# University of Groningen

## A receding-horizon approach to the nonlinear H∞ control problem

Magni, L.; Nijmeijer, H.; Schaft, A.J. van der

*Published in:*
Automatica

**IMPORTANT NOTE: You are advised to consult the publisher's version (publisher's PDF) if you wish to cite from it. Please check the document version below.**

*Document Version*
Publisher's PDF, also known as Version of record

*Publication date:*
2001

[Link to publication in University of Groningen/UMCG research database](#)

*Citation for published version (APA):*
Magni, L., Nijmeijer, H., & Schaft, A. J. V. D. (2001). A receding-horizon approach to the nonlinear H∞ control problem. *Automatica*, *37*(3), 429-435.

Brief Paper

# A receding–horizon approach to the nonlinear $H_\infty$ control problem[☆]

## L. Magni[a,*], H. Nijmeijer[b], A. J. van der Schaft[c]

[a]*Dipartimento di Informatica e Sistemistica, Università di Pavia, via Ferrata 1, 27100 Pavia, Italy*
[b]*Faculty of Mechanical Engineering, Dynamics and Control Group, Eindhoven University of Technology, WH 0.139/141,*
*P.O. Box 513, 5600 MB Eindhoven, Netherlands*
[c]*Faculty of Mathematical Sciences, University of Twente, P.O. Box 217, 7500 AE Enschede, Netherlands*

## Abstract

The receding–horizon (RH) methodology is extended to the design of a robust controller of $H_\infty$ type for nonlinear systems. Using the nonlinear analogue of the Fake $H_\infty$ algebraic Riccati equation, we derive an inverse optimality result for the RH schemes for which increasing the horizon causes a decrease of the optimal cost function. This inverse optimality result shows that the input–output map of the closed-loop system obtained with the RH control law has a bounded $L_2$-*gain*. Robustness properties of the nonlinear $H_\infty$ control law in face of dynamic input uncertainty are considered. © 2001 Elsevier Science Ltd. All rights reserved.

*Keywords:* Receding–horizon control; $H_\infty$ control; Nonlinear control; HJI equation; Robustness

## 1. Introduction

Nonlinear receding–horizon (RH) control has received much interest in the academic community in the last years (Keerthi & Gilbert, 1988; Mayne & Michalska, 1990; Michalska & Mayne, 1993; Parisini & Zoppoli, 1995; De Nicolao, Magni, & Scattolini, 1996, 1997, 1998; Chen & Allgöwer, 1998; Scokaert, Rawlings, & Meadows, 1997), due to the capacity of obtaining a stabilizing state feedback controller based on the solution of a finite horizon optimal control problem. The main advantage of such a scheme is its ability to handle nonlinear multivariable systems that are subject to constraints in the state and/or in the control variables. In Magni & Sepulchre (1997), it is shown that all these control laws, although based on an (open-loop) solution of a finite horizon optimal control problem, also yield

a (feedback) solution of an associated infinite horizon optimal control problem. This inverse optimality result establishes an important robustness property of receding horizon control since the control laws are shown to possess stability margins of optimal control laws (Glad, 1987; Jacobson, 1977; Sepulchre, Jankovic, & Kokotovic, 1996). Nevertheless, disturbance attenuation specifications are not directly considered in the formulation of the problem. From this motivation comes the idea to consider a game theoretic approach to nonlinear RH control. This approach can be seen as a way to consider disturbance attenuation specifications in the synthesis of the RH control law but also as a possible way to achieve a solution of nonlinear $H_\infty$ problems. It is well known that $H_\infty$ theory provides an excellent theoretical framework for dealing with nonlinear stability and robustness issues. On the other hand, the computational effort of the infinite horizon formulation for nonlinear systems makes the application to real systems often almost impossible (van der Schaft, 1992; Isidori & Astolfi, 1992).

In the present paper the RH methodology is extended to design robust controllers of $H_\infty$ type for nonlinear systems. A RH approach for the solution of an $H_\infty$ control problem was first proposed in Tadmor (1992) and Lall and Glover (1994) for linear unconstrained system and recently in Scokaert and Mayne (1998) for linear constrained systems. The RH control law is based on the solution of a closed-loop finite horizon differential game

* Corresponding author. Fax: + 39-382-505373.
*E-mail address:* magni@conpro.unipv.it (L. Magni).

with two different players (inputs) that try, respectively, to minimize and to maximize a suitable finite horizon cost function. Based on the derivation of a stationary Hamilton–Jacobi–Isaacs (HJI) equation, which is the nonlinear analogous of the FHARE (fake $H_\infty$ algebraic Riccati equation) (De Nicolao & Bitmead, 1997), it is shown that the $H_\infty$ RH control law is the solution of an associated infinite horizon $H_\infty$ control problem, for the RH schemes for which increasing the horizon causes a decrease of the optimal cost function. In this way, it is easy to show that this RH control law has the same robustness properties as the standard $H_\infty$ control law. For another recent approach to the inverse optimality problem see Isidori and Lin (1997), where for a restrictive class of (triangular form) nonlinear systems the problem was dealt with.

