# Computing invariant manifolds

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*Publication date:*
1996

# ABSTRACT

Invariant manifolds are important in the global study of dynamical systems. This thesis describes the computation of invariant manifolds using the constructive proofs of existence theorems. Our methods are variations of the graph transform. We restrict to discrete dynamical systems, given by maps and focus on normally hyperbolic manifolds and their stable and unstable manifolds.

A normally hyperbolic invariant manifold is compact, for example a circle or a torus. In the simplest case it is a point, called a hyperbolic fixed point. The tangent space at the normally hyperbolic manifold can be split into stable and unstable subspaces, together with the subspace tangent to the manifold, such that this splitting is invariant under the derivative map. This implies that attraction and expansion have a linear nature. The manifold is normally hyperbolic if the attraction and expansion in the stable and unstable subspaces are stronger than the possible attraction and expansion on the manifold, respectively. (For a hyperbolic fixed point, it simply means that the eigenvalues of the derivative matrix at the fixed point are all off the unit circle.)

The stable manifold of a normally hyperbolic invariant manifold is the subset of points in the state space that converge to an orbit on the normally hyperbolic manifold under forward iteration. If the normally hyperbolic manifold consists of only one (fixed) point, the points on the stable manifold necessarily converge to the fixed point. The unstable manifold is the stable manifold of the inverse map.

The Invariant Manifold Theorem states that a normally hyperbolic invariant manifold, together with its local stable and unstable manifolds, persists under small perturbations. The Local Stable Manifold Theorem for a hyperbolic fixed point is a special case of the above theorem. The proofs of both theorems use the graph transform. The idea of the graph transform is to consider the unknown manifold as the graph of a function from the unperturbed manifold to a space with dimension equal to the codimension of the unperturbed manifold. The graph transform operates on the space of such continuous functions and can be proven to be a contraction by using a proper metric. The Banach Fixed Point Theorem now states that the graph transform has a unique fixed point that can be approximated with any given accuracy.

In this thesis the graph transform is adapted such that the algorithm can be implemented on a computer. The implementation is not straightforward, since the domain of the graph transform, the space of continuous functions, is infinite dimensional. We describe how to discretize the graph transform and prove the correctness of our algorithm. Moreover, we find estimates for the speed of convergence.

We present two variants of our method: one for the computation of local stable and unstable manifolds of a hyperbolic fixed point, and the other for normally hyperbolic invariant manifolds. The latter method can also be used for the computation of stable and unstable manifolds of a normally hyperbolic invariant manifold. We have implemented these methods giving an algorithm that works for the computation of manifolds with arbitrary finite dimensions. This is illustrated with several examples.