Equality

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Chapter 8

A simple measure of inequality

8.1 Introduction

In the previous chapter I arrived at a list of properties that a satisfactory measure reflecting the moral badness of inequality should satisfy. They are:

1. A restricted Pigou Dalton transfer principle, meaning that a transfer from a poor person to a rich person will worsen inequality, under the assumption that the poor is worse off than all could have been simultaneously.

2. Symmetry restricted to equalisanda, meaning that changing who gets which part of the equalisanda, or what comes to the same, take a permutation of the distribution, will make no difference for the index of inequality.

3. Restricted homogeneity, expressing the idea that the index is independent of the arbitrary units of measuring the amount of equalisanda

4. Principle of independence of no complaints, adding persons who have no complaints, who are not worse off than they all could have been, does not change the index.

5. Weak independence, meaning that the ordering of inequality based on the distribution of a subgroup cannot be reversed by the welfare of others: $(\vec{x}_g, \vec{x}_{\neg g}) \preceq_I (\succeq_I)(\vec{x}'_g, \vec{x}_{\neg g}) \Longleftrightarrow (\vec{x}_g, \vec{y}_{\neg g}) \preceq_I (\succeq_I)(\vec{x}'_g, \vec{y}_{\neg g})$, which can be strengthened to independence if the reference, representing how well of all could have been, is fixed.

Constructions of a satisfactory index or measure $I(\vec{x})$ such that it is greater if and only if distribution $\vec{x}$ is worse with respect to inequality, so far discussed, failed. This failure did raise some doubts about the possibility of a complete ordering of inequality in a distribution problem. The failure had to be explained and these explanations turned into arguments against a simple measure representing a complete ordering in distribution problems. This line of reasoning is blocked if we can determine a simple measure. In this chapter such a simple measure $I(\vec{x})$ is suggested which is determined up to a monotone increasing transformation. It is sufficient to support the idea that in distribution problems there is a complete ordering regarding inequality.

I start the determination of the measure of inequality by discussing the measure of equalisanda in section 8.2.1. A proper measure of equalisanda is a prerequisite for a satisfactory measure of inequality. Problems concerning the measure of equalisanda will infect a measure of inequality. In section 8.2.2, I will determine the form of the measure of inequality. The properties: independence, the restricted transfer principle and symmetry, and the invariance to units of a measurent of the equalisandum, lead to a measure which is additive separable with respect to the amount of equalisandum allocated to persons. By attending to iso-inequality curves the measure will be determined further. In section 8.2.3 it will be argued that from this class of functions characterised in the previous section one can choose one particular class that has some convenient properties the others lack. It is the class of functions that are monotone transformations of the Euclidean distance from a distribution to the ideal reference, restricted to those being worse off than they could have been in the ideal reference.

Next, I show in section 8.3, how this measure meets the arguments of Sen and Temkin against simple measures. It is argued that on a closer examination the different judgements on the series that Temkin presents as evidence against a simple measure, support the idea of a simple measure. Finally, in section 8.4, I have some remarks on the aggregation of the measures of inequality concerning the diverse equalisanda.

8.2 The construction of a simple measure

8.2.1 The measure of equalisanda

One part of a proper measure of inequality concerns the measure of equalisanda. A measure of inequality presupposes that the equalisanda can be ordered and measured. Problems with a measure of the equalisanda will lead to problems in the ordering regarding inequality. If the ordering with respect to the equalisanda is not clear then an ordering of inequality will of course be problematic too.

In chapter 3 the possibility of interpersonal comparability was already explained. Denying interpersonal comparability was like being a strange creature, a solipsist.[1] By the argument there, comparability of levels of equalisanda was established. Having a certain amount of distribuenda leads to a certain amount of equalisandum which can be compared to the amount of equalisandum another enjoys. But it was not established that differences of amounts of equalisanda could be compared between persons. The latter is of course a desirable property for a measure of inequality.[2] But as was made clear by Ng it is plausible that if interpersonal comparability of levels is possible then interpersonal comparability of differences in equalisanda is possible [Ng, 1984]. Comparability of differences will follow from level comparability if some conditions are satisfied. These are:

1. existence of overlapping individuals, meaning that there are distributions such that a person is better off regarding the equalisandum with one distribution than another person is with another distribution, and there are distributions such that it is the other way around. Furthermore, persons are not equally well off with regard to the equalisandum in all these distributions.

2. semi-connectedness and continuity, meaning that between any pair of distributions in which a person is differently well off with respect to the equalisandum, there is a continuum of distributions connecting the indifference curve of one distribution with the other.

The first condition is satisfied in contexts in which inequality has some relevance. If the condition was not satisfied, some persons would be, whatever the distributions, worse off than all other persons. But then there would be no inequality with respect to these distributions. Because all but

[1] See chapter 3 p. 101.

[2] If comparing the equalisanda different persons enjoy was not possible the orderings would be restricted to those of maximising the minima [Sen, 1973, p. 44]. Improving the second worst-off would not mean an improvement with respect to inequality which is not always according to our intuitions.

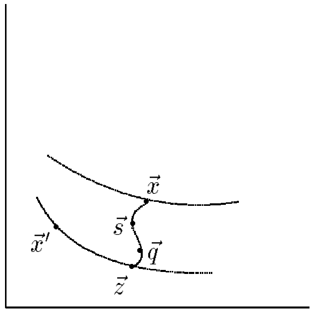

Figure 8.1: Indifference curves for i in Ng's argument

the worst-off would be better off than the reference and the latter would not be below the ideal reference, there would not be any relevant inequality with respect to this equalisandum.

The second condition is also seen to be satisfied in the situations in which there is some inequality. In case of discrete equalisanda while the second condition cannot be satisfied, it is a 'have or have not' situation like for example having or lacking a normal physiological apparatus for vision. In such cases the amount of the equalisandum one person enjoys can be taken arbitrarily and level comparability is directly difference comparability.

