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A PRIMER TO RELATIVISTIC MOND THEORY

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Abstract. We first review the nonrelativistic lagrangian theory as a framework for the MOND equation. Obstructions to a relativistic version of it are discussed leading up to TeVeS, a relativistic tensor-vector-scalar field theory which displays both MOND and Newtonian limits. The whys for its particular structure are discussed and its achievements so far are summarized.

1 Introduction

The success of Milgrom's MOND paradigm (Milgrom 1983) in explaining the shapes of disk galaxy rotation curves, and correlations between the parameters of such systems (e.g. Tully-Fisher law) with no appeal to dark haloes is by now well known—for a modern review see Sanders & McGaugh (2003; henceforth S&McG). This nonrelativistic scheme is summed up by a novel relation between acceleration **a** of each element of an extragalactic system and the local Newtonian field $-\nabla\Phi_N$,

$$\mu(|\mathbf{a}|/a_0)\mathbf{a} = -\nabla\Phi_N, \tag{1.1}$$

where $a_0 \approx 10^{-10}$m s^{-2}, and the μ function interpolates between $\mu(x) = x$ for $x \ll 1$ and $\mu(\infty) = 1$. It is this last proviso which guarantees correspondence of MOND with Newtonian gravity in the laboratory or the solar system, both of which are supra a_0 situations.

MOND lends itself to reformulation as a gravitational field theory alternative to Newton's. In this lagrangian based theory, named AQUAL (Bekenstein & Milgrom 1984; henceforth B&M), the Poisson equation is replaced by

$$\nabla \cdot [\mu(|\nabla\Phi|/a_0)\nabla\Phi] = 4\pi G\rho, \tag{1.2}$$

with Φ the physical gravitational potential (so that $\mathbf{a} = -\nabla\Phi$). In spherically symmetric situations Eq. (1.2) has Eq. (1.1) as a unique solution. But when

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symmetry is lower, an extra term appears on the r.h.s. It is this corrected MOND equation which is to be used in place of Eq. (1.1): deriving as it does from a lagrangian, it guarantees that momentum, energy and angular momentum will be conserved in MOND dynamics. The original MOND formula failed to assure this. AQUAL has a number of further advantages over plain MOND (B&M, S&McG).

AQUAL is thus a fruitful tool for exploring non dark matter (DM) resolutions of the missing mass problems. But it has shortcomings. For example, like the naive MOND formula, it fails to do away with the need for DM in clusters of galaxies, although its requirements for the dark stuff are certainly lower than in Newtonian gravity. And since it is basically nonrelativistic, AQUAL is unable to address a number of key subjects with relativistic overtones: gravitational lensing, cosmological evolution and the growth of structure, to name a few. How to cast the MOND idea in a relativistic mold while retaining the benefits of AQUAL ?

2   Relativistic MOND theories leading up to TeVeS

A relativistic version of AQUAL (call it RAQUAL) was offered in its defining paper (B&M). The basic idea was to replace the physical potential Φ by a scalar field ϕ whose equation is a relativistic reflection of Eq. (1.2). This equation is written on a metric $g_{\alpha\beta}$ obeying Einstein's equations (with a contribution to the energy momentum tensor from ϕ). One then defines the physical metric as

$$\tilde{g}_{\alpha\beta} \equiv e^{2\phi} g_{\alpha\beta}. \tag{2.1}$$

This last describes a spacetime which has been locally stretched in every direction with respect to that described by $g_{\alpha\beta}$. All matter and radiations are supposed to sense only $\tilde{g}_{\alpha\beta}$, and the requisite equations must contain only $\tilde{g}_{\alpha\beta}$ and never $g_{\alpha\beta}$ by itself. RAQUAL reproduces nonrelativistic AQUAL, and so recovers much of the successful MOND phenomenology.

However, RAQUAL fails on two counts. It permits superluminal propagation of ϕ waves (B&M). And it is unable to give an account of gravitational lensing in agreement with the basic observation that lensing by galaxy clusters is anomalously strong compared to what was to be expected in view of their galaxies and gas content. This last problem is rooted in the conformal relation (2.1) between the Einstein and physical metrics. Conformally related metrics share the same tracks for photons (or electromagnetic wave packets). And because ϕ's contribution to the source of Einstein's equations is weak in clusters of galaxies, the metric relevant for lensing in RAQUAL is nearly that in GR with no DM.

