State maps of general behaviors, their lattice structure and bisimulations

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1 Introduction

In this paper we study the so called *state maps* in the behavioral approach to systems theory. The discussion in this paper continues some preliminary development presented in [5] and it also can be thought of as a generalization of the work in [10].

The concept of *states* or *state variables* is present in almost all branches of dynamical systems theory. In areas as remotely connected as discrete event systems and linear time invariant systems we can observe that the notion of *states* is present. One may think that this is a mere coincidence, but this is not true. The different notions of states have something in common. They are all connected by the so called *state property* or the *axiom of state*. In short (and perhaps rather inaccurately), one can say that a quantity or variable possesses the state property (or satisfies the axiom of state) if it captures the necessary information about the evolution of the dynamical system. There is of course a more mathematically formal and rigor formulation of this property, for example in [9]. In this paper, we will make use of this concept intensively, as will be revealed in the following sections.

It is quite a common view in the field of dynamical systems that states are understood to be internal. When the system is interconnected with other systems, states usually do not appear explicitly in the description of the interconnection. Nevertheless, they play an arguably central role in characterizing the compatibility of the interconnection. We shall not discuss this issue further, and the interested reader is referred to [13] and [5].

Following the earlier development in [5, 10], our point of view, which is based on the behavioral approach, is that states are constructed out of the system trajectories (the behavior). In the behavioral point of view, the behavior (i.e. the collection of all possible trajectories) defines/identifies the system. Of course, it is required that the trajectories bear all information/variables on everything relevant to the discussion. Irrelevant variables/information, which in the case of linear behaviors are called *latent variables*, generally can be eliminated. See [9] for more about this.

Having stated the previous paragraph, we should also remark that external behavior (i.e. the collection of external trajectories) does not always define the system. In some analysis, for example in the field of discrete event systems, the states are also relevant. In this case, the external behavior and the states define the system.

In this paper, we also discuss the concept of minimal state map from the general behavioral point of view. This is done by introducing a simple definition of partial ordering among state maps. With this partial ordering, comes the lattice structure of state maps, and with that minimal and maximal state maps are defined. The definition of minimality here coincides with well known definitions, for example the state space with minimal dimension in linear systems. We shall also discuss some additional properties of state maps, and how state maps that satisfy the properties are positioned in the lattice.

In this paper, we shall also study bisimulations between systems, cast in the general behavioral framework. Bisimulation as a notion of equivalence between dynamical systems has its root in discrete event systems. See, for example, the excellent introduction in [7]. The concept is centered around equivalence between the states of the systems. Therefore, when studying bisimulation, the states and the external trajectories define the systems under study.

Following recent increase in interest in hybrid systems, some efforts have been made to extend known theories in both discrete event systems and continuous dynamical systems to the new field of hybrid systems. Similarly, there also has been a traffic of cross applications of theories between discrete event systems and continuous dynamical systems. Bisimulation is one of them. Extending the notion of bisimulation to cover hybrid systems is done, for example in [6, 1]. Its extension to continuous dynamical systems is done, for example in [8, 12]. Studies of bisimulation in the general systems framework, encompassing more than one class of systems, has also been done before. The reader is referred to, for example, the work by Haghverdi *et.al.* in the category theoretical framework [2]; and to the more recent work by the same authors [3].

A comparison between [2, 3] and this paper is that in the former the analysis is geared towards general bisimulations for general systems, while in the latter a particular (non abstract) bisimulation is considered for general systems. We shall justify our definition of bisimulation relation by showing its relation with some existing bisimulations for special classes of systems.

To make it explicitly clear, the motivation behind this paper is to study two things in a general framework. The first is state constructions from trajectories of the systems. The second is the possible role played by the states in the notion of external systems equivalence. The setup of the study will be general, but at the end we shall relate the obtained results to results from more concrete classes of systems, i.e. linear time invariant systems and discrete event systems.

The rest of this paper is organized as follows. Section 2 contains some mathematical preliminaries including the construction of the lattice structure of the state maps. In Section 3 we define the bisimulation being studied, and show some important properties of it. In Section 4, we present some results on the relationship between the bisimulation and the lattice structure of the state maps. In Section 5 we define a time-independent version of the bisimulation defined in Section 3 (which is time-dependent) and show its relation to other notions of bisimulation.

2 The lattice structure of dynamic maps and state maps

Recall the definition of dynamical systems given in e.g. [9]. A dynamical system Σ is defined as a triple $(\mathbb{T}, \mathbb{W}, \mathfrak{B})$, where $\mathbb{T}$ is the *time axis*, $\mathbb{W}$ is the *signal space* and $\mathfrak{B}$ is the *behavior* of the system, which is a collection of all possible trajectories. A trajectory is a mapping from the time axis to the signal space. We assume that $\mathbb{T}$ is totally ordered by $>$.

A dynamic map of the system Σ is a map that has $\mathfrak{B} \times \mathbb{T}$ as its domain. For example, $\phi : \mathfrak{B} \times \mathbb{T} \to \Phi$ is a dynamic map. We always assume that dynamic maps are surjective. Elements of the codomain of a dynamic map are called *points*. We also equip the set of dynamic maps with the partial ordering $\preccurlyeq$, define as the following.

Definition 2.1. *Let ϕ and γ be two dynamic maps of Σ. We say $\phi \preccurlyeq \gamma$ if and only if for any $t \in \mathbb{T}$, and $w_1, w_2 \in \mathfrak{B}$ the following implication holds.*

$$(\gamma(w_1, t) = \gamma(w_2, t)) \Rightarrow (\phi(w_1, t) = \phi(w_2, t)). \tag{1}$$

We denote the inverse of $\preccurlyeq$ as $\succcurlyeq$. With that we can define equivalence of dynamic maps (with respect to $\preccurlyeq$) as

$$(\phi \approx \gamma) :\Leftrightarrow (\phi \preccurlyeq \gamma) \text{ and } (\phi \succcurlyeq \gamma). \tag{2}$$

Remark 2.2. *Notice that dynamic maps are tightly connected with the notion of* projection *of behaviors as presented in [4]. Indeed, the collection of trajectories of the dynamic map ϕ, as described above, constitutes a behavior of type $(\mathbb{T}, \Phi)$, which can be seen as the result of projecting $\mathfrak{B}$ according to some projection operation.*

For any system Σ there exists a *unique maximal dynamic map* (up to $\approx$), namely the identity map (or any other isomorphism) from $\mathfrak{B} \times \mathbb{T}$ to itself. There also exists a *unique minimal dynamic map* (up to $\approx$), namely the one that maps $\mathfrak{B} \times \mathbb{T}$ to a singleton.

We can introduce some properties to define subclasses of dynamic maps.

Definition 2.3. *A dynamic map $\phi : (\mathfrak{B}, \mathbb{T}) \to \Phi$ is called* Markovian *or is said to possess the* Markov property *if for any $w_1, w_2 \in \mathfrak{B}$ and $\tau \leq \tau' \in \mathbb{T}$, the following implication holds.*

$$\left\{(\phi(w_1, \tau) = \phi(w_2, \tau)) \;\; and \;\; \left(w_1|_{[\tau,\tau']} = w_2|_{[\tau,\tau']}\right)\right\} \Rightarrow \{\phi(w_1, \tau') = \phi(w_2, \tau')\}. \tag{3}$$

Notice that both the maximal and minimal dynamic maps are Markovian.

Definition 2.4. *A dynamic map* $\phi : (\mathfrak{B}, \mathbb{T}) \to \Phi$ *is called* past-induced *if for any* $w_1, w_2 \in \mathfrak{B}$ *and* $\tau \in \mathbb{T}$, *the following implication holds.*

$$(w_1|_{t \leq \tau} = w_2|_{t \leq \tau}) \Rightarrow (\phi(w_1, \tau) = \phi(w_2, \tau)) \,. \tag{4}$$

Dually, ϕ *is called* future-induced *if the following implication holds.*

$$(w_1|_{t \geq \tau} = w_2|_{t \geq \tau}) \Rightarrow (\phi(w_1, \tau) = \phi(w_2, \tau)) \,.$$

The definition of past and future inducedness have appeared earlier in the literatures, e.g. [11].