Ng's reasoning is as follows, it is illustrated with the help of the figures 8.1 and 8.2. Take for example two overlapping individuals i and j and distributions $\vec{x}, \vec{x}\,', \vec{y}, \vec{y}\,'$ such that in $\vec{x}$ person i is better off than person j is in $\vec{y}$ with respect to the equalisandum, $x_i > y_j$, and in $\vec{x}\,'$ i is worse off than j in $\vec{y}\,'$; $x'_i < y'_j$, and furthermore $x_i > x'_i$ and $y'_j > y_j$. This is possible because of the first condition, the existence of overlapping individuals. Because of semi-connectedness and continuity there exists a distribution $\vec{z}$ such that $z_i = x'_i$, and a continuity of distributions between x_i and x'_i, whether or not via z_i. Similarly, there exists a continuity of distributions between y'_j and y_j whether or not via a distribution $\vec{w}$ such that $w_j = y_j$. Take some $\vec{s}$ such that $y_i < s_i < x_i$ than there is distribution $\vec{t}$ because of semi-connectedness and continuity such that $y_j < t_j < y'_j$ and $s_i = t_j$. But because of continuity there is also a distribution $\vec{q}$ such that $x'_i < q_i < s_i$ and also a distribution $\vec{r}$ such that $q_i = r_j < t_j$. Now because we have $s_i = t_j$ and $s_i > q_i = r_j < t_j$, we can conclude that $s_i - q_i = t_j - r_j$ which

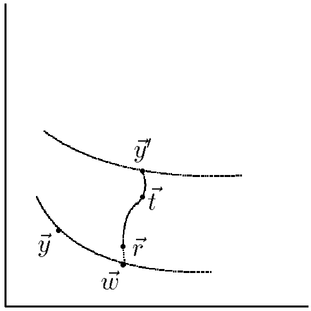

Figure 8.2: Indifference curves for j in Ng's argument

is interpersonal comparison of differences of equalisanda. This reasoning is applicable to all persons, consequently we have level comparability and comparability of differences.

Because for all persons one could easily accept that there is one and the same distribution leading to zero of an equalisandum, namely having zero of the distribuenda, this implies a complete ordering of an interpersonal comparable equalisandum given the equalisandum is intrapersonally a complete ordering, which can be represented with a ratio scale. In such a scale the measure of the equalisanda is determined up to multiplication by a positive real number.[3] The units of measuring the amount of equalisanda can be chosen arbitrarily. This was the reason for homogeneity as one of the properties a measure of inequality should have. The structure of the measure of inequality should be invariant to the units chosen.[4] So far the measurement of the equalisandum is discussed, let me turn to the measure of inequality.

8.2.2 Independence

An index of inequality is of course dependent on the distribution $x_1, \ldots, x_n$ or $\vec{x}$ of equalisanda in which x_i represents the amount of the equalisandum allocated to person $i = 1, \ldots, n$. Such an index can be represented by $I(\vec{x})$. The ordering regarding inequality is ordinal in which only better or worse

[3] See also [Roberts, 1977, p. 23].

[4] It will appear in the next section that the measure of inequality is a homothetic function

with respect to inequality counts, so the measure $I()$ will be determined up to monotone increasing transformations.

In the previous chapters, it was also made clear that what matters regarding inequality is that some are worse off than all simultaneously could have been. That some are better off is not important at all. The moral badness of inequality is dependent on the complaints of those worse off than they could have been. So the index of inequality is dependent on $x_{ref} - x_i$ for those persons i for which $x_i < x_{ref}$, x_{ref} represents how well off with respect to the equalisandum all could have been, i.e. the ideal reference. If $x_i > x_{ref}$, more or less x_i will have no influence. So the relevant complaints of person i can be defined by $c_i = x_{ref} - min\{x_i, x_{ref}\}$. The complaints of those persons having more than in the reference, is zero. The index of inequality is dependent on complaints, so it can be represented by $I(\vec{c})$.

One of the properties a measure should satisfy is independence. It was first restricted to weak independence because whether an increase in the equalisanda resulted in a decrease of inequality was not independent of how well off the others were.[5] It was argued to be dependent on how well off the others were, because the reference could be different. But once the reference is fixed, one can accept independence. This means that if the complaints of one group g increase or decrease, then independent of the complaints of the others who belong to the group $\neg g$, the badness of the inequality varies with the complaints in group g. This will mean that a certain increase in the complaint of one person in group g can be equivalent to an increase of the complaint of another person in group g independent how well off those in $\neg g$ are. With equivalence is meant here that the inequality is the same. The index does not change if one interchanges or substitutes the increase of the complaint of one person with an increase of the complaint of another one. In other words an increase of a complaint of one person which can be offset by the decrease of another such that the moral seriousness remains the same, is independent of how well off others are. How they should be compared is independent of how well off the persons in $\neg g$ are. This property called independence has as a consequence that the index of inequality is of the separate additive form $M(\sum_{i=1}^{n} f_i(c_i))$ ($M()$ being a monotone increasing transformation function).

That this separate additive form is implied can be seen in the following way. Suppose the index can be differentiated twice with respect to the complaints; an assumption which is quite acceptable. The rate of substitution of complaints between two individuals is independent on how well off others are. More formally:

[5] See p. 238 chapter 7.

$$\forall i, j, k : \frac{d(\frac{dI(\vec{c})}{dc_i} / \frac{dI(\vec{c})}{dc_j})}{dc_k} = 0, i, j \neq k$$

By a theorem of Leontief it can be shown that $I()$ will be of the separate additive form Kolm[Kolm, 1977, p. 10].[6] An example with three individuals will suffice to give the idea of how the result is reached. With three individuals there will be three equations of the following form arrived at by the differentiation rule for quotients:

$$\frac{d^2I}{dc_1 dc_3} . \frac{1}{\frac{dI}{dc_2}} - \frac{d^2I}{dc_2 dc_3} . \frac{dI}{dc_1} = 0$$

$$\frac{d^2I}{dc_1 dc_2} . \frac{1}{\frac{dI}{dc_3}} - \frac{d^2I}{dc_3 dc_2} . \frac{dI}{dc_1} = 0$$

$$\frac{d^2I}{dc_2 dc_1} . \frac{1}{\frac{dI}{dc_3}} - \frac{d^2I}{dc_3 dc_1} . \frac{dI}{dc_2} = 0$$

