Control in a behavioral context

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Chapter 4

More on interconnection

In this chapter we look into several diverse issues that arise when dealing with control in a behavioral setup. In section 4.1 we bring out the relation between regular interconnection and the issue of implementability by feedback. Roughly speaking, regularity of interconnection means that there is a signal flow graph of the kind that exists in 'intelligent control' (see figure 1.1). One can describe a regular controller as feeding back the sensor signals of the plant suitably into the actuator inputs. Feedback interconnection in the behavioral framework has been broached in Willems [74]. We prove some results related to feedback in the setting we have adopted, namely, the control variables being different from the to-be-controlled variables. In section 4.2 we show an important connection between disturbances, their freeness and regularity of interconnection. We prove that when an interconnection is not regular, there is a potential situation in which our controller is putting restrictions on disturbances (against the spirit that disturbances get chosen by the environment and are hence *free* for the controlled system). Section 4.3 deals with the filtering problem and this is formulated and solved. The results here are similar to those in Valcher & Willems [66]. In section 4.4 the pole placement and the stabilization results of the previous chapter are specialized to the case that we want these specifications on, not just the variables w, but on the control variables c also. In other words, the case that all the plant variables are to be controlled (stabilized, etc). In section 4.5 we dwell upon the concept of canonical controller as introduced in van der Schaft [48, 47]. We derive the conditions under which this controller is a regular controller. Section 4.6 deals with the question under what conditions we can control a plant with a *controllable* controller. Many of the results of this chapter are utilized in chapter 7 when we deal with the synthesis of dissipative systems.

4.1 Feedback interconnection

This section considers the issue of implementability of a controller in a feedback configuration. This issue is important when we want to attach a controller to a plant in a way such that the controller takes the measured outputs of the plant as inputs and such that the controller outputs are fed back into the control inputs of the plant. Thus the i/o partition of the controller gets fixed by the plant. It is an important question whether a controller that we have obtained (using the results of the previous chapter, for example) adheres to such an i/o configuration. In case our controller does not adhere, then the question arises whether there exists one that does adhere to the given i/o partition. Regularity of the interconnection is related to this question, as was shown in Willems [75]. Before we define what a feedback configuration is, we recall the necessary results about an i/o partition of the variables of a behavior.

Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ and let $R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ be a minimal kernel representation. Then there exists a partition $w = (u, y)$ (after perhaps a permutation of the components of w) such that $\mathfrak{B}$ is represented minimally by $P(\frac{\mathrm{d}}{\mathrm{d}t})y = Q(\frac{\mathrm{d}}{\mathrm{d}t})u$. It was explained in section 2.9 that $w = (u, y)$ is an i/o partition if and only if $\det(P) \neq 0$.

We now recall the definition of feedback interconnection as defined in Willems [75] for the case of *full interconnection*. The interconnection of $\mathfrak{B}^1$ and $\mathfrak{B}^2 \in \mathfrak{L}^{\mathtt{w}}$ is said to be a *feedback interconnection* if, after permutation of components, there exists a partition of w into $w = (u, y_1, y_2)$ such that

- in $\mathfrak{B}^1$, (u, y_1) is input and y_2 output,
- in $\mathfrak{B}^2$, (u, y_2) is input and y_1 output, and
- in $\mathfrak{B}^1 \cap \mathfrak{B}^2$, u is input and (y_1, y_2) output.

Here by 'input', we mean in the $\mathfrak{C}^\infty$-sense. Note how, in a feedback interconnection, the output cardinalities add up, thus showing that a feedback interconnection is a regular interconnection, i.e. $\mathtt{p}(\mathfrak{B}^1) + \mathtt{p}(\mathfrak{B}^2) = \mathtt{p}(\mathfrak{B}^1 \cap \mathfrak{B}^2)$. It was shown in Willems [75] that the converse is also true: if the full interconnection of $\mathfrak{B}^1$ and $\mathfrak{B}^2$ is regular, then the interconnection is also a feedback interconnection. Thus the classical method of attaching a controller that feeds back the plant outputs into the plant inputs is 'regular'.

In this section we shall extend the results to the case that the controller acts on just the control variables. For this case we shall consider the question under what conditions the control variables c can be partitioned in such a way that the controller can be considered as a feedback controller. Under the assumption that the manifest controlled behavior

$\mathcal{K}$ is autonomous, and under regularity assumption on the interconnection, the following theorem shows that such a partition always exists. This is illustrated in figure 4.1.

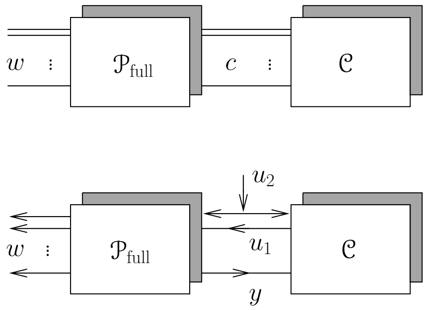

Figure 4.1: Feedback interconnection of $\mathcal{P}$ and $\mathcal{C}$

Theorem 4.1.1 : *Let $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$ and let $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ be autonomous and regularly implementable through c. Let $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ be a controller that regularly implements $\mathcal{K}$. Then, after permuting the components of c, there exists a partition of c into $c = (y, u_1, u_2)$ such that:*

(i) for $(w, y, u_1, u_2) \in \mathcal{P}_{\text{full}}$, (u_1, u_2) is input and (w, y) is output,
(ii) for $(y, u_1, u_2) \in \mathcal{C}$, (y, u_2) is input and u_1 is output,
(iii) for $(w, y, u_1, u_2) \in \mathcal{K}_{\text{full}}$, u_2 is input and (w, y, u_1) is output.

Proof : Let $R_1(\frac{\mathrm{d}}{\mathrm{d}t})w + R_2(\frac{\mathrm{d}}{\mathrm{d}t})c = 0$ be a minimal kernel representation of $\mathcal{P}_{\text{full}}$. Let U be a unimodular matrix such that $UR_1 = \operatorname{col}(G, 0)$, with G full row rank. Accordingly partition

$$UR_2 = \begin{bmatrix} R_{21} \\ R_{22} \end{bmatrix}.$$

Then

$$\begin{bmatrix} G & R_{21} \\ 0 & R_{22} \end{bmatrix} \begin{bmatrix} w \\ c \end{bmatrix} = 0$$

is a minimal kernel representation of $\mathcal{P}_{\text{full}}$. Let $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ regularly implement $\mathcal{K}$. Assume $C(\frac{\mathrm{d}}{\mathrm{d}t})c = 0$ is a minimal kernel representation of $\mathcal{C}$. Then a minimal kernel representation of the corresponding $\mathcal{K}_{\text{full}}$ is given by

$$\begin{bmatrix} G & R_{21} \\ 0 & R_{22} \\ 0 & C \end{bmatrix} \begin{bmatrix} w \\ c \end{bmatrix} = 0. \tag{4.1}$$

The submatrix $\mathrm{col}(R_{22}, C)$ has full row rank, hence (after a permutation of its columns, and accordingly, of the components of c), there exists a partition of this submatrix into

$$\begin{bmatrix} P_1 & Q_1 \\ P_2 & Q_2 \end{bmatrix}$$

such that $\mathrm{col}(P_1, P_2)$ is square and nonsingular. Due to the nonsingularity, again after possibly permuting the columns, we can partition

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix},$$

with P_{11} and P_{22} square and nonsingular. Such a partition exists because of Lagrange's formula which expresses the determinant as a sum of the products of the determinants of its minors of suitable dimensions.

Summarizing, partitioning $c = (y, u_1, u_2)$, we have now found the following minimal representation of $\mathcal{K}_{\mathrm{full}}$:

$$\begin{bmatrix} G & * & * & * \\ 0 & P_{11} & P_{12} & Q_1 \\ 0 & P_{21} & P_{22} & Q_2 \end{bmatrix} \begin{bmatrix} w \\ y \\ u_1 \\ u_2 \end{bmatrix} = 0,$$

with the $*$'s denoting the corresponding blocks of R_{21}. Note that if $\mathcal{K}$ is autonomous, so is $\mathcal{N}$. Since $G(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ is a minimal kernel representation of $\mathcal{N}$, G must be square and nonsingular. Since

$$\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

is square and nonsingular, we infer that in $\mathcal{K}_{\mathrm{full}}$, u_2 is input and (w, y, u_1) is output. By nonsingularities of P_{11} and G, it follows that in $\mathcal{P}_{\mathrm{full}}$, (u_1, u_2) is input and (w, y) is output. Finally from the nonsingularity of P_{22} we obtain that in $\mathcal{C}$, (y, u_2) is input and u_1 is output. This proves the theorem. □

In the special case that $\mathcal{K}_{\mathrm{full}}$ is autonomous there are no inputs and the matrix

$$\begin{bmatrix} G & R_{21} \\ 0 & R_{22} \\ 0 & C \end{bmatrix}$$

in equation (4.1) is square and nonsingular. The partitioning still works, except that u_2 is absent. Figure 4.1 depicts how the control variables are partitioned into inputs and outputs in order to implement the controller behavior in a feedback configuration. Note that the transfer functions of both the plant and the controller with respect to the given i/o partitions are in general singular, i.e. possibly nonproper. This is also the situation in the following theorem.