In the second part of the paper robustness properties of the $H_\infty$ control law are analyzed. In particular, following the arguments in Sepulchre et al. (1996), dynamic input uncertainty is considered. This analysis is an extension of the results obtained in van der Schaft (1993) for static input uncertainty. Moreover, it is shown that the nonlinear $H_\infty$ control guarantees the same stability margins in the face of dynamic input uncertainty as the $H_2$ optimal control law with an additional robustness margin.

The paper is organized as follows. Section 2 introduces a game theoretic approach to RH control. The inverse optimality via a fake HJI equation is derived in Section 3. The robustness analysis is reported in Section 4, and Section 5 contains the conclusions.

## 2. Receding–horizon strategy

We consider a nonlinear system (NS)

$$\dot{x} = a(x) + b(x)u + g(x)d, \qquad (1)$$

$$z = \begin{bmatrix} h(x) \\ u \end{bmatrix}, \qquad (2)$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $d \in \mathbb{R}^p$, $h(x) \in \mathbb{R}^q$, $a(0) = 0$, $h(0) = 0$.

In the sequel a nonlinear (RH) approach is used to solve the nonlinear sub-optimal $H_\infty$ problem, i.e. the problem to find a control law that guarantees a finite disturbance attenuation level.

The RH control law in this setting is based, instead of only on the standard minimization problem, on a finite horizon differential game, where $u(t)$ is the input of the minimizing player (the controller) and $d(t)$ is the input of the maximizing player (the nature). The solution of the differential game will be in sets of piecewise continuous time-varying feedback-type functions $\mathscr{K} = \{\kappa : [0, L] \times \mathbb{R}^n \to \mathbb{R}^m\}$ and $\mathscr{N} = \{v : [0, L] \times \mathbb{R}^n \to \mathbb{R}^p\}$. These spaces are the strategy spaces that we shall call as $\kappa$ and $v$ to distinguish them from signals $u$ and $d$. Differently from

the standard RH approach, in this case it is in general not sufficient to consider only open-loop control laws $u(t)$ and $d(t)$ since the control action of a given player would not account for changes in the state, due to unpredictable control actions of the other player (see also Scokaert and Mayne, 1998). In the following, according to the RH paradigm, at each time instant $\tau$, we will focus on a finite interval, i.e. $t \in [\tau, \tau + L]$.

The *finite horizon optimal differential game* (FHODG) at time $\tau$ consists of the minimization with respect to $\kappa(t - \tau, x(t)) \in \mathscr{K}$, $t \in [\tau, \tau + L]$, and the maximization with respect to $v(t - \tau, x(t)) \in \mathscr{N}$, $t \in [\tau, \tau + L]$, of the cost function

$$J(\bar{x}, \kappa, v, L) = \int_\tau^{\tau + L} (\|z(t)\|^2 - \gamma^2 \|d(t)\|^2) \, dt$$
$$+ V_f(x(\tau + L)), \quad x(\tau + L) \in X_f \subset \mathbb{R}^n \quad (3)$$

subject to (1) and (2), with $x(\tau) = \bar{x}$, $u(t) = \kappa(t - \tau, x(t))$ and $d(t) = v(t - \tau, x(t))$ where $V_f(x)$ is a $C^2$ nonnegative function with $V_f(0) = 0$. Here $\gamma$ is a constant, which can be interpreted as the disturbance attenuation level. □

For a given initial condition $\bar{x} \in \mathbb{R}^n$, if a feedback saddle-point solution exists, we denote this solution of the FHODG as $\kappa^*(t - \tau, x(t))$ and $v^*(t - \tau, x(t))$, $\tau \le t \le \tau + L$. In the following, the optimal value of the FHODG will be denoted by $V(x, L)$, i.e. $V(\bar{x}, L) := J(\bar{x}, \kappa^*, v^*, L)$. In receding horizon control, at each time $\tau$, the resulting feedback control at state $\bar{x}$ is obtained by solving the FHODG and setting

$$\kappa^{\mathrm{RH}}(\bar{x}) = \kappa^*(0, \bar{x}). \qquad (4)$$

We now introduce the following definitions.

**Definition 1.** Let $\mathscr{U}(x(\tau), L)$ be the set of all strategy $\kappa$ such that starting from $x(\tau)$, $x(\tau + L) \in X_f$ for every admissible strategy $v \in \mathscr{N}$.

**Definition 2** (*Playable set* (*Vincent & Grantham, 1997*)). Let $\Omega^{\mathrm{RH}}(L)$ the set of initial states $x(\tau)$ such that $\mathscr{U}(x(\tau), L)$ is nonempty.

**Assumption 1.** The control law (4) is continuously differentiable and the value function $V(x, L)$ is two times continuously differentiable function with respect to all its arguments.