Because $\frac{d^2I}{dc_i dc_j} = \frac{d^2I}{dc_j dc_i}$, else $I()$ would not be differentiable twice [Apostol2, 1961, p. 278] the equations are of the form:

$$a . \frac{1}{\frac{dI}{dc_2}} - b . \frac{dI}{dc_1} = 0$$

$$k . \frac{1}{\frac{dI}{dc_3}} - b . \frac{dI}{dc_1} = 0$$

$$k . \frac{1}{\frac{dI}{dc_3}} - a . \frac{dI}{dc_2} = 0$$

while

$$a = \frac{d^2I}{dc_1 dc_3}; b = \frac{d^2I}{dc_2 dc_3}; k = \frac{d^2I}{dc_1 dc_2}$$

After substituting the results from the second and the third equation in the first equation one arrives at:

$$a . \left(\frac{1}{\frac{dI}{dc_2}} - \frac{dI}{dc_2}\right) = 0$$

[6] This property is also used in the informal proof given by Broome in [Broome, 1991, p. 83]. There are also some other proofs see for example [Fishburn, 1970].

Meaning $a = 0$ resulting in $b = 0$ and $k = 0$, because the other factors cannot be zero, $I()$ is varying with the complaints. The other solution $\frac{dI}{dc_2} = 1$ means also that $b = 0$, and consequently $a = 0$.

So, we can state that the partial derivatives are functions f'_i of those components belonging to i only. So the gradient of $M(\vec{c})$ is:

$$f'_1(c_1)\vec{e}_1 + f'_2(c_2)\vec{e}_2 + f'_3(c_3)\vec{e}_3$$

($\vec{e}_i$ is a unit vector on axis i)

This means that the index is of the separate additive form. That can be seen by integration via the path $(0,0,0);(c_1,0,0);(c_1,c_2,0);(c_1,c_2,c_3)$ which is $\sum_{i=1}^{3} f_i(c_i)$ in which f'_i is the derivative of f_i.[7]

The minimum of the index is reached if the complaints c_i are zero for all. The greater the value of this sum of f_i's, the greater the inequality. Because we are after a representation of an ordinal ordering the index of inequality will be of the separate additive form $M(\sum_{i=1}^{n} f_i(c_i))$ in which M is a monotone increasing transformation function.

So far, we did not use all the properties enumerated above and a natural question is what can be said more about the index because of these properties beyond that it is of a separate additive form. Symmetry with respect to the equalisanda will have as a consequence that the index will not be different if the positions in the tuple $\vec{x}$ describing the distribution change. The dependence of f_i on i, i.e. the position, is not allowed by symmetry, so $\forall i, j, c : f_i(c) = f_j(c) = f(c)$. Hence, the index will be of the form: $M(\sum_{i=1}^{n} f(x_{ref} - min\{x_i, x_{ref}\}))$

Furthermore, the principle of transfers of Pigou Dalton, will lead to the restriction that the possible functions f are convex.[8] If it was not convex, a transfer from a relatively well off person below the reference to another worse off person below the reference, would not lead to a decrease of the index of inequality. But can we say still more about that function f?

By different f's we have different forms of the inequality equivalents, or iso-inequality curves in the space of distributions X^n. Two distributions on such a curve, defined by $I(\vec{c}) = a$ and a is constant, are equivalent regarding inequality. The form of this curve can be derived by the gradient because the direction of the gradient is: $\sum_{i=1}^{n} f'(c_i)\vec{e}_i$.[9] ($f'$ is the derivative of f), is normal to this curve [Apostol2, 1961, p. 265]. The direction of the gradient

[7] This line integral is independent of the path, because $(f'_1, \ldots, f'_n)$ can be taken as a continuous gradient field [Apostol2, 1961, p. 339] and so the index of (x_1, x_2, x_3) is well defined by this integral.

[8] A function is convex if all the points of a line between two points belonging to the graph of the function, are above the graph.

[9] The gradient is $\sum_{i=1}^{n} \frac{dM(\sum_{i=1}^{n} f(c_i))}{d(\sum_{i=1}^{n} f(c_i))} f'(c_i)\vec{e}_i$ but the direction of the gradient is

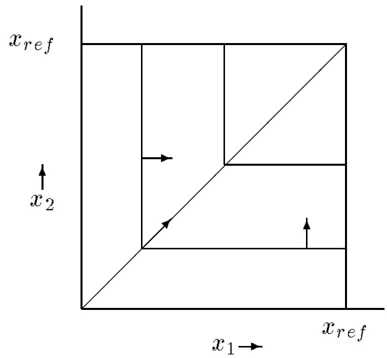

Figure 8.3: Iso-inequality curves expressing almost the maximin measure

is the direction in which a variation of inequality is maximal. Some figures can be illuminating.

Suppose three persons of which person 3 has more than the reference. The iso-inequality curves could for example be as in figure 8.3. It shows an example in which the greatest change in the index of inequality is made by changing the complaints of the worst-off. Figure 8.4 is an example in which the greatest change in the inequality is made by an equal change of the complaints of all. Figure 8.55 shows the case in which the direction of the greatest change is in the direction of the reference, i.e. a change of complaints proportional to the complaints persons have.

This class of measures can be characterised more precisely.[10] Because the equalisandum is measured on a ratio scale the structure of the measure of inequality, the form of the iso-inequality curves, should be invariant to the units used to measure the equalisandum. Because the inequality is measured up to a monotone increasing transformation, the direction of the gradient should be similar on each ray through the reference. They are not dependent on the length of the line between the distribution and the reference, but only on the ratios of the complaints, because the length of the line on a ray has no absolute meaning at all. It could mean a particular amount of inequality, but also any other amount of inequality. The direction

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similar to the one mentioned in the text because one can divide by the common factor $\frac{dM(\sum_{i=1}^{n} f(c_i))}{d(\sum_{i=1}^{n} f(c_i))}$.