The above theorem *assigns* an i/o partition without modifying the controller itself. Often, we are not allowed to choose such a partition, because we are given *a priori* that some variables are sensors, while others are actuators. Hence, necessarily, the sensors are plant outputs and should, correspondingly, be controller inputs. The actuators then are inputs to the plant. In the following theorem we show that if our plant $\mathcal{P}_{\text{full}}$ has an *a priori* given i/o structure with respect to sensors and actuators, and if $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ is regularly implementable and autonomous, then $\mathcal{K}$ can be regularly implemented by a controller $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ that takes the sensors as input, and actuates part of the plant actuators. Since $\mathcal{K}_{\text{full}}$ is again not necessarily autonomous, some control variables remain free. These can be interpreted as plant actuators which are not being used for the control of the to-be-controlled variables.

Theorem 4.1.2 : *Let $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{y}+\mathtt{u}}$ with to-be-controlled variable w and control variable $c = (y, u)$. Assume, in $\mathcal{P}_{\text{full}}$, u is input and (w, y) is output. Then for every regularly implementable, autonomous $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$, there exist a controller $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ that implements $\mathcal{K}$ through c, and a partition $u = (u_1, u_2)$ such that*

- *in $\mathcal{C}$, (y, u_2) is input and u_1 is output,*
- *in $\mathcal{K}_{\text{full}}$, u_2 is input and (w, y, u_1) is output.*

Proof : The proof of this theorem closely mimics the proof of the previous theorem. Let $\mathcal{P}_{\text{full}}$ be represented by the minimal kernel representation

$$\begin{bmatrix} G & R_{21y} & R_{21u} \\ 0 & R_{22y} & R_{22u} \end{bmatrix} \begin{bmatrix} w \\ y \\ u \end{bmatrix} = 0,$$

with G square and nonsingular (again because $\mathcal{N}$ is autonomous). Let $C_y y + C_u u = 0$ be a minimal kernel representation of a controller $\mathcal{C}' \in \mathfrak{L}^{\mathtt{c}}$ that regularly implements $\mathcal{K}$ through c. Hence we have $\mathcal{K}_{\text{full}}$ given by the following minimal kernel representation

$$\begin{bmatrix} G & R_{21y} & R_{21u} \\ 0 & R_{22y} & R_{22u} \\ 0 & C_y & C_u \end{bmatrix} \begin{bmatrix} w \\ y \\ u \end{bmatrix} = 0.$$

The submatrix $\begin{bmatrix} R_{22y} & R_{22u} \\ C_y & C_u \end{bmatrix}$ has full row rank. Further, R_{22y} is square and nonsingular because (w, y) is output in $\mathcal{P}_{\text{full}}$. This implies that $\begin{bmatrix} R_{22y} \\ C_y \end{bmatrix}$ has full column rank. Hence, it is possible to partition (after a permutation) u into $u = (u_1, u_2)$ such that $\mathcal{K}_{\text{full}}$ is represented as follows

$$\begin{bmatrix} G & R_{21y} & R_{21u_1} & R_{21u_2} \\ 0 & R_{22y} & R_{22u_1} & R_{22u_2} \\ 0 & C_y & C_{u_1} & C_{u_2} \end{bmatrix} \begin{bmatrix} w \\ y \\ u_1 \\ u_2 \end{bmatrix} = 0 \tag{4.2}$$

with $\begin{bmatrix} R_{22y} & R_{22u_1} \\ C_y & C_{u_1} \end{bmatrix}$ square and nonsingular. This allows us to choose u_2 as input to $\mathcal{K}_{\text{full}}$ and the rest of the variables as output. In order to have u_1 as output of the controller, we require that C_{u_1} be nonsingular. From $\mathcal{C}'$ we shall construct a $\mathcal{C} \in \mathfrak{L}^{\mathrm{c}}$ to obtain the necessary nonsingularity. We have that $\begin{bmatrix} R_{22u_1} \\ C_{u_1} \end{bmatrix}$ has full column rank. Hence there exists a $T \in \mathbb{R}^{\bullet\times\bullet}[\xi]$ such that $\det(C_{u_1} + TR_{22u_1}) \neq 0$. Once such a T is found, we define $\mathcal{C} \in \mathfrak{L}^{\mathrm{c}}$ by the kernel representation

$$(C_y + TR_{22y})y + (C_{u_1} + TR_{22u_1})u_1 + (C_{u_2} + TR_{22u_2})u_2 = 0,$$

with output u_1 and input (y, u_2). This $\mathcal{C}$ also implements $\mathcal{K}_{\text{full}}$ regularly. This completes the proof. □

4.2 Disturbances and freeness

This section analyzes the way the freeness of any unmodeled disturbances in a controlled system is connected to the regularity of the interconnection. The precise meaning of 'unmodeled' is made clear in what follows.

Consider again the problems of stabilization and pole placement for a given plant $\mathcal{P}_{\text{full}}$ with to-be-controlled variable w and control variable c (as formulated in sections 3.3 and 3.4). In most system models, an unknown external disturbance variable d, also occurs. The stabilization problem is then to find a controller acting on c such that whenever $d(t) = 0$ $(t \geqslant 0)$, we have $w(t) \to 0$ $(t \to \infty)$. Typically, the disturbance d is assumed to be free, in the sense that every $\mathfrak{C}^\infty$ function d is compatible with the equations of the model. As an example, think of a model of a car suspension system given by $R_1(\frac{\mathrm{d}}{\mathrm{d}t})w + R_2(\frac{\mathrm{d}}{\mathrm{d}t})c + R_3(\frac{\mathrm{d}}{\mathrm{d}t})d = 0$, where d is the road profile as a function of time. In the stabilization problem, one puts $d = 0$ and solves the stabilization problem for the full plant $\mathcal{P}_{\text{full}}$ represented by $R_1(\frac{\mathrm{d}}{\mathrm{d}t})w + R_2(\frac{\mathrm{d}}{\mathrm{d}t})c = 0$. In doing this, one should make sure that the stabilizing controller $\mathcal{C}$: $C(\frac{\mathrm{d}}{\mathrm{d}t})c = 0$, when connected to the actual model, *does not put restrictions on d.*

Thus in general when one encounters a to-be-controlled plant $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$, $\mathcal{P}_{\text{full}}$ can be extended to $\mathcal{P}^{\text{ext}}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}+\mathtt{d}}$, which includes a representation of the disturbance behavior as well. We say the disturbance variables were unmodeled in the original system $\mathcal{P}_{\text{full}}$. Equivalently, $\mathcal{P}^{\text{ext}}_{\text{full}}$ is an extension of $\mathcal{P}_{\text{full}}$ such that the disturbances also have been modeled in $\mathcal{P}^{\text{ext}}_{\text{full}}$. We now define precisely how an extension is related to the original model.

Consider the full plant behavior $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$. An *extension* of $\mathcal{P}_{\text{full}}$ is a behavior $\mathcal{P}^{\text{ext}}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}+\mathtt{d}}$ (with $\mathtt{d}$ an arbitrary positive integer), with variables (w, c, d), such that

1. d is free in $\mathcal{P}^{\text{ext}}_{\text{full}}$,
2. $\mathcal{P}_{\text{full}} = \{(w, c) \mid (w, c, 0) \in \mathcal{P}^{\text{ext}}_{\text{full}}\}$.

Thus, $\mathcal{P}^{\text{ext}}_{\text{full}}$ being an extension of $\mathcal{P}_{\text{full}}$ formalizes that $\mathcal{P}_{\text{full}}$ contains exactly those signals (w, c) that are compatible with the disturbance $d = 0$ in $\mathcal{P}^{\text{ext}}_{\text{full}}$. Of course, a given full behavior $\mathcal{P}_{\text{full}}$ has many extensions.

For a given extension $\mathcal{P}^{\text{ext}}_{\text{full}}$ and a given controller $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$, we define the extended controlled behavior by

$$\mathcal{K}^{\text{ext}}_{\text{full}} = \{(w, c, d) \mid (w, c, d) \in \mathcal{P}^{\text{ext}}_{\text{full}} \text{ and } c \in \mathcal{C}\}.$$

During the process of controller design, one would need that the controller has the following property: the disturbance d remains free in $\mathcal{K}^{\text{ext}}_{\text{full}}$, *for any possible extension* $\mathcal{P}^{\text{ext}}_{\text{full}}$. It turns out that this is guaranteed exactly by the regularity of the interconnection of $\mathcal{P}_{\text{full}}$ and $\mathcal{C}$!

Theorem 4.2.1 : *Let $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$ and $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$. The following two conditions are equivalent.*

1. *The interconnection of $\mathcal{P}_{\text{full}}$ and $\mathcal{C}$ is regular,*
2. *for any extension $\mathcal{P}^{\text{ext}}_{\text{full}}$ of $\mathcal{P}_{\text{full}}$, d is free in the extended controlled behavior $\mathcal{K}^{ext}_{\text{full}}$.*

Proof : *((1) ⇒ (2):)* Suppose $\mathcal{P}^{\text{ext}}_{\text{full}}$ is represented minimally by $R_1(\frac{\mathrm{d}}{\mathrm{d}t})w + R_2(\frac{\mathrm{d}}{\mathrm{d}t})c + R_3(\frac{\mathrm{d}}{\mathrm{d}t})d = 0$. Then $\mathcal{P}_{\text{full}}$ is represented by $R_1(\frac{\mathrm{d}}{\mathrm{d}t})w + R_2(\frac{\mathrm{d}}{\mathrm{d}t})c = 0$.

We first claim that $[R_1\ R_2]$ also has full row rank. Indeed, assume this matrix did not have full row rank. Then after premultiplication by a unimodular matrix, $\mathcal{P}_{\text{full}}^{\text{ext}}$ is represented minimally by

$$\begin{bmatrix} R_1' & R_2' & R_3' \\ 0 & 0 & R_3'' \end{bmatrix} \begin{bmatrix} w \\ c \\ d \end{bmatrix} = 0, \tag{4.3}$$

with $R_3'' \neq 0$. Equation (4.3) has $R_3''(\frac{\mathrm{d}}{\mathrm{d}t})d = 0$, and this means that d is not free (against our assumption). Thus $[R_1\ R_2]$ has full row rank, as claimed.

