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Schaft, A.J. van der; Maschke, B.M.

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# Hamiltonian formulation of distributed-parameter systems with boundary energy flow

A.J. van der Schaft [a,*], B.M. Maschke [b]

[a] *Faculty of Mathematical Sciences, University of Twente, P.O. Box 217, 7500 AE Enschede, The Netherlands*

[b] *Lab. d'Automatique et de Genie des Procedes, Universite Claude Bernard Lyon-1, F-69622 Villeurbanne Cedex, France*

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**Abstract**

A Hamiltonian formulation of classes of distributed-parameter systems is presented, which incorporates the energy flow through the boundary of the spatial domain of the system, and which allows to represent the system as a boundary control Hamiltonian system. The system is Hamiltonian with respect to an infinite-dimensional Dirac structure associated with the exterior derivative and based on Stokes' theorem. The theory is applied to the telegraph equations for an ideal transmission line, Maxwell's equations on a bounded domain with non-zero Poynting vector at its boundary, and a vibrating string with traction forces at its ends. Furthermore, the framework is extended to cover Euler's equations for an ideal fluid on a domain with permeable boundary. Finally, some properties of the Stokes–Dirac structure are investigated, including the analysis of conservation laws.


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## 1. Introduction

The Hamiltonian formulation of classes of distributed-parameter systems has been a challenging and fruitful area of research for quite some time. (A nice introduction, especially

* Corresponding author. Tel.: +31-53-489-3449; fax: +31-53-434-0733.
*E-mail address:* a.j.vanderschaft@math.utwente.nl (A.J. van der Schaft).

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with respect to systems stemming from fluid dynamics, can be found in [24], where also a historical account is provided.) The identification of the underlying Hamiltonian structure of sets of PDEs has been instrumental in proving all sorts of results on integrability, the existence of soliton solutions, stability, reduction, etc. and in unifying existing results, see e.g. [1,9,13,14,22,23].

Recently, there has been also a surge of interest in the *design* and *control* of non-linear distributed-parameter systems, motivated by various applications. At the same time, it is well-known from *finite-dimensional* non-linear control systems [7,19,25,26,29–31,33] that the (generalized) Hamiltonian formulation may be very helpful in the control design, and even more is to be expected in the distributed-parameter case. However, in extending the Hamiltonian theory as for instance exposed in [24] to distributed-parameter control systems a fundamental difficulty arises in the treatment of *boundary conditions*. Indeed, the treatment of infinite-dimensional Hamiltonian systems in the literature seems mostly focused on systems with infinite spatial domain, where the variables go to zero for the spatial variables tending to infinity, or on systems with boundary conditions such that the energy exchange through the boundary is *zero*. On the other hand, from a control and interconnection point of view it is essential to be able to describe a distributed-parameter system with varying boundary conditions inducing *energy exchange through the boundary*, since in many applications, interaction with the environment (e.g. actuation or measurement) takes place through the boundary of the system. Clear examples are the telegraph equations (describing the dynamics of a transmission line), where the boundary of the system is described by the behavior of the voltages and currents at both ends of the transmission line, or a vibrating string (or, more generally, a flexible beam), where it is natural to consider the evolution of the forces and velocities at the ends of the string. Furthermore, in both examples it is obvious that in general the boundary exchange of power (voltage times current in the transmission line example, and force times velocity for the vibrating string) will be non-zero, and that in fact one would like to consider the voltages and currents or forces and velocities as additional *boundary variables* of the system, which can be interconnected to other systems. Also for numerical integration and *simulation* of complex distributed-parameter systems it is essential to be able to describe the complex system as the interconnection or coupling of its subsystems via their boundary variables; for example in the case of coupled fluid–solid dynamics.

From a mathematical point of view, it is not obvious how to incorporate non-zero energy flow through the boundary in the existing Hamiltonian framework for distributed-parameter systems. The problem is already illustrated by the Hamiltonian formulation of e.g. the Korteweg-de Vries equation (see e.g. [24]). Here for zero boundary conditions a *Poisson bracket* can be formulated with the use of the differential operator $(\mathrm{d}/\mathrm{d}x)$, since by integration by parts this operator is obviously skew-symmetric. However, for boundary conditions corresponding to non-zero energy flow the differential operator is not skew-symmetric anymore (since after integrating by parts the remainders are not zero). Also the interesting paper [12] does not really solve this problem, since it is concerned with the modification of the Poisson bracket in case of a free boundary.

In the present paper we provide a framework to overcome this fundamental problem by using the notion of a *Dirac structure*; extending and generalizing a preliminary and partial treatment of this framework in [20,21]. Dirac structures were originally introduced in

[6,8] as a geometric structure generalizing both *symplectic* and *Poisson* structures. Later on (see e.g. [2,7,18,31]) it was realized that in the finite-dimensional case Dirac structures can be naturally employed to formalize Hamiltonian systems with *constraints* as *implicit* Hamiltonian systems. It will turn out that in order to allow the inclusion of boundary variables in distributed-parameter systems the concept of Dirac structure again provides the right type of generalization with respect to the existing framework using Poisson structures. (In fact, already in [8] Dirac structures were employed for the Hamiltonian representation of certain evolution equations. However, this treatment did not involve the inclusion of boundary variables, and, in fact, the employed Dirac structures are equivalent to Poisson structures.)

The Dirac structure for distributed-parameter systems used in this paper has a specific form by being defined on certain spaces of differential forms on the spatial domain of the system and its boundary, and making use of Stokes' theorem. Its construction emphasizes the geometrical content of the physical variables involved, by identifying them as differential $k$-forms, for appropriate $k$. This interpretation is rather well-known (see e.g. [11]) in the case of Maxwell's equations (and actually directly follows from Faraday's law and Ampère's law), but seems less well-known for the telegraph equations and the description of the Euler's equations for an ideal isentropic fluid. (Although, very much related formulations of systems of partial differential equations have been studied within the general context of conservation laws.)

From the systems and control point of view, the present paper can be seen as providing the extension of the generalized Hamiltonian framework established for lumped-parameter systems in [4,7,26,28–31] to the distributed-parameter case. In the lumped-parameter case this Hamiltonian framework has been successfully employed in the consistent (modular) modeling and simulation of complex *interconnected* lumped-parameter physical systems, including (actuated) multi-body systems with kinematic constraints and electro-mechanical systems [7,18,30,31], and in the design and *control* of such systems, exploiting the Hamiltonian and passivity structure in a crucial way [19,25,26,29,30,33]. Similar developments can be pursued in the distributed-parameter case; see already [27,32] for some initial ideas in this direction.

The present paper is organized as follows. The main framework is established in Section 2. After a general introduction to Dirac structures in Section 2.1 the definition of a Stokes–Dirac structure is treated in Section 2.2. This paves the way for the Hamiltonian formulation of distributed-parameter systems with boundary variables in Section 2.3. In Section 3 this is applied to Maxwell's equations on a bounded domain (Section 3.1), the telegraph equations for an ideal transmission line (Section 3.2), and the vibrating string (Section 3.3). Furthermore, by modifying the Stokes–Dirac structure with an additional term corresponding to three-dimensional convection, Euler's equations for an ideal isentropic fluid are treated in Section 3.4. Finally, in Section 4 the properties of Stokes–Dirac structures are further analyzed: Section 4.1 deals with the pseudo-Poisson bracket associated to the Stokes–Dirac structure, Section 4.2 sets up the basic notions of conservation laws and Casimir functions captured by the Stokes–Dirac structure, while Section 4.3 deals with a covariant definition of Stokes–Dirac structures and the resulting Hamiltonian systems. Finally, Section 5 contains the conclusions.

## 2. Hamiltonian formulation of distributed-parameter systems with boundary energy flow

### 2.1. Dirac structures

The notion of a Dirac structure was originally introduced in [6,8] as a geometric structure generalizing both *symplectic* and *Poisson* structures. In [2,4,7,18,28–31], it was employed as the geometrical notion formalizing general *power-conserving interconnections*, thereby allowing the Hamiltonian formulation of interconnected and constrained mechanical and electrical systems.

A definition of Dirac structures (which is actually slightly more general than the one in [6,8]) can be given as follows. Let $\mathcal{F}$ and $\mathcal{E}$ be linear spaces, equipped with a pairing, that is, a bilinear operation:

$$\mathcal{F} \times \mathcal{E} \to L \tag{1}$$

with $L$ a linear space. The pairing will be denoted by $\langle e|f\rangle \in L$, $f \in \mathcal{F}, e \in \mathcal{E}$. By symmetrizing the pairing we obtain a symmetric bilinear form $\langle\langle\cdot\rangle\rangle$ on $\mathcal{F} \times \mathcal{E}$, with values in $L$, defined as

$$\langle\langle (f_1, e_1), (f_2, e_2)\rangle\rangle := \langle e_1|f_2\rangle + \langle e_2|f_1\rangle, \quad (f_i, e_i) \in \mathcal{F} \times \mathcal{E} \tag{2}$$

**Definition 2.1.** Let $\mathcal{F}$ and $\mathcal{E}$ be linear spaces with a pairing $\langle|\rangle$. A Dirac structure is a linear subspace $D \subset \mathcal{F} \times \mathcal{E}$ such that $D = D^{\perp}$, with $\perp$ denoting the orthogonal complement with respect to the bilinear form $\langle\langle\cdot\rangle\rangle$.

**Example 2.1.** Let $\mathcal{F}$ be a linear space over $\mathbb{R}$. Let $\mathcal{E}$ be given as $\mathcal{F}^*$ (the space of linear functionals on $\mathcal{F}$), with pairing $\langle|\rangle$ the duality product $\langle e|f\rangle \in \mathbb{R}$.

(a) Let $J : \mathcal{E} \to \mathcal{F}$ be a skew-symmetric map. Then graph $J \subset \mathcal{F} \times \mathcal{E}$ is a Dirac structure.
(b) Let $\omega : \mathcal{F} \to \mathcal{E}$ be a skew-symmetric map. Then graph $\omega \subset \mathcal{F} \times \mathcal{E}$ is a Dirac structure.
(c) Let $V \subset \mathcal{F}$ be a finite-dimensional linear subspace. Then $V \times V^{\text{orth}} \subset \mathcal{F} \times \mathcal{E}$ is a Dirac structure, where $V^{\text{orth}} \subset \mathcal{E}$ is the annihilating subspace of $V$. The same holds if $\mathcal{F}$ is a topological vector space, $\mathcal{E}$ is the space of linear continuous functionals on $\mathcal{F}$, and $V$ is a *closed* subspace of $\mathcal{F}$.

**Example 2.2.** Let $M$ be a finite-dimensional manifold. Let $\mathcal{F} = V(M)$ denote the Lie algebra of smooth vector fields on $M$, and let $\mathcal{E} = \Omega^1(M)$ be the linear space of smooth one-forms on $M$. Consider the usual pairing $\langle\alpha|X\rangle = i_X\alpha$ between one-forms $\alpha$ and vector fields $X$; implying that $L$ is the linear space of smooth functions on $M$.

