A modern perspective on Fano's approach to linear differential equations

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4 Fano Groups of Linear Differential Equations

In what follows k is a differential field whose field of constants C is assumed to be algebraically closed of characteristic 0. Consider a linear differential equation $L(y) = y^{(n)} + \sum_{i=1}^{n-1} a_i y^{(n-i)} = 0$, $a_i \in k$ of degree n. Our aim is to compare the differential Galois group G of L over k with a similar group introduced by G. Fano in 1900 in his paper [Fa].

4.1 Notations and introduction

The differential operator L induces a differentiation on the polynomial ring

$$R_0 = k[X_i^{(j)}, \frac{1}{W}] \text{ (with } 1 \leq i \leq n,\ 0 \leq j \leq n-1, \text{ and } W = \det(X_i^{(j)}))$$

by

$$(X_i^{(j)})' = \begin{cases} X_i^{(j+1)} & if\ j < n-1 \\ -\sum_{l=1}^{n-1} a_i X_i^{(n-l)} & if\ j = n-1. \end{cases}$$

In terms of this, the Picard-Vessiot ring of L over k is $R = R_0/I$, where $I \subset R_0$ is a maximal differential ideal. Put $y_i = X_i^{(0)} \bmod I$, then $V = Cy_1 + \cdots + Cy_n$ is the solution space of the equation $L(y) = 0$.

Consider $J = I \cap C[X_1^{(0)}, \ldots, X_n^{(0)}]$. Then $C[V] = C[X_1^{(0)}, \ldots, X_n^{(0)}]/J$ is the subalgebra of the Picard-Vessiot ring R generated by the solution space V.

Any $\sigma \in \mathrm{GL}(V)$ can be extended uniquely to a k-algebra automorphism of R_0 commuting with the derivation, by $\sigma(X_i^{(j)}) = \sigma(X_i)^{(j)}$. By definition, the differential Galois group of L over k is

$$G = \{\sigma \in \mathrm{GL}(V) | \sigma I \subset I\} = \{\sigma \in \mathrm{GL}(V) | \sigma I = I\}.$$

In a century-old paper of Fano [Fa], not the group G is considered but a group we denote as G^+ here, which is defined as follows.

Any $\sigma \in \mathrm{GL}(V)$ also acts on the subalgebra $C[X_1^{(0)}, \ldots, X_n^{(0)}]$ of R_0 and we define G^+ to be the group of all $\sigma \in \mathrm{GL}(V)$ which induce C-linear automorphisms of $C[V]$, i.e.

$$G^+ = \{\sigma \in \mathrm{GL}(V) | \sigma J \subset J\} = \{\sigma \in \mathrm{GL}(V) | \sigma J = J\}.$$

We will call G^+ the *Fano group* of a linear differential equation.

Remark 4.1.1. G^+ is an algebraic subgroup of $\mathrm{GL}(V)$.

Remark 4.1.2. Note that the notation G^+ also appears in Chapters 1 and 2. However, its meaning is quite different here.

The aim of this chapter is to study the relation between G and G^+. Obviously, $G \subset G^+$. In Section 4.2 we prove

Theorem 4.1.3. *$G = G^+$ if and only if I equals the radical of the minimal differential ideal in R_0 containing J.*

In Section 4.3 we present examples. In particular, for second order equations, we show that if a differential Galois group has dimension at least 2 then the corresponding Fano group is GL_2. Moreover we give various examples for differential Galois groups of dimensions 0 and 1. In fact, Fano [Fa] mostly uses the projective group PG^+, defined as the image of G^+ in $\mathrm{PGL}(V)$. Our examples show that in general $PG \neq PG^+$.

More precisely, Fano considers the ideal $H \subset C[X_1^{(0)}, \ldots, X_n^{(0)}]$, generated by the homogeneous polynomials in the variables $X_1^{(0)}, \ldots, X_n^{(0)}$ that belong to the ideal I. Since I is a prime ideal, H is a homogeneous prime ideal and defines an irreducible projective variety $S \subset \mathbb{P}(V) \cong \mathbb{P}^{n-1}$. Fano formulates this as follows: "*the solutions of L lie on S*". The interpretation seems to be as follows.