Assume $C(\frac{\mathrm{d}}{\mathrm{d}t})c = 0$ is a minimal kernel representation of the controller $\mathcal{C}$. Since $\mathcal{P}_{\text{full}}$ and $\mathcal{C}$ are interconnected regularly, $\begin{bmatrix} R_1 & R_2 \\ 0 & C \end{bmatrix}$ also has full row rank.

Consider the following minimal kernel representation of the extended controlled behavior $\mathcal{K}_{\text{full}}^{\text{ext}}$:

$$\begin{bmatrix} R_1 & R_2 & R_3 \\ 0 & C & 0 \end{bmatrix} \begin{bmatrix} w \\ c \\ d \end{bmatrix} = 0\ , \quad \text{or} \quad \begin{bmatrix} R_1 & R_2 \\ 0 & C \end{bmatrix} \begin{bmatrix} w \\ c \end{bmatrix} = \begin{bmatrix} -R_3 \\ 0 \end{bmatrix} d. \tag{4.4}$$

Because of the full row rank condition on $\begin{bmatrix} R_1 & R_2 \\ 0 & C \end{bmatrix}$, (see proposition 2.9.4), d is free in the $\mathfrak{C}^\infty$ sense in $\mathcal{K}_{\text{full}}^{\text{ext}}$ also.

((2) ⇒ (1):) Let $R_1(\frac{\mathrm{d}}{\mathrm{d}t})w + R_2(\frac{\mathrm{d}}{\mathrm{d}t})c = 0$ be a minimal representation of $\mathcal{P}_{\text{full}}$. One particular extension $\mathcal{P}_{\text{full}}^{\text{ext}}$ of $\mathcal{P}_{\text{full}}$, is represented by $R_1 w + R_2 c + d = 0$. Let $\mathcal{C}$ be given by the minimal kernel representation $Cc = 0$. Then we have that d is free in

$$\begin{aligned} R_1 w + R_2 c + d &= 0, \\ Cc &= 0. \end{aligned} \tag{4.5}$$

We now show that $\begin{bmatrix} R_1 & R_2 \\ 0 & C \end{bmatrix}$ has full row rank. Suppose this matrix did not have full row rank. Then there exists a polynomial row vector $[p_1\ p_2] \neq [0\ 0]$, such that

$$[p_1 \;\; p_2] \begin{bmatrix} R_1 & R_2 \\ 0 & C \end{bmatrix} = [0 \;\; 0]\,.$$

Now we claim that $p_1 \neq 0$. For, otherwise, we get $p_2 C = 0$, and this implies $p_2 = 0$ too, since C has full row rank. Hence, as claimed, $p_1 \neq 0$. From equation (4.5) we get that for all (w, c, d) in $\mathcal{K}_{\text{full}}^{\text{ext}}$, we have

$$[p_1 \ \ p_2] \begin{bmatrix} R_1 & R_2 & I \\ 0 & C & 0 \end{bmatrix} \begin{bmatrix} w \\ c \\ d \end{bmatrix} = 0.$$

This implies that d satisfies the differential equation $p_1(\frac{\mathrm{d}}{\mathrm{d}t})d = 0$, which would mean that d is not free in $\mathcal{K}_{\text{full}}$. Hence $[p_1 \ \ p_2] \neq [0 \ \ 0]$ leads to a contradiction. This means that

$$\begin{bmatrix} R_1 & R_2 \\ 0 & C \end{bmatrix}$$

has full row rank, so we have shown that the interconnection of $\mathcal{P}_{\text{full}}$ and $\mathcal{C}$ is indeed regular. □

The above theorem shows that regularity of interconnection is equivalent to the controller allowing the disturbances in every extended model to be free. By interconnecting regularly, we are introducing new laws that are *just enough* to control the plant, without imposing any restrictions, whatsoever, on disturbances in any extended plant model.

4.3 Filtering and estimation

Our general problem formulation of finding, for a given $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$, a regularly implementable, stable $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ includes also a problem that is, at first sight, not a control problem, but rather a *filtering problem.*

Consider the set-up of figure 4.2. The *observed plant* $\mathcal{P}_{\text{obs}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{y}}$ has two types of variables, w and y. w is a variable that we want to *estimate* and y is a variable that we *measure.*

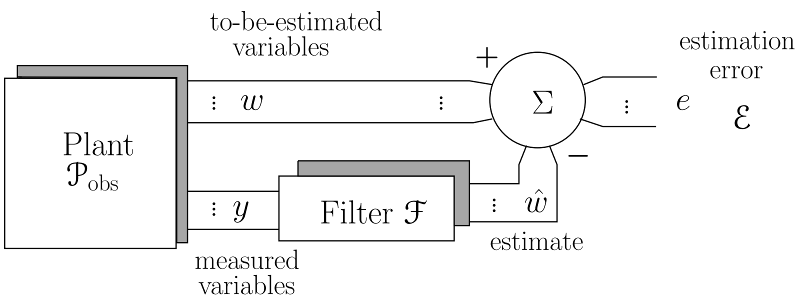

Figure 4.2: Plant and observer configuration

A *filter* is a system $\mathcal{F} \in \mathfrak{L}^{\mathtt{w}+\mathtt{y}}$, with variables $(y, \hat{w})$. The idea is to find a filter $\mathcal{F}$ such that in the interconnection of $\mathcal{P}_{\text{obs}}$ and $\mathcal{F}$ through y (the measured variable), $\hat{w}$ becomes an estimate of w. In order to formalize this, for a given filter $\mathcal{F}$ we define the associated *estimation error behavior* $\mathcal{E}$ by

$$\begin{aligned} \mathcal{E} = \ & \{e \in \mathcal{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid \exists\, w, \hat{w}, y \text{ such that:} \\ & \quad (w, y) \in \mathcal{P}_{\text{obs}},\ (y, \hat{w}) \in \mathcal{F} \text{ and } e = w - \hat{w}\}\ . \end{aligned} \tag{4.6}$$

If $\mathcal{E}$, $\mathcal{P}_{\text{obs}}$ and $\mathcal{F}$ are related via equation (4.6), we say that $\mathcal{E}$ is *implemented by the filter* $\mathcal{F}$. Given $\mathcal{P}_{\text{obs}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{y}}$, a given behavior $\mathcal{E} \in \mathfrak{L}^{\mathtt{w}}$ is called *implementable* (with respect to $\mathcal{P}_{\text{obs}}$) if there exists a filter $\mathcal{F} \in \mathfrak{L}^{\mathtt{w}+\mathtt{y}}$ such that $\mathcal{E}$ is implemented by $\mathcal{F}$. The question what $\mathcal{E}$'s are implementable is answered in the following lemma. In the following, let $\mathcal{N}$ be the hidden behavior associated with $\mathcal{P}_{\text{obs}}$, i.e.

$$\mathcal{N} = \{w \mid (w, 0) \in \mathcal{P}_{\text{obs}}\}.$$

Lemma 4.3.1 : *Let $\mathcal{P}_{\text{obs}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{y}}$. Then we have:*

1. *The behavior $\mathcal{E} \in \mathfrak{L}^{\mathtt{w}}$ is implementable if and only if $\mathcal{N} \subseteq \mathcal{E}$.*
2. *If $\mathcal{E}$ is autonomous and implementable, it can be implemented by a filter $\mathcal{F} \in \mathfrak{L}^{\mathtt{w}+\mathtt{y}}$ such that, in $\mathcal{F}$, y is input and $\hat{w}$ output.*

Proof : Let $R_w(\frac{\mathrm{d}}{\mathrm{d}t})w + R_y(\frac{\mathrm{d}}{\mathrm{d}t})y = 0$ be a minimal kernel representation of $\mathcal{P}_{\text{obs}}$. Then $R_w(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ is a kernel representation of $\mathcal{N}$.
(1 – only if:) Assuming $\mathcal{E}$ is implementable, we show that $\mathcal{N} \subseteq \mathcal{E}$. We have that there exists an $\mathcal{F} \in \mathfrak{L}^{\mathtt{w}+\mathtt{y}}$ that implements $\mathcal{E}$. Let $F_1(\frac{\mathrm{d}}{\mathrm{d}t})y + F_2(\frac{\mathrm{d}}{\mathrm{d}t})\hat{w} = 0$ be a kernel representation of $\mathcal{F}$. Writing down the various representations in (4.6) into a matrix, we get the following latent variable representation of $\mathcal{E}$ (with latent variable $(w, y, \hat{w})$):

$$\begin{bmatrix} 0 \\ I \\ 0 \end{bmatrix} e = \begin{bmatrix} R_w & R_y & 0 \\ I & 0 & -I \\ 0 & F_1 & F_2 \end{bmatrix} \begin{bmatrix} w \\ y \\ \hat{w} \end{bmatrix}. \tag{4.7}$$