Lemma 2.5. *Let* ϕ *and* γ *be two dynamic maps of* Σ. *Suppose that* $\phi \succcurlyeq \gamma$. *If* ϕ *is a past-induced (resp. future-induced), then* γ *is also past-induced (resp. future-induced).*

It can be proven that for any system Σ, there exists a *unique maximal past-induced dynamic map* (up to $\approx$). In the literatures, this map is called the *Nerode state construction.* By symmetry, there also exists a *unique maximal future-induced dynamic map* (up to $\approx$). We shall call this map the *Dual Nerode state construction.* Unique minimal past-induced and future-induced dynamic maps also exist. As a consequence of Lemma 2.5, they coincide with the minimal dynamic map. Notice that we have used the word "state", whose meaning is about to be explained in the following.

Some dynamic maps possess a special property, called the *state property.* This property is generally known and discussed in many basic systems theory literatures. It is also known as the *axiom of state.*

Definition 2.6. *A state map* $x : (\mathfrak{B}, \mathbb{T}) \to \mathbb{X}$ *is a dynamic map such that, for any* $w_1, w_2 \in \mathfrak{B}$ *and* $\tau \in \mathbb{T}$, *the following implication holds.*

$$\begin{aligned} \{x(w_1, \tau) = x(w_2, \tau)\} \Rightarrow \{&w_3 := (w_1 \wedge_\tau w_2) \in \mathfrak{B} \text{ and} \\ &x(w_3, \tau) = x(w_1, \tau) = x(w_2, \tau)\} \end{aligned} \tag{5}$$

The set $\mathbb{X}$ *is called the* state space *of the state map* x.

The symbol $\wedge_\bullet$ signifies the patching/concatenation operation, where

$$(w_1 \wedge_\tau w_2)(t) := \begin{cases} w_1(t), & t < \tau \\ w_2(t), & t \geq \tau. \end{cases} \tag{6}$$

Since state maps are basically dynamic maps, they also inherit the ordering $\preccurlyeq$. Elements of the codomain of a state map are called *states.* Thus *states* are a special case of *points.* The following lemma says something about the structure of state maps in the lattice of dynamic maps.

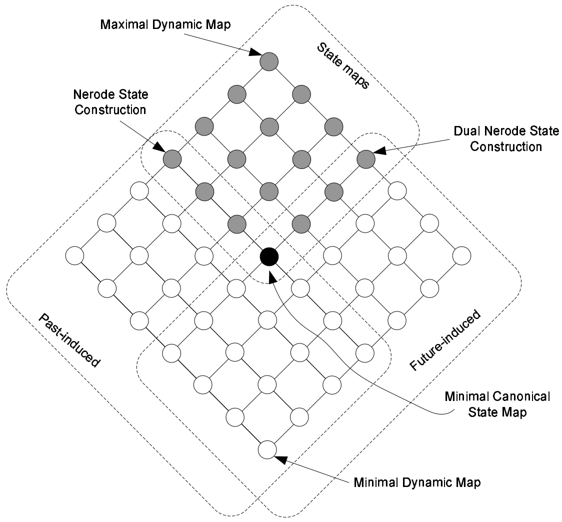

Figure 1: An illustration of the relations between maps in the lattice, when the canonical minimal state map exists. White circles represent dynamic maps. Shaded circles represent state maps. The black circle represents the minimal state map.

Lemma 2.7. *Let ϕ and γ be two dynamic maps of Σ. Suppose that $\phi \preccurlyeq \gamma$. If ϕ is a state map, then γ is also a state map.*

In fact, we can always guarantee that for any system Σ, there always exists a state map, as we can (trivially) prove that the maximal dynamic map, the Nerode state construction and the Dual Nerode state construction are state maps. It is obvious that the maximal dynamic map also acts as the unique maximal state map. The minimal state map(s) is a more interesting object to study. For example, about its uniqueness.

Some necessary and sufficient conditions for the existence of a unique minimal state map (up to $\approx$) were given in [5]. If such state map exists, we shall call it the *canonical minimal* state map. It is the only state map (again, up to $\approx$) that satisfies the canonical state property, where the implication in (5) is replaced with a bi-implication. It can also be proven that the canonical minimal state map is the only state map that possesses both the past-inducedness and future-inducedness properties.

Figure 1 and 2 illustrate the results we have discussed so far. In both figures, the lattice of dynamic maps is portrayed as a planar lattice, which is inaccurate. But still, they capture the main story. Figure 1 depicts the situation where the canonical minimal state map exists, and in Figure 2, it does not.

3 Time dependent bisimulation and its properties

Let Σ_1 and Σ_2 be two dynamical systems characterized the triple $(\mathbb{T}, \mathbb{V} \times \mathbb{D}_i, \mathfrak{B}_i)$, $i = 1, 2$. The variables v and d_i are the external and internal variables of Σ_i, respectively.

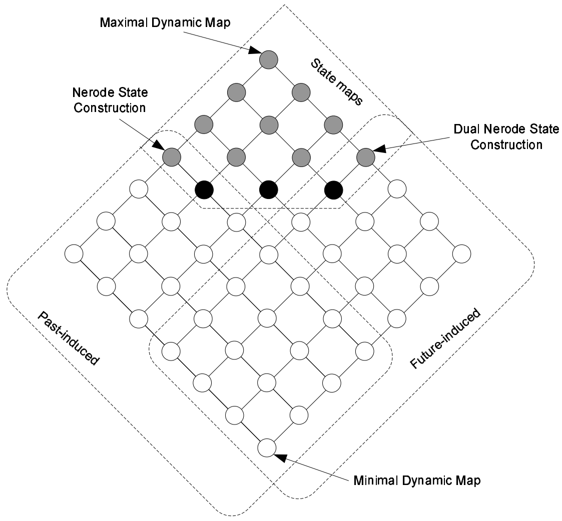

Figure 2: An illustration of the relations between maps in the lattice, when the canonical minimal state map does not exist. White circles represent dynamic maps. Shaded circles represent state maps. The black circles represent the minimal state maps.

Definition 3.1. *The* external behavior *is obtained by projecting the full behavior (i.e. including the hidden variables) to the external variables. A state map of the external behavior is called an* external state map*.*

Bisimulation of dynamical systems, can be defined in the following way.

Definition 3.2. *Given two dynamical systems Σ_1 and Σ_2 as defined above, with dynamic maps x_1 and x_2 respectively. A time dependent relation $\mathcal{R} : \mathbb{T} \to 2^{\mathbb{X}_1 \times \mathbb{X}_2}$ is a* bisimulation relation *if for any $\tau \in \mathbb{T}$, and any pair $(\xi_1, \xi_2) \in \mathcal{R}(\tau)$, the following holds.*
Given any $w_1 := (v_1, d_1) \in \mathfrak{B}_1$ such that $x_1(w_1, \tau) = \xi_1$, there exists a $w_2 := (v_2, d_2) \in \mathfrak{B}_2$ such that $x_2(w_2, \tau) = \xi_2$ and for all $\tau' \geq \tau$,

$$(x_1(w_1, \tau'), x_2(w_2, \tau')) \in \mathcal{R}(\tau'), \tag{7}$$

$$v_1(\tau') = v_2(\tau'), \tag{8}$$

and vice versa.

Definition 3.3. *Two dynamical systems Σ_1 and Σ_2 with dynamic maps x_1 and x_2 respectively, are said to be* bisimilar *if there exists a bisimulation relation $\mathcal{R}$ between them such that for all $t \in \mathbb{T}$, $\pi_i \mathcal{R}(t) = x_i(\mathfrak{B}_i, t)$, $i = 1, 2$. Here π_i denotes the canonical projection to $\mathbb{X}_i$.*

Remark 3.4. *It is important to remember that a bisimulation relation is not defined between two dynamical systems, but rather between two dynamical systems together with their respective dynamic maps. Later in the discussion, we can see that two equal behaviors might not be bisimilar if the dynamic maps do not satisfy certain conditions.*

In the subsequent discussion, for brevity, we shall denote the system Σ equipped with the dynamic map x as the pair (Σ, x).