Take a point z_0 in the complex plane where the equation L has n independent local, meromorphic solutions $f_1, \ldots, f_n$. For z in a neighborhood D of z_0, there is a well-defined analytic map $m : D \to \mathbb{P}^{n-1}$, given by the formula $z \mapsto (f_1(z) : f_2(z) : \cdots : f_n(z))$. The smallest projective subspace of $\mathbb{P}^{n-1}$, containing the image of m, can be seen to be S.

The group that Fano considers is the algebraic subgroup of $\mathrm{PGL}(V)$ consisting of the elements A with $A(S) = S$. This group contains the above group PG^+.

We remark that the case of third order linear differential equations with finite order differential Galois group was recently considered by C. S. Malagón in his master's thesis [Ma] including several examples.

4.2 Equality of G and G^+

Let J_1 be the radical ideal of the differential ideal generated by J in R_0. By [Kap, Lemma 1.8], as R_0 is a Ritt algebra, J_1 is a differential ideal. Our aim is to prove Theorem 4.1.3, i.e.,

$$G = G^+ \text{ if and only if } I = J_1.$$

Proof. The "if" part is clear, since for an automorphism $\sigma \in \mathrm{GL(V)}$, if it fixes J, then it will also fix $J_1 = I$.

For the "only if" part, we first assume for the moment the following lemma.

Lemma 4.2.1. *Let $\tilde{I}$ be a maximal differential ideal of R_0. There exists $\sigma \in \mathrm{GL(V)}$ such that $\sigma(I) = \tilde{I}$.*

By this lemma and the fact that $G = G^+$ we conclude that I is the unique maximal differential ideal in R_0 containing J_1. Indeed, if $\tilde{I}$ be a maximal differential ideal of R_0 containing J_1, then by the lemma there exists $\sigma \in \mathrm{GL(V)}$ such that $\sigma(I) = \tilde{I}$. We have

$$J \subset \tilde{I} \cap C[X_1^{(0)}, \ldots, X_n^{(0)}] = \sigma(I) \cap C[X_1^{(0)}, \ldots, X_n^{(0)}] = \sigma(J).$$

If $J \subsetneq \sigma(J)$ then we would have the following nonstop chain

$$J \subsetneq \sigma(J) \subsetneq \sigma^2(J) \subsetneq \sigma^3(J) \subsetneq \ldots$$

of ideals in $C[X_1^{(0)}, \ldots, X_n^{(0)}]$. This is a contradiction, as $C[X_1^{(0)}, \ldots, X_n^{(0)}]$ is a noetherian ring. Thus $J = \sigma(J)$, or $\sigma \in G^+ = G$. Therefore $\tilde{I} = \sigma(I) = I$.

Now by [vdP-S, Lemmas 1.23, 1.29], there is a bijection between the differential ideals of R_0 and the G-invariant ideals of $C[X_i^{(j)}, \frac{1}{W}]$. Under this bijection, radical differential ideals in R_0 correspond to G-invariant radical ideals in $C[X_i^{(j)}, \frac{1}{W}]$. Let $I_1 \subset C[X_i^{(j)}, \frac{1}{W}]$ correspond to I under this bijection. By the maximality of I, I_1 is not contained in any strictly larger G-invariant ideal of $C[X_i^{(j)}, \frac{1}{W}]$. Therefore I_1 is a radical ideal and its zero set $Z(I_1) \subset \mathrm{GL}(V)$ is a minimal Zariski closed G-invariant set. By the minimality of $Z(I_1)$, there exists $\xi \in \mathrm{GL}(V)$ such that $Z(I_1) = \overline{G \cdot \xi}$. Similarly, let Z be the Zariski closed subset of $\mathrm{GL}(V)$ corresponding to J_1. We have $Z(I_1) = \overline{G \cdot \xi} \subset Z$, and because J_1 is radical, $I(Z)$ is the G-invariant ideal in $C[X_i^{(j)}, \frac{1}{W}]$ corresponding to J_1 under the above bijection. We claim that $\overline{G \cdot \xi} = Z$. Indeed, suppose that there exists $\eta \in Z(J_1) \backslash \overline{G \cdot \xi}$, then the differential ideal in R_0 corresponding to $I(\overline{G \cdot \eta}) \subset C[X_i^{(j)}, \frac{1}{W}]$ would be another maximal differential ideal containing J_1. This contradicts the uniqueness of I. Thus $Z(I) = \overline{G \cdot \xi} = Z$, and therefore $I = J_1$. □