Let $e \in \mathcal{N}$. Then $R_w(\frac{\mathrm{d}}{\mathrm{d}t})e = 0$. Hence, in equation (4.7), e is supported by the latent variable $(w, y, \hat{w}) = (e, 0, 0)$. We infer that $e \in \mathcal{E}$.
(1 – if:) We assume $\mathcal{N} \subseteq \mathcal{E}$. We shall prove that this $\mathcal{E}$ is implementable. Clearly, there exists an $F \in \mathbb{R}^{\bullet\times\bullet}[\xi]$ such that $FR_we = 0$ is a kernel representation of $\mathcal{E}$. We claim that

$$FR_w\hat{w} + FR_yy = 0, \tag{4.8}$$

is a kernel representation of an $\mathcal{F}$ that implements $\mathcal{E}$. Indeed, the estimation error behavior, $\mathcal{E}'$, implemented by $\mathcal{F}$ is represented by the latent variable representation

$$\begin{bmatrix} 0 \\ I \\ 0 \end{bmatrix} e = \begin{bmatrix} R_w & R_y & 0 \\ I & 0 & -I \\ 0 & FR_y & FR_w \end{bmatrix} \begin{bmatrix} w \\ y \\ \hat{w} \end{bmatrix}. \tag{4.9}$$

To obtain a kernel representation of $\mathcal{E}'$, we eliminate w, y and $\hat{w}$ from the above equation. By premultiplication of (4.9) by the unimodular matrix

$$\begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ F & FR_w & -I_3 \end{bmatrix}$$

we find that $\mathcal{E}'$ is also represented by

$$\begin{bmatrix} 0 \\ I \\ FR_w \end{bmatrix} e = \begin{bmatrix} R_w & R_y & 0 \\ I & 0 & -I \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} w \\ y \\ \hat{w} \end{bmatrix}.$$

Since $\begin{bmatrix} R_w & R_y & 0 \\ I & 0 & -I \end{bmatrix}$ has full row rank, we see that $\mathcal{E}'$ is represented by $F(\frac{\mathrm{d}}{\mathrm{d}t})R_w(\frac{\mathrm{d}}{\mathrm{d}t})e = 0$, whence $\mathcal{E}' = \mathcal{E}$. It follows that $\mathcal{F}$ indeed implements $\mathcal{E}$, completing the proof of the first part of the lemma.

(2:) Now, supposing $\mathcal{E}$ is autonomous, we can take the F to be such that FR_w is square and nonsingular. Hence from equation (4.8), it follows that in $\mathcal{F}$, $\hat{w}$ is output and y is input. □

The problem we want to consider in this section is to find a filter that makes the estimation error behavior *stable*. The following theorem states when such a filter exists.

Theorem 4.3.2 : *Let $\mathcal{P}_{\mathrm{obs}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{y}}$. There exists a filter $\mathcal{F} \in \mathfrak{L}^{\mathtt{w}+\mathtt{y}}$ such that the estimation error $\mathcal{E}$ is stable if and only if, in $\mathcal{P}_{\mathrm{obs}}$, w is detectable from y. In that case, there exists a filter such that the measured variable y is input and the estimate $\hat{w}$ is output.*

Proof : *(⇒:)* Assume a filter $\mathcal{F}$ exists such that $\mathcal{E}$ is stable. Since $\mathcal{E}$ is implementable, from lemma 4.3.1 we have that $\mathcal{N} \subseteq \mathcal{E}$. Hence $\mathcal{N}$ is stable, equivalently w is detectable from y.

(⇐:) Assume $\mathcal{N}$ is stable. Obviously, $\mathcal{E} := \mathcal{N}$ is implementable by a filter $\mathcal{F}$. Further, $\mathcal{N}$ is stable and hence autonomous, so it is also implementable by a filter which has input y and output $\hat{w}$. □

Remark : Another relevant problem is to find, for a given $\mathcal{P}_{\text{obs}}$, a filter such that $\mathcal{E} = 0$. Obviously, such an $\mathcal{F}$ exists if and only if, in $\mathcal{P}_{\text{obs}}$, w is observable from y.

4.4 Full plant: stabilization and pole placement

In the previous chapter we have studied the stabilization problem, where we wanted to find a controller acting on the control variables c such that the w-trajectories in the controlled system tend to 0 as $t \to \infty$. Also in the case of pole placement, only the w-trajectories were required to satisfy certain specifications. While the to-be-controlled variables are steered to zero, it is often important to steer the control variables to zero as well. This amounts to demanding a stable $\mathcal{K}_{\text{full}}$. We discuss here the conditions on $\mathcal{P}_{\text{full}}$ under which there exists a stable, regularly implementable $\mathcal{K}_{\text{full}}$. We also discuss pole placement of $\mathcal{K}_{\text{full}}$. As mentioned, the problem of finding a controller such that both w and c satisfy the desired specifications is a special case of the results in chapter 3.

We continue with the constraint that we can apply control only through the c-variable. Recall that, given a $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$, we call $\mathcal{K}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$ implementable w.r.t. $\mathcal{P}_{\text{full}}$ if there exists a $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ that implements $\mathcal{K}_{\text{full}}$ through c, i.e.

$$\mathcal{K}_{\text{full}} = \{(w,c) \in \mathcal{P}_{\text{full}} \mid c \in \mathcal{C}\}.$$

$\mathcal{K}_{\text{full}}$ is called regularly implementable if such a $\mathcal{C}$ exists with the property that the interconnection of $\mathcal{P}_{\text{full}}$ and $\mathcal{C}$ is regular. We formulate the problems now.

Full stabilization problem : Given $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$, find conditions for the existence of $\mathcal{K}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$ that is stable and regularly implementable w.r.t. $\mathcal{P}_{\text{full}}$.

Full pole placement problem : Given $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$, find conditions under which, for each monic polynomial $r \in \mathbb{R}[\xi]$, there exists a $\mathcal{K}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$ such that $\chi_{\mathcal{K}_{\text{full}}} = r$ and $\mathcal{K}_{\text{full}}$ is regularly implementable w.r.t. $\mathcal{P}_{\text{full}}$.

The following theorem solves the full stabilization problem. The solution to these problems involves again the hidden behavior as defined in equation (3.4)

$$\mathcal{N} = \{w \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid (w,0) \in \mathcal{P}_{\text{full}}\}.$$

We need stabilizability of $\mathcal{P}_{\text{full}}$ and stability of $\mathcal{N}$ for the existence of a stable, regularly implementable $\mathcal{K}_{\text{full}}$. Note that stability of $\mathcal{N}$ is equivalent to detectability of w from c in $\mathcal{P}_{\text{full}}$,

Theorem 4.4.1 : *Let $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$. There exists a stable, regularly implementable $\mathcal{K}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$ if and only if the following two conditions are satisfied:*

1. *$\mathcal{N}$ is stable, i.e. in $\mathcal{P}_{\text{full}}$, w is detectable from c, and*
2. *$\mathcal{P}_{\text{full}}$ is stabilizable.*

Proof : The idea is to include c in the to-be-controlled variable. Thus we define the to-be-controlled variable w' by $w' := (w, c)$. Let $\mathcal{P}^{\text{aux}}_{\text{full}}$ be the auxiliary system obtained from $\mathcal{P}_{\text{full}}$ as follows

$$\mathcal{P}^{\text{aux}}_{\text{full}} = \{(w', c) \mid w' = (w, c) \text{ and } (w, c) \in \mathcal{P}_{\text{full}}\}.$$

Then the w'-behavior $\mathcal{P}^{\text{aux}}$ associated with $\mathcal{P}^{\text{aux}}_{\text{full}}$ is equal to $\mathcal{P}_{\text{full}}$, while the hidden behavior $\mathcal{N}^{\text{aux}}$, is given by

$$\mathcal{N}^{\text{aux}} = \{w' \mid w' = (w, 0) \text{ and } w' \in \mathcal{P}_{\text{full}}\}.$$

It follows that $w' \in \mathcal{N}^{\text{aux}}$ if and only if $w' = (w, 0)$ with $w \in \mathcal{N}$. This leads to the equivalence of the stability of $\mathcal{N}^{\text{aux}}$ and that of $\mathcal{N}$. Also the $\mathcal{K}_{\text{full}}$ that we are seeking is a stable and regularly implementable $\mathcal{K}^{\text{aux}}$. By using the stabilization theorem – theorem 3.4.1, we get that there exists a $\mathcal{K}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$ satisfying the problem statement if and only if $\mathcal{N}^{\text{aux}}$ is stable and $\mathcal{P}^{\text{aux}}$ is stabilizable. This completes the proof. □

The following theorem establishes the corresponding conditions for the full pole placement problem. Here, observability of w from c comes into picture, which from lemma 3.1.3, is equivalent to $\mathcal{N} = 0$.

Theorem 4.4.2 : *Let $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$. For every $r \in \mathbb{R}[\xi]$, there exists a regularly implementable $\mathcal{K}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$ such that $\chi_{\mathcal{K}_{\text{full}}} = r$ if and only if the following two conditions are satisfied:*

1. *$\mathcal{N} = 0$, i.e. in $\mathcal{P}_{\text{full}}$, w is observable from c, and*
2. *$\mathcal{P}_{\text{full}}$ is controllable.*

We omit the proof of this theorem since it is similar to that of the previous one.

4.5 Canonical controller

This section contains a study of the concept of canonical controller as was introduced in van der Schaft [47, 48]. We look into implementability issues for this elegant method of constructing a controller. We also prove necessary and sufficient conditions for the canonical controller to be a regular controller.

Throughout our discussion on control, we have emphasized that it is the controlled behavior $\mathcal{K}$ that is of interest, and we have not paid much attention to constructing controllers that implement a given controlled behavior. We have been satisfied with just making sure that $\mathcal{K}$ is implementable (i.e. making sure that $\mathcal{N} \subseteq \mathcal{K} \subseteq \mathcal{P}$) and invoking the controller implementability theorem to guarantee that there exists a controller $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ that implements $\mathcal{K}$. Often there are many controllers that implement a given $\mathcal{K}$.