(a) Let $J$ be a Poisson structure on $M$, defining a skew-symmetric mapping $J : \Omega^1(M) \to V(M)$. Then graph $J \subset V(M) \times \Omega^1(M)$ is a Dirac structure.
(b) Let $\omega$ be a presymplectic structure on $M$, defining a skew-symmetric mapping $\omega$: $V(M) \to \Omega^1(M)$. Then graph $\omega \subset V(M) \times \Omega^1(M)$ is a Dirac structure.
(c) Let $V$ be a constant-dimensional distribution on $M$, and let ann $V$ be its annihilating co-distribution. Then $V \times$ ann $V$ is a Dirac structure.

**Remark 2.1.** Usually in Example 2.2 an additional integrability condition is imposed on the Dirac structure, cf. [6,8]. In part (a) this condition is equivalent to the *Jacobi-identity* for the Poisson structure; in part (b) it is equivalent to the *closedness* of the presymplectic structure, while in part (c) it is equivalent to the *involutivity* of the distribution $D$. Integrability is equivalent to the existence of canonical coordinates, cf. [6–8]. Various formulations of integrability of Dirac structures and their implications have been worked out in [7]. For the developments of the current paper the notion of integrability is not crucial; see however the conclusions section for some comments in this direction.

From the defining property $D = D^{\perp}$ of a Dirac structure it directly follows that for any $(f, e) \in D$

$$0 = \langle\langle (f, e), (f, e)\rangle\rangle = 2\langle e|f\rangle \tag{3}$$

Thus, if $(f, e)$ is a pair of *power variables*(e.g. currents and voltages in an electric circuit context, or forces and velocities in a mechanical context), then the condition $(f, e) \in D$ implies *power-conservation* $\langle e|f\rangle = 0$ (as do Kirchhoff's laws or Newton's third law). This is the starting point for the geometric formulation of general power-conserving interconnections in physical systems by Dirac structures as alluded to above.

### 2.2. Stokes–Dirac structures

In this subsection we treat the underlying geometric framework for the Hamiltonian formulation of distributed-parameter systems on a bounded spatial domain, with non-zero energy flow through the boundary. The key concept is the introduction of a special type of Dirac structure on suitable spaces of differential forms on the spatial domain and its boundary, making use of Stokes' theorem. A preliminary treatment of this Dirac structure has been given in [20,21].

Throughout, let $Z$ be an $n$-dimensional smooth manifold with smooth $(n-1)$-dimensional boundary $\partial Z$, representing the space of *spatial variables*.

Denote by $\Omega^k(Z)$, $k = 0, 1, \ldots, n$, the space of exterior $k$-forms on $Z$, and by $\Omega^k(\partial Z)$, $k = 0, 1, \ldots, n-1$, the space of $k$-forms on $\partial Z$. (Note that $\Omega^0(Z)$, respectively $\Omega^0(\partial Z)$, is the space of smooth functions on $Z$, respectively $\partial Z$.) Clearly, $\Omega^k(Z)$ and $\Omega^k(\partial Z)$ are (infinite-dimensional) linear spaces (over $\mathbb{R}$). Furthermore, there is a natural pairing between $\Omega^k(Z)$ and $\Omega^{n-k}(Z)$ given by

$$\langle \beta|\alpha\rangle := \int_Z \beta \wedge \alpha, \quad (\in \mathbb{R}) \tag{4}$$

with $\alpha \in \Omega^k(Z)$, $\beta \in \Omega^{n-k}(Z)$, where $\wedge$ is the usual wedge product of differential forms yielding the $n$-form $\beta \wedge \alpha$. In fact, the pairing (4) is *non-degenerate* in the sense that if $\langle \beta|\alpha\rangle = 0$ for all $\alpha$, respectively, for all $\beta$, then $\beta = 0$, respectively $\alpha = 0$.

Similarly, there is a pairing between $\Omega^k(\partial Z)$ and $\Omega^{n-1-k}(\partial Z)$ given by

$$\langle \beta|\alpha\rangle := \int_{\partial Z} \beta \wedge \alpha \tag{5}$$

with $\alpha \in \Omega^k(\partial Z)$, $\beta \in \Omega^{n-1-k}(\partial Z)$. Now let us define the linear space

$$\mathcal{F}_{p,q} := \Omega^p(Z) \times \Omega^q(Z) \times \Omega^{n-p}(\partial Z) \tag{6}$$

for any pair $p, q$ of positive integers satisfying

$$p + q = n + 1 \tag{7}$$

and correspondingly let us define

$$\mathcal{E}_{p,q} := \Omega^{n-p}(Z) \times \Omega^{n-q}(Z) \times \Omega^{n-q}(\partial Z) \tag{8}$$

Then the pairing (4) and (5) yields a (non-degenerate) pairing between $\mathcal{F}_{p,q}$ and $\mathcal{E}_{p,q}$ (note that by Eq. (7) $(n-p)+(n-q) = n-1$). As before (see Eq. (2)), symmetrization of this pairing yields the following bilinear form on $\mathcal{F}_{p,q} \times \mathcal{E}_{p,q}$ with values in $\mathbb{R}$:

$$\begin{aligned}&\langle\langle (f_p^1, f_q^1, f_b^1, e_p^1, e_q^1, e_b^1), (f_p^2, f_q^2, f_b^2, e_p^2, e_q^2, e_b^2)\rangle\rangle \\ &\quad := \int_Z [e_p^1 \wedge f_p^2 + e_q^1 \wedge f_q^2 + e_p^2 \wedge f_p^1 + e_q^2 \wedge f_q^1] + \int_{\partial Z} [e_b^1 \wedge f_b^2 + e_b^2 \wedge f_b^1]\end{aligned} \tag{9}$$

where for $i = 1, 2$

$$\begin{aligned} &f_p^i \in \Omega^p(Z), && f_q^i \in \Omega^q(Z) \\ &e_p^i \in \Omega^{n-p}(Z), && e_p^i \in \Omega^{n-q}(Z) \\ &f_b^i \in \Omega^{n-p}(\partial Z), && e_b^i \in \Omega^{n-q}(\partial Z) \end{aligned} \tag{10}$$

The spaces of differential forms $\Omega^p(Z)$ and $\Omega^q(Z)$ will represent the energy variables of two different physical energy domains interacting with each other, while $\Omega^{n-p}(\partial Z)$ and $\Omega^{n-q}(\partial Z)$ will denote the boundary variables whose (wedge) product represents the boundary energy flow. For example, in Maxwell's equations (Section 3.1) we will have $n = 3$ and $p = q = 2$; with $\Omega^p(Z) = \Omega^2(Z)$, respectively $\Omega^q(Z) = \Omega^2(Z)$, being the space of electric field inductions, respectively magnetic field inductions, and $\Omega^{n-p}(\partial Z) = \Omega^1(\partial Z)$ denoting the electric and magnetic field intensities at the boundary, with product the Poynting vector.

**Theorem 2.1.** *Consider $\mathcal{F}_{p,q}$ and $\mathcal{E}_{p,q}$ given in* Eqs. (6) and (8) *with $p, q$ satisfying* Eq. (7), *and bilinear form $\langle\langle\cdot\rangle\rangle$ given by* Eq. (9). *Define the following linear subspace $D$ of $\mathcal{F}_{p,q} \times \mathcal{E}_{p,q}$*

$$\begin{aligned} D = \Bigg\{ (f_p, f_q, f_b, e_p, e_q, e_b) \in \mathcal{F}_{p,q} \times \mathcal{E}_{p,q} \,|\, & \begin{bmatrix} f_p \\ f_q \end{bmatrix} = \begin{bmatrix} 0 & (-1)^r d \\ d & 0 \end{bmatrix} \begin{bmatrix} e_p \\ e_q \end{bmatrix}, \\ & \begin{bmatrix} f_b \\ e_b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -(-1)^{n-q} \end{bmatrix} \begin{bmatrix} e_{p|\partial Z} \\ e_{q|\partial Z} \end{bmatrix} \Bigg\} \end{aligned} \tag{11}$$

*where $|_{\partial Z}$ denotes restriction to the boundary $\partial Z$, and $r := pq + 1$. Then $D = D^\perp$, that is, $D$ is a Dirac structure.*

**Proof.** First we show $D \subset D^\perp$, and secondly $D^\perp \subset D$.

(i) $D \subset D^\perp$ : let $(f_p^1, f_q^1, f_b^1, e_p^1, e_q^1, e_b^1) \in D$, and consider any $(f_p^2, f_q^2, f_b^2, e_p^2, e_q^2, e_b^2) \in D$. By substitution of Eq. (11) into Eq. (9) the right-hand side of Eq. (9) becomes

$$\int_Z [(-1)^r e_p^1 \wedge \mathrm{d}e_q^2 + e_q^1 \wedge \mathrm{d}e_p^2 + (-1)^r e_p^2 \wedge \mathrm{d}e_q^1 + e_q^2 \wedge \mathrm{d}e_p^1] - (-1)^{n-q} \int_{\partial Z} [e_q^1 \wedge e_p^2 + e_q^2 \wedge e_p^1] \tag{12}$$

By the properties of the exterior derivative:

$$\begin{aligned} \mathrm{d}(e_q^2 \wedge e_p^1) &= \mathrm{d}e_q^2 \wedge e_p^1 + (-1)^{n-q} e_q^2 \wedge \mathrm{d}e_p^1 \\ \mathrm{d}(e_q^1 \wedge e_p^2) &= \mathrm{d}e_q^1 \wedge e_p^2 + (-1)^{n-q} e_q^1 \wedge \mathrm{d}e_p^2 \end{aligned} \tag{13}$$

and by the properties of the wedge product:

$$\begin{aligned} e_p^1 \wedge \mathrm{d}e_q^2 &= (-1)^{(n-p)(n-q+1)} \mathrm{d}e_q^2 \wedge e_p^1 \\ e_p^2 \wedge \mathrm{d}e_q^1 &= (-1)^{(n-p)(n-q+1)} \mathrm{d}e_q^1 \wedge e_p^2 \end{aligned} \tag{14}$$

Hence, the first and fourth term in the $\int_Z$ integral in Eq. (12) can be rewritten as

$$\begin{aligned} &(-1)^r e_p^1 \wedge \mathrm{d}e_q^2 + e_q^2 \wedge \mathrm{d}e_p^1 \\ &\quad = (-1)^{r+(n-p)(n-q+1)} \mathrm{d}e_q^2 \wedge e_p^1 + e_q^2 \wedge \mathrm{d}e_p^1 \\ &\quad = (-1)^{n-q} \mathrm{d}e_q^2 \wedge e_p^1 + e_q^2 \wedge \mathrm{d}e_p^1 = (-1)^{n-q} \mathrm{d}(e_q^2 \wedge e_p^1) \end{aligned} \tag{15}$$

since by $p+q = n+1$ and $r = pq+1$, $r+(n-p)(n-q+1) = r+(q-1)p = 2pq-p+1$ and $(-1)^{2pq-p+1} = (-1)^{1-p} = (-1)^{n-q}$.