Without any lost of generality, we can assume that a time-independent bisimulation relation $\mathcal{R}$ as in Definition 3.2 possesses a kind of *homogeneity* property, such that for all $t \in \mathbb{T}$,

$$\{\xi_1 \mathcal{R}(t) \xi_2 \wedge \xi_1 \mathcal{R}(t) \eta_2 \wedge \eta_1 \mathcal{R}(t) \xi_2\} \Rightarrow \{\eta_1 \mathcal{R}(t) \eta_2\},$$

for all $\xi_i, \eta_i \in \mathbb{X}_i$, $i = 1, 2$. This is due to the fact that we can prove that the *homogenized bisimulation relation* $\mathcal{R}_{\text{hom}}$, defined as

$$\mathcal{R}_{\text{hom}}(t) := \mathcal{R}(t) \circ \overline{(\mathcal{R}^{-1}(t) \circ \mathcal{R}(t))}, \forall t \in \mathbb{T}, \tag{9}$$

is a bisimulation relation, if $\mathcal{R}$ is a bisimulation relation. The overbar in (9) indicates equivalence closure.

Proposition 3.5. *Given two systems $\Sigma_i := (\mathbb{T}, \mathbb{V} \times \mathbb{D}_i, \mathfrak{B}_i)$, $i = 1, 2$, and their respective dynamic maps x_1 and x_2. If a time-independent relation $\mathcal{R}$ is a bisimulation, then the homogenized relation $\mathcal{R}_{hom}$ is a bisimulation relation as well.*

Proof. For any $t \in \mathbb{T}$, take any $(\xi_1, \xi_2) \in \mathcal{R}_{\text{hom}}(t)$. Let $w_1 := (v_1, d_1)$ such that $x_1(w_1, t) = \xi_1$. By construction (9), there exists a finite sequence $\eta_k|_{1 \leq k \leq 2n}$, $n \geq 0$, such that

$$\begin{gathered} \eta_1 = \xi_1 \text{ and } \eta_{2n} = \xi_2, \\ \eta_i \mathcal{R}(t) \eta_{i+1}, \text{ if } i \text{ is odd and} \\ \eta_{i+1} \mathcal{R}(t) \eta_i, \text{ if } i \text{ is even.} \end{gathered}$$

By following this chain of bisimilar points, we can infer that there exists a $w_2 := (v_2, d_2)$ such that $x_2(w_2, t) = \xi_2$, and for all $t' \geq t$,

$$\begin{gathered} w_1(t') = w_2(t'), \\ x_1(w_1, t') \mathcal{R}_{\text{hom}}(t') x_2(w_2, t'). \end{gathered}$$

□

Notice that the notion of bisimulation that we use here is different from, for example the ones used in [7, 8]. The main difference is that here we assume that bisimulation is time dependent. We do not say that two points are bisimilar, rather we say two points are bisimilar at a certain time. The reason behind this difference is that the time axis we use in the discussion is so minimally structured that only total ordering is assumed. With so little structure, we cannot talk about, for example, time-shifting. This is somewhat restrictive. For some classes of behaviors, for example LTI systems with infinite time axis and discrete event systems, the time axes are such that time-shifting is well defined. Later in this paper we shall also consider a time-independent version of the bisimulation relation.

In the definition, it is clear that bisimilarity between systems is reflexive and commutative. This means every pair (Σ, x) is bisimilar to itself and that if (Σ_1, x_1) is bisimilar to (Σ_2, x_2) then (Σ_2, x_2) is bisimilar to (Σ_1, x_1). In the literatures, it is well known that bisimilarity between automata is also transitive. Formally, since we are working with general behaviors, we still have to prove that this is also the case with the definition that we formulate above.

Proposition 3.6 (transitivity). *Consider three pairs of systems and their respective dynamic map,* (Σ_i, x_i), $i = 1, 2, 3$. *If* (Σ_1, x_1) *is bisimilar to* (Σ_2, x_2) *and* (Σ_2, x_2) *is bisimilar to* (Σ_3, x_3) *then* (Σ_1, x_1) *is bisimilar to* (Σ_3, x_3).

Proof. Let $\mathcal{R}_{12}(t)$ be the bisimulation relation between (Σ_1, x_1) and (Σ_2, x_2), and $\mathcal{R}_{23}(t)$ be that of (Σ_2, x_2) and (Σ_3, x_3). We shall prove that

$$\mathcal{R}_{13}(t) := \mathcal{R}_{12}(t) \circ \mathcal{R}_{23}(t), \;\; t \in \mathbb{T}, \tag{10}$$

is a bisimulation relation between (Σ_1, x_1) and (Σ_3, x_3). For any $t \in \mathbb{T}$, take any $(\xi_1, \xi_3) \in \mathcal{R}_{13}(t)$. By construction (10), there is a $\xi_2 \in \mathbb{X}_2$ such that

$$(\xi_1, \xi_2) \in \mathcal{R}_{12}(t) \text{ and } (\xi_2, \xi_3) \in \mathcal{R}_{23}(t).$$

Since $(\xi_1, \xi_2) \in \mathcal{R}_{12}(t)$, it follows that for any $w_1 := (v_1, d_1) \in \mathfrak{B}_1$ such that $x_1(w_1, t) = \xi_1$, there exists a $w_2 := (v_2, d_2) \in \mathfrak{B}_2$ such that $x_2(w_2, t) = \xi_2$ and for all $t' \geq t$,

$$\begin{aligned} (x_1(w_1, t'), x_2(w_2, t')) &\in \mathcal{R}_{12}(\tau'), \\ v_1(\tau') &= v_2(\tau'). \end{aligned}$$

However, since $(\xi_2, \xi_3) \in \mathcal{R}_{23}(t)$, it also follows that there exists a $w_3 := (v_3, d_3) \in \mathfrak{B}_3$ such that $x_3(w_3, t) = \xi_3$ and for all $t' \geq t$,

$$\begin{aligned} (x_2(w_2, t'), x_3(w_3, t')) &\in \mathcal{R}_{23}(\tau'), \\ v_2(\tau') &= v_3(\tau'). \end{aligned}$$

This implies

$$\begin{aligned} (x_1(w_1, t'), x_3(w_3, t')) &\in \mathcal{R}_{13}(\tau'), \\ v_1(\tau') &= v_3(\tau'). \end{aligned}$$

Proof for the converse follows straight from the proof above, so it won't be given. Formally we still have to prove that for all $t \in \mathbb{T}$, $\pi_i \mathcal{R}_{13}(t) = x_i(\mathfrak{B}_i, t)$, $i = 1, 3$. But it can be verified that it follows from the construction (10). $\square$

Now we have established the fact that bisimilarity between systems is an equivalence relation. We shall use the following shorthand notation throughout the paper.

Notation 3.7. *We write* $(\Sigma_1, x_1) \approx_{tdb} (\Sigma_2, x_2)$ *to indicate that* (Σ_1, x_1) *is time-dependent bisimilar to* (Σ_2, x_2).

Before we continue with the discussion on the properties of the bisimulation relation, consider the following definition of *complete behaviors*.

Definition 3.8. *A behavior* $\mathfrak{B} \subset \mathbb{W}^{\mathbb{T}}$ *is* complete *if for any trajectory* $w \in \mathbb{W}^{\mathbb{T}}$, *the following holds.*

$$\{w \notin \mathfrak{B}\} \Leftrightarrow \left\{\exists a, b \in \mathbb{T} \text{ s.t. } a \leq b \in \mathbb{T} \text{ and } w|_{[a,b]} \notin \mathfrak{B}|_{[a,b]}\right\}. \tag{11}$$

In words, a behavior is complete if for any trajectory it rejects, there is a finite interval in which the rejected trajectory is distinguishable from any accepted trajectory.