Given $\mathcal{P}_{\text{full}}$ and an implementable $\mathcal{K}$, there is a representation-free method of constructing a controller that implements $\mathcal{K}$. It was shown in van der Schaft [47] that one could interconnect the desired controlled behavior $\mathcal{K}$ with $\mathcal{P}_{\text{full}}$ through the to-be-controlled variables w to obtain a controller behavior $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ that implements $\mathcal{K}$. This is illustrated in figure 4.3. This controller has been called the canonical controller.

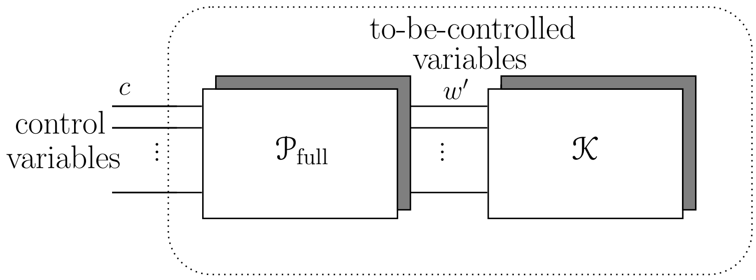

Figure 4.3: The canonical controller

The canonical controller is particularly attractive because it contains all the features of a controller that is based on the principle of an *internal model* of the plant. In requiring a desired controlled behavior $\mathcal{K}$ from the plant, what could be easier than specifying precisely this $\mathcal{K}$ to the plant? The following figure (figure 4.4) illustrates the canonical controller interconnected with the plant.

More precisely, we arrive at the following definition of the canonical controller.

Definition 4.5.1 : Suppose $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$ and let $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ satisfy $\mathcal{N} \subseteq \mathcal{K} \subseteq \mathcal{P}$. The canonical controller $\mathcal{C}_{\text{can}} \in \mathfrak{L}^{\mathtt{c}}$ is defined as

$$\mathcal{C}_{\text{can}} := \{c \mid \exists\, w' \in \mathcal{K} \text{ such that } (w', c) \in \mathcal{P}_{\text{full}}\}. \tag{4.10}$$

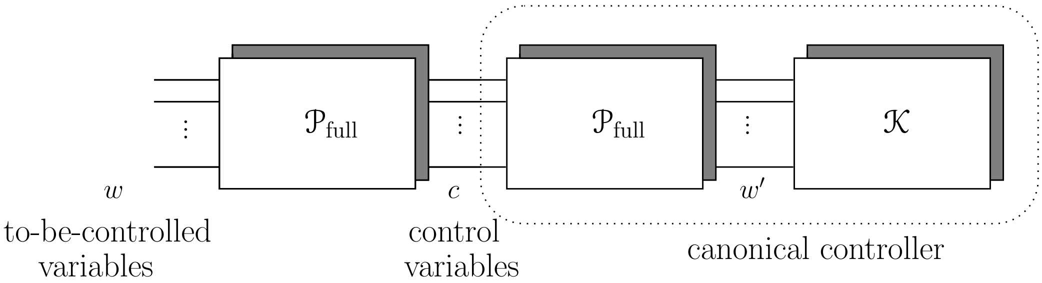

Figure 4.4: The canonically controlled system

Compare the above equation of $\mathcal{C}_{\text{can}}$ with the condition in equation (3.3), which expresses that a controller $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ implements $\mathcal{K}$:

$$\mathcal{K} = \{w \mid \exists\, c \in \mathcal{C} \text{ such that } (w, c) \in \mathcal{P}_{\text{full}}\}.$$

The reversal of the roles that w and c play in the above two equations leads us to interpret $\mathcal{C}_{\text{can}}$ as the controller behavior implemented by $\mathcal{K}$, by interconnection with $\mathcal{P}_{\text{full}}$ through w. We shall use this remark later in corollary 4.6.4 below.

The following proposition from van der Schaft [47] makes sure that the canonical controller implements $\mathcal{K}$ precisely when $\mathcal{K}$ is implementable.

Proposition 4.5.2 : *Let $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$ and $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$. Let $\mathcal{C}_{\text{can}}$ be the associated canonical controller. Then $\mathcal{C}_{\text{can}}$ implements $\mathcal{K}$ if and only if $\mathcal{N} \subseteq \mathcal{K} \subseteq \mathcal{P}$.*

Before we continue studying further properties of the canonical controller $\mathcal{C}_{\text{can}}$ that is constructed from $\mathcal{K}$, with $\mathcal{K}$ satisfying the assumption $\mathcal{N} \subseteq \mathcal{K} \subseteq \mathcal{P}$, we shall digress into the role that $\mathcal{N} \subseteq \mathcal{K} \subseteq \mathcal{P}$ plays. Obviously, definition 4.5.1 of $\mathcal{C}_{\text{can}}$ applies to any $\mathcal{K}$, and not just to those $\mathcal{K}$'s satisfying $\mathcal{N} \subseteq \mathcal{K} \subseteq \mathcal{P}$. The following theorem shows that, in general, $\mathcal{C}_{\text{can}}$ implements $\mathcal{N} + \mathcal{K} \cap \mathcal{P}$.

Theorem 4.5.3 : *Let $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$ and $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ be given. Let $\mathcal{C}_{\text{can}}$ be the canonical controller defined in equation* (4.10) *above. Let $\hat{\mathcal{K}} \in \mathfrak{L}^{\mathtt{w}}$ be the manifest controlled behavior obtained by the interconnection of $\mathcal{P}_{\text{full}}$ and $\mathcal{C}_{\text{can}}$ through c. Then*

$$\hat{\mathcal{K}} = \mathcal{N} + \mathcal{K} \cap \mathcal{P}.$$

Proof : *($\hat{\mathcal{K}} \supseteq \mathcal{N} + \mathcal{K} \cap \mathcal{P}$:)* Since $\hat{\mathcal{K}}$ is implementable (by $\mathcal{C}_{\text{can}}$, for instance) it follows that $\hat{\mathcal{K}} \supseteq \mathcal{N}$. Now let $w \in \mathcal{K} \cap \mathcal{P}$. Then there exists c such that $(w, c) \in \mathcal{P}_{\text{full}}$. $w \in \mathcal{K}$ implies that $c \in \mathcal{C}_{\text{can}}$. Since $(w, c) \in \mathcal{P}_{\text{full}}$ and $c \in \mathcal{C}_{\text{can}}$, by definition of $\hat{\mathcal{K}}$ we infer that $w \in \hat{\mathcal{K}}$. Thus $\hat{\mathcal{K}} \supseteq \mathcal{K} \cap \mathcal{P}$. By linearity we get $\hat{\mathcal{K}} \supseteq \mathcal{N} + \mathcal{K} \cap \mathcal{P}$.

($\hat{\mathcal{K}} \subseteq \mathcal{N} + \mathcal{K} \cap \mathcal{P}$:) Let $\tilde{w} \in \hat{\mathcal{K}}$. This implies that there exists $c \in \mathcal{C}_{\text{can}}$ such that $(\tilde{w}, c) \in \mathcal{P}_{\text{full}}$. By definition (equation (4.10)), $c \in \mathcal{C}_{\text{can}}$ means there exists $w \in \mathcal{K}$ such that $(w, c) \in \mathcal{P}_{\text{full}}$. This implies that $w \in \mathcal{P}$ and hence $w \in \mathcal{K} \cap \mathcal{P}$. Thus it only remains to show that $\tilde{w} - w \in \mathcal{N}$. This follows since both (w, c) and $(\tilde{w}, c) \in \mathcal{P}_{\text{full}}$ thus implying $(\tilde{w} - w, 0) \in \mathcal{P}_{\text{full}}$. Thus every $\tilde{w} \in \hat{\mathcal{K}}$ can be written as the sum of $\tilde{w} - w \in \mathcal{N}$ and $w \in \mathcal{K} \cap \mathcal{P}$ proving $\hat{\mathcal{K}} \subseteq \mathcal{N} + \mathcal{K} \cap \mathcal{P}$. □

Note that as a special case of the above theorem, if $\mathcal{N} \subseteq \mathcal{K} \subseteq \mathcal{P}$, we indeed obtain $\hat{\mathcal{K}} = \mathcal{K}$. We now address the issue whether $\mathcal{C}_{\text{can}}$ is a *regular* controller. The conditions under which the canonical controller is regular turns out to depend on just $\mathcal{P}_{\text{full}}$ and not on the desired controlled behavior $\mathcal{K}$. We shall show that the canonical controller is regular if and only if the control variables are free in $\mathcal{P}_{\text{full}}$.

Define the *control variable plant behavior* $\mathcal{P}_c \in \mathfrak{L}^{\mathtt{c}}$ as the behavior obtained from $\mathcal{P}_{\text{full}}$ by eliminating w

$$\mathcal{P}_c := \{c \mid \exists\, w \text{ such that } (w, c) \in \mathcal{P}_{\text{full}}\}. \tag{4.11}$$

$\mathcal{P}_c = \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{c}})$ is equivalent to c being free in $\mathcal{P}_{\text{full}}$. (In fact, this was the way freeness was defined in section 2.9.) The following theorem characterizes the conditions on $\mathcal{P}_{\text{full}}$ under which *every* controller is a regular controller.