Similarly, the second term together with third term can be written as

$$e_q^1 \wedge \mathrm{d}e_p^2 + (-1)^r e_p^2 \wedge \mathrm{d}e_q^1 = (-1)^{n-q} \mathrm{d}(e_q^1 \wedge e_p^2) \tag{16}$$

Substitution of Eqs. (15) and (16) in the $\int_Z$ integral in Eq. (12) then yields by Stokes' theorem that this integral is equal to

$$(-1)^{n-q} \int_Z \mathrm{d}(e_q^2 \wedge e_p^1) + \mathrm{d}(e_q^1 \wedge e_p^2) = (-1)^{n-q} \int_{\partial Z} [e_q^1 \wedge e_p^2 + e_q^2 \wedge e_p^1] \tag{17}$$

showing that Eq. (12) is zero, and thus $D \subset D^\perp$.

(ii) $D^\perp \subset D$ : let $(f_p^1, f_q^1, f_b^1, e_p^1, e_q^1, e_b^1) \in D^\perp$, implying that for all elements $(f_p^2, f_q^2, f_b^2, e_p^2, e_q^2, e_b^2) \in D$ the right-hand side of Eq. (9) is zero, and hence by substitution of Eq. (11)

$$\int_Z [(-1)^r e_p^1 \wedge \mathrm{d}e_q^2 + e_q^1 \wedge \mathrm{d}e_p^2 + e_p^2 \wedge f_p^1 + e_q^2 \wedge f_q^1] + \int_{\partial Z} [e_b^1 \wedge e_p^2 - (-1)^{n-q} e_q^2 \wedge f_b^1] = 0 \tag{18}$$

for all $e_p^2, e_q^2$. Now, consider first $e_p^2, e_q^2$ which are zero on the boundary $\partial Z$, implying that

$$\int_Z [(-1)^r e_p^1 \wedge \mathrm{d}e_q^2 + e_q^1 \wedge \mathrm{d}e_p^2 + e_p^2 \wedge f_p^1 + e_q^2 \wedge f_q^1] = 0 \tag{19}$$

for all $e_p^2, e_q^2$ with $e_p^2|_{\partial Z} = e_q^2|_{\partial Z} = 0$. By the first line of Eqs. (13) and (14)

$$\begin{aligned}(-1)^r e_p^1 \wedge \mathrm{d}e_q^2 &= (-1)^{r+(n-p)(n-q+1)} \mathrm{d}e_q^2 \wedge e_p^1 = (-1)^{n-q} \mathrm{d}e_q^2 \wedge e_p^1 \\ &= (-1)^{n-q} \mathrm{d}(e_q^2 \wedge e_p^1) - e_q^2 \wedge \mathrm{d}e_p^1\end{aligned} \tag{20}$$

Similarly, by the second line of Eqs. (13) and (14)

$$\begin{aligned}e_q^1 \wedge \mathrm{d}e_p^2 &= (-1)^{n-q} \mathrm{d}(e_q^1 \wedge e_p^2) - (-1)^{n-q} \mathrm{d}e_q^1 \wedge e_p^2 \\ e_p^2 \wedge f_p^1 &= (-1)^{(n-p)p} f_p^1 \wedge e_p^2\end{aligned} \tag{21}$$

Since $e_p^2|_{\partial Z} = e_q^2|_{\partial Z} = 0$, substitution of Eqs. (20) and (21) into Eq. (19) then yields by Stokes' theorem

$$\int_Z [-e_q^2 \wedge \mathrm{d}e_p^1 - (-1)^{n-q} \mathrm{d}e_q^1 \wedge e_p^2 + (-1)^{(n-p)p} f_p^1 \wedge e_p^2 + e_q^2 \wedge f_q^1] = 0 \tag{22}$$

for all $e_p^2, e_q^2$ with $e_p^2|_{\partial Z} = e_q^2|_{\partial Z} = 0$. Clearly, this implies

$$\begin{aligned}f_q^1 &= \mathrm{d}e_p^1 \\ (-1)^{(n-p)p} f_p^1 &= (-1)^{(n-q)} \mathrm{d}e_q^1\end{aligned} \tag{23}$$

where the last equality is easily seen to be equivalent to

$$f_p^1 = (-1)^r \mathrm{d}e_q^1 \tag{24}$$

Finally, substitute Eqs. (23) and (24) into Eq. (18) to obtain

$$\begin{aligned}&\int^Z [(-1)^r e_p^1 \wedge \mathrm{d}e_q^2 + e_q^2 \wedge \mathrm{d}e_p^1 + e_q^1 \wedge de_p^2 + (-1)^r e_p^2 \wedge \mathrm{d}e_q^1] \\ &\quad + \int_{\partial Z} [e_b^1 \wedge e_p^2 - (-1)^{n-q} e_q^2 \wedge f_b^1] = 0\end{aligned} \tag{25}$$

for all $e_p^2, e_q^2$. Substituting again Eq. (20) and the first line of Eq. (21), noting that $(-1)^{n-q} \mathrm{d}e_q^1 \wedge e_p^2 = (-1)^r e_p^2 \wedge \mathrm{d}e_q^1$, this yields

$$\begin{aligned}&\int_Z [(-1)^{n-q} \mathrm{d}(e_q^2 \wedge e_p^1) + (-1)^{n-q} \mathrm{d}(e_q^1 \wedge e_p^2)] \\ &\quad + \int_{\partial Z} [e_b^1 \wedge e_p^2 - (-1)^{n-q} e_q^2 \wedge f_b^1] = 0\end{aligned} \tag{26}$$

and hence by Stokes' theorem

$$\int_{\partial Z} [(-1)^{n-q} e_q^2 \wedge e_p^1 - (-1)^{n-q} e_q^2 \wedge f_b^1 + (-1)^{n-q} e_q^1 \wedge e_p^2 + e_b^1 \wedge e_p^2] = 0 \tag{27}$$

for all $e_p^2, e_q^2$, implying that

$$\begin{aligned} f_b^1 &= e_p^1|_{\partial Z} \\ e_b^1 &= -(-1)^{n-q} e_q^1|_{\partial Z} \end{aligned} \tag{28}$$

showing that indeed $(f_p^1, f_q^1, f_b^1, e_p^1, e_q^1, e_b^1) \in D$. □

**Remark 2.2.** The spatial *compositionality* properties of the Stokes–Dirac structure immediately follow from its definition. Indeed, let $Z_1$, $Z_2$ be two $n$-dimensional manifolds with boundaries $\partial Z_1$, $\partial Z_2$, such that

$$\begin{aligned} \partial Z_1 &= \Gamma \cup \Gamma_1, \quad \Gamma \cap \Gamma_1 = \phi \\ \partial Z_2 &= \Gamma \cup \Gamma_2, \quad \Gamma \cap \Gamma_2 = \phi \end{aligned} \tag{29}$$

for certain $(n-1)$-dimensional manifolds $\Gamma$, $\Gamma_1$, $\Gamma_2$ (that is, $Z_1$ and $Z_2$ have boundary $\Gamma$ in common). Then the Stokes–Dirac structures $D_1$, $D_2$ on $Z_1$, respectively $Z_2$, compose to the Stokes–Dirac structure on the manifold $Z_1 \cup Z_2$ with boundary $\Gamma_1 \cup \Gamma_2$ if we equate on $\Gamma$ the boundary variables $f_b^1$ (corresponding to $D_1$) with $-f_b^2$ (corresponding to $D_2$), or if we reverse orientation. (Note that a minus sign is inserted in order to ensure that the power flowing *into* $Z_1$ via $\Gamma$ is equal to the power flowing *out* of $Z_2$ via $\Gamma$.)

### *2.3. Distributed-parameter port-Hamiltonian systems*

The definition of a distributed-parameter Hamiltonian system with respect to a Stokes–Dirac structure can now be stated as follows. Let $Z$ be an $n$-dimensional manifold with boundary $\partial Z$, and let $D$ be a Stokes–Dirac structure as in Section 2.2. Consider furthermore a *Hamiltonian density* (energy per volume element)

$$\mathcal{H} : \Omega^p(Z) \times \Omega^q(Z) \times Z \to \Omega^n(Z) \tag{30}$$

resulting in the total energy

$$H := \int_Z \mathcal{H} \in \mathbb{R} \tag{31}$$

Recall, see Eq. (4), that there exists a non-degenerate pairing between $\Omega^p(Z)$ and $\Omega^{n-p}(Z)$, respectively between $\Omega^q(Z)$ and $\Omega^{n-q}(Z)$. This means that $\Omega^{n-p}(Z)$ and $\Omega^{n-q}(Z)$ can be regarded as *dual spaces* to $\Omega^p(Z)$, respectively $\Omega^q(Z)$ (although strictly contained in their functional analytic duals). Let now $\alpha_p, \partial\alpha_p \in \Omega^p(Z)$, $\alpha_q, \partial\alpha_q \in \Omega^q(Z)$. Then under weak smoothness conditions on $\mathcal{H}$

$$\begin{aligned} H(\alpha_p + \partial\alpha_p, \alpha_q + \partial\alpha_q) &= \int_Z \mathcal{H}(\alpha_p + \partial\alpha_p, \alpha_q + \partial\alpha_q, z) \\ &= \int_Z \mathcal{H}(\alpha_p, \alpha_q, z) + \int_Z [\delta_p H \wedge \partial\alpha_p + \delta_q H \wedge \partial\alpha_q] \\ &\quad + \text{higher order terms in } \partial\alpha_p, \partial\alpha_q \end{aligned} \tag{32}$$

for certain differential forms

$$\begin{aligned} \delta_p H &\in \Omega^{n-p}(Z) \\ \delta_q H &\in \Omega^{n-q}(Z) \end{aligned} \tag{33}$$

Furthermore, from the non-degeneracity of the pairing between $\Omega^p(Z)$ and $\Omega^{n-p}(Z)$, respectively between $\Omega^q(Z)$ and $\Omega^{n-q}(Z)$, it immediately follows that these differential forms are uniquely determined. This means that $(\delta_p H, \delta_q H) \in \Omega^{n-p}(Z) \times \Omega^{n-q}(Z)$ can be regarded as the (partial) *variational derivatives* (see e.g. [24]) of $H$ at $(\alpha_p, \alpha_q) \in \Omega^p(Z) \times \Omega^q(Z)$. Throughout this paper we shall assume that the Hamiltonian $H$ admits variational derivatives satisfying Eq.(32).