For the remaining parts of this paper we are going to assume that the class of behaviors we discuss is such that at least one of the following is satisfied.

AS1. The external behaviors involved in the discussion are *complete*.

AS2. The time axis $\mathbb{T}$ has a *minimal element*.

Notice that although these assumptions are somewhat restrictive, it still allows for incorporating some important classes of behaviors in the discussion. For example, LTI behaviors with $\mathbb{R}_+$ as the time axis and discrete event systems.

Theorem 3.9. *If* $(\Sigma_1, x_1) \approx_{tdb} (\Sigma_2, x_2)$ *then* Σ_1 *and* Σ_2 *have the same external behavior.*

Proof. Let $\mathcal{R}$ be the bisimulation relation between (Σ_1, x_1) and (Σ_2, x_2) as in Definition 3.3. We denote the external behaviors of the systems as $\mathfrak{B}_1^{\text{ext}}$ and $\mathfrak{B}_2^{\text{ext}}$ respectively. Take any $v_1 \in \mathfrak{B}_1^{\text{ext}}$. By definition, there is a d_1 such that $(v_1, d_1) \in \mathfrak{B}_1$. Now, following the definition of $\mathcal{R}$ in Definition 3.2, for any $t \in \mathbb{T}$, there exists a $(v_2^t, d_2^t) \in \mathfrak{B}_2$ such that v_1 and v_2^t coincide for all time $t' \geq t$. Based on the underlying assumption we made earlier, $v_1 \in \mathfrak{B}_2^{\text{ext}}$. We have proven that $\mathfrak{B}_1^{\text{ext}} \subset \mathfrak{B}_2^{\text{ext}}$. The converse is analogous. $\square$

Theorem 3.9 suggests that bisimilarity is generally a stronger notion of external equivalence between systems than external behavior equivalence. We confirm this suggestion by showing that the converse of Theorem 3.9 is generally not true. Consider the following counterexample.

Example 3.10. *Consider two finite state automata shown in Figure 3. Suppose that we define the respective external behavior of each automaton as the collection of all finished word of executions. Clearly these two automata share the same external behavior. However, it can also be verified that they are not bisimilar.*

Now that we know that bisimilar dynamical systems (with respect to certain dynamic maps) have equal external behavior, whenever we want to discuss bisimulation between two dynamical systems, we can restrict our attention to systems with equal external behaviors.

The study of bisimulation between dynamical systems can be cast as follows. Given two dynamical systems Σ_1 and Σ_2 with equal external behavior $\mathfrak{B}^{\text{ext}}$, equipped with two dynamic maps x_1 and x_2 respectively. Study the conditions such that Σ_1 equipped with x_1 is bisimilar to Σ_2 equipped with x_2.

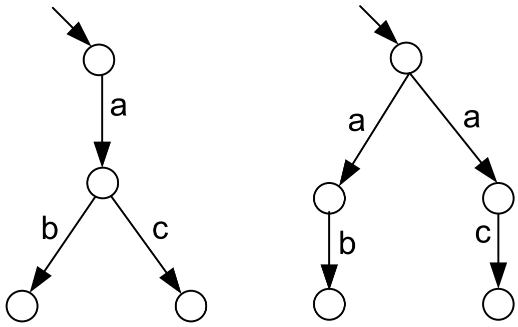

Figure 3: Two automata discussed in Example 3.10.

4 Bisimulation and the lattice structure

In this section, we shall investigate the relation between bisimulation relations and the lattice structure of dynamic maps. First, we shall see how bisimulation relation can be related to consistent reduction of dynamic maps. By consistent reduction we mean construction of another dynamic map, which is smaller than the initial one with respect to the lattice structure, while preserving the bisimulation.

Proposition 4.1. *Given two systems $\Sigma_i := (\mathbb{T}, \mathbb{V} \times \mathbb{D}_i, \mathfrak{B}_i)$, $i = 1, 2$, and two dynamic maps x_1 and x_2 such that $(\Sigma_1, x_1) \approx_{tdb} (\Sigma_2, x_2)$. Let $\mathcal{R}$ be the homogeneous time-dependent bisimulation relation. We can create another dynamic map x_1' for Σ_1, whose equivalence class (that is, its position on the lattice) is determined according to*

$$\forall w_1, w_2 \in \mathfrak{B}_1, \forall t \in \mathbb{T}, \{x_1'(w_1, t) = x_1'(w_2, t)\} \Leftrightarrow \left\{(x_1(w_1, t), x_1(w_2, t)) \in \mathcal{R}(t) \circ \mathcal{R}^{-1}(t)\right\}.$$

We then have that $x_1' \preccurlyeq x_1$ and $(\Sigma_1, x_1') \approx_{tdb} (\Sigma_2, x_2)$.

Proof. First of all, we need to prove that for all $t \in \mathbb{T}$, $\mathcal{R}(t) \circ \mathcal{R}^{-1}(t)$ is an equivalence relation covering the whole $x_1(\mathfrak{B}_1, t)$. Its reflexive and commutative properties are obvious. We shall prove that it is transitive as well. Take any pairs (ξ_1, ξ_2) and (ξ_2, ξ_3), both are in $\mathcal{R}(t) \circ \mathcal{R}^{-1}(t)$. This implies the existence of ξ' and ξ'' such that (ξ_1, ξ'), (ξ_2, ξ'), (ξ_2, ξ''), and (ξ_3, ξ'') are all in $\mathcal{R}(t)$. By the homogeneity property, we have that (ξ_1, ξ'') is also in $\mathcal{R}(t)$. Therefore $(\xi_1, \xi_3) \in \mathcal{R}(t) \circ \mathcal{R}^{-1}(t)$. The fact that $\mathcal{R}(t) \circ \mathcal{R}^{-1}(t)$ covers the whole $x_1(\mathfrak{B}_1, t)$ follows from the fact that $\pi_1 \mathcal{R}(t) = x_1(\mathfrak{B}_1, t)$ (see Definition 3.3). We then have to prove $x_1' \preccurlyeq x_1$. But this follows straightforwardly from the fact that $\mathcal{R}(t) \circ \mathcal{R}^{-1}(t)$ contains the identity relation. Finally, we have to prove that $(\Sigma_1, x_1') \approx_{\mathrm{tdb}} (\Sigma_2, x_2)$. We shall do this by providing the time-dependent bisimulation relation between the two pairs. Notice that each element in $\mathbb{X}_1'$, that is, the codomain of x_1' corresponds to a partition in $\mathbb{X}_1$. We can therefore create a relation $\mathcal{R}' \subset \mathbb{X}_1' \times \mathbb{X}_2$ by requiring that $(\xi_1', \xi_2) \in \mathcal{R}'(t)$ if and only if the partition corresponding to ξ_1' is related to ξ_2 by $\mathcal{R}(t)$. By the homogeneity property we know that each partition is either totally related or not at all related to ξ_2. $\square$

$$\begin{tikzcd}
(\Sigma_1, x_1) \arrow[r, phantom, "\overset{\mathcal{R}}{\sim}"' , "\text{bisimilar}"] \arrow[d, "\text{reduction}"'] & (\Sigma_2, x_2) \arrow[d, "\text{reduction}"] \\
(\Sigma_1, x_1') \arrow[r, phantom, "\overset{\mathcal{R}'}{\sim}"', "\text{bisimilar}"] & (\Sigma_2, x_2')
\end{tikzcd}$$

Figure 4: Diagram illustrating consistent reduction of dynamic maps. The bisimulation relation $\mathcal{R}'$ is an isomorphism.

The proposition above tells us that we can use the bisimulation relation to reduce the dynamic map of the systems while maintaining the bisimilarity. For the case of linear time invariant systems, this has been discussed in [8, 12].

Notice that although the proposition only stipulates that the first system can be state reduced, by symmetry so can the second. Furthermore, by the transitive property of bisimulation, the state reduced systems are again bisimilar. See the diagram in Figure 4. Also notice that, although not explicitly stated in the diagram, all pairs are bisimilar to each other.