Theorem 4.5.4 : *Let $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$ be given and let $\mathcal{P}_c \in \mathfrak{L}^{\mathtt{c}}$ be the control variable plant behavior (as defined in equation* (4.11)*). Then every controller $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ is a regular controller if and only if $\mathcal{P}_c = \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{c}})$.*

Proof : Let $R(\frac{\mathrm{d}}{\mathrm{d}t})w + M(\frac{\mathrm{d}}{\mathrm{d}t})c = 0$ be a minimal kernel representation of $\mathcal{P}_{\text{full}}$. We begin by noting that $\mathcal{P}_c = \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{c}})$ is equivalent to R having full row rank.
(if :) Suppose $\mathcal{P}_c = \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{c}})$. Let $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ be given by a minimal kernel representation $C(\frac{\mathrm{d}}{\mathrm{d}t})c = 0$. Then

$$\begin{bmatrix} R & M \\ 0 & C \end{bmatrix} \begin{bmatrix} w \\ c \end{bmatrix} = 0$$

is a kernel representation of $\mathcal{K}_{\text{full}}$.
Then it follows that the interconnection is regular since both R and C have full row rank themselves. This shows that $\mathcal{C}$ is a regular controller.
(only if :) We need to show that if every controller is regular then $\mathcal{P}_c = \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{c}})$. Assume, to the contrary, $\mathcal{P}_c \subsetneqq \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{c}})$. This implies that R does not have full row rank.

We premultiply the equations $R(\frac{d}{dt})w + M(\frac{d}{dt})c = 0$ to obtain the following equivalent minimal kernel representation of $\mathcal{P}_{\rm full}$

$$\begin{bmatrix} 0 & M_1 \\ R_2 & M_2 \end{bmatrix} \begin{bmatrix} w \\ c \end{bmatrix} = 0 \tag{4.12}$$

with R_2 and M_1 having full row rank. We see that the controller $\mathcal{C} \in \mathfrak{L}^{\tt c}$ with minimal kernel representation $M_1(\frac{d}{dt})c = 0$ is a controller that is not regular. This contradiction establishes that $\mathcal{P}_c = \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\tt c})$. □

One of the immediate consequences of the above theorem is that, under the condition $\mathcal{P}_c = \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\tt c})$, every implementable $\mathcal{K}$ is also regularly implementable. This is shown as follows. Suppose $\mathcal{P}_c = \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\tt c})$. Let $\mathcal{K}$ be implementable. Then there exists a $\mathcal{C} \in \mathfrak{L}^{\tt c}$ which implements $\mathcal{K}$. From the above theorem, this $\mathcal{C}$ is also a regular controller. This implies that $\mathcal{K}$ is regularly implementable.

We now come to regularity of the canonical controller. Consider the the canonical controller $\mathcal{C}_{\rm can} \in \mathfrak{L}^{\tt c}$ as defined in equation (4.10). The following theorem shows that $\mathcal{P}_c = \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\tt c})$ is both necessary and sufficient for $\mathcal{C}_{\rm can}$ to be a regular controller. Relating this to theorem 4.5.4, we notice that the canonical controller is regular if and only if every controller is regular.

Theorem 4.5.5 : *Let $\mathcal{P}_{\rm full} \in \mathfrak{L}^{\tt w+c}$ and let $\mathcal{K} \in \mathfrak{L}^{\tt w}$ satisfy $\mathcal{N} \subseteq \mathcal{K} \subseteq \mathcal{P}$ where $\mathcal{N}$ and $\mathcal{P}$ are the hidden and the manifest plant behaviors, respectively. Let $\mathcal{C}_{\rm can} \in \mathfrak{L}^{\tt c}$ be the canonical controller. $\mathcal{C}_{\rm can}$ implements $\mathcal{K}$ regularly if and only if $\mathcal{P}_c = \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\tt c})$.*

Proof : *(if :)* If $\mathcal{P}_c = \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\tt c})$, then every controller is regular from the previous lemma. In particular, the canonical controller is also regular.
(only if :) Without loss of generality, we assume $\mathcal{P}_{\rm full}$ to have a minimal kernel representation of the form in equation (4.12) above, with R_2 and M_1 having full row rank. Since $\mathcal{N} \subseteq \mathcal{K}$, we can assume a minimal kernel representation of $\mathcal{K}$ to have the form $F(\frac{d}{dt})R_2(\frac{d}{dt})w = 0$. Then the following is a latent variable representation of $\mathcal{K}_{\rm full}$ (with latent variable w').

$$\begin{array}{l} \mathcal{P}_{\rm full} \left\{ \vphantom{\begin{matrix} 0 \\ 0 \end{matrix}} \right. \\ \mathcal{C}_{\rm can} \left\{ \vphantom{\begin{matrix} 0 \\ 0 \\ 0 \end{matrix}} \right. \end{array} \begin{bmatrix} 0 & M_1 \\ R_2 & M_2 \\ \hdashline 0 & M_1 \\ 0 & M_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} w \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \hdashline 0 \\ R_2 \\ FR_2 \end{bmatrix} w' \tag{4.13}$$

Eliminating w' from the above equation we see that a kernel representation of the canonical controller $\mathcal{C}_{\text{can}}$ is of the form:

$$\begin{bmatrix} M_1 \\ FM_2 \end{bmatrix} c = 0 \ . \tag{4.14}$$

We see that $\mathcal{C}_{\text{can}}$ always repeats some laws of $\mathcal{P}_{\text{full}}$, namely the rows in M_1. Thus $\mathcal{C}_{\text{can}}$ is a regular controller only if $M_1 = 0$. This is equivalent to $\mathcal{P}_c = \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{c}})$. This proves the result. □

Remarks:

(1:) We note here that the condition $\mathcal{P}_c = \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{c}})$ is not particularly restrictive. It is satisfied in several standard classical control problems where it is assumed that the additive 'noise' influences the measured output surjectively. For example, see Trentelman, et al [60], theorem 11.14 in the $\mathcal{H}_2$ optimal control setup and theorem 14.1 in the context of $\mathcal{H}_\infty$ control, or see Zhou [83], assumption (ii) in section 13.5 regarding the $\mathcal{H}_2$ optimal control problem. These assumptions ensure that $\mathcal{P}_c = \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{c}})$.

(2:) A second remark about the canonical controller is relevant. Suppose a given $\mathcal{K}$ is regularly implementable w.r.t. $\mathcal{P}_{\text{full}}$ through c. This means that there exists a controller that regularly implements $\mathcal{K}$. However, every controller that implements $\mathcal{K}$ need not be a regular controller. From theorem 3.2.2, we know that the property of regular implementability of $\mathcal{K}$ means that the autonomy of $\mathcal{P}$ is retained within $\mathcal{K}$. Hence, every controller that implements $\mathcal{K}$, though not necessarily regularly, at least ensures that the autonomy of $\mathcal{P}$ is retained within $\mathcal{K}$. In this sense the (possible) non-regularity of the canonical controller need not make it any less attractive.

4.6 Controllability of the controller

In this section we address the issue about the controllability of the controller that we utilize to implement $\mathcal{K}$ in, for example, the pole placement problem. This property of $\mathcal{K}$ which allows us to implement it by a controllable controller is defined below as $\mathcal{K}$ being *controllably* implementable.

Definition 4.6.1 : Let $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$ and $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$. $\mathcal{K}$ is called *controllably* implementable if there exists a $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}_{\text{cont}}$ that implements $\mathcal{K}$.

Note that the word 'controllably' in the definition above refers to the controllability of $\mathcal{C}$ and *not* to the controllability of $\mathcal{K}$. Suppose $\mathcal{K}$ is implementable, then controllability of $\mathcal{K}$ is not necessary for $\mathcal{K}$ to be controllably implementable. However, it turns out that controllability of $\mathcal{K}$ is sufficient, i.e. if $\mathcal{K}$ is controllable and implementable, then $\mathcal{K}$ is controllably implementable. This is shown within the following theorem, which contains a slightly more general result. We will use this theorem later in chapter 7. We first need the following straightforward lemma.

Lemma 4.6.2 : *Let $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$ and assume $\mathcal{C}^1$ and $\mathcal{C}^2$ implement $\mathcal{K}^1$ and $\mathcal{K}^2$, respectively. Then*

$$\mathcal{C}^1 \subseteq \mathcal{C}^2 \qquad \Rightarrow \qquad \mathcal{K}^1 \subseteq \mathcal{K}^2.$$

Proof : Let $w_1 \in \mathcal{K}^1$. Hence there exists $c_1 \in \mathcal{C}^1$ such that $(w_1, c_1) \in \mathcal{P}_{\text{full}}$. Since $\mathcal{C}^1 \subseteq \mathcal{C}^2$, we have $c_1 \in \mathcal{C}^2$ also. Hence $w_1 \in \mathcal{K}^2$. This proves $\mathcal{K}^1 \subseteq \mathcal{K}^2$. □

In other words, the smaller a controller behavior becomes, the more it restricts. This fact is utilized in proving the second statement of the following theorem.