Now consider a time-function

$$(\alpha_p(t), \alpha_q(t)) \in \Omega^p(Z) \times \Omega^q(Z), \quad t \in \mathbb{R} \tag{34}$$

and the Hamiltonian $H(\alpha_p(t), \alpha_q(t))$ evaluated along this trajectory. It follows that at any time $t$

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \int_Z \left[ \delta_p H \wedge \frac{\partial \alpha_p}{\partial t} + \delta_q H \wedge \frac{\partial \alpha_q}{\partial t} \right] \tag{35}$$

The differential forms $\partial\alpha_p/\partial t$, $\partial\alpha_q/\partial t$ represent the generalized velocities of the energy variables $\alpha_p, \alpha_q$. They are connected to the Stokes–Dirac structure $D$ by setting

$$\begin{aligned} f_p &= -\frac{\partial \alpha_p}{\partial t} \\ f_q &= -\frac{\partial \alpha_q}{\partial t} \end{aligned} \tag{36}$$

(again the minus sign is included to have a consistent energy flow description). Since the right-hand side of Eq. (35) is the rate of increase of the stored energy $H$, we set

$$\begin{aligned} e_p &= \delta_p H \\ e_q &= \delta_q H \end{aligned} \tag{37}$$

(In network modeling terminology $\delta_p H$ and $\delta_q H$ are called the *co-energy* variables, which are set equal to the effort variables $e_p, e_q$.) Now we come to the general Hamiltonian description of a distributed-parameter system with boundary energy flow. In order to emphasize that the boundary variables are regarded as *interconnection variables*, which can be interconnected to other systems and whose product represents *power*, we call these models *port*-Hamiltonian systems. (This terminology comes from network modeling, see e.g. [16,31,30].)

**Definition 2.2.** The *distributed-parameter port-Hamiltonian system* with $n$-dimensional manifold of spatial variables $Z$, state space $\Omega^p(Z) \times \Omega^q(Z)$ (with $p + q = n + 1$), Stokes–Dirac structure $D$ given by Eq. (11), and Hamiltonian $H$, is given as

(with $r = pq + 1$)

$$\begin{bmatrix} -\dfrac{\partial \alpha_p}{\partial t} \\ -\dfrac{\partial \alpha_q}{\partial t} \end{bmatrix} = \begin{bmatrix} 0 & (-1)^r d \\ d & 0 \end{bmatrix} \begin{bmatrix} \delta_p H \\ \delta_q H \end{bmatrix},$$
$$\begin{bmatrix} f_b \\ e_b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -(-1)^{n-q} \end{bmatrix} \begin{bmatrix} \delta_p H|_{\partial Z} \\ \delta_q H|_{\partial Z} \end{bmatrix} \tag{38}$$

By the power-conserving property (3) of any Dirac structure it immediately follows that for any $(f_p, f_q, f_b, e_p, e_q, e_b)$ in the Stokes–Dirac structure $D$

$$\int_Z [e_p \wedge f_p + e_q \wedge f_q] + \int_{\partial Z} e_b \wedge f_b = 0 \tag{39}$$

Hence, by substitution of Eqs. (36) and (37) and using Eq. (35) we obtain

**Proposition 2.1.** *Consider the distributed-parameter port-Hamiltonian system* Eq. (38). *Then*

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \int_{\partial Z} e_b \wedge f_b, \tag{40}$$

*expressing that the increase in energy on the domain $Z$ is equal to the power supplied to the system through the boundary $\partial Z$.*

The system (38) can be called a (non-linear) *boundary control* system in the sense of e.g. [10]. Indeed, we could interpret $f_b$ as the boundary control inputs to the system, and $e_b$ as the measured outputs (or the other way around). In Section 3 we shall further elaborate on this point of view.

Energy exchange through the boundary is not the only way a distributed-parameter system may interact with its environment. An example of this is provided by Maxwell's equations (Section 3.1), where interaction may also take place via the current density $J$, which directly affects the electric charge distribution in the domain $Z$. In order to cope with this situation we augment the spaces $\mathcal{F}_{p,q}, \mathcal{E}_{p,q}$ as defined in Eqs. (6) and (8) to

$$\begin{aligned} \mathcal{F}^a_{q,p} &:= \mathcal{F}_{p,q} \times \Omega^d(S) \\ \mathcal{E}^a_{q,p} &:= \mathcal{E}_{p,q} \times \Omega^{n-d}(S) \end{aligned} \tag{41}$$

for some $m$-dimensional manifold $S$ and some $d \in \{0, 1, \ldots, m\}$, with $f^d \in \Omega^d(S)$ denoting the externally supplied distributed control flow, and $e^d \in \Omega^{n-d}(S)$ the conjugate distributed quantity, corresponding to an energy exchange

$$\int_S e^d \wedge f^d \tag{42}$$

The Stokes–Dirac structure (11) is now extended to

$$\begin{bmatrix} f_p \\ f_q \end{bmatrix} = \begin{bmatrix} 0 & (-1)^r d \\ d & 0 \end{bmatrix} \begin{bmatrix} e_p \\ e_q \end{bmatrix} + G(f_d),$$
$$\begin{bmatrix} f_b \\ e_b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -(-1)^{n-q} \end{bmatrix} \begin{bmatrix} e_{p|\partial Z} \\ e_{q|\partial Z} \end{bmatrix}, \qquad e_d = -G^* \begin{bmatrix} e_p \\ e_q \end{bmatrix} \tag{43}$$

with $G$ denoting a linear map

$$G = \begin{pmatrix} G_p \\ G_q \end{pmatrix} : \Omega^d(S) \to \Omega^p(Z) \times \Omega^q(Z) \tag{44}$$

with dual map (again we consider $\Omega^{n-p}(Z)$ and $\Omega^{n-q}(Z)$ as dual spaces to $\Omega^p(Z)$, respectively $\Omega^{n-q}(Z)$)

$$G^* = (G_p^*, G_q^*) : \Omega^{n-p}(Z) \times \Omega^{n-q}(Z) \to \Omega^{n-d}(S) \tag{45}$$

satisfying

$$\int_Z [e_p \wedge G_p(f_d) + e_q \wedge G_q(f_d)] = \int_S [G_p^*(e_p) + G_q^*(e_q)] \wedge f_d \tag{46}$$

for all $e_p \in \Omega^{n-p}(Z)$, $e_q \in \Omega^{n-q}(Z)$, $f_d \in \Omega^d(S)$.

The following proposition can be easily checked.

**Proposition 2.2.** Eq. (43) *determine a Dirac structure* $D^a \subset \mathcal{F}^a_{p,q} \times \mathcal{E}^a_{p,q}$ *with respect to the augmented bilinear form on* $\mathcal{F}^a_{p,q} \times \mathcal{E}^a_{p,q}$ *which is obtained by adding to the bilinear form* (9) *on* $\mathcal{F}_{p,q} \times \mathcal{E}_{p,q}$ *the term*

$$\int_S [e_d^1 \wedge f_d^2 + e_d^2 \wedge f_d^1] \tag{47}$$

By making now the substitutions (36) and (37) into $D^a$ given by Eq. (43) we obtain a port-Hamiltonian system with external variables $(f_b, f_d, e_b, e_d)$, with $f_b, e_b$ the *boundary* external variables and $f_d, e_d$ the *distributed* external variables. Furthermore, the energy balance (40) extends to

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \int_{\partial Z} e_b \wedge f_b + \int_S e_d \wedge f_d \tag{48}$$

with the first term on the right-hand side denoting the power flow through the boundary, and the second term denoting the distributed power flow.

Finally, *energy dissipation* can be incorporated in the framework of distributed-parameter port-Hamiltonian systems by *terminating* some of the ports (boundary or distributed) with a *resistive relation*. For example, for distributed dissipation, let $R : \Omega^{n-d}(S) \to \Omega^d(S)$ be a map satisfying

$$\int_S e_d \wedge R(e_d) \geq 0, \quad \forall e_d \in \Omega^{n-d}(S) \tag{49}$$

Then by adding the relation

$$f_d = -R(e_d) \tag{50}$$

to the port-Hamiltonian system defined with respect to the Dirac structure $D^a$, we obtain a port-Hamiltonian system *with dissipation*, satisfying the energy inequality

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \int_{\partial Z} e_b \wedge f_b - \int_S e_d \wedge R(e_d) \leq \int_{\partial Z} e_b \wedge f_b \tag{51}$$

## 3. Examples

In this section we show how the framework of distributed-parameter port-Hamiltonian systems admits the representation of Maxwell's equations, the telegraph equations of an ideal transmission line, the vibrating string, and the Euler equations of an ideal isentropic fluid.

### 3.1. Maxwell's equations

We closely follow the formulation of Maxwell's equations in terms of differential forms as presented in [11], and show how this directly leads to the formulation as a distributed-parameter port-Hamiltonian system.

Let $Z \subset \mathbb{R}^3$ be a three-dimensional manifold with boundary $\partial Z$, defining the spatial domain, and consider the electromagnetic field in $Z$. The energy variables are the *electric field induction* two-form $\alpha_p = \mathcal{D} \in \Omega^2(Z)$:

$$\mathcal{D} = \tfrac{1}{2} D_{ij}(t, z)\, \mathrm{d}z^i \wedge \mathrm{d}z^j \tag{52}$$

and the *magnetic field induction* two-form $\alpha_q = \mathcal{B} \in \Omega^2(Z)$:

$$\mathcal{B} = \tfrac{1}{2} B_{ij}(t, z)\, \mathrm{d}z^i \wedge \mathrm{d}z^j \tag{53}$$

The corresponding Stokes–Dirac structure ($n = 3$, $p = 2$, $q = 2$) is given as (cf. Eq. (11))

$$\begin{bmatrix} f_p \\ f_q \end{bmatrix} = \begin{bmatrix} 0 & -d \\ d & 0 \end{bmatrix} \begin{bmatrix} e_p \\ e_q \end{bmatrix}, \qquad \begin{bmatrix} f_b \\ e_b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e_{p|\partial Z} \\ e_{q|\partial Z} \end{bmatrix} \tag{54}$$

Usually in this case one does *not* start with the definition of the total energy (Hamiltonian) $H$, but instead with the co-energy variables $\delta_p H$, $\delta_q H$, given, respectively, as the electric field intensity $\mathcal{E} \in \Omega^1(Z)$:

$$\mathcal{E} = E_i(t, z)\, \mathrm{d}z^i \tag{55}$$

and the magnetic field intensity $\mathcal{H} \in \Omega^1(Z)$:

$$\mathcal{H} = H_i(t, z)\, \mathrm{d}z^i \tag{56}$$

They are related to the energy variables through the constitutive relations of the medium (or material equations)

$$\begin{aligned} *\mathcal{D} &= \epsilon\mathcal{E} \\ *\mathcal{B} &= \mu H \end{aligned} \tag{57}$$

with the scalar functions $\epsilon(t,z)$ and $\mu(t,z)$ denoting the electric permittivity, respectively magnetic permeability, and $*$ denoting the Hodge star operator (corresponding to a Riemannian metric on $Z$), converting two- into one-forms. Then one *defines* the Hamiltonian $H$ as

$$H = \int_Z \tfrac{1}{2}(\mathcal{E}\wedge\mathcal{D} + \mathcal{H}\wedge\mathcal{B}) \tag{58}$$

and one immediately verifies that $\delta_p H = \delta_D H = \mathcal{E}$, $\delta_q H = \delta_B H = \mathcal{H}$.