[10] The idea of the following explanation is derived from the argument of Hicks in [Hicks, 1965, p. 336].

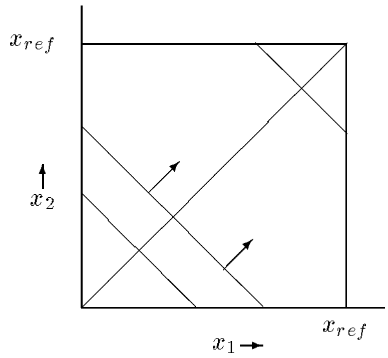

Figure 8.4: Iso-inequality curves expressing almost the total deviation from the reference as measure

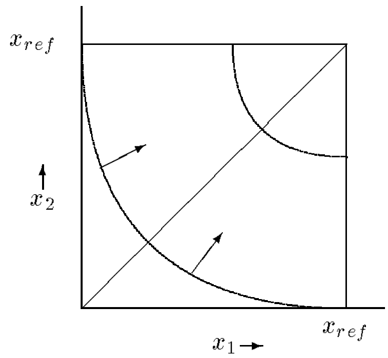

Figure 8.5: Iso-inequality curves expressing the Euclidean distance to the reference as measure

of gradient should not be dependent on the monotone transformation and the particular units used in measuring the equalisandum. In other words the ratios of the parts of gradient should be independent of the units. This means for the gradient:

$$\forall c_i, c_j, \lambda : \frac{M'(\sum_{i=1}^{n} f(c_i)).f'(c_i)}{M'(\sum_{i=1}^{n} f(c_i)).f'(c_j)} = \frac{M'(\sum_{i=1}^{n} f(\lambda.c_i)).f'(\lambda.c_i)}{M'(\sum_{i=1}^{n} f(\lambda.c_i)).f'(\lambda.c_j)}$$

Hence,

$$\forall c_i, c_j, \lambda : \frac{f'(c_i)}{f'(\lambda.c_i)} = \frac{f'(c_j)}{f'(\lambda.c_j)} = k(\lambda)$$

$k(\lambda)$ is a value dependent on λ but independent of c_i. Differentiating with respect to λ gives:

$$0 - \frac{f'(c_i).f''(\lambda.c_i).c_i}{(f'(\lambda.c_i)).(f'(\lambda.c_i))} = k'(\lambda)$$

This is similar to:

$$f''(\lambda.c_i).c_i = f'(\lambda.c_i). - 1.\frac{k'(\lambda)}{k(\lambda)}$$

Take $\lambda = 1$, we have:

$$f''(c_i).c_i = f'(c_i).a$$

(a is constant with respect to c_i) The solution to this is:

$$f'(c) = a_1 c^p + a_2$$

This means for f that it is of the form:[11]

$$f(c) = b_1 c^q + b_2.c + b_3$$

Because of ordinality the constant b_1 can be incorporated in the monotone transformation $M()$ of the index of inequality, we arrive at:

$$f(c) = c^q + d_1.c + d_2$$

Because adding more people which have no complaint does not affect the inequality, (the principle of independence of no complaints) implies $f(0) = 0$, so $d_2 = 0$.

[11] Strictly spoken, one should add by convention $logc + d_1.c + d_2$ if $p = -1$. This is however excluded because f should be convex because of the restricted Pigou Dalton principle. This form violates this condition.

Once again using the fact that the equalisanda are determined up to a ratio scale it will be clear that $d_1 = 0$.

$$\forall c_i, \lambda : \frac{q.(c_i)^{q-1} + d_1}{q.(c_j)^{q-1} + d_1} = \frac{q.\lambda.(c_i)^{q-1} + d_1}{q.\lambda.(c_j)^{q-1} + d_1}$$

But this is only possible if $d_1 = 0$, $(q > 1)$. So the class of indices can be described as:

$$M(\sum_{i=1}^{n}(c_i)^q)$$

In which $q > 1$, because of convexity to f. If $q \to \infty$ figure 8.3 will result [Varian, 1992, p. 20 ff.].

All functions belonging to this class are possible. The question is whether the form of the iso-inequalities can be determined further, or what comes to the same, whether f can be determined.

8.2.3 Euclidean distance as basis

The indices of inequality are of the form:

$$M(\sum_{i=1}^{n}(c_i)^q), (q > 1)$$

An index is determined by a value of q. The question is: can the indeterminacy of the iso-inequality curves be resolved? This indeterminacy of the iso-inequalities was the reason for the gaps in the evaluations, whether a distribution is more or less unequal than another, discussed in the previous chapter.[12] The indeterminacy can be seen to represent the uncertainty about how much an increase in complaints of one person can be set off against a decrease in complaints of another without changing the seriousness of the inequality. By determining this form of iso- inequalities the uncertainty seems to disappear. But this disappearing uncertainty is contrary to our experience of uncertainty. It could be argued that a further specification of the form of the iso-inequalities is not to be expected and even not desirable, because if it was determined precisely, the index would not represent the judgements adequately by denying this uncertainty.

Although this reasoning seems rather convincing, it is not valid. Our experience of uncertainty is not contradicted by a further determination of the iso-inequalities. It is denied that this indeterminacy and uncertainty is an inherent feature of inequality. Uncertainty can mean that although,

[12] See chapter 7 p. 231.

some distribution is worse than another it is not seen by us to be so, for example because of lack of knowledge. This can be the case for example in case we are not sure about the amount of equalisanda someone has. We can be uncertain about the exact dependency of the equalisandum on the distribuenda.[13] It is also possible that the reference is not determined precisely. In that case, the direction of the gradient is not determined exactly because $\frac{(x_{ref}-x_i)^q}{(x_{ref}-x_j)^q}$ varies with x_{ref}. It can be concluded that a further determination of f is not contradicting the possibility of lack of a clear evaluation in a particular case, uncertainty remains possible.