Theorem 4.6.3 : *Let $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$.*

1. *Let $\mathcal{C}^1$ and $\mathcal{C}^2 \in \mathfrak{L}^{\mathtt{c}}$ implement $\mathcal{K}^1$ and $\mathcal{K}^2 \in \mathfrak{L}^{\mathtt{w}}$, respectively.*
   Then $\mathcal{C}^1_{\text{cont}} = \mathcal{C}^2_{\text{cont}} \Rightarrow \mathcal{K}^1_{\text{cont}} = \mathcal{K}^2_{\text{cont}}$.
2. *Suppose $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ implements $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ and assume $\mathcal{K}$ is controllable, then $\mathcal{C}_{\text{cont}}$ also implements $\mathcal{K}$.*

Proof : *(1:)* Let $\mathcal{C}^0 := \mathcal{C}^1_{\text{cont}} = \mathcal{C}^2_{\text{cont}}$. Further, let $\mathcal{K}^0_{\text{full}}$, $\mathcal{K}^1_{\text{full}}$ and $\mathcal{K}^2_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$ be the full controlled behaviors resulting from interconnection of $\mathcal{P}_{\text{full}}$ with $\mathcal{C}^0$, $\mathcal{C}^1$ and $\mathcal{C}^2$, respectively. Since $\mathcal{C}^1_{\text{cont}} = \mathcal{C}^2_{\text{cont}}$, there exist $C_0, C_1, C_2 \in \mathbb{R}^{\bullet\times\bullet}[\xi]$ such that C_1 and C_2 are nonsingular, and $C_1(\frac{\mathrm{d}}{\mathrm{d}t})C_0(\frac{\mathrm{d}}{\mathrm{d}t})c = 0$, $C_2(\frac{\mathrm{d}}{\mathrm{d}t})C_0(\frac{\mathrm{d}}{\mathrm{d}t})c = 0$ and $C_0(\frac{\mathrm{d}}{\mathrm{d}t})c = 0$ are minimal kernel representations of $\mathcal{C}^1$, $\mathcal{C}^2$ and $\mathcal{C}^0$, respectively. Let $R_w(\frac{\mathrm{d}}{\mathrm{d}t})w + R_c(\frac{\mathrm{d}}{\mathrm{d}t})c = 0$ be a minimal kernel representation of $\mathcal{P}_{\text{full}}$. We obtain the following kernel representation of $\mathcal{K}^1_{\text{full}}$:

$$\begin{bmatrix} R_w & R_c \\ 0 & C_1 C_0 \end{bmatrix} \begin{bmatrix} w \\ c \end{bmatrix} = 0.$$

Let $(\mathcal{K}^1_{\text{full}})_{\text{cont}}$ have an observable image representation:

$$\begin{bmatrix} w \\ c \end{bmatrix} = \begin{bmatrix} M_w \\ M_c \end{bmatrix} \ell\ .$$

This results in $C_1C_0M_c = 0$ and by nonsingularity of C_1, we also get $C_0M_c = 0$. This means that $\mathcal{K}^0_{\text{full}}$ has the same controllable part as $\mathcal{K}^1_{\text{full}}$. Similarly, by nonsingularity of C_2, we obtain that $\mathcal{K}^2_{\text{full}}$ also has the same controllable part as $\mathcal{K}^0_{\text{full}}$. It follows that $\mathcal{K}^1_{\text{cont}} = \mathcal{K}^2_{\text{cont}} = \text{Im}(M_w(\frac{\text{d}}{\text{d}t}))$. This proves statement 1.
(2:) Define $\mathcal{C}^1 := \mathcal{C}$ and let $\mathcal{C}^2 := \mathcal{C}_{\text{cont}}$. Assume $\mathcal{K}^1$ and $\mathcal{K}^2 \in \mathfrak{L}^{\mathtt{w}}$ are the behaviors that $\mathcal{C}^1$ and $\mathcal{C}^2$ implement, respectively. Then $\mathcal{K}^1 = \mathcal{K}$. We need to show that $\mathcal{K}^2 = \mathcal{K}$. From statement 1 of this theorem, we already have $\mathcal{K}^2_{\text{cont}} = \mathcal{K}$. Now since $\mathcal{C}^2 \subseteq \mathcal{C}^1$, we use lemma 4.6.2 to infer that $\mathcal{K}^2 \subseteq \mathcal{K}$. This implies $\mathcal{K}^2 = \mathcal{K}$. □

Theorem 4.6.3 has a useful extension in the context of canonical controllers. Let $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$ and let $\mathcal{K}$ be a given implementable behavior (i.e. $\mathcal{N} \subseteq \mathcal{K} \subseteq \mathcal{P}$). Suppose $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ is the canonical controller that implements $\mathcal{K}$. We noted that the canonical controller $\mathcal{C}$ is the controller behavior that $\mathcal{K}$ implements by interconnection with $\mathcal{P}_{\text{full}}$ through w. Hence, using theorem 4.6.3, $\mathcal{K}_{\text{cont}}$ implements $\mathcal{C}_{\text{cont}}$. Moreover, suppose $\mathcal{K}_{\text{cont}}$ also is implementable, i.e. $\mathcal{N} \subseteq \mathcal{K}_{\text{cont}} \subseteq \mathcal{P}$, then using theorem 4.6.3 again we easily infer that $\mathcal{C}_{\text{cont}}$ implements $\mathcal{K}_{\text{cont}}$. We state this into the following corollary for easy reference.

Corollary 4.6.4 : *Let $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$ and let $\mathcal{N}, \mathcal{P}$ be its hidden and manifest plant behaviors, respectively. Let $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ be an implementable behavior. Suppose $\mathcal{K}_{\text{cont}}$ is also implementable. Let $\mathcal{C}$ be the canonical controller that implements $\mathcal{K}$. Then $\mathcal{C}_{\text{cont}}$ implements $\mathcal{K}_{\text{cont}}$.*

The significance of the above corollary is as follows. Suppose we have an implementable $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ and we are actually interested in implementing $\mathcal{K}_{\text{cont}}$ (assumed implementable). Instead of using $\mathcal{K}_{\text{cont}}$ to construct a controller that implements it, we can first construct the canonical controller $\mathcal{C}$ from $\mathcal{K}$ and then use $\mathcal{C}_{\text{cont}}$ to implement $\mathcal{K}_{\text{cont}}$. Thus the operation of 'taking-the-controllable-part' can be done *after* we construct a canonical controller. We shall need this later in subsection 7.3.3 in the context of synthesis of dissipative systems.

We now come to the issue of regular implementability of $\mathcal{K}$ by a controllable controller. The definition of controllably regularly implementable behavior is a straightforward extension of definition 4.6.1.

Definition 4.6.5 : Let $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$ and $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$. $\mathcal{K}$ is called *controllably* regularly implementable if there exists a $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}_{\text{cont}}$ that regularly implements $\mathcal{K}$.

Another extension of theorem 4.6.3 is the following corollary. This result admits as a special case that if $\mathcal{K}$ is regularly implementable and if it is controllable then $\mathcal{K}$ is controllably regularly implementable. The proof of the corollary is very similar to that of theorem 4.6.3. The fact that $\mathtt{p}(\mathcal{C}) = \mathtt{p}(\mathcal{C}_{\text{cont}})$ makes sure that if $\mathcal{C}$ is a regular controller then $\mathcal{C}_{\text{cont}}$ is also a regular controller. We omit the straightforward proof.

Corollary 4.6.6 : *Let $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$.*

1. *Let $\mathcal{C}^1$ and $\mathcal{C}^2 \in \mathfrak{L}^{\mathtt{c}}$ regularly implement $\mathcal{K}^1$ and $\mathcal{K}^2 \in \mathfrak{L}^{\mathtt{w}}$, respectively.* *Then $\mathcal{C}^1_{\text{cont}} = \mathcal{C}^2_{\text{cont}} \Rightarrow \mathcal{K}^1_{\text{cont}} = \mathcal{K}^2_{\text{cont}}$.*
2. *Assume $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ regularly implements $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}$ and $\mathcal{K}$ is controllable, then $\mathcal{C}_{\text{cont}}$ also regularly implements $\mathcal{K}$. In other words, if $\mathcal{K} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ is regularly implementable then $\mathcal{K}$ is controllably regularly implementable.*

We have seen that controllability of $\mathcal{K}$ is sufficient (in addition to, of course, regular implementability) for $\mathcal{K}$ to be controllably regularly implementable. We now come to the issue that $\mathcal{K}$ is not controllable but autonomous. Consider the problem of pole placement. From the pole placement theorem (theorem 3.3.1) we know under what conditions on $\mathcal{P}_{\text{full}}$, for each monic $r \in \mathbb{R}[\xi]$, there exists a $\mathcal{K}$ with characteristic polynomial r such that $\mathcal{K}$ is regularly implementable with respect to $\mathcal{P}_{\text{full}}$ through c. We shall now consider the question under what conditions on $\mathcal{P}_{\text{full}}$, for each monic polynomial r, there exists a *controllably* regularly implementable $\mathcal{K}$ with characteristic polynomial r. Before we consider the general case we shall state and prove the result for the full interconnection case.