Nevertheless there are other cases (corresponding to a *non-linear* theory of the electromagnetic field, such as the Born-infield theory, see e.g. [11]) where one starts with a more general Hamiltonian $H = \int_Z h$, with the energy density $h(\mathcal{D}, B)$ being a more general expression than $1/2(\epsilon^{-1} * \mathcal{D}\wedge\mathcal{D} + \mu^{-1} * \mathcal{B}\wedge\mathcal{B})$.

Assuming that there is no current in the medium Maxwell's equations can now be written as (see [11])

$$\begin{aligned} \frac{\partial\mathcal{D}}{\partial t} &= \mathrm{d}\mathcal{H} \\ \frac{\partial\mathcal{B}}{\partial t} &= -\mathrm{d}\mathcal{E} \end{aligned} \tag{59}$$

Explicitly taking into account the behavior at the boundary, Maxwell's equations on a domain $Z \subset \mathbb{R}^3$ are then represented as the port-Hamiltonian system with respect to the Stokes–Dirac structure given by Eq. (54), as

$$\begin{bmatrix} -\dfrac{\partial\mathcal{D}}{\partial t} \\ -\dfrac{\partial B}{\partial t} \end{bmatrix} = \begin{bmatrix} 0 & -d \\ d & 0 \end{bmatrix} \begin{bmatrix} \delta_D H \\ \delta_B H \end{bmatrix}, \qquad \begin{bmatrix} f_b \\ e_b \end{bmatrix} = \begin{bmatrix} \delta_D H|_{\partial Z} \\ \delta_B H|_{\partial Z} \end{bmatrix} \tag{60}$$

Note that the first line of Eq. (59) is nothing else than (the differential version of) Ampère's law, while the second line of Eq. (59) is Faraday's law. Hence, the Stokes–Dirac structure in Eqs. (59) and (60) expresses the basic physical laws connecting $\mathcal{D}$, $\mathcal{B}$, $\mathcal{H}$ and $\mathcal{E}$.

The energy-balance equation (Eq. (40)) in the case of Maxwell's equations takes the form

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \int_{\partial Z} \delta_B H \wedge \delta_D H = \int_{\partial Z} \mathcal{H}\wedge\mathcal{E} = -\int_{\partial Z} \mathcal{E}\wedge\mathcal{H} \tag{61}$$

with $\mathcal{E}\wedge\mathcal{H}$ a two-form corresponding to the *Poynting vector* (see [11]).

In the case of a non-zero *current density* we have to modify the first matrix equation of (60) to

$$\begin{bmatrix} -\dfrac{\partial D}{\partial t} \\ -\dfrac{\partial B}{\partial t} \end{bmatrix} = \begin{bmatrix} 0 & -d \\ d & 0 \end{bmatrix} \begin{bmatrix} \delta_D H \\ \delta_B H \end{bmatrix} + \begin{bmatrix} I \\ 0 \end{bmatrix} J \tag{62}$$

with $I$ denoting the identity operator from $J \in \Omega^2(Z)$ to $\Omega^2(Z)$. (Thus, in the notation of Eq. (44), $f_d = J$, $S = Z$, and $\Omega^d(S) = \Omega^2(Z)$.) Furthermore, we add the equation

$$\begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} \delta_D H \\ \delta_B H \end{bmatrix} = -\mathcal{E} \tag{63}$$

yielding the augmented energy balance

$$\frac{\mathrm{d}H}{\mathrm{d}t} = -\int_{\partial Z} \mathcal{E} \wedge \mathcal{H} - \int_Z \mathcal{E} \wedge J \tag{64}$$

which is known as *Poynting's theorem*.

Finally, in order to incorporate energy dissipation we write $J = J_d + \bar{J}$, and we impose Ohm's law

$$*J_d = \sigma \mathcal{E} \tag{65}$$

with $\sigma(t, z)$ the specific conductivity of the medium.

### 3.2. Telegraph equations

Consider an ideal lossless transmission line with $Z = [0, 1] \subset \mathbb{R}$. The energy variables are the charge density one-form $Q = Q(t, z)\,\mathrm{d}z \in \Omega^1([0, 1])$, and the flux density one-form $\varphi = \varphi(t, z)\,\mathrm{d}z \in \Omega^1([0, 1])$; thus $p = q = n = 1$. The total energy stored at time $t$ in the transmission line is given as

$$H(Q, \varphi) = \int_0^1 \frac{1}{2} \left( \frac{Q^2(t, z)}{C(z)} + \frac{\varphi^2(t, z)}{L(z)} \right) \mathrm{d}z \tag{66}$$

with co-energy variables

$$\begin{aligned} \delta_Q H &= \frac{Q(t, z)}{C(z)} = V(t, z), \quad \text{(voltage)} \\ \delta_\varphi H &= \frac{\varphi(t, z)}{L(z)} = I(t, z), \quad \text{(current)} \end{aligned} \tag{67}$$

where $C(z)$, $L(z)$ are, respectively, the distributed capacitance and distributed inductance of the line.

The resulting port-Hamiltonian system is given by the telegraph equations

$$\begin{aligned} \frac{\partial Q}{\partial t} &= -\frac{\partial I}{\partial z} \\ \frac{\partial \varphi}{\partial t} &= -\frac{\partial V}{\partial z} \end{aligned} \tag{68}$$

together with the boundary variables

$$\begin{aligned} f_b^0(t) &= V(t, 0), \quad f_b^1(t) = V(t, 1) \\ e_b^0(t) &= -I(t, 0), \quad e_b^1(t) = -I(t, 1) \end{aligned} \tag{69}$$

The resulting energy-balance is

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \int_{\partial([0,1])} e_b f_b = -I(t,1)V(t,1) + I(t,0)V(t,0), \tag{70}$$

in accordance with Eq. (40).

### 3.3. Vibrating string

Consider an elastic string subject to traction forces at its ends. The spatial variable $z$ belongs to the interval $Z = [0,1] \subset \mathbb{R}$. Let us denote by $u(t,z)$ the displacement of the string. The elastic potential energy is a function of the *strain* given by the one-form

$$\alpha_q(t) = \epsilon(t,z)\,\mathrm{d}z = \frac{\partial u}{\partial z}(t,z)\,\mathrm{d}z \tag{71}$$

The associated co-energy variable is the *stress* given by the zero-form

$$\sigma = T * \alpha_q \tag{72}$$

with $T$ the elasticity modulus and $*$ the Hodge star operator. Hence, the potential energy is the quadratic function

$$U(\alpha_q) = \int_0^1 \sigma\alpha_q = \int_0^1 T * \alpha_q \wedge \alpha_q = \int_0^1 T\left(\frac{\partial u}{\partial z}\right)^2 \mathrm{d}z \tag{73}$$

and $\sigma = \delta_q U$.

The kinetic energy $K$ is a function of the kinetic *momentum* defined as the one-form

$$\alpha_p(t) = p(t,z)\,\mathrm{d}z \tag{74}$$

given by the quadratic function

$$K(\alpha_p) = \int_0^1 \frac{p^2}{\mu}\,\mathrm{d}z \tag{75}$$

The associated co-energy variable is the *velocity* given by the zero-form

$$v = \frac{1}{\mu} * \alpha_p = \delta_p K \tag{76}$$

In this case the Dirac structure is the Stokes–Dirac structure for $n = p = q = 1$, with an *opposite sign* convention leading to the equations (with $H := U + K$)

$$\begin{bmatrix} -\dfrac{\partial \alpha_p}{\partial t} \\ -\dfrac{\partial \alpha_q}{\partial t} \end{bmatrix} = \begin{bmatrix} 0 & -d \\ -d & 0 \end{bmatrix} \begin{bmatrix} \delta_p H \\ \delta_q H \end{bmatrix}, \qquad \begin{bmatrix} f_b \\ e_b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \delta_p H|_{\partial Z} \\ \delta_q H|_{\partial Z} \end{bmatrix} \tag{77}$$

or, in more down-to-earth notation

$$\begin{aligned}
&\frac{\partial p}{\partial t} = \frac{\partial \sigma}{\partial z} = \frac{\partial}{\partial z}(T\epsilon)\\
&\frac{\partial \epsilon}{\partial t} = \frac{\partial v}{\partial z} = \frac{\partial}{\partial z}\left(\frac{1}{\mu}p\right)\\
&f_b = v|_{\{0,1\}}\\
&e_b = \sigma|_{\{0,1\}}
\end{aligned} \tag{78}$$

with boundary variables the velocity and stress at the ends of the string. Of course, by substituting $\epsilon = \partial u/\partial z$ into the second equation of (78) one obtains $\partial/\partial z(\partial u/\partial t - p/\mu) = 0$, implying that

$$p = \mu\frac{\partial u}{\partial t} + \mu f(t) \tag{79}$$

for some function $f$, which may be set to zero. Substitution of Eq. (79) into the first equation of (78) then yields the wave equation

$$\mu\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial z}\left(T\frac{\partial u}{\partial z}\right) \tag{80}$$

### *3.4. Ideal isentropic fluid*

Consider an ideal compressible isentropic fluid in three-dimensions, described in Eulerian representation by the standard Euler equations

$$\begin{aligned}
&\frac{\partial \rho}{\partial t} = -\nabla\cdot(\rho v)\\
&\frac{\partial v}{\partial t} = -v\cdot\nabla v - \frac{1}{\rho}\nabla p
\end{aligned} \tag{81}$$

with $\rho(z,t) \in \mathbb{R}$ the mass density at the spatial position $z \in \mathbb{R}^3$ at time $t$, $v(z,t) \in \mathbb{R}^3$ the (Eulerian) velocity of the fluid at spatial position $z$ and time $t$, and $p(z,t)$ the pressure function, derivable from an internal energy function $U(\rho)$ as

$$p(z,t) = \rho^2(z,t)\frac{\partial U}{\partial \rho}(\rho(z,t)) \tag{82}$$

Much innovative work has been done regarding the Hamiltonian formulation of Eq.(81) and more general cases; we refer in particular to [1,13,14,22,23]. However, in these treatments only *closed* fluid dynamical systems are being considered with no energy exchange through the boundary of the spatial domain. As a result, a formulation in terms of Poisson structures can be given, while as argued before, the general inclusion of boundary variables necessitates the use of Dirac structures.