Proof of Lemma 4.2.1. Observe that $R_0/\tilde{I}$ is a Picard-Vessiot ring of L over k. By the uniqueness of the Picard-Vessiot ring, there is a differential ring isomorphism

$$\phi : R_0/I \longrightarrow R_0/\tilde{I}.$$

This isomorphism maps V to the solution space of $L(y)=0$ in $R_0/\tilde{I}$. Hence there is an invertible matrix $(a_{ij})_{n\times n}$ with entries in C such that

$$\phi(X_i \bmod I) = \sum_{j=1}^{n} a_{ij}X_j \bmod \tilde{I},$$

for all $i=1,\ldots,n$. The matrix $(a_{ij})_{n\times n}$ defines a C-automorphism $\sigma \in \mathrm{GL(V)}$ of V and we have $\sigma(I)=\tilde{I}$. □

4.3 G^+ in $\mathrm{GL}_2(C)$: Examples

Suppose that $L(y)=0$ over $k=C(z)$ is a second order linear differential equation with differential Galois group $G\subset \mathrm{GL}_2(C)$. We consider some possibilities for G and G^+, based on the dimension of G.

4.3.1 $\dim(G)\geq 3$

Proposition 4.3.1. *If* $\dim(G)\geq 3$ *then* $G^+=\mathrm{GL}_2$.

Proof. If $\dim(G)=4$, then $G=\mathrm{GL}_2$ and $G\subset G^+$ implies $G^+=\mathrm{GL}_2$. Suppose that $\dim(G)=3$, then either $\mathrm{SL}_2(C)$ is a subgroup of G of finite index, or G is the Borel group B'.

In the first case, a maximal differential ideal $I\subset R^0$ is of the form $I=((X_1X_2'-X_2X_1')^n-f)$ for some nonzero $f\in C(z)$. This implies $J=0$. Hence $G^+=\mathrm{GL}_2$.

In the second case, a maximal differential ideal $I\subset R^0$ is of the form $I=(X_1'-fX_1)$ for some nonzero $f\in C(z)$. This also implies $J=0$, hence $G^+=\mathrm{GL}_2$. □

Remark 4.3.2. If $G=B'$, then $PG\neq PG^+$. If $\mathrm{SL}_2(C)$ is a subgroup of finite index in G, then $PG=PG^+$ and $G\neq G^+$.

4.3.2 $\dim(G)=2$

In case $\dim(G)=2$, it follows from the classification as presented in Appendix A that $G\subset B'$. The short exact sequence

$$0 \to \mathbf{G}_a \to B' \to \mathbf{G}_m \times \mathbf{G}_m \to 1$$

shows that if $\mathbf{G}_a \not\subset G$, then G is diagonal, hence a maximal differential ideal $I \subset R^0$ is of the form $I = ((X_1' - fX_1, X_2' - gX_2)$ for some $f, g \in C(z)$, such that the equation $H' = (\lambda f + \mu g)H$ with $\lambda, \mu \in \mathbb{Q}$, not both zero, has in the field $C(z)$ only the trivial solution $H = 0$. This implies $J = 0$. Thus $G^+ = \mathrm{GL}_2$.

In the case that $\mathbf{G}_a \subset G$, we may write

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} | a^k c^l = 1 \right\} \text{ with } k,l \in \mathbb{Z} \text{ and } (k,l) \neq (0,0).$$

One computes in this case that a maximal differential ideal in R_0 has generators $X_2' - kfX_2$ and $X_1' + lfX_1 + gX_2$ for certain functions f,g. It follows that $J = 0$, hence $G^+ = \mathrm{GL}_2$. So we have proven

Proposition 4.3.3. *If* $\dim(G) \geq 2$ *then* $G^+ = \mathrm{GL}_2$.

4.3.3 $\dim(G) = 1$

We will consider only two cases, $G = \mathbb{G}_a \subset \mathrm{SL}_2$ and $G = \mathbb{G}_m \subset \mathrm{SL}_2$. In the first case, the second order equation L has a nonzero rational solution s. Suppose first that $L = \partial^2 - f$ and so $f = \frac{s''}{s}$. Using variation of constants we obtain a second independent solution $x_2 = c \cdot s$ with $c' = \frac{1}{s^2}$. Hence $x_2' = \frac{1}{s} + x_2 \frac{s'}{s}$. Since the differential Galois group is $\mathbb{G}_a$, one has that c is transcendental over $C(z)$. Moreover, a maximal differential ideal is of the form $(X_1 - s, X_1' - s', X_2' - \frac{1}{s} - \frac{s'}{s} X_2)$. This gives $J = 0$, hence $G^+ = \mathrm{GL}_2$. An explicit s having the required property is $s = \frac{z+1}{z}$.