Theorem 4.6.7 : *Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}}$. For each monic $r \in \mathbb{R}[\xi]$, there exists a controllably regularly implementable $\mathcal{K}$ such that $\chi_{\mathcal{K}} = r$ if and only if the following two conditions are satisfied:*

1. *$\mathcal{P}$ is controllable and $\mathtt{m}(\mathcal{P}) \geqslant 1$,*
2. *$\mathtt{m}(\mathcal{P}) < \mathtt{w}$, i.e. $\mathcal{P} \subsetneqq \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$.*

Proof : *(if :)* Without loss of generality we shall assume that a minimal kernel representation of $\mathcal{P}$ is already in Smith form: $[I_{\mathtt{p}} \;\; 0]\, w = 0$, with $I_{\mathtt{p}}$ of size $\mathtt{p}(\mathcal{P})$. Partition w accordingly, into $w = (w_p, w_m)$. We partition w_m further into $w_m = (w_{m1}, w_{m2})$ such that w_{m1} has $\mathtt{m}(\mathfrak{B}) - 1$ components and w_{m2} is the single remaining component of w_m. We define $\mathcal{K}$ as that having the following minimal kernel representation:

$$\begin{bmatrix} I_{\mathtt{p}} & 0 & 0 \\ 0 & I_{\mathtt{m}-1} & 0 \\ 0 & 0 & r(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix} \begin{bmatrix} w_p \\ w_{m1} \\ w_{m2} \end{bmatrix} = 0 \; .$$

The controller that would regularly implement this $\mathcal{K}$ can be easily defined as the one defined by the last $\mathtt{m}(\mathcal{P})$ rows of the above equation. However, this controller is not controllable. We define $\mathcal{C} \in \mathfrak{L}^{\mathtt{w}}$ by the following minimal kernel representation:

$$\begin{array}{r} \mathtt{m}-1 \text{ rows} \\ 1 \text{ row} \end{array} \begin{array}{l} \{ \\ \{ \end{array} \begin{bmatrix} 0 & 0 & \cdots & 0 & I_{\mathtt{m}-1} & 0 \\ 1 & 0 & \cdots & 0 & 0 & r \end{bmatrix} \begin{bmatrix} w_p \\ w_{m1} \\ w_{m2} \end{bmatrix} = 0 \ .$$

It is easy to see that this $\mathcal{C}$ regularly implements the $\mathcal{K}$ defined above, and that this $\mathcal{C}$ is controllable.
(only if :) We already know from the pole placement theorem that controllability of $\mathcal{P}$ and $\mathtt{m}(\mathcal{P}) \geqslant 1$ are both necessary for pole placement. We shall show that $\mathcal{P} \subsetneqq \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ is necessary for the existence of a controllable controller. Suppose $\mathcal{P} = \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$. Then to implement any $\mathcal{K}$ we have to take $\mathcal{C} = \mathcal{K}$. If $\mathcal{K}$ is not controllable, neither can $\mathcal{C}$ be. This proves the necessity of $\mathtt{m}(\mathcal{P}) < \mathtt{w}$. This completes the proof. □

Having considered the full interconnection case, we now consider the general case. The sufficiency part is easily formulated as the following result. Note that controllability of $\mathcal{P}$ is implicit when we assume $\mathcal{P}_{\text{full}}$ to be controllable.

Theorem 4.6.8 : *Let $\mathcal{P}_{\text{full}} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$. For each monic $r \in \mathbb{R}[\xi]$, there exists a controllably regularly implementable $\mathcal{K}$ such that $\mathcal{K}$ has characteristic polynomial r if the following four conditions are satisfied*

1. $\mathcal{N} = 0$, *equivalently, in $\mathcal{P}_{\text{full}}$, w is observable from c,*
2. $\mathcal{P}_{\text{full}}$ *is controllable,*
3. $\mathtt{m}(\mathcal{P}) \geqslant 1$, *i.e. $\mathcal{P}$ is not autonomous, and*
4. $\mathtt{w} + \mathtt{m}(\mathcal{P}) \leqslant \mathtt{p}(\mathcal{P}_{\text{full}})$.

Proof : We note that for the general case, we are allowed only some kinds of isomorphisms on the space $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \times \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{c}})$. The kind of isomorphisms have to retain the structure that the 'interconnection through c' constraint requires. Let $R_w(\frac{\mathrm{d}}{\mathrm{d}t})w + R_c(\frac{\mathrm{d}}{\mathrm{d}t})c = 0$ be a kernel representation of $\mathcal{P}_{\text{full}}$. Let $U_w \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ and $U_c \in \mathbb{R}^{\mathtt{c}\times\mathtt{c}}[\xi]$ be unimodular matrices. Then we are allowed to manipulate $[R_w \ \ R_c]$ by postmultiplication by unimodular matrices of the kind:

$$\begin{bmatrix} U_w & 0 \\ 0 & U_c \end{bmatrix} .$$

For premultiplication of $[R_w \;\; R_c]$ there is no particular structure in the kind of unimodular matrices allowed.

Let $R_w(\frac{\mathrm{d}}{\mathrm{d}t})w + R_c(\frac{\mathrm{d}}{\mathrm{d}t})c = 0$ be a minimal kernel representation of $\mathcal{P}_{\text{full}}$. Using the above remark, we shall assume $\mathcal{P}_{\text{full}}$ to have a simpler representation. Since $\mathcal{P}_{\text{full}}$ is controllable, so is $\mathcal{P}$. We assume $\mathcal{P}$ to have a kernel representation $[I_{\mathtt{p}} \;\; 0]\, w = 0$. Further, since $\mathcal{N} = 0$ we obtain that R_w is left invertible. This brings us to the following minimal kernel representation of $\mathcal{P}_{\text{full}}$ (after perhaps an isomorphism on $\mathcal{P}_{\text{full}}$ of the kind described above).

$$\begin{bmatrix} 0 & 0 & 0 & R_{c1} \\ 0 & I_{\mathtt{m}-1} & 0 & R_{c2} \\ 0 & 0 & 1 & R_{c3} \\ I_{\mathtt{p}} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} w_p \\ w_{m1} \\ w_{m2} \\ c \end{bmatrix} = 0$$

with $\operatorname{col}(R_{c1}, R_{c2}, R_{c3})$ having full row rank and $w = (w_p, w_{m1}, w_{m2})$ partitioned such that $\dim(w_p) = \mathtt{p}(\mathcal{P}), \dim(w_{m1}) = \mathtt{m}(\mathfrak{B}) - 1$ and $\dim(w_{m2}) = 1$. In other words, the partition is just like in the 'if part' of the proof of theorem 4.6.7 (the full interconnection analog). Now, since $\mathcal{P}_{\text{full}}$ is controllable we obtain that $R_{c1}(\lambda)$ has constant rank for all $\lambda \in \mathbb{C}$. Moreover, since we started with a minimal representation, R_{c1} has full row rank. From the above equation we see that $\operatorname{rowdim}(R_{c1}) = \mathtt{p}(\mathcal{P}_{\text{full}}) - \mathtt{w} =: \mathtt{p}_{\mathtt{c}}$ (say). (Notice that $\mathcal{P}_c \in \mathfrak{L}^{\mathtt{c}}$ defined by $\mathcal{P}_c := \Pi_c(\mathcal{P}_{\text{full}})$ is given by a minimal kernel representation $R_{c1}(\frac{\mathrm{d}}{\mathrm{d}t})c = 0$, and $\mathcal{P}_c$ has output cardinality $\mathtt{p}_{\mathtt{c}}$.) Let U and V be unimodular matrices such that R_{c1} is written in its Smith form $U\,[I_{p_c} \;\; 0]\, V = R_{c1}$. Let $V(\frac{\mathrm{d}}{\mathrm{d}t})c = (c_1, c_2)$ be partitioned accordingly. Hence by using an isomorphism induced by a unimodular matrix on the variable c, we obtain the following minimal kernel representation of $\mathcal{P}_{\text{full}}$:

$$\begin{bmatrix} 0 & 0 & 0 & I_{\mathtt{p_c}} & 0 \\ 0 & I_{\mathtt{m}-1} & 0 & 0 & R'_{c4} \\ 0 & 0 & 1 & 0 & R'_{c5} \\ I_{\mathtt{p}} & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} w_p \\ w_{m1} \\ w_{m2} \\ c_1 \\ c_2 \end{bmatrix} = 0 \; . \tag{4.15}$$

(The zeros below $I_{\mathtt{p_c}}$ are a result of premultiplication by a unimodular matrix.) We define the controller $\mathcal{C}_1 \in \mathfrak{L}^{\mathtt{c}}$ by

$$\begin{matrix} \mathtt{m}-1 \text{ rows} \\ 1 \text{ row} \end{matrix} \begin{Bmatrix} \\ \\ \end{Bmatrix} \begin{bmatrix} 0 & R'_{c4} \\ 0 & r\,R'_{c5} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = 0 \; .$$

Appending the above controller equations to equation (4.15), and making a few row operations, we see that $\mathcal{C}_1$ indeed regularly implements $\mathcal{K}$ with $\chi_{\mathcal{K}} = r$. However, this controller need not be controllable. We need to add the first few rows of equation (4.15) to the equations of $\mathcal{C}_1$ and this operation will yield us the desired $\mathcal{C}_2 \in \mathfrak{L}^{\mathtt{c}}$ which is controllable. Define $\mathcal{C}_2$ by the following minimal kernel representation:

$$\begin{bmatrix} I_{\mathtt{m}-1} & 0 & 0 & R'_{c2} \\ 0 & 1 & 0 & r\,R'_{c3} \end{bmatrix} \begin{bmatrix} c_{11} \\ c_{12} \\ c_{13} \\ c_2 \end{bmatrix} = 0 \ .$$

where $c_1 = (c_{11}, c_{12}, c_{13})$ is partitioned as follows: $\dim(c_{11}) = \mathtt{m}(\mathcal{P}) - 1$, $\dim(c_{12}) = 1$, $\dim(c_{13}) = \mathtt{p_c} - \mathtt{m}(\mathcal{P})$ (and as before, $\dim(c_2) = \mathtt{c} - \mathtt{p_c}$). For this procedure to be possible, we need $\mathtt{p_c} - \mathtt{m}(\mathcal{P}) \geqslant 0$. From $\mathtt{p_c} - \mathtt{m}(\mathcal{P}) = \mathtt{p}(\mathcal{P}_{\text{full}}) - \mathtt{w} - \mathtt{m}(\mathcal{P})$, it follows that this is equivalent to $\mathtt{w} + \mathtt{m}(\mathcal{P}) \leqslant \mathtt{p}(\mathcal{P}_{\text{full}})$. □

It is possible to show that the (sufficient) conditions as in the theorem above are not necessary. The precise kind of necessary conditions remains to be formulated. Similarly, it is interesting to know under what conditions a stabilizing controller can be chosen to be controllable. These issues are currently under investigation.