Is there a way of specifying or singling out one of the indices? Of course one could choose a value of q and indeed this particular feature is separating an index from the rest. But this is like using names or dates of birth as relevant features in moral matters.[14] It does not meet the argument of arbitrariness. If particular values of q can be used as singling out the index from the rest, then any particular value could be used as such and indeterminacy remains.

Looking at the figures above it is remarkable that all the indices but the one with $q = 2$ have the feature that a change of inequality in the direction of the gradient, results in a change of the direction of the gradient of the distribution arrived at. This change of the direction of the gradient is dependent on the former change in inequality as is illustrated in figure 8.6. But this dependency will, whatever the way it is constructed, arbitrary. A problem the index with $q = 2$ does not suffer from, because there is no such dependency. The direction of the gradient does not change if there is a change of inequality in the direction of the gradient. The index with $q = 2$ will not suffer from arbitrary dependency of the direction of the gradient on former changes in the direction of the gradient.

The index with $q = 2$ is also the only one such that the direction of the maximal change, i.e. the gradient, at any point is directed to the ideal reference, a property that is likely to be expected from a measure. This index is the only one such that the distribution of the total amount of equalisandum necessary to reach the reference, is following the direction of the maximal change of inequality. This can be seen as follows.

The total amount of the equalisandum necessary to make all the complaints zero is $\sum_{i=1}^{n} \triangle x_i$ such that $\sum_{i=1}^{n} \triangle x_i + \sum_{i=1}^{n} c_i = 0$. Suppose now the distribution represented by $\vec{\alpha}$ of this amount is in the direction of the gradient, $\frac{\alpha_i}{\alpha_j} = \frac{f'(c_i)}{f'(c_j)}$ and $\sum_i^n \alpha_i = 1$, $(\sum_{i=1}^{n} \alpha_i (\sum_{i=1}^{n} \triangle x_i) = \sum_{i=1}^{n} \triangle x_i)$

[13] See for example the footnote on p. 205 in chapter 6.

[14] See chapter 2 p. 24.

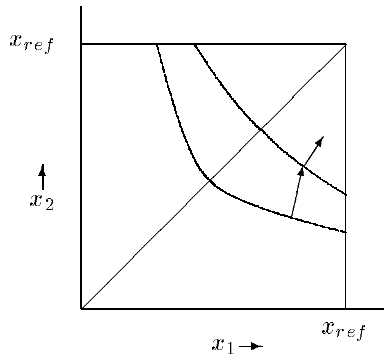

Figure 8.6: Illustration of the change of gradients dependent on former changes in inequality in the direction of gradients

Because $\forall i : c_i + \alpha_i(\sum_{i=1}^{n} \triangle x_i) = 0$ we can state:

$$\frac{f'(c_i)}{f'(c_j)} = \frac{\alpha_i}{\alpha_j} = \frac{\alpha_i . \sum_{i=1}^{i} \triangle x_i}{\alpha_j . \sum_{i=1}^{i} \triangle x_i} = \frac{-c_i}{-c_j} = \frac{c_i}{c_j}$$

In other words $\frac{f'(c_i)}{c_i} = k$ (k is a constant independent from the complaints c_i). So $f'(c_i) = k.c_i$. Integrating this equality shows $f(c)$ is of the form c^2. If the total amount of equalisanda necessary to reach egalitarian reference is distributed along the gradient then the reference will not be reached unless $q = 2$. If $q \neq 2$ the reference will not be reached because of wasting the equalisandum to those already better off than the reference. Only the function with $q = 2$ has the property that an improvement along the gradient at a point will result in reaching the reference.

Because of these properties, a change along the gradient will lead to the reference, and the change of the direction of the gradient is not dependent on the change of the inequality, the index with $q = 2$ can be given priority to the other indices. So, the index reflecting the moral badness of inequality in a proper way is:

$$I(\vec{x}) = M(\sum_{i=1}^{n}(x_{ref} - min\{x_{ref}, x_i\})^2)$$

The inequality is worse, the greater the Euclidean distance between the reference and the distribution restricted to those that are worse off than all could have been.

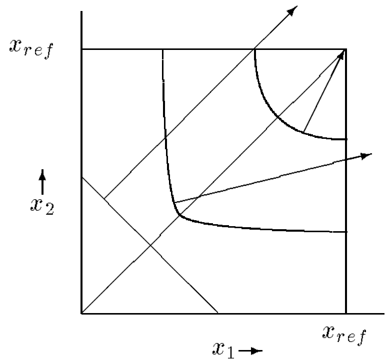

Figure 8.7: Illustration of changes in inequality in the direction of gradients

This index has some resemblance to the variance and the coefficient of variation. There is some relation with the descriptive statistical measures. It would of course be very surprising if there was no relation at all. After all, all the measures that are suggested try to capture some aspect of inequality. Their disadvantage however, was that they lacked the proper connection with what is morally wrong with inequality. The index presented here has the advantage that it is not arbitrary, it is based on the meaning of inequality reflecting what is morally wrong with inequality, namely that some persons are worse off than all could have been simultaneously.

On the other hand because there is such a close connection with the coefficient of variation we should be careful and check whether it does not suffer the drawbacks of the this coefficient, in other words whether it holds against the arguments against the measures discussed in the previous chapters. Because if it could not, there would be a powerful reductio ad absurdum argument against the possibility of a simple measure of inequality.

8.3 The arguments for complexity revisited

It could easily be argued that one important reason for the complexity of inequality has its origin in the assumption that there is one equalisandum. This assumption is responsible for some problems about determining the meaning of equality and also about the measure of inequality. If it is assumed that there is one equalisandum it seems that equality is complex while in fact the problem is located at the attempt to search for 'the most important equalisandum' in all situations. If there is no ordering with respect to this equalisandum, there will not be an ordering of inequality either. But as will be clear, it is not the measure of inequality which is complex but the ordering regarding this 'most important equalisandum'. Once moral value pluralism is accepted, the problem or the lack of the ordering of this 'most important equalisandum' can be bypassed and it is no longer an argument for the complexity of the measure of inequality.