Breaking the conformal relation has been the central theme in attempts to fix this problem. For example, it is possible to add to Eq. (2.1) the symmetric tensor $\phi_{,\alpha}\,\phi_{,\beta}$ multiplied by some scalar (Bekenstein & Sanders 1994). It turns out that in order for gravitational waves—perturbations of $g_{\alpha\beta}$—to propagate causally, i.e. within the light cone of $\tilde{g}_{\alpha\beta}$, the mentioned scalar factor must be negative, and it is then found to *weaken* gravitational lensing, rather than enhancing it as intended.

A way out of the *impasse* (Sanders 1997) is to introduce a constant vector field which points in the time direction, and then to stretch spacetime along the vector

by a factor $e^{2\phi}$ while shrinking it by the same factor orthogonally to the vector. This does the trick: provided the scalar field ϕ used comes from a RAQUAL-type equation, MOND phenomenology is recovered while the lensing is augmented to the right proportions. However, this "Stratified" theory is not covariant (one needs the vector to be in the time direction, and this cannot be true in all coordinate systems). Covariance of equations of a physical theory is so ingrained in modern physics that we have to look further.

3 The TeVeS field theory

TeVeS is a tensor-vector-scalar covariant field theory (Bekenstein 2004; henceforth B04) incorporating various motifs already mentioned. Just as in RAQUAL, the tensor is an Einstein metric $g_{\alpha\beta}$ out of which is built the standard Einstein-Hilbert action. One passes to the physical metric as in the stratified theory, but now using an everywhere timelike *dynamical* normalized vector field $\mathcal{U}_\alpha$. In equations

$$\tilde{g}_{\alpha\beta} = e^{-2\phi}(g_{\alpha\beta} + \mathcal{U}_\alpha\mathcal{U}_\beta) - e^{2\phi}\mathcal{U}_\alpha\mathcal{U}_\beta, \tag{3.1}$$

together with the normalization requirement $g^{\alpha\beta}\mathcal{U}_\alpha\mathcal{U}_\beta = -1$.

The dynamics of $\mathcal{U}_\alpha$ is dictated by the action

$$S_v = -\frac{K}{32\pi G} \int \left[g^{\alpha\beta}g^{\mu\nu}\mathcal{U}_{[\alpha,\mu]}\mathcal{U}_{[\beta,\nu]} - 2(\lambda/K)(g^{\mu\nu}\mathcal{U}_\mu\mathcal{U}_\nu + 1) \right](-g)^{1/2}d^4x, \tag{3.2}$$

where the square brackets denote antisymmetrization. The K here is a *dimensionless* coupling constant, one of the parameters of the theory. The λ is a Lagrange multiplier field included to enforce the normalization of the vector field; λ is to be determined as the equations are solved. The kinetic part of S_v is that of an Abelian gauge field, but the unit norm constraint deprives $\mathcal{U}_\alpha$ of the proverbial gauge freedom. In addition, $\mathcal{U}_\alpha$ is not coupled to a current of any sort, so it would be incorrect to view the vector as mediating a repulsive Coulomb-like force.

The dynamical scalar field ϕ required in Eq. (3.1) is assigned the action

$$S_s = -\frac{1}{2} \int \left[\sigma^2 h^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta} + \frac{1}{2}G\ell^{-2}\sigma^4 F(kG\sigma^2) \right](-g)^{1/2}d^4x, \tag{3.3}$$

which involves an auxiliary *nondynamical* scalar field σ, a new dimensionless parameter, k, a scale parameter, ℓ, and an unspecified function F. In the kinetic part of S_s, $\phi_{,\alpha}\phi_{,\beta}$ is contracted not with $g^{\alpha\beta}$, but with the tensor $h^{\alpha\beta} \equiv g^{\alpha\beta} - \mathcal{U}^\alpha\mathcal{U}^\beta$; this is introduced to cope with the problem of superluminal propagation of scalar perturbations evidenced by RAQUAL. With $h^{\alpha\beta}$ the propagation is found to be subluminal with respect to the physical metric on condition that $\phi > 0$ (B04). Contrary to appearances, S_s is not a quadratic action. First σ^2 must be expressed in terms of $\phi_{,\alpha}$ by varying S_s with respect to σ^2, and inverting the resulting equation. When the result is substituted in both terms of S_s, the action becomes aquadratic in $\phi_{,\alpha}$, resembling RAQUAL's action, with the scalar $h^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta}$ replacing $g^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta}$. The form (3.3) is, however, more convenient for calculation.