Another equation with differential Galois group $\mathbb{G}_a$ is $zy'' + y' = 0$ which has 1 and $\log z$ as solutions. In this case a maximal differential ideal is of the form $(X_1 - 1, X_1', X_2 - \frac{1}{z})$. This gives us the ideal $J = (X_1 - 1)$, hence $G^+ = \left(\begin{smallmatrix} 1 & * \\ 0 & * \end{smallmatrix}\right)$.

We conclude from the above examples that G^+ is not always completely determined by G.

For the case $G = \mathbb{G}_m \subset \mathrm{SL}_2$, one considers a transcendental element s over $C(z)$ satisfying $s' = s$. The vector space $Cs + Cs^{-1}$ is the solution space for the equation $L = \partial^2 - 1$. The differential Galois group of L is $\mathbb{G}_m$. Clearly $(X_1X_2 - 1, X_1 - X_1', X_2 + X_2')$ is a maximal differential ideal and $J = (X_1X_2 - 1)$. Hence G^+ is the infinite dihedral group D_∞, i.e.,

$$D_\infty := \left\{ \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} | c \in C^* \right\} \cup \left\{ \begin{pmatrix} 0 & -d \\ d^{-1} & 0 \end{pmatrix} | d \in C^* \right\}.$$

Similarly, the monic operator L of degree two with solution space $Cs + Czs^{-1}$ is $\partial^2 + \frac{2}{1-2z}\partial + \frac{2z-3}{1-2z}$. One obtains $J = 0$, and therefore $G^+ = \mathrm{GL}_2$.

4.3.4 $\dim(G) = 0$

Example 4.3.4. Consider the differential equation $y'' = -\frac{y'}{2z}$ over $C(z)$. This equation has $\{1, \sqrt{z}\}$ as fundamental solutions and the maximal differential ideal in the Picard-Vessiot ring is $I = (X_2 - 1,\ X_1^2 - z,\ 2zX_1' - X_1)$. The differential Galois group G of the equation has order two and the same holds for PG. Further, $J = I \cap C[X_1, X_2] = (X_2 - 1)$. The group G^+ is the stabilizer of the line $X_2 = 1$ and therefore $G^+ = \{\begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} | a,b \in C\} = \mathbb{G}_a \ltimes \mathbb{G}_m$ $(PG^+ = PB)$.

Example 4.3.5. Consider the differential equation $y'' - \frac{2}{z}y' + \frac{2}{z^2}y = 0$ over $C(z)$. This equation has $\{z, z^2\}$ as fundamental solutions and the maximal differential ideal in the Picard-Vessiot ring is $I = (X_1 - z,\ X_2 - z^2,\ X_1' - 1, X_2' - 2z)$. Then the differential Galois group of the equation is $G = \{1\}$ $(PG = \{1\})$. We observe that $(X_1^2 - X_2) = J = I \cap C[X_1, X_2]$. The group G^+ leaves the curve $X_2 = X_1^2$ invariant, therefore $G^+ = \{\left(\begin{smallmatrix} a & 0 \\ 0 & a^2 \end{smallmatrix}\right) |\ a \in C^*\}$.

The following example shows that $G = G^+ = \{1\}$ is a possibility.

Example 4.3.6. Consider the differential equation $y'' - \frac{6z^2}{2z^2+z}y' + \frac{6z+2}{2z^3+z^2}y = 0$ over $C(z)$. The solution space is $Cz + C(z^2 + z^3)$ and the maximal differential ideal in the Picard-Vessiot ring is $I = (X_1 - z,\ X_2 - z^2 - z^3,\ X_1' - 1,\ X_2' - 2z - 3z^2)$. The differential Galois group is $G = \{1\}$. One finds $(X_1^2 + X_1^3 - X_2) = J = I \cap C[X_1, X_2]$. The group G^+ is the stabilizer of the curve $X_2 = X_1^2 + X_1^3$ and thus $G^+ = \{1\}$.

It can be shown that if $\dim G = 0$, then $\dim G^+ \leq 2$.