But even if such a plurality is accepted and one turns to one among several equalisanda, even then there are some arguments against a simple measure. These were discussed in the previous chapter.[15] So it is checked whether the index presented here, which has some resemblance with the variance or coefficient of variation, does not suffer from the disadvantages of these measures.

Sen mentions against the coefficient of variation that it is arbitrary and the principle of diminishing transfers is violated, furthermore only comparisons with the mean count.[16] These arguments are not valid against the measure presented here.

First, although it could already be questioned whether the variance was based on a comparison with the mean only, for the index presented here it is certainly not valid.[17] There is no comparison with the mean but with a reference which represents the situation in which all are simultaneously equally as well off as could be with respect to the equalisandum. Sometimes this could happen to be the mean, but it is not necessarily so. This argument of Sen is not valid against the index presented here.

Second, Sen argued that the principle of diminishing transfers is violated by the coefficient of variation. Again, this argument is not valid for the index presented here, because as was argued, the only way in which an ordinal scale could exhibit this principle of diminishing transfers is by denying that in the higher regions of the distribution, above the reference, a transfer has no effect at all.[18] More or less influence cannot be described in another way than either some influence or no influence at all, because differences in inequality cannot be compared. This idea is incorporated in the index by only measuring the gaps between those who are worse off than they might have been in the reference. Those better off do not count at all.

Third, Sen argues that taking the square of the gap in order to satisfy the principle of transfers of Pigou Dalton is just arbitrary, why could not another convex function be taken. But this argument too is answered by this index. It was shown that taking the square was not arbitrary at all, it

[15] In section 7.3.2.

[16] See chapter 7 p. 224.

[17] See the remark by Kakwani cited in footnote on p. 224 in chapter 7.

[18] See p. 233 of chapter 7.

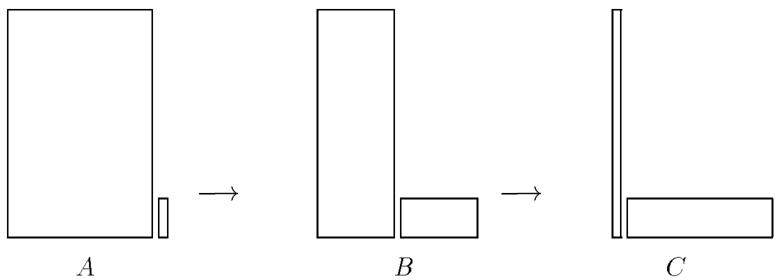

Figure 8.8: Temkin's series

was the only one with the property that an improvement along the gradient at a point would result in reaching the reference. Of course, it could happen to appear that another index can be singled out, but for the time being there is no such reason and the most plausible index is the one following the Euclidean distance. So Sen's arguments can be answered. What about the ones presented by Temkin?

The arguments presented by Temkin against the coefficient of variation or the variance are similar to those of Sen [Temkin, 1993, p. 123 ff.], which have been dealt with in the previous paragraph, but they are embedded in a more general argument for complexity, namely his claim that there are several different aspects of inequality, which is a thread to the idea of a simple measure. His argument is mainly based on differences in evaluations of a series of situations in which there become more and more people worse off illustrated in figure 8.8.

There are the following evaluations with regard to inequality if the number of the poor increases in the series above:

1. better and better
2. worse and worse
3. all equivalent
4. first worse, then better

It is remarkable that the evaluation: 'first better, then worse', is missing. The possibility of these different judgements is explained by Temkin by different principles. Now we should check whether the index presented here can account for these different judgements. How is the series to be evaluated by this index?

In the series, there are two groups, the rich and the poor people, receiving x_{rich} and x_{poor} respectively. Further the reference x_{ref} is equal or below what the rich receive.[19] Suppose the number of the poor is m. Further, the reference will generally be dependent on m so, $x_{ref} = g(m)$ while $g'(m) \leq 0$, ($g'(m)$ is the derivative of $g(m)$) because if m increases the reference will decrease or remain constant. An increasing reference is not plausible. The inequality index will be a monotone increasing transformation of $m(g(m) - x_{poor})^2$.

The judgement 'better and better' means that this index decreases, in other words, the derivative is negative and remains negative. So let us turn to the derivative. This is: $(g(m) - x_{poor})^2 + 2m(g(m) - x_{poor})g'(m) = (g(m) - x_{poor})(g(m) - x_{poor} + 2mg'(m))$. The judgement better and better means that $g(m) - x_{poor} + 2mg'(m) < 0$ for all m which is possible.

The judgement 'worse and worse' means $g(m) - x_{poor} + 2mg'(m) > 0$ for all m which is also possible.

The judgement that all would be equivalent would be the case if $g(m) - x_{poor} + 2mg'(m) = 0$ for all m, this would be the case if $g(m)$ is of the form $m^{-\frac{1}{2}} - x_{poor}.m + k$ or if $g(m) = x_{poor}$.

'First worse, then better' would be the case if there was a change of sign from positive to negative. If the sign of the second derivative is negative, this will not be a problem. The sign of the second derivative is negative, i.e. $g'(m) + 2g'(m) + 2mg''(m) < 0$ if $g''(m) < 0$, in other words if $g(m)$ is concave. $g(m)$ being concave means that the changes in the reference will be less and less as m, the number of the poor, increases. Hence the judgement of 'first worse, then better' will be possible under quite reasonable assumptions.[20]

What about the judgement 'first better, then worse'? Although this is possible by the index presented here, the function will behave rather peculiar. This can be seen if the series is extended with the extremes on both sides, one in which all are equally well off as the rich people and one in which all are equally well off as the poor people.