These pieces of the total action must be supplemented with the matter action S_m. In accordance with the Einstein equivalence principle (Will 1993) it must be formed by replacing $g_{\alpha\beta} \mapsto \tilde{g}_{\alpha\beta}$ everywhere in the matter action in use in general relativity (GR), and modifying all covariant derivatives accordingly. In this way all types of matter respond to gravitational fields in the same way, and it could be said that matter and radiation delineate the physical metric. Now because $\tilde{g}_{\alpha\beta}$ contains ϕ as well as $\mathcal{U}_\alpha$, the variations of the total action with respect to these quantities engender variations of S_m which create sources for the equations of ϕ and $\mathcal{U}_\alpha$. No direct coupling of these fields to the matter variables is thus required, which is pleasant as no principles are evident for determining such couplings.

Is TeVeS unique ? Not at all. For example, one could choose to write S_s and S_v with $\tilde{g}_{\alpha\beta}$ instead of $g_{\alpha\beta}$. The present approach seems preferable in that evidently $g_{\alpha\beta}$, ϕ and $\mathcal{U}_\alpha$ are all gravitational fields, so their actions should not involve the "matter's metric" $\tilde{g}_{\alpha\beta}$. Further, one could add to S_s a kinetic term for σ such as is included in Phase Coupled Gravity (PCG), a theory devised in a partially successful attempt to forestall superluminal propagation (Bekenstein 1986, 1990). In addition one could redefine $h^{\alpha\beta}$ as $g^{\alpha\beta} - \zeta\mathcal{U}^\alpha\mathcal{U}^\beta$ (ζ should be positive but not small in order to obviate superluminality; it could be a constant). An additional freedom is the choice of function F in Eq. (3.3); each such choice defines a distinct theory. Some of the freedoms mentioned have been used to make a variant of TeVeS (Sanders 2005a, 2005b; henceforth S05).

4   Results from TeVeS

TeVeS has GR as a limit for $K \to 0$ and $\ell \to \infty$ (B04). Thus if $|K| \ll 1$ and the size of the system of interest falls well short of ℓ, predictions of TeVeS and GR should not be that different. To verify that TeVeS can also give MOND in some other limit one must make a choice for the function F in action (3.3). B04 makes the choice which is equivalent to the relation

$$k\ell^2 h^{\mu\nu}\phi_{,\mu}\phi_{,\nu} = (3/4)k^2 G^2 \sigma^4 (kG\sigma^2 - 2)^2 (1 - kG\sigma^2)^{-1}. \qquad (4.1)$$

This is to be regarded as defining a toy theory. A better choice would have the r.h.s. change sign smoothly through $kG\sigma^2 = 1$, a transition corresponding to the passage between a quasistatic system such as a cluster of galaxies ($h^{\mu\nu}\phi_{,\mu}\phi_{,\nu} > 0$) and cosmology ($h^{\mu\nu}\phi_{,\mu}\phi_{,\nu} < 0$) (see B04).

Imagine linearizing the Einstein-like equations of TeVeS about flat spacetime ($g_{\alpha\beta} = \eta_{\alpha\beta} + \cdots$) in a quasistatic situation. The final result for the physical metric after suitable choice of units is (B04)

$$ds^2 = -(1 + 2\Phi)dt^2 + (1 - 2\Phi)\delta_{ij}dx^i dx^j; \qquad \Phi \equiv \Xi\Phi_N + \phi. \qquad (4.2)$$

Here Φ_N is the usual Newtonian potential of the baryonic matter and Ξ is a constant depending on K and on the asymptotic (cosmological) value of the scalar field ϕ; Ξ is expected to be close to unity. With Φ interpreted as the physical gravitational potential, metric (4.2) is precisely that used in GR to discuss both

dynamics *and* gravitational lensing. An immediate conclusion is that if GR succeeds in describing both the dynamics of, say, a galaxy, and its lensing properties with a *single* assumed DM distribution, and if TeVeS succeeds in predicting the dynamics everywhere from the observed baryonic distribution, then TeVeS will predict the same lensing pattern as GR. Thus the problem that accosted RAQUAL and variants of it (Sec. 2) is solved: lensing can be described using MOND ideas. This much was clear earlier from the non-covariant framework (Sanders 1997). Of course the problem with galaxy cluster dynamics (Sec. 1) still looms. But it is important to realize that the two mentioned problems are distinct.