Given Proposition 4.1, a natural question that arises is whether any comparable dynamic maps (of the same system) are bisimilar. Generally, the answer to this question is no. However, if the dynamic maps involved possess some certain properties, then comparable dynamic maps are indeed bisimilar.

Proposition 4.2. *Given a system $\Sigma := (\mathbb{T}, \mathbb{V} \times \mathbb{D}, \mathfrak{B})$ with a Markovian state map x_1 and a past-induced Markovian state map x_2 such that $x_1 \preccurlyeq x_2$, then $(\Sigma, x_1) \approx_{tdb} (\Sigma, x_2)$.*

Proof. We have to construct a candidate for the bisimulation relation between the state space $\mathbb{X}_1$ and $\mathbb{X}_2$. Consider the following construction. For any $t \in \mathbb{T}$, $\xi_i \in \mathbb{X}_i, i = 1, 2$, we define $\xi_1 \mathcal{R}(t) \xi_2$ if and only if there is a $w \in \mathfrak{B}$ such that $x_1(w, t) = \xi_1$ and $x_2(w, t) = \xi_2$. We shall now verify that $\mathcal{R}$ is indeed a bisimulation relation. For any $t \in \mathbb{T}$, take any $(\xi_1, \xi_2) \in \mathcal{R}(t)$. Let $w_1 := (v_1, d_1)$ such that $x_1(w_1, t) = \xi_1$. By construction, there should exists a $w' := (v', d') \in \mathfrak{B}$ such that

$$\begin{aligned} x_1(w', t) &= \xi_1, \\ x_2(w', t) &= \xi_2. \end{aligned}$$

Now, define $w_2 := w' \wedge_t w_1$. Because of the past-inducedness property we have that $x_2(w_2, t) = \xi_2$, and because of the state property, $x_1(w_2, t) = \xi_1$. We also have that for all $t' \geq t$, $w_1(t') = w_2(t')$. Moreover, because of the Markovian property of x_1, we have that for all $t' \geq t$, $x_1(w_1, t') = x_1(w_2, t')$. Hence $x_1(w_1, t') \mathcal{R}(t') x_2(w_2, t')$. Now we verify the other direction of the bisimulation. For any $t \in \mathbb{T}$, take any $(\xi_1, \xi_2) \in \mathcal{R}(t)$. Let $w_2 := (v_2, d_2)$

such that $x_2(w_2, t) = \xi_2$. By construction, there should exists a $w' := (v', d') \in \mathfrak{B}$ such that

$$x_1(w', t) = \xi_1,$$
$$x_2(w', t) = \xi_2.$$

Since $x_1 \preccurlyeq x_2$, we have that $x_1(w_2, t) = \xi_1$. Obviously, if we take $w_1 := w_2$, we then obtain that for all $t' \geq t$,

$$w_1(t') = w_2(t'),$$
$$x_1(w_1, t')\mathcal{R}(t')x_2(w_2, t').$$

Formally, we still have to prove that for all $t \in \mathbb{T}$, $\pi_i \mathcal{R}(t) = x_i(\mathfrak{B}_i, t)$, $i = 1, 2$. However, it can be verified that this is true, due to the construction of $\mathcal{R}$. $\square$

Proposition 4.2 indicates that although bisimulation is related to consistent state map reduction, strict reduction (provided that the state map is not yet minimal) is only guaranteed for Markovian past induced state maps. Notice that although in the proposition, we don't require that x_1 is past-induced, it follows from Lemma 2.5 that it is past-induced. We can see in the proof of Proposition 4.2 that the Markovian and past-inducedness properties are needed to guarantee strict reduction. However, this does not mean that they are necessary conditions.

Although requiring the state maps to have both Markovian and past-induced properties seems restrictive, notice that the usual state constructions for LTI systems and deterministic automata possess these properties. Therefore, for those large classes of systems, we can still assert that state reduction corresponds to bisimulation, despite of this restriction.

An immediate consequence of Proposition 4.2 is given below.

Proposition 4.3. *Given two systems $\Sigma_i := (\mathbb{T}, \mathbb{V} \times \mathbb{D}_i, \mathfrak{B}_i)$, $i = 1, 2$, and two past-induced Markovian state maps x_1 and x_2 such that $(\Sigma_1, x_1) \approx_{tdb} (\Sigma_2, x_2)$. For any $\tilde{x}_i$, a Markovian past-induced state map of Σ_i such that $x_i \preccurlyeq \tilde{x}_i$, $i = 1, 2$, we have that $(\Sigma_1, \tilde{x}_1) \approx_{tdb} (\Sigma_2, \tilde{x}_2)$.*

Now, consider the dynamic map corresponding to taking the past of the projection to external variables.

Definition 4.4. *For a system $\Sigma := (\mathbb{T}, \mathbb{V} \times \mathbb{D}, \mathfrak{B})$, we define the* past external projection *π_{pe} as*

$$\pi_{pe}(w, t) := v(\tau)|_{\tau \leq t}, \forall w \in \mathfrak{B}, t \in \mathbb{T},$$

where v is the external part of w.

Proposition 4.3 tells us that given two systems with equal external behavior and a Markovian past-induced state map for each of the systems, such that the systems equipped with the state maps are bisimilar, we can replace the state maps with other Markovian past-induced state maps smaller than the original ones while maintaining bisimilarity. The past external projection is a dynamic map that is Markovian and past-induced. Further, we can obtain the following result.

Proposition 4.5. *Given two systems $\Sigma_i := (\mathbb{T}, \mathbb{V} \times \mathbb{D}_i, \mathfrak{B}_i)$, $i = 1, 2$, with equal external behavior. We have that $(\Sigma_1, \pi^1_{pe}) \approx_{tdb} (\Sigma_2, \pi^2_{pe})$, where π^i_{pe} is the past external projection of Σ_i.*

Proof. Notice that π^1_{pe} and π^2_{pe} share the same codomain. We denote the codomain by Π. We construct the (candidate) bisimulation relation $\mathcal{R}(t)$ as the identity relation in Π. That is, $(\xi_1, \xi_2) \in \mathcal{R}(t)$ if and only if $\xi_1 = \xi_2$. Now we shall verify that $\mathcal{R}$ is indeed a bisimulation relation. For any $t \in \mathbb{T}$, take any $(\xi_1, \xi_2) \in \mathcal{R}(t)$. Let $w_1 := (v_1, d_1)$ such that $\pi^1_{\text{pe}}(w_1, t) = \xi_1$. Since the systems have the same external behavior, there exists a d_2 such that $w_2 := (v_1, d_2) \in \mathfrak{B}_2$. By construction, $\xi_1 = \xi_2$. We also have that for all $\tau \geq t$, $\pi^1_{\text{pe}}(w_1, \tau)\mathcal{R}(\tau)\pi^2_{\text{pe}}(w_2, t)$. The other half of the proof is analogous to this one. Formally, we still have to prove that all points of the dynamic maps are related by $\mathcal{R}$, but this fact is a direct consequence of $\mathcal{R}$ being the identity relation in Π. $\square$

Combining the results of Proposition 4.3 and 4.5, we can obtain the following theorem.

Theorem 4.6. *Given two systems $\Sigma_i := (\mathbb{T}, \mathbb{V} \times \mathbb{D}_i, \mathfrak{B}_i)$, $i = 1, 2$, with equal external behavior. We use π^i_{pe} to denote the past external projection of Σ_i. Suppose that π^i_{pe} possesses the state property for system Σ_i, $i = 1, 2$, and that both systems admit a minimal canonical state map. We denote the minimal canonical state maps as x^1_{min} and x^2_{min} respectively. We then have that $(\Sigma_1, x^1_{min}) \approx_{tdb} (\Sigma_2, x^2_{min})$.*

Proof. The proof follows immediately from Proposition 4.3 and 4.5, using the fact that the canonical minimal state map is Markovian and past-induced. $\square$

We shall now apply this result to investigate bisimulation of systems of the following state space representation.

$$\dot{x} = Ax + Bu + Gd, \tag{12a}$$
$$y = Cx. \tag{12b}$$

To be precise, we mean that the represented behaviors are collections of the locally integrable solutions of the differential equation. In the introductory text to the behavioral systems theory [9], this is called the *weak solution* of the differential equation.