The formulation of Eq. (81) as a port-Hamiltonian system is given as follows. Let $\mathcal{D} \subset \mathbb{R}^3$ be a given domain, filled with the fluid. We assume the existence of a Riemannian metric $\langle\cdot\rangle$

on $\mathcal{D}$; usually the standard Euclidean metric on $\mathbb{R}^3$. Let $Z \subset \mathcal{D}$ be any three-dimensional manifold with boundary $\partial Z$.

We identify the mass-density $\rho$ with a three-form on $Z$ (see e.g. [13,14]), that is, with an element of $\Omega^3(Z)$. Furthermore, we identify the Eulerian vector field $v$ with a one-form on $Z$, that is, with an element of $\Omega^1(Z)$. (By the existence of the Riemannian metric on $Z$ we can, by "index raising" or "index lowering", identify vector fields with one-forms and vice versa.) The precise motivation for this choice of variables will become clear later on. As a result we consider as the carrier spaces for the port-Hamiltonian formulation of Eq. (81) the linear spaces $\mathcal{F}_{p,q}$ and $\mathcal{E}_{p,q}$ for $n = 3, p = 3, q = 1$; that is

$$\mathcal{F}_{p,q} = \Omega^3(Z) \times \Omega^1(Z) \times \Omega^0(\partial Z) \tag{83}$$

and

$$\mathcal{E}_{p,q} = \Omega^0(Z) \times \Omega^2(Z) \times \Omega^2(\partial Z) \tag{84}$$

Since $p + q = n + 1$ we can define the corresponding Stokes–Dirac structure $D$ given by Eq. (11) on $\mathcal{F}_{p,q} \times \mathcal{E}_{p,q}$. However, as will become clear later on, due to three-dimensional convection we need to *modify* this Stokes–Dirac structure with an additional term into the following modified Stokes–Dirac structure

$$\begin{aligned} D^m := \Bigg\{ & (f_p, f_v, f_b, e_\rho, e_v, e_b) \in \Omega^3(Z) \times \Omega^1(Z) \times \Omega^0(\partial Z) \times \Omega^0(Z) \times \Omega^2(Z) \\ & \times \Omega^2(\partial Z), \qquad \begin{bmatrix} f_\rho \\ f_v \end{bmatrix} = \begin{bmatrix} \mathrm{d}e_v \\ \mathrm{d}e_\rho + \dfrac{1}{*\rho} * ((*\mathrm{d}v) \wedge (*e_v)) \end{bmatrix} \\ & \times \begin{bmatrix} f_b \\ e_b \end{bmatrix} = \begin{bmatrix} e_{\rho|\partial Z} \\ -e_{v|\partial Z} \end{bmatrix} \Bigg\} \end{aligned} \tag{85}$$

where $*$ denotes the Hodge star operator (corresponding to the Riemannian metric on $Z$), converting $k$-forms on $Z$ to $(3-k)$-forms. A fundamental difference of the modified Stokes–Dirac structure $D^m$ with respect to the standard Stokes–Dirac structure $D$ is that $D^m$ explicitly depends on the energy variables $\rho$ and $v$ (via the terms $*\rho$ and $\mathrm{d}v$ in the additional term $(1/*\rho) * ((*\mathrm{d}v) \wedge (*e_v))$).

Completely similar to the proof of Theorem 2.1 it is shown that $(D^m(\rho, v))^\perp = D^m(\rho, v)$ for all $\rho, v$; the crucial additional observation is that the expression

$$e_v^2 \wedge *((*\mathrm{d}v) \wedge (*e_v^1)) \tag{86}$$

is *skew-symmetric* in $e_v^1, e_v^2 \in \Omega^2(Z)$.

**Remark 3.1.** In the standard Euclidean metric, identifying via the Hodge star operator two-forms $\beta_i$ with one-forms, and representing one-forms as vectors, we have in vector calculus notation the equality

$$\beta_2 \wedge *(\alpha \wedge *\beta_1) = \alpha \cdot (\beta_1 \times \beta_2) \tag{87}$$

for all two-forms $\beta_1, \beta_2$ and one-forms $\alpha$. This shows clearly the skew-symmetry of Eq. (86).

The Eulerian equations (81) for an ideal isentropic fluid are obtained in the port-controlled Hamiltonian representation by considering the Hamiltonian

$$H(\rho, v) := \int_Z [\tfrac{1}{2}\langle v^\sharp, v^\sharp\rangle \rho + U(*\rho)\rho] \tag{88}$$

with $v^\sharp$ the vector field corresponding to the one-form $v$ ("index lowering"), and $U(*\rho)$ the internal energy. Indeed, by making the substitutions (36) and (37) in $D^m$, and noting that

$$\text{grad}\, H = (\delta_\rho H, \delta_v H) = \left(\frac{1}{2}\langle v^\sharp, v^\sharp\rangle + \frac{\partial}{\partial\tilde{\rho}}(\tilde{\rho}U(\tilde{\rho})),\ i_{v^\sharp}\rho\right) \tag{89}$$

with $\tilde{\rho} := *\rho$, the port-Hamiltonian system takes the form

$$\begin{aligned} -\frac{\partial\rho}{\partial t} &= \mathrm{d}(i_{v^\sharp}\rho) \\ -\frac{\partial v}{\partial t} &= \mathrm{d}\left(\frac{1}{2}\langle v^\sharp, v^\sharp\rangle + w(*\rho)\right) + \frac{1}{*\rho}\left((*\mathrm{d}v)\wedge(*i_{v^\sharp}\rho)\right) \\ f_b &= \left[\tfrac{1}{2}\langle v^\sharp, v^\sharp\rangle + w(*\rho)\right]_{|\partial Z} \\ e_b &= -i_{v^\sharp}\rho_{|\partial Z} \end{aligned} \tag{90}$$

with

$$w(\tilde{\rho}) := \frac{\partial}{\partial\tilde{\rho}}(\tilde{\rho}U(\tilde{\rho})) \tag{91}$$

the *enthalpy*. The expression $\delta_\rho H = (1/2)\langle v^\sharp, v^\sharp\rangle + w(\tilde{\rho})$ is known as the *Bernoulli function*.

The first two equations of (90) can be seen to represent the Eulerian equations (81). The first equation corresponds to the basic law of *mass-balance*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\varphi_t(V)}\rho = 0 \tag{92}$$

where $V$ denotes an arbitrary volume in $Z$, and $\varphi_t$ is the flow of the fluid (transforming the material volume $V$ at $t = 0$ to the volume $\varphi_t(V)$ at time $t$). Indeed, Eq. (92) for any $V$ is equivalent to

$$\frac{\partial\rho}{\partial t} + L_{v^\sharp}\rho = 0 \tag{93}$$

Since by Cartan's magical formula $L_{v^\sharp}\rho = \mathrm{d}(i_{v^\sharp}\rho) + i_{v^\sharp}\mathrm{d}\rho = \mathrm{d}(i_{v^\sharp}\rho)$ (since $\mathrm{d}\rho = 0$) this yields the first line of Eq. (90). It also makes clear the interpretation of $\rho$ as a three-form on $Z$.

For the identification of the second equation of (90) with the second equation of (86) we note the following (see [32] for further details). Interpret $\nabla\cdot$ in Eq. (81) as the covariant

derivative corresponding to the assumed Riemannian metric $\langle\cdot\rangle$ on $Z$. For a vector field $u$ on $Z$, let $u^\flat$ denote the corresponding one-form $u^\flat := i_u\langle\cdot\rangle$ ("index raising"). The covariant derivative $\nabla$ is related to the Lie derivative by the following formula (see for a proof [1], p. 202)

$$L_u u^\flat = (\nabla_u u)^\flat + \tfrac{1}{2}\,\mathrm{d}\langle u, u\rangle \tag{94}$$

Since by Cartan's magical formula $L_u u^\flat = i_u \mathrm{d}u^\flat + \mathrm{d}(i_u u^\flat) = i_u \mathrm{d}u^\flat + \mathrm{d}\langle u, u\rangle$, Eq. (94) can be also written as

$$(\nabla_u u)^\flat = i_u \mathrm{d}u^\flat + \tfrac{1}{2}\,\mathrm{d}\langle u, u\rangle \tag{95}$$

(This is the coordinate-free analog of the well-known vector calculus formula $u \cdot \nabla u = \operatorname{curl} u \times u + (1/2)\nabla|u|^2$.) Furthermore, we have the identity

$$i_{v^\sharp}\mathrm{d}v = \frac{1}{*\rho} * ((*\mathrm{d}v) \wedge (*i_{v^\sharp}\rho)) \tag{96}$$

Finally, we have the following well-known relation between enthalpy and pressure (obtained from Eqs. (82) and (91))

$$\frac{1}{\tilde{\rho}}\mathrm{d}p = \mathrm{d}(w(\tilde{\rho})) \tag{97}$$

Hence, by Eq. (95)–(97) (with $u = v^\sharp$), we may rewrite the second equation of (90) as

$$-\frac{\partial v}{\partial t} = \left(\nabla_{v^\sharp} v^\sharp\right)^\flat + \frac{1}{*\rho}\,\mathrm{d}p \tag{98}$$

which is the coordinate-free formulation of the second equation of (81).

The boundary variables $f_b$ and $e_b$ given in Eq. (90) are respectively the *stagnation pressure* at the boundary divided by $\rho$, and the (incoming) *mass flow* through the boundary. The energy-balance Eq. (40) can be written out as

$$\begin{aligned}\frac{\mathrm{d}H}{\mathrm{d}t} &= \int_{\partial Z} e_b \wedge f_b = -\int_{\partial Z} i_{v^\sharp}\rho \wedge \left[\tfrac{1}{2}\langle v^\sharp, v^\sharp\rangle + w(*\rho)\right] \\ &= -\int_{\partial Z} i_{v^\sharp}\left[\tfrac{1}{2}\langle v^\sharp, v^\sharp\rangle\rho + w(*\rho)\rho\right] \\ &= -\int_{\partial Z} i_{v^\sharp}\left[\tfrac{1}{2}\langle v^\sharp, v^\sharp\rangle\rho + U(*\rho)\rho\right] - \int_{\partial Z} i_{v^\sharp}(*p)\end{aligned} \tag{99}$$

where for the last equality we have used the relation (following from Eqs. (82) and (91))

$$w(*\rho)\rho = U(*\rho)\rho + *p \tag{100}$$

The first term in the last line of Eq. (99) corresponds to the convected energy through the boundary $\partial Z$, while the second term is (minus) the external work (static pressure times velocity).