Because the extremes show no inequality, the judgement 'better and better'would be 'first worse, then better'. The judgements 'worse and worse' would be 'first worse, then better' at the end. The judgement 'all equivalent' would remain all equivalent, if all could have been not better off than the

[19] See the discussion on this in chapter 6 p. 204.

[20] The arguments are similar if the index was determined by $q \neq 2 > 1$. In the formulas in the text the 2's can be replaced by q.

worst-off, or it would be first worse then equivalent and subsequently better, as in an on-off situation. The judgement 'first worse, then better' is just 'first worse, then better'.

The evaluation 'first better, then worse' would be in fact, 'first worse, then better, then worse again and finally better'. This is a rather peculiar behaviour of an index of inequality. This strange behaviour is consistent with the lack of the judgement 'first better, then worse'. The lack of this judgement can be accounted for by assuming that a proper index gives the underlying judgement 'first worse, then better'. So, an index of inequality as presented in this chapter can account for the judgements which Temkin cites as evidence for complexity. But it does something more, it can account also for the lack of judgement 'first better, then worse'. In Temkin's account of the judgements, the missing of the judgement 'first better, then worse' just happens. Actually the acceptance of an index is more in line with the judgements on the series than the account of Temkin himself. The judgements on the series do support the idea that there is a simple measure instead of the idea that (in)equality is complex.

Summarising, the assumption that there is an index satisfying the properties an index of inequality should satisfy led to an index which follows the Euclidean distance of a distribution to the ideal reference. In this it has a resemblance with the well-known measures as the variance and the coefficient of variation. It did not exhibit the drawbacks these suffered from. It could account for the judgements of the series Temkin presented as evidence for the complexity of inequality, it could even account for the implausibiliy of a particular judgement, 'first better then worse'. This judgement would mean that the judgements regarding inequality would vary in a very peculiar way in Temkin's series.

8.4 Specificity and aggregation

Finally, one issue about the index of inequality has to be looked at. In the introduction of this study, I wondered whether it is possible to accept the importance of equality as an ideal within a particularistic framework. I argued that the simple index appeared to be possible because of this framework and not despite of it. The moral universalism, implying that like cases should a priori be treated similar, or what came to the same, if something is worthwhile in one situation it is a priori also in another, although the way and extent in which can differ, blocked a simple measure of inequality. This idea that increase in one value will mean an increase in the total value, led to a quasi-ordering.[21] Sen argued that the extent to which inequality

[21] See chapter 1 p.7.

matters is not independent from the other values and principles [Sen, 1973, p. 75]. For example, the situation in which there is an equal distribution of one good or equalisandum which precludes the survival of at least one person, is not preferable to the situation in which one survives but in which the distribution of the equalisandum is unequal. It is not preferable even not from an egalitarian point of view. In this way moral universalism led to a quasi-ordering instead of an ordering. By accepting moral particularism and moral value pluralism, it becomes possible to analyse the situation differently.

By accepting the plurality of equalisanda, equality is not linked to one equalisandum or one good which was supposed to be an aggregation of several goods. It became possible to analyse a situation such that although a distribution is better regarding inequality with respect to one equalisandum it is not so regarding equality with respect to another equalisandum. Precisely this possibility blocks specificity of equality with respect to one equalisandum.

Kolm proved that if inequality exhibited specificity, i.e. separability with respect to equalisanda, there would be a general index of inequality $I(X) = F(h^1(I^1(\vec{x}^1)) + \ldots + h^m(I^m(\vec{x}^m)))$ (in which the superscripts indicate a particular equalisandum), which is separate additive with respect to the inequality indices each concerning one equalisandum. Now it can be shown that specificity does not apply.[22]

If specificity was valid, it would mean, because of the separate additive form of aggregation, that less inequality with respect to one aspect, would be better regarding inequality in general, disregarding the amount of inequality of the other equalisanda. But the example concerning survival mentioned above, shows it does not. In the situation described above the value of inequality with respect to the other equalisanda does not count, because in this case it could be argued that survival should be the proper equalisandum. Because only one person can survive, equality has nothing to order. In this respect, the survival of only one person is not worse regarding inequality than no survival at all; the reference is no survival at all. Equality with respect to the distribution of the other equalisanda does not make it better regarding inequality. The functions h^j could not be adjusted so that the foregoing situation is treated properly by this separate form of aggregation, namely, that equality with respect to the other equalisanda has nothing to contribute to the equality.[23] An aggregation in this way is

[22] Because of individualism and specificity this index would be possible [Kolm, 1977].

[23] The function h^j could be adjusted in such a way that the total inequality did not allow the repellent conclusion, described on p. 235 in chapter 7, such that inequality with respect to one equalisandum no matter how large, could be set off against the inequality of an equalisandum which became so large because of the numbers it concerned, no

excluded and because of that, specificity is excluded. Lack of specificity is of course no surprise.

That an aggregation in a separate additive way of indices of equalisanda into one index of inequality is not possible, was to be expected. One serious problematic aspect of the traditional views on equality concerned the assumption that there is just one equalisandum. This equalisandum was of course the supervalue or the common comparing value which incorporated all evaluations. The problems with this supervalue, which is, either not leading to an ordering of the equalisandum, or is of no help in determining the content of equality, infected the measure of inequality.[24] Because of these problems with the equalisandum, it seemed that equality was complex. But once multiple equalisanda are allowed, it is possible to formulate a simple measure of inequality.[25] A simple separate additive aggregation of these idices however is not to be expected. Because moral universalism is left it is not any longer a priori possible. The only possibility remaining, is that it could happen to be possible. But the argument given by Sen, although given in order to question the possibility of a simple index of inequality, shows that such a simple aggregation does not happen to exist.[26]

8.5 Summary

In this chapter, I argued contrary to Sen's and Temkin's opinion that a simple measure that represents in a distribution problem a complete ordering regarding inequality is possible. Their argument was based mainly on the lack of a measure which incorporated the most basic ideas of inequality. The measures they considered were not satisfactory, they did not represent the moral badness of inequality in a proper way. That led them to the idea

matter how small the difference is between the better-off and the worse off.