The input u and the output y are considered to be the external variables, while d is considered to be the internal one. The following setup is used in [12], and the special case with $B = 0$ is used in e.g. [8].

Since we are working with dynamic maps, whose trajectories are *observable* from those of the behavior, we are going to assume that the systems we are working with are observable. That is, we assume that the pair (A, C) in (12) is an *observable pair.* Later in the next section we are going to work on state space systems without this assumption.

Proposition 4.7. *Given two observable state space systems represented by*

$$\dot{x}_i = A_i x_i + B_i u + G_i d_i, \tag{13a}$$
$$y = C_i x_i, \tag{13b}$$

with $i = 1, 2$. *Suppose that for both systems, the largest controlled invariant subspace (with* d_i *seen as the control input) of* $\dot{x}_i = A_i x_i + G_i d_i$ *contained in the kernel of* C_i *is* $\{0\}$. *We then have that* $(\Sigma_1, x_1) \approx_{tdb} (\Sigma_2, x_2)$ *if and only if the external behaviors are equal.*

Proof. First of all, notice that x_i is the minimal canonical state map of Σ_i. We can apply Theorem 4.6 and complete proof, provided that the past external projections are state maps. We can prove that the past external projections are indeed state maps by showing that it is larger (with respect to $\preccurlyeq$) than the canonical minimal state maps. In other words, we need to show that for each system, two trajectories with equal past external projections are mapped to the same state. This is equivalent to the fact that the following equation is solved only if $x_i \equiv 0$.

$$\dot{x}_i = A_i x_i + B_i u + G_i d_i, \tag{14a}$$

$$y = C_i x_i = 0, \tag{14b}$$

$$u = 0. \tag{14c}$$

Following the fact that the largest controlled invariant subspace (with d_i seen as the control input) of $\dot{x}_i = A_i x_i + G_i d_i$ contained in the kernel of C_i is $\{0\}$, we conclude that the past external projections are indeed state maps. □

5 Stronger state property and time-independent bisimulation

Recall that the only assumption that we impose on the time axis $\mathbb{T}$ is just that it is totally ordered, with respect to $<$. With just this assumption, some concepts that we normally encounter in systems theory do not make any sense, for example *time invariant* systems and *time shifting* of trajectories. If we assume that the the axis $\mathbb{T}$ possesses some additional properties, time shifting can be made formal.

The following technicalities, put inside the box, are steps taken to make time shifting formal, so that a stronger state property can be formulated. The reader who is not interested in the details can skip them without affecting the flow of the discussion.

We introduce the difference operation "$-$" as a binary operation, such that for all $t \leqq t' \in \mathbb{T}$, the difference between them is denoted as $(t' - t)$. The collection of all differences is denoted as $\mathbb{T}^+$. Notice that although we might intuitively expect that $\mathbb{T}^+ \subset \mathbb{T}$, this is not necessarily true.
Next, we define the addition operation "$+$" as a commutative binary operation, $+ : \mathbb{T} \times \mathbb{T}^+ \to \mathbb{T}$, such that for all $t \in \mathbb{T}$ and $\delta \in \mathbb{T}^+$, the sum $t + \delta = \delta + t$ is well defined in $\mathbb{T}$. We require the operations to satisfy the following set of axioms.

(a1) For all $t', t \in \mathbb{T}$, $t' + (t - t) = (t - t) + t' \equiv t'$.

(a2) For all $t' > t \in \mathbb{T}$, $t + (t' - t) = (t' - t) + t \equiv t'$.

(a3) For all $t > s > r \in \mathbb{T}$, $s + (t - r) \equiv t + (s - r)$.

(a4) For all $t \in \mathbb{T}, \delta \in \mathbb{T}^+,\ t + \delta \geq t$.

(a5) For all $t \in \mathbb{T}, \delta, \delta' \in \mathbb{T}^+,\ (t + \delta) + \delta' \equiv (t + \delta') + \delta$.

(a6) For all $t \in \mathbb{T}, \delta, \delta' \in \mathbb{T}^+,\ ((t + \delta) = (t + \delta')) \Rightarrow (\delta = \delta')$.

Notice that using these axioms, it is possible to extend the commutative operator $+$ to act on a pair of elements of $\mathbb{T}^+$, by defining for all $t \in \mathbb{T}, \delta, \delta' \in \mathbb{T}^+$,

$$(t + \delta) + \delta' =: t + (\delta + \delta'). \tag{15}$$

Definition 5.1. *A dynamic map x is said to possess the* time independent state property *if for any w_1 and w_2 in $\mathfrak{B}$ and t_1 and t_2 in $\mathbb{T}$, the following holds.*

$$\begin{aligned} \{x(w_1, t_1) = x(w_2, t_2)\} \Rightarrow \{w_3 := \left(w_1 \wedge_{t_2}^{t_1} w_2\right) \in \mathfrak{B} \ \textit{and} \\ x(w_3, t_1) = x(w_1, t_1) = x(w_2, t_2)\} \end{aligned} \tag{16}$$

The symbol $\wedge_{t_2}^{t_1}$ denotes the shift-concatenation operation,

$$\left(w_1 \wedge_{t_2}^{t_1} w_2\right)(t) = \begin{cases} w_1(t), & t < t_1, \\ w_2(t - t_1 + t_2), & t \geq t_1. \end{cases} \tag{17}$$

Definition 5.2. *A dynamic map x is said to possess the* time independent Markovian property *if for any w_1, w_2 in $\mathfrak{B}$, and t_1, t_2 in $\mathbb{T}$, and $\Delta \in \mathbb{T}^+$ the following holds.*

$$\left\{(x(w_1, t_1) = x(w_2, t_2)) \ \textit{and} \ \left(w_1|_{[t_1, t_1 + \Delta]} = w_2|_{[t_2, t_2 + \Delta]}\right)\right\} \Rightarrow \{x(w_1, t_1 + \Delta) = x(w_2, t_2 + \Delta)\}. \tag{18}$$

Notice that the bracketed term on the second line of the right hand side of (17) can be interpreted without confusion, because of axiom (a3).

Lemma 5.3. *Take any* $w_1,w_2 \in \mathfrak{B}$ *and* $t_1, t_2 \in \mathbb{T}$. *Denote* $w_3 := w_1 \wedge_{t_2}^{t_1} w_2$. *The following holds.*

$$w_2 = w_2 \wedge_{t_1}^{t_2} w_3. \tag{19}$$

Proof. By definition,

$$w_3(t) = \begin{cases} w_1(t), & t < t_1, \\ w_2(t - t_1 + t_2), & t \geq t_1. \end{cases} \tag{20}$$

Now denote $w_4 := w_2 \wedge_{t_2}^{t_1} w_3$. We shall prove that $w_4 = w_2$. First, we have that

$$w_4(t) = \begin{cases} w_2(t), & t < t_2, \\ w_3(t - t_2 + t_1), & t \geq t_2. \end{cases} \tag{21}$$

From here we see that w_4 and w_2 agree for all $t < t_2$. From axiom (a4), we know that $((t - t_2) + t_1) \geq t_1$. Hence,

$$\begin{aligned} w_3((t - t_2) + t_1) &= w_2 \left[((t - t_2) + t_1) - t_1 + t_2 \right], \\ &\overset{(a6)}{=} w_2 \left[(t - t_2) + t_2 \right], \\ &\overset{(a2)}{=} w_2(t). \end{aligned} \tag{22}$$

Therefore $w_4 = w_2$. □

Lemma 5.3 tells us that the shift operation defined in the shift-concatenation operation in (17) is invertible. No information about the shifted trajectory is lost. This seems trivial if we think of $\mathbb{T}$ as being $\mathbb{R}$ or $\mathbb{Z}$, but since we only rely on the set of axioms (a1) - (a6), we have to be careful.