Usually, the second line of the Euler equations (81) (or equivalently equation (98)) is obtained from the basic conservation law of momentum-balance together with the first line

of Eq. (81). Alternatively, emphasizing the interpretation of $v$ as a one-form, we may obtain it from *Kelvin's circulation theorem*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\varphi_t(C)} v = 0 \tag{101}$$

where $C$ denotes any *closed* contour. Indeed, Eq. (101) for any closed $C$ is equivalent to the one-form $\partial v/\partial t + L_{v^\sharp} v$ being *closed*. By Eq. (94) this is equivalent to requiring

$$\frac{\partial v}{\partial t} + (\nabla_{v^\sharp} v^\sharp)^\flat \tag{102}$$

to be closed, that is

$$\frac{\partial v}{\partial t} + (\nabla_{v^\sharp} v^\sharp)^\flat = -\mathrm{d}k \tag{103}$$

for some (possibly locally defined) $k : Z \to \mathbb{R}$. Now additionally requiring that this function $k$ depends on $z$ through $\rho$, that is

$$k(z) = w(\rho(z)) \tag{104}$$

for some function $w$, we recover Eq. (98) with $(1/*\rho)\,\mathrm{d}p$ replaced by $\mathrm{d}w$ (the differential of the enthalpy).

**Remark 3.2.** In the case of a one- or two-dimensional fluid flow the extra term in the Dirac structure $D^m$ as compared with the standard Stokes–Dirac structure $D$ vanishes, and so in these cases we are back to the standard definition of a distributed-parameter port-Hamiltonian system (with $\rho$ being a one-form, respectively, a two-form).

Furthermore, if in the three-dimensional case the two-form $\mathrm{d}v(t)$ happens to be zero at a certain time-instant $t = t_0$ (*irrotational flow*), then it *continues* to be zero for all time $t \geq t_0$. Hence, also in this case the extra term Eq. (86) in the modified Stokes–Dirac structure $D^m$ vanishes, and the port-Hamiltonian system describing the Euler equations reduces to the standard distributed-parameter port-Hamiltonian system given in Definition 2.2.

## 4. Properties of Stokes–Dirac structures

### *4.1. Poisson brackets associated to Stokes–Dirac structures*

Although, Dirac structures strictly generalize Poisson structures we can associate a pseudo-Poisson structure to any Dirac structure as defined in Section 2.1. Indeed, let $D \subset \mathcal{F} \times \mathcal{E}$ be a Dirac structure as given in Definition 2.1. Then we can define a skew-symmetric bilinear form on a subspace of $\mathcal{E}$; basically following [6,8]. First, define the space of "admissible efforts"

$$\mathcal{E}_{\mathrm{adm}} = \{e \in \mathcal{E} | \exists f \in \mathcal{F} \text{ such that } (f, e) \in D\} \tag{105}$$

Then we define on $\mathcal{E}_{\mathrm{adm}}$ the bilinear form

$$[e_1, e_2] := \langle e_1 | f_2 \rangle \in L \tag{106}$$

where $f_2 \in \mathcal{F}$ is such that $(f_2, e_2) \in D$. This bilinear form is well-defined, since for any other $f_2' \in \mathcal{F}$ such that $(f_2', e_2) \in D$ we obtain by linearity $(f_2 - f_2', 0) \in D$, and hence

$$0 = \langle\langle (f_1, e_1), (f_2 - f_2', 0)\rangle\rangle = \langle e_1 | f_2\rangle - \langle e_1 | f_2'\rangle \tag{107}$$

Furthermore, $[\cdot]$ is *skew-symmetric* since for any $(f_1, e_1), (f_2, e_2) \in D$

$$0 = \langle\langle (f_1, e_1), (f_2, e_2)\rangle\rangle = \langle e_1 | f_2\rangle + \langle e_2 | f_1\rangle \tag{108}$$

Now, let us define on $\mathcal{F}$ the set of *admissible mappings*

$$\begin{aligned} &K_{\text{adm}} = \{k : \mathcal{F} \to L | \forall a \in \mathcal{F} \ \exists e(k,a) \in \mathcal{E}_{\text{adm}}, \\ &\text{such that, for all } \partial a \in \mathcal{F}, \quad k(a + \partial a) = k(a) + \langle e(k,a) | \partial a\rangle + \mathrm{O}(\partial a)\} \end{aligned} \tag{109}$$

Note that $e(k,a)$ (if it exists) is uniquely defined modulo the following linear subspace of $\mathcal{E}$

$$\mathcal{E}_0 = \{e \in \mathcal{E} | \langle e | f\rangle = 0 \quad \text{for all } f \in \mathcal{F}\} \tag{110}$$

We call $e(k,a)$ (in fact, its equivalence class) the *derivative* of $k$ at $a$, and we denote it by $\delta k(a)$. We define on $K_{\text{adm}}$ the following bracket:

$$\{k_1, k_2\}_D(a) := [\delta k_1(a), \delta k_2(a)], \quad k_1, k_2 \in K_{\text{adm}} \tag{111}$$

which is clearly independent from the choice of the representants $\delta k_1(a), \delta k_2(a)$. By skew-symmetry of $[\cdot]$ it immediately follows that also $\{\cdot\}$ is *skew-symmetric*. The Jacobi-identity for $\{\cdot\}_D$, however, is not automatically satisfied, and we call therefore $\{\cdot\}_D$ a *pseudo-Poisson bracket*.

For the Stokes–Dirac structure $D$ of Theorem 2.1, given in Eq. (11), the bracket takes the following form. The set of admissible functions $K_{\text{adm}}$ consists of those functions

$$k : \Omega^p(Z) \times \Omega^q(Z) \times \Omega^{n-p}(\partial Z) \to \mathbb{R} \tag{112}$$

whose derivatives

$$\delta k(z) = (\delta_p k(z), \delta_q k(z), \delta_b k(z)) \in \Omega^{n-p}(Z) \times \Omega^{n-q}(Z) \times \Omega^{n-q}(\partial Z) \tag{113}$$

satisfy (cf. the last line of Eq. (11))

$$\delta_b k(z) = -(-1)^{n-q} \delta_q k(z)|_{\partial Z} \tag{114}$$

Furthermore, the bracket on $K_{\text{adm}}$ is given as (leaving out the arguments $z$)

$$\begin{aligned} \{k^1, k^2\}_D = &\int_Z [\delta_p k^1 \wedge (-1)^r \, \mathrm{d}(\delta_q k^2) + (\delta_q k^1) \wedge \mathrm{d}(\delta_p k^2)] \\ &- \int_{\partial Z} (-1)^{n-q} (\delta_q k^1) \wedge (\delta_p k^2) \end{aligned} \tag{115}$$

It follows from the general considerations above that this bracket is skew-symmetric. (This can be also directly checked using Stokes' theorem.) Furthermore, in this case it is straightforward to check that $\{\cdot\}_D$ also satisfies the Jacobi-identity

$$\{\{k^1, k^2\}_D, k^3\}_D + \{\{k^2, k^3\}_D, k^1\} + \{\{k^3, k^1\}_D, k^2\}_D = 0 \tag{116}$$

for all $k^i \in K_{\text{adm}}$.

For the modified Stokes–Dirac structure $D^m$ given in Eq. (85) the space $K_{\rm adm}$ is the same, but the resulting skew-symmetric bracket has an additional term:

$$\{k^1, k^2\}D^m = \int_Z [(\delta_\rho k^1) \wedge (-1)^r \mathrm{d}(\delta_q k^2) + (\delta_q k^1) \wedge \mathrm{d}(\delta_p k^2) + \frac{1}{*\rho}\delta_v k^1 \wedge *((*\mathrm{d}v) \wedge (*\delta_v k^2))] - \int_{\partial Z} (-1)^{n-q}(\delta_q k^1) \wedge (\delta_p k^2) \tag{117}$$

(For the skew-symmetry of the additional term see Eq. (86) and Remark 3.1.)

### 4.2. Conservation laws of port-Hamiltonian systems

Let us consider the distributed-parameter port-Hamiltonian system $\Sigma$, as defined in Definition 2.2, on an $n$-dimensional spatial domain $Z$ having state space $\Omega^p(Z) \times \Omega^q(Z)$ (with $p + q = n + 1$) and Stokes–Dirac structure $D$ given by Eq. (11).

*Conservation laws* for $\Sigma$, which are *independent* from the Hamiltonian $H$, are obtained as follows. Let

$$C : \Omega^p(Z) \times \Omega^q(Z) \times Z \to \mathbb{R} \tag{118}$$

be a function satisfying

$$\mathrm{d}(\delta_p C) = 0, \quad \mathrm{d}(\delta_q C) = 0 \tag{119}$$

where $\mathrm{d}(\delta_p C)$, $d(\delta_q C)$ are defined similarly to Eq. (33). Then the time-derivative of $C$ along the trajectories of $\Sigma$ is given as (in view of Eq. (119), and using similar calculations as in the proof of Theorem 2.1)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}C &= \int_Z \delta_p C \wedge \dot{\alpha}_p + \int_Z \delta_q C \wedge \dot{\alpha}_q \\ &= -\int_Z \delta_p C \wedge (-1)^r \mathrm{d}(\delta_q H) - \int_Z \delta_q C \wedge \mathrm{d}(\delta_p H) \\ &= -(-1)^{n-q}\int_Z \mathrm{d}(\delta_q H \wedge \delta_p C) - (-1)^{n-q}\int_Z \mathrm{d}(\delta_q C \wedge \delta_p H) \\ &= \int_{\partial Z} e_b \wedge f_b^C + \int_{\partial Z} e_b^C \wedge f_b\end{aligned} \tag{120}$$

where we have denoted, in analogy with Eq. (11),

$$f_b^C := \delta_p C|_{\partial Z}, \qquad e_b^C := -(-1)^{n-q}\delta_q C|_{\partial Z} \tag{121}$$

In particular, if additionally to Eq. (119) the function $C$ satisfies

$$\delta_p C|_{\partial Z} = 0, \qquad \delta_q C|_{\partial Z} = 0 \tag{122}$$

then $\mathrm{d}C/\mathrm{d}t = 0$ along the system trajectories of $\Sigma$ for any Hamiltonian $H$. Therefore, a function $C$ satisfying Eqs. (119) and (122) is called a *Casimir* function. If $C$ satisfies

Eq. (119) but not Eq. (122) then $C$ is called a *conservation law* for $\Sigma$: its time-derivative is determined by the boundary conditions of $\Sigma$.

