{B}$ we denote the class of Borelsets of $(-\infty,\infty)$. A distribution F is a probability measure on $\mathcal{B}$. The distribution function (p.d.f) of a distribution F is indicated with the same symbol and defined by $F(x) = F\{(-\infty, x)\}$, $-\infty < x < \infty$. We call F right one-sided if $F\{(-\infty, 0)\} = 0$. The characteristic function (ch.f.) of a p.d.f. F, denoted by φ, is defined by

(1.3) $$\varphi(u) = \int e^{iux}\, dF(x).$$