Since time independent state maps are special cases of state maps, they also inherit the partial ordering relation $\preccurlyeq$. In fact, the following lemma holds.

Lemma 5.4. *Let* ϕ *and* γ *be two dynamic maps of* Σ. *Suppose that* $\phi \preccurlyeq \gamma$. *If* ϕ *is a time independent state map, then* γ *is also a time independent state map.*

Also notice that both the Nerode state construction and the dual Nerode state construction are both time independent state maps.

With the additional structure on the time axis, we can formulate a *time-independent* bisimulation relation.

Definition 5.5. *Given two dynamical systems* Σ_1 *and* Σ_2, *with dynamic maps* x_1 *and* x_2 *respectively. A (time independent) relation* $\mathcal{R} \subset 2^{\mathbb{X}_1 \times \mathbb{X}_2}$ *is a (time independent)* bisimulation relation *if for any pair* $(\xi_1, \xi_2) \in \mathcal{R}$, *the following holds.*

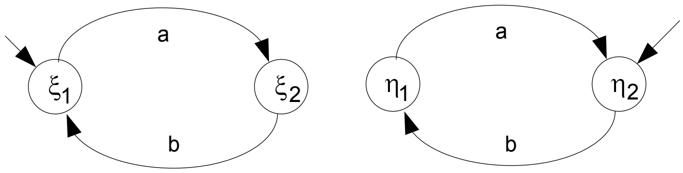

Figure 5: The automata discussed in Example 5.7.

Given any $w_1 := (v_1, d_1) \in \mathfrak{B}_1$ *and* $t_1 \in \mathbb{T}$ *such that* $x_1(w_1, t_1) = \xi_1$. *If* $t_2 \in \mathbb{T}$ *is such that there exists a* $w' \in \mathfrak{B}_2$ *such that* $x_2(w', t_2) = \xi_2$, *then there exists a* $w_2 := (v_2, d_2) \in \mathfrak{B}_2$ *such that* $x_2(w_2, t_2) = \xi_2$ *and for all* $\tau \geq t_2$,

$$v_1(\tau - t_2 + t_1) = v_2(\tau), \tag{23}$$

$$(x_1(v_1, \tau - t_2 + t_1), x_2(v_2, \tau)) \in \mathcal{R}. \tag{24}$$

and vice versa.

In words, the time-independent bisimulation requires that from any two bisimilar points it should be possible to proceed with equal external trajectory while visiting points that are bisimilar. No reference whatsoever is made to the time instant at which the starting points are reached. This formulation will be closer to the one defined in [7] for discrete event systems and also with those defined for some classes of time invariant dynamical systems, e.g. [8][12].

Definition 5.6. *Two dynamical systems* Σ_1 *and* Σ_2 *with dynamic maps* x_1 *and* x_2 *respectively, are said to be (time-independent)* bisimilar *if there exists a (time - independent) bisimulation relation* $\mathcal{R}$ *between them such that* $\pi_i \mathcal{R} = \mathbb{X}_i$, $i = 1, 2$. *Here* π_i *denotes the canonical projection to* $\mathbb{X}_i$.

Contrary to the case of time-dependent bisimulation, time-independent bisimulation does **not** imply external behavior equivalence. The following is an (counter)example.

Example 5.7. *Consider the two automata depicted in Figure 5. We can easily verify that* $\mathcal{R} := \{(\xi_1, \eta_1), (\xi_2, \eta_2)\}$ *is a time-independent bisimulation relation, and that the two systems are time independent bisimilar. However, the external behaviors (i.e. the language generated by the automata) are not equal. In terms of regular expressions, the language generated by the automaton on the left and on the right are* $(ab)^*$ *and* $(ba)^*$ *respectively.*

The counterexample above motivates us to formulate the following. Suppose that the class of behaviors we study are such that the time axis $\mathbb{T}$ has a minimal element t_0 (see Assumption AS2 in Section 3). Suppose that x is a dynamic map of the behavior $\mathfrak{B}$. The *initial points* of $\mathfrak{B}$ is defined as

$$x^{\text{init}} := x(\mathfrak{B}, t_0).$$

Definition 5.8. *Under the assumption that the time axis* $\mathbb{T}$ *has a minimal element* t_0*, two dynamical systems* Σ_1 *and* Σ_2 *with dynamic maps* x_1 *and* x_2 *respectively, are said to be (time-independent)* rooted bisimilar *if there exists a (time - independent) bisimulation relation* $\mathcal{R}$ *between them such that*

$$\left\{\xi \in \mathbb{X}_2 \mid \exists \xi' \in x_1^{init} \text{ s.t. } \xi' \mathcal{R} \xi\right\} \supset x_2^{init},$$
$$\left\{\xi \in \mathbb{X}_1 \mid \exists \xi' \in x_2^{init} \text{ s.t. } \xi \mathcal{R} \xi'\right\} \supset x_1^{init}.$$

Theorem 5.9. *If* Σ_1 *and* Σ_2*, equipped with state maps* x_1 *and* x_2 *are (time-independent) rooted bisimilar then*
(i) their external behaviors are equal.
(ii) they are also (time-independent) bisimilar.

This theorem is given without proof, as the proof can be constructed quite easily, for example following a similar line as in the proof of Theorem 3.9. This result states that rooted bisimilarity is a stronger notion than bisimilarity. In hindsight, we can observe that the two automata in Example 5.7 are not rooted bisimilar, although they are bisimilar. For the remaining of this section, we shall assume that the time axis $\mathbb{T}$ has a minimal element t_0.

We can prove that the rooted bisimulation also possesses the transitivity property, analogous to that of Proposition 3.6 for time dependent bisimulation. Hence rooted bisimulation is also an equivalence relations, which we shall denote by $\approx_{\mathrm{rb}}$. We can perform analysis on rooted bisimulation as we have done to the time dependent bisimulation. But for now, let us focus our attention to the analog of Proposition 4.2 for time independent state maps.

Proposition 5.10. *Given a system* $\Sigma := (\mathbb{T}, \mathbb{V}\times\mathbb{D}, \mathfrak{B})$ *with a time independent Markovian state map* x_1 *and a past-induced time independent Markovian state map* x_2 *such that* $x_1 \preccurlyeq x_2$*, then* $(\Sigma, x_1) \approx_{rb} (\Sigma, x_2)$*.*

Proof. We have to construct a candidate for the time independent bisimulation relation between the state space $\mathbb{X}_1$ and $\mathbb{X}_2$. Consider the following construction. For any $\xi_i \in \mathbb{X}_i, i = 1, 2$, we define $\xi_1 \mathcal{R} \xi_2$ if and only if there is a $w \in \mathfrak{B}$ and $t \in \mathbb{T}$ such that $x_1(w, t) = \xi_1$ and $x_2(w, t) = \xi_2$. We shall now verify that $\mathcal{R}$ is indeed a bisimulation relation. Take any $(\xi_1, \xi_2) \in \mathcal{R}$. Let $w_1 := (v_1, d_1)$ and $t \in \mathbb{T}$ be such that $x_1(w_1, t) = \xi_1$. By construction, there should exist a $w' := (v', d') \in \mathfrak{B}$ and $t' \in \mathbb{T}$ such that

$$x_1(w', t') = \xi_1,$$
$$x_2(w', t') = \xi_2.$$

Now, let $t'' \in \mathbb{T}$ be such that there exists a $w'' := (v'', d'')$ such that $x_2(w'', t'') = \xi_2$. Since $x_1 \preccurlyeq x_2$, we also have that $x_1(w'', t'') = \xi_1$. We construct $w_2 := w'' \wedge_t^{t''} w_1$. Because of the past-inducedness property we have that $x_2(w_2, t'') = \xi_2$, and because of the state property, $x_1(w_2, t'') = \xi_1$. We also have that for all $\tau \geq t''$,

$$w_1(\tau - t'' + t) = w_2(\tau),$$
$$x_1(w_1, \tau - t'' + t) \mathcal{R} x_2(w_2, \tau).$$