[24] One set of problems is for example seen in determining freedom in general as the equalisandum. The budgetset is suggested as an index of freedom. A greater budgetset means more freedom. However there are problems with this, because of the possibility of intersection of budgetsets instead of inclusion. See for example figure 3.5 in chapter 3. A problem pointed to by Sen and LeGrand and recently by Vallentyne against the suggestion by van Parijs [Sen, 1973, p. 67 ff.] [LeGrand, 1991] [Vallentyne, 1996] [Van Parijs, 1995].

[25] See also p.146 chapter 4.

[26] It is not excluded that one could construct a complete ordering regarding equality over all situations. One could think of such an ordering as a construction of the orderings of the situations in the distribution problems whereby the weighings of the values of the equalisanda take care for the ordering between these problems. It is also not excluded that such an ordering will be incomplete, but the incompleteness will be due to the general problem of weighing different values and will not be located in the complexity of the ideal of equality as was assumed by Temkin en Sen. It is however highly dubious whether such an ordering over all situations makes sense.

that a simple measure was not possible and proposed some arguments to explain this idea. In this chapter, I have shown that a measure is possible. I assumed the existence of a measure and I have shown on the basis of the properties a measure should have, that it is of a separate additive form of which the terms are a function of the complaints of those who are worse off than all could have been regarding the equalisandum. Next, I have shown that, because the equalisandum can be represented on a ratio scale, the function with the complaints as arguments was a simple power function. Of this class, it was possible to point at a particular one to be the index, because it was the only one such that an improvement in the direction of the gradient, the direction of maximal change in inequality, was in the direction of the reference. The index constructed in this chapter is of the form:

$$I(\vec{x}) = M(\sum_{i=1}^{n}(x_{ref} - min\{x_{ref}, x_i\})^2)$$

(in which M is a monotone increasing transformation function)

I considered the arguments of Sen and Temkin again, to check whether this index could handle the arguments they stated for their view that a simple measure was not possible. It appeared that the index was immune to the arguments of Sen and could better account for the judgements Temkin based his view on, than Temkin's own explanation.

Finally, at the end of this chapter it is possible to answer the three questions posed in the introduction:

1. What is the equalisandum?
   There are several equalisanda instead of one, each with a different urgency in a particular problem.

2. Why is equality desirable?
   Inequality matters because some are being worse off than all could have been simultaneously. Denying equality to have any value at all is only possible by practical solipsists, although not impossible, difficult to defend.

3. What is the ordering according to the ideal of equality?
   The ordering in a distribution problem is one represented by

   $$I(\vec{x}) = M(\sum_{i=1}^{n}(x_{ref} - min\{x_{ref}, x_i\})^2)$$

   (in which M is a monotone increasing function)

These answers became possible in a moral realistic framework which is characterised further by moral particularism, realistic individualism and moral pluralism, based on the Wittgenstein-Davidson approach on language and interpretation. This framework was argued to be a fruitful alternative to the traditional background of moral universalism, volitional individualism and moral monism for the development of an ideal of equality.

In the next chapter, I will illuminate some consequences of the answers for current political issues, such as questions concerning health and the distribution of health care.

8.6 Appendix 1

In the previous chapter, I argued that decomposability was not acceptable, because the inequality in a group which is better off than all could have been, would not add to the inequality. Strictly spoken, additivity is not consistent with the idea of equality as it is developed in this study. But just as in case of independence, one could keep the ideal reference fixed for the whole group and admit that indeed inequality in the group of those better off than the reference, does not count, because there is no relevant inequality for that group. So, the reference used in the subgroups for the determination of the inequality within the groups is not above the reference of the whole group.

But what should the references be in the subgroups? It should not be such that improving the best-off in such a subgroup would improve the inequality within the subgroup. The most likely reference is the level of the best-off in the subgroup. All in that subgroup could be as well off as the best-off in a subgroup unless the reference of the whole group is lower than the best-off of the subgroup.

What about the inequality between the groups? This inequality should be determined with the reference of the whole group, and something which represents how well off the groups are. That could be the level of how well off all could have been in that group. Decomposability might be defined in a way that is consistent with the idea of why inequality is wrong. The question is whether the measure as it was developed in this chapter is decomposable. The answer is no, simply because the measure is determined up to monotone transformations, consequently the addition of values of inequality has no meaning. The next question would be, is there a monotone increasing transformation such that index is decomposable?[27] But also this question is to be answered negatively.

Suppose there would be an index which is decomposable. Decompose the whole group in such a way that all strictly worse off than the reference are in one group and the others in another. If the index is decomposable the iso-inequalities in the non-decomposed index should be equal to the iso-inequalities in the decomposed index. If the index is

$$I(\vec{x}) = M(\sum_{i=1}^{n}(x_{ref} - min\{x_{ref}, x_i\})^2)$$

[27] This question is more in line with [Shorrocks, 1984].

If x_2 is the maximum of those worse off than x_{ref} we have also

$$I(\vec{x}) = M(\sum_{i=1}^{n_g}(x_2 - x_i)^2) + 0 + M(\sum_{i=1}^{n}(x_{ref} - x_2)^2)$$

The iso-inequalities should be equal of these indices. This means that the direction of the gradient should be equal. In other words:

$$\frac{(x_{ref} - x_1)}{(x_{ref} - x_3)} = \frac{M'_g.2.(x_2 - x_1) + 0}{M'_g.2.(x_2 - x_3) + 0}$$

(M'_g is the first derivative with respect to its argument which is $\sum_{i=1}^{n_g}(x_2 - x_i)^2$) This relation should be valid for all $x_i < x_2$. This will be satisfied only if:

$$\frac{(x_{ref} - x_i)}{(x_2 - x_i)} = k$$

This will not be true in general. So there is no decomposable index.[28]

[28] Taking the mean as representant will not do either.