Obviously, to determine Φ requires one to calculate ϕ from the baryonic mass distribution using the TeVeS analog of Eq. (1.2). In the linearized framework mentioned and assuming spherical symmetry, one can be immediately integrate the said equation once to get (we assume $k \ll 1$ as well as $\Xi \approx 1$)

$$\mu \nabla \Phi = \nabla \Phi_N, \qquad \mu \approx \frac{-1 + \sqrt{1 + 4|\nabla\Phi|/a_0}}{1 + \sqrt{1 + 4|\nabla\Phi|/a_0}}; \qquad a_0 \approx \frac{\sqrt{3k}}{4\pi\ell}. \qquad (4.3)$$

Thus we get the MOND equation (1.1), as well as a formula for μ showing the appropriate asymptotic behavior at small and large $|\nabla\Phi|/a_0$, and the value of a_0 expressed in terms of the TeVeS parameters. MOND is recovered. Famaey and Binney (2005) have noted that the Galaxy's rotation curve is not fully consistent with this μ. Apparently a better F is needed.
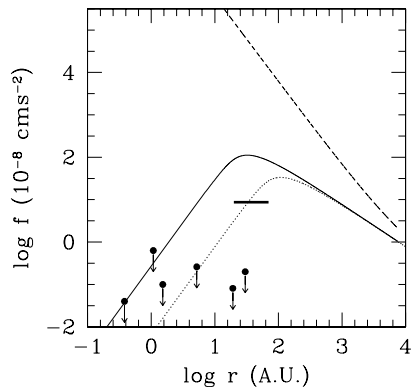
Fig. 1. $|\nabla\Phi|$ (dashed), $|\nabla(\Phi - \Phi_N)|$ (in units of 10^{-8} cms^{-2}) with $k = 0.03$ (solid) or $k = 0.1$ (dotted) from TeVeS plotted vs the distance from the Sun. Observed constraints on the non-Newtonian part of the acceleration, $|\nabla(\Phi - \Phi_N)|$, are (left to right): from the precession of perihelion of Mercury, and of Icarus, from variation of Kepler's constant between Earth and Mars, between inner planets and Jupiter, Uranus or Neptune, respectively. The solid bar is the Pioneer anomaly range.

The approximation inherent in Eq. (4.3) is good for all extragalactic applica-

tions, but begins to fail when $|\nabla\Phi| > (\pi/k)^2 a_0$. In such situations, e.g. the solar system, it is best to start from the first term of the Laurent expansion of the r.h.s. of Eq. (4.1) about $kG\sigma^2 = 1$. One then obtains

$$\mu \approx \frac{1}{\Xi + k/4\pi}\left(1 - \frac{16\pi^3}{k^3}\frac{a_0{}^2}{|\nabla\Phi|^2}\right). \tag{4.4}$$

The fact that μ asymptotes to $(\Xi + k/4\pi)^{-1}$ as $|\nabla\Phi| \to \infty$ means that for arbitrarily strong fields gravity is Newtonian, except that the effective gravitational constant is $G_N = (\Xi + k/4\pi)\,G$. It should be mentioned that the salient post-Newtonian parameters (Will 1993), important for inner solar system celestial dynamics, agree with those of GR; in particular $\beta = 1$ and $\gamma = 1$ (Giannios 2004, B04).

Notice the slow approach of μ to its asymptotic value as $|\nabla\Phi|$ strengthens. One consequence is a slow increase of the Kepler constant a^3/P^2 with distance from the Sun. Fig. 1 illustrates this effect and compares it with the constraints set by measurements of the orbits of diverse planets. Although a better form of F is clearly required, the significant point is that, in such theories, the total force in the outer Solar System will inevitably deviate from inverse square. This is of considerable interest in view of the Pioneer anomaly (Anderson et al. 1998, Turyshev 2005).

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