The second line of the equation above is due to the time independent Markovian property. Now we verify the other direction of the bisimulation. Take any $(\xi_1, \xi_2) \in \mathcal{R}$. Let $w_2 := (v_2, d_2)$ and $t \in \mathbb{T}$ be such that $x_2(w_2, t) = \xi_2$. By construction, there should exist a $w' := (v', d') \in \mathfrak{B}$ and $t' \in \mathbb{T}$ such that

$$\begin{aligned} x_1(w', t') &= \xi_1, \\ x_2(w', t') &= \xi_2. \end{aligned}$$

Now, let $t'' \in \mathbb{T}$ be such that there exists a $w'' := (v'', d'')$ such that $x_1(w'', t'') = \xi_1$.Since $x_1 \preccurlyeq x_2$, we have that $x_1(w_2, t) = \xi_1$. Obviously, if we take $w_1 := w'' \wedge_t^{t''} w_2$, we then obtain that for all $\tau \geq t''$,

$$\begin{aligned} w_2(\tau - t'' + t) &= w_1(\tau), \\ x_1(w_1, \tau)\mathcal{R}x_2(w_2, \tau - t'' + t)&. \end{aligned}$$

Again, the second line of the equation above is due to the time independent Markovian property. Formally, we still have to prove that

$$\begin{aligned} \left\{\xi \in \mathbb{X}_2 \mid \exists \xi' \in x_1^{\text{init}} \text{ s.t. } \xi'\mathcal{R}\xi\right\} &\supset x_2^{\text{init}}, \\ \left\{\xi \in \mathbb{X}_1 \mid \exists \xi' \in x_2^{\text{init}} \text{ s.t. } \xi\mathcal{R}\xi'\right\} &\supset x_1^{\text{init}}. \end{aligned}$$

However, it can be verified that this is true, due to the construction of $\mathcal{R}$. ☐

Furthermore, we can also obtain the analog of Proposition 4.5, which will be given without proof as follows.

Proposition 5.11. *Given two systems $\Sigma_i := (\mathbb{T}, \mathbb{V} \times \mathbb{D}_i, \mathfrak{B}_i)$, $i = 1, 2$, with equal external behavior. We have that $(\Sigma_1, \pi_{pe}^1) \approx_{rb} (\Sigma_2, \pi_{pe}^2)$, where π_{pe}^i is the past external projection of Σ_i.*

Following Proposition 5.10 and 5.11, we can formulate the analog of Theorem 4.6, as follows

Theorem 5.12. *Given two systems $\Sigma_i := (\mathbb{T}, \mathbb{V} \times \mathbb{D}_i, \mathfrak{B}_i)$, $i = 1, 2$, with equal external behavior. We use π_{pe}^i to denote the past external projection of Σ_i. Suppose that π_{pe}^i possesses the time independent state property for system Σ_i, $i = 1, 2$, and that both systems admit a minimal canonical time independent state map. We denote the minimal canonical time independent state maps as x_{min}^1 and x_{min}^2 respectively. We then have that $(\Sigma_1, x_{min}^1) \approx_{rb} (\Sigma_2, x_{min}^2)$.*

Proof. The proof follows immediately from Proposition 5.10 and 5.11, using the fact that both the past external projection and the canonical minimal time independent state map are time independent Markovian and past-induced. ☐

We shall now revisit the bisimulation of state space systems represented by

$$\dot{x} = Ax + Bu + Gd, \tag{25a}$$
$$y = Cx. \tag{25b}$$

Again, we consider the behavior associated to the weak solution of the differential equation. Recall that in the previous section we require *observability* of the state space system. Now we are going to apply the result of Theorem 5.12 to systems represented by (25), with non-observable states. For that we have to go through some certain procedure.

We shall make use of a result in linear systems theory that any unobservable state space representation of the form (25) can be brought to the form shown in (26), by means of invertible transformation of the states. See for example, Corollary 5.3.14 in [9].

$$\dot{x}^{\text{obs}} = \tilde{A}_{11}x^{\text{obs}} + \tilde{B}_1 u + \tilde{G}_1 d, \tag{26a}$$
$$\dot{x}^{\text{non}} = \tilde{A}_{21}x^{\text{obs}} + \tilde{A}_{22}x^{\text{non}} + \tilde{B}_2 u + \tilde{G}_2 d, \tag{26b}$$
$$y = \tilde{C}x^{\text{obs}}, \tag{26c}$$

with $(\tilde{A}_{11}, \tilde{C})$ as an observable pair. Here we can see that the transformation split the states into the observable and unobservable parts. Recall from the previous section that y and u are external variables and d can be thought of as the internal variable [8, 12].

We claim that removing (26b) from (26) will give us an equivalent behavior with respect to u, y, and d. However, by doing so we obtain an observable state space representation of the same behavior (in form of (26a) and (26c)). Therefore x^{obs} is a state map of the behavior. We denote the observable subspace of the state space as $\mathbb{X}^{\text{obs}}$.

Bisimulation between systems expressed in the form of (25) is given, for example in [12] as

Definition 5.13. *Given two dynamical systems of the form*

$$\dot{x}_i = A_i x_i + B_i u_i + G_i d_i,$$
$$y_i = C_i x_i, \ i = 1, 2.$$

Denote the state space as $\mathbb{X}_i$, $i = 1, 2$. A relation $\mathcal{R} \subset \mathbb{X}_1 \times \mathbb{X}_2$ is a bisimulation relation if for any $(x_{10}, x_{20}) \in \mathcal{R}$ and $u_1(\cdot) = u_2(\cdot)$, the following holds. For any $d_1(\cdot)$, there exists a $d_2(\cdot)$ such that the resulting state solution trajectories $x_1()$, with $x_1(0) = x_{10}$ and $x_2(0) = x_{20}$ satisfy
(i) $(x_1(t), x_2(t)) \in \mathcal{R}$, for all $t \geq 0$,
(ii) $y_1(t) = y_2(t)$, for all $t \geq 0$.
In addition, the two systems are bisimilar if $\pi_i\mathcal{R} = \mathbb{X}_i$, $i = 1, 2$.

Finally, we are going to give an analog of Proposition 4.7 bisimulation (in the sense of Definition 5.13) between systems expressed in the form of (25).

Proposition 5.14. *Given two state space systems represented by*

$$\dot{x}_i = A_i x_i + B_i u + G_i d_i, \tag{28a}$$
$$y = C_i x_i, \tag{28b}$$

with $i = 1, 2$. Suppose that for both systems, the largest controlled invariant subspace (with d_i seen as the control input) of $\dot{x}_i = A_i x_i + G_i d_i$ contained in the kernel of C_i is $\{0\}$. We then have that $(\Sigma_1, x_1) \approx_{rb} (\Sigma_2, x_2)$ if and only if the external behaviors are equal.

Proof. (sketch) First we apply the procedure above to obtain the observable states of the respective systems as external state maps. Hence we split the state spaces such that

$$\mathbb{X}_i = \mathbb{X}_i^{\text{obs}} \oplus \mathbb{X}_i^{\text{uno}},\ i = 1, 2.$$

Analogous to the proof of Proposition 4.7, we have that the past external projections are time independent state maps. Therefore, according to Theorem 5.12 the systems with the observable states are rooted bisimilar (in the sense of Definition 5.8). Let $\mathcal{R} \subset \mathbb{X}_1^{\text{obs}} \times \mathbb{X}_2^{\text{obs}}$ be the corresponding rooted bisimulation relation. It can be proven that $\tilde{\mathcal{R}} \subset \mathbb{X}_1 \times \mathbb{X}_2$, constructed as the following

$$\xi_1 \tilde{\mathcal{R}} \xi_2 :\Leftrightarrow \xi_1^{\text{obs}} \mathcal{R} \xi_2^{\text{obs}},$$

is a bisimulation relation for the two systems, in the sense of Definition 5.13. □

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