**Example 4.1.** In the case of the telegraph equations (Section 3.2) the total charge

$$C_Q = \int_0^1 Q(t,z)\,\mathrm{d}z$$

as well as the total magnetic flux

$$C_\varphi = \int_0^1 \varphi(t,z)\,\mathrm{d}z$$

are both conservation laws. Indeed

$$\frac{\mathrm{d}}{\mathrm{d}t}C_Q = -\int_0^1 \frac{\partial I}{\partial z} = I(0) - I(1)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}C_\varphi = -\int_0^1 \frac{\partial V}{\partial z}\mathrm{d}z = V(0) - V(1)$$

Similarly, in the case of the vibrating string (Section 3.3) conservation laws are

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \epsilon(t,z)\,\mathrm{d}z = \frac{\mathrm{d}}{\mathrm{d}t}(u(t,1) - u(t,0)) = v(t,1) - v(t,0)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 p(t,z)\,\mathrm{d}z = \sigma(t,1) - \sigma(t,0)$$

Conservation laws $C$ for $\Sigma$ which are *dependent* on the Hamiltonian $H$ are obtained by replacing Eq. (119) by the weaker condition

$$\delta_q H \wedge \mathrm{d}(\delta_p C) + (-1)^r \delta_p H \wedge \mathrm{d}(\delta_q C) = 0 \tag{123}$$

Indeed, it immediately follows from the computation in Eq. (120) that under this condition (120) continues to hold.

In the case of the modified Stokes–Dirac structure $D^m$ defined in Eq. (85), for any function $C : \Omega^3(Z) \times \Omega^1(Z) \times Z \to \mathbb{R}$ satisfying

$$\delta_v H \wedge \mathrm{d}(\delta_p C) + \delta_\rho H \wedge \mathrm{d}(\delta_v C) = 0, \quad \rho \in \Omega^3(Z), v \in \Omega^1(Z) \tag{124}$$

Eq. (120) takes the form

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}C &= \int_Z \delta_\rho C \wedge \mathrm{d}(\delta_v H) + \int_Z \delta_v C \wedge \left[\mathrm{d}(\delta_\rho H) + \frac{1}{*\rho} * ((*\mathrm{d}v) \wedge (*\delta_v H))\right] \\ &= \int_{\partial Z} \delta_\rho C \wedge \delta_v H + \int_{\partial Z} \delta_v C \wedge \delta_\rho H + \int_Z \frac{1}{*\rho}\delta_v C * ((*\mathrm{d}v) \wedge (*\delta_v H))\end{aligned} \tag{125}$$

Hence, we conclude that in order to obtain a conservation law we need to impose an extra condition eliminating the last $\int_Z$ integral. A specific example of a conservation law in this case is the *helicity*

$$C = \int_Z v \wedge \mathrm{d}v \tag{126}$$

with time-derivative

$$\frac{\mathrm{d}}{\mathrm{d}t}C = -\int_{\partial Z} f_b \wedge \mathrm{d}v \tag{127}$$

A *second* class of conserved quantities corresponding to the Stokes–Dirac structure $D$ (Eq. (11)) is identified by noting that by Eq. (38)

$$\begin{aligned} -\mathrm{d}\left(\frac{\partial \alpha_p}{\partial t}\right) &= (-1)^r \mathrm{d}(\mathrm{d}\delta_q H) = 0 \\ -\mathrm{d}\left(\frac{\partial \alpha_q}{\partial t}\right) &= \mathrm{d}(\mathrm{d}\delta_p H) = 0 \end{aligned} \tag{128}$$

and thus the differential forms $\mathrm{d}\alpha_p$ and $\mathrm{d}\alpha_q$ do not depend on time. Therefore, the component functions of $\mathrm{d}\alpha_p$ and $\mathrm{d}\alpha_q$ are conserved quantities of any port-Hamiltonian system corresponding to $D$.

**Example 4.2.** In the case of Maxwell's equations (Section 3.1) this yields that $\mathrm{d}\mathcal{D}$ and $\mathrm{d}\mathcal{B}$ are constant three-forms. The three-form $\mathrm{d}\mathcal{D}$ is the *charge density* (Gauss' electric law), while by Gauss' magnetic law $\mathrm{d}\mathcal{B}$ is actually zero.

In the case of an ideal isentropic fluid (Section 3.4) for which the vorticity $\mathrm{d}v(t_0, z)$ is *zero* at a certain time $t_0$ we obtain by the same reasoning (since the additional term in the Stokes–Dirac structure $D^m$ is zero for $t_0$) that $\mathrm{d}v(t, z)$ is *zero* for all $t \geq t_0$ (irrotational flow); cf. Remark 3.2.

### *4.3. Covariant formulation of port-Hamiltonian systems*

A covariant formulation of distributed-parameter port-Hamiltonian systems can be obtained following a construction which is well-known for Maxwell's equations (see [11]), and directly generalizes to port-Hamiltonian systems (38) defined with respect to a general Stokes–Dirac structure $D$.

Define on $Z \times \mathbb{R}$ with coordinates $(z, t)$ (that is, space–time) the $p$-, respectively $q$-form

$$\begin{aligned} \gamma_p &:= \alpha_p + (-1)^r \delta_q H \wedge \mathrm{d}t \\ \gamma_q &:= \alpha_q + \delta_p H \wedge \mathrm{d}t \end{aligned} \tag{129}$$

Then the first part of the Eq. (38) can be equivalently stated as

$$\begin{aligned} L_{\partial/\partial t} \bar{\mathrm{d}} \gamma_p &= 0 \\ L_{\partial/\partial t} \bar{\mathrm{d}} \gamma_q &= 0 \end{aligned} \tag{130}$$

with $\bar{\mathrm{d}}$ denoting the exterior derivative on $Z \times \mathbb{R}$ (with respect to $z$ *and* $t$).

Indeed, Eq. (130) means that $\bar{\mathrm{d}}\gamma_p$ and $\bar{\mathrm{d}}\gamma_q$ do not depend on $t$, that is,

$$\begin{aligned} \bar{\mathrm{d}}\gamma_p &= \beta_p \\ \bar{\mathrm{d}}\gamma_q &= \beta_q \end{aligned} \tag{131}$$

for certain $(p+1)$- and $(q+1)$-forms $\beta_p$, respectively $\beta_q$, not depending on $t$. Writing out Eq. (131) yields (with 'd' denoting the exterior derivative with respect to $z$)

$$\begin{aligned} \mathrm{d}\alpha_p + \frac{\partial \alpha_p}{\partial t} \wedge \mathrm{d}t + (-1)^r \mathrm{d}(\delta_q H) \wedge \mathrm{d}t &= \beta_p \\ \mathrm{d}\alpha_q + \frac{\partial \alpha_q}{\partial t} \wedge \mathrm{d}t + \mathrm{d}(\delta_p H) \wedge \mathrm{d}t &= \beta_q \end{aligned} \tag{132}$$

resulting in the equations of a port-Hamiltonian system (38)

$$\begin{aligned} -\frac{\partial \alpha_p}{\partial t} &= (-1)^r \mathrm{d}(\delta_q H) \\ -\frac{\partial \alpha_q}{\partial t} &= \mathrm{d}(\delta_p H) \end{aligned} \tag{133}$$

together with the conserved quantities (cf. Section 4.2) $\mathrm{d}\alpha_p = \beta_p$, $\mathrm{d}\alpha_q = \beta_q$.

Furthermore, the boundary variables of the port-Hamiltonian system (38) can be reformulated as

$$\begin{aligned} (i_{\partial/\partial t}\gamma_q)_{|\partial Z} &= f_b \\ (i_{\partial/\partial t}\gamma_p)_{|\partial Z} &= (-1)^q e_b \end{aligned} \tag{134}$$

## 5. Conclusions and final remarks

The main results of this paper concern the definition of a Dirac structure which allows the Hamiltonian formulation of a large class of distributed-parameter systems with boundary energy-flow, including the examples of the telegraph equations, Maxwell's equations, vibrating strings and ideal isentropic fluids. It has been argued that in order to incorporate boundary variables into this formulation the notion of a Dirac structure provides the appropriate generalization of the more commonly used notion of a Poisson structure for evolution equations. The employed Dirac structure is based on Stokes' theorem, and emphasizes the geometrical content of the variables as being differential $k$-forms.

From a physical point of view the Stokes–Dirac structure captures the *balance laws* inherent to the system, like Faraday's and Ampère's law (in Maxwell's equations), or mass-balance (in the case of an ideal fluid). This situation is quite similar to the lumped-parameter case where the Dirac structure incorporates the topological interconnection laws (Kirchhoff's laws, Newton's third law) and other interconnection constraints (see e.g. [18,17,31]).

Hence, the starting point for the Hamiltonian description in this paper is different from the more common approach of *deriving* Hamiltonian equations from a variational principle and its resulting Lagrangian equations, or (very much related) a Hamiltonian formulation

starting from a state space being a co-tangent bundle endowed with its natural symplectic structure. In the case of Maxwell's equations this results in the use of the basic physical variables $D$ and $B$ (the electric and magnetic field inductions), instead of the use of the variable $D$ (or $E$) together with the *vector potential* $A$ (with $\mathrm{d}A = B$) in the symplectic formulation of Maxwell's equations. It should be of interest to compare both approaches more closely, also in the context of the natural multi-symplectic structures which have been formulated for the Hamiltonian formulation of Lagrangian field equations; see e.g. [5,15]. Another related issue in this context is the *"canonicity"* of the Stokes–Dirac structure (as compared with the canonicity of the symplectic structure on cotangent bundles and the multi-symplectic structures). Indeed, the Stokes–Dirac structure as defined in Theorem 2.1 satisfies the usual integrability condition for Dirac structures [6–8], since it is a constant Dirac structure. Thus, one could expect to be able to find "canonical coordinates" for the Stokes–Dirac structure, in which it takes (almost) the form of a canonical symplectic form. (The modified Stokes–Dirac structure $D^m$ defined in Eq. (85) is not constant anymore, but still is conjectured to be integrable.)

A very prominent and favorable property of Dirac structures is that they are *closed under power-conserving interconnection*. This has been formally proven in the finite-dimensional case in [18,29], but the result should carry through to the infinite-dimensional case as well. It is a property of fundamental importance since it enables to link port-Hamiltonian systems (lumped- or distributed-parameter) to each other to obtain an interconnected port-Hamiltonian system with total energy being the sum of the Hamiltonians of its constituent parts. Clearly, this is very important in modeling (coupling e.g. solid components with fluid components, or finite-dimensional electric components with transmission lines), as well as in control. First of all, it enables to formulate directly distributed-parameter systems with *constraints* as (implicit) Hamiltonian systems, like this has been done in the finite-dimensional case for mechanical systems with kinematic constraints [7,31], multi-body systems [3,18], and general electrical networks [2,31]. Secondly, from the control perspective the notion of feedback control can be understood on its most basic level as the coupling of given physical components with additional control components (being themselves physical systems, or software components linked to sensors and actuators). A preliminary study from this point of view of a control scheme involving transmission lines has been provided in [27]. Among others, this opens up the way for the application of passivity-based control techniques, which have been proven to be very effective for the control of lumped-parameter physical systems modeled as port-Hamiltonian systems.

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