## 3. Inverse optimality via a Fake HJI equation

As it is clarified for the linear case (Poubelle, Bitmead, & Gevers, 1988), the "fake" algebraic Riccati equation (ARE) is a useful tool to analyze the properties of a RH control scheme. In fact, the solution at time $t$ of the differential Riccati equation (DRE), associated with the finite horizon problem, is viewed as the steady state

solution of a new suitably defined ARE. For design purposes, the "fake" Riccati analysis suggests the use of finite horizon cost functions with a terminal-state penalty $V_f$ and a terminal region $X_f$ properly chosen so as to ensure monotonicity of the solution of the associated DRE from its initial condition. In the following we show that the value function $V(x, L)$ satisfies a "fake" Hamilton–Jacobi–Isaacs equation that is the nonlinear version of the linear "fake" $H_\infty$ ARE (De Nicolao & Bitmead, 1997).

**Theorem 1.** *Assume that* $-(\partial/\partial L)V(x, L)$ *is nonnegative and that Assumption* 1 *holds. Then, the control law* $u = \kappa^{\mathrm{RH}}(x)$ *solves the state feedback* $H_\infty$ *optimal control problem associated with the cost function*

$$J_{IH}(x(\tau), u, d) = \int_\tau^\infty (\|\bar{z}(t)\|^2 - \gamma^2 \|d(t)\|^2)\, \mathrm{d}t \tag{5}$$

*with*

$$\bar{z} = \begin{bmatrix} \bar{h}(x) \\ u \end{bmatrix} \tag{6}$$

*and*

$$\bar{h}(x) = \begin{bmatrix} h(x) \\ (-(\partial/\partial L)V(x, L))^{1/2} \end{bmatrix} \tag{7}$$

**Proof.** Given $\bar{v}(t - \tau, x(t)) = 0$, $t \in [\tau, \tau + L]$, for every $\bar{\kappa}(\cdot, x(\cdot))$ we have $J(\bar{x}, \bar{\kappa}, \bar{v}, L) \geq 0$ and than $V(x, L) \geq J(\bar{x}, \bar{\kappa}, \bar{v}, L) \geq 0$.

We now show that the value function $V(x, L)$ satisfies the HJI equation

$$0 = \bar{h}(x)' \bar{h}(x) + V_x(x, L)a(x)$$
$$- \frac{1}{4}V_x(x, L)\left[b(x)b(x)' - \frac{1}{\gamma^2}g(x)g(x)'\right]V_x(x, L)' \tag{8}$$

with boundary condition $V(0, L) = 0$ and with $\bar{h}(x)$ given by (7). From standard results on dynamic programming, the value function $V$ satisfies the equation

$$-\frac{\partial}{\partial t}V(\bar{x}, L - t + \tau)$$
$$= \min_\kappa \max_\nu \{h(x(t))'h(x(t)) + u(t)'u(t) - \gamma^2 d(t)'d(t)$$
$$+ V_x(x(t), L - t + \tau)[a(x(t)) + b(x(t))u(t) + g(x(t))d(t)]\},$$
$$t \in [\tau, \tau + L].$$

In particular, we have for $t = \tau$

$$-\frac{\partial}{\partial t}V(\bar{x}, L - t + \tau)|_{t=\tau}$$
$$= h(\bar{x})'h(\bar{x}) + V_x(\bar{x}, L)a(\bar{x})$$
$$- \frac{1}{4}V_x(\bar{x}, L)\left[b(\bar{x})b(\bar{x})' - \frac{1}{\gamma^2}g(\bar{x})g(\bar{x})'\right]V_x(\bar{x}, L)'.$$

Note that

$$V(\bar{x}, L - (t + \Delta t) + \tau) = V(\bar{x}, (L - \Delta t) - t + \tau)$$

which implies

$$-\frac{\partial}{\partial t}V(\bar{x}, L - t + \tau)|_{t=\tau} = \frac{\partial}{\partial L}V(\bar{x}, L)$$

and then (8) holds. Finally, since $V_f(0) = 0$, $V(x, L) \geq 0$ and $(\partial/\partial L)V(x, L) \leq 0$ it follows that $V(0, L) = 0$ for all $L \geq 0$. Note also that this implies that $\bar{h}(0) = 0$. Then, in view of the assumption that $-(\partial/\partial L)V(x, L)$ is nonnegative, the proof follows from standard results (van der Schaft, 1996). $\quad\square$

The receding horizon control scheme does not guarantee that $-(\partial/\partial L)V(x, L) \geq 0$ unless the final-state penalty $V_f(\cdot)$ and the terminal region $X_f$ are chosen appropriately. This condition is essential to guarantee that $\bar{h}(x)$ is well defined. One way to achieve the monotonicity property, with a particular class of uncertainties, can be derived by Chen, Scherer, and Allgöwer (1997), where a quadratic terminal penalty and a terminal region define as the interior of a suitable level set of such a quadratic function is used. In Chen et al. (1997) an open-loop optimization problem is solved to derive the RH control law. To do this a precompensation feedback control law is used. The main drawback of this approach is that the playable set can be very small and it is not even possible to guarantee that it is larger than the terminal region $X_f$. On the other hand, in the approach presented here the optimization is carried out in an infinite-dimensional space, due to the use of feedback strategies, and this complicates the complexity of the optimal control problem.

Furthermore, assume that the uncertainty is such that

$$\|d(t)\|^2 \leq \frac{1}{\gamma^2}\|z(t)\|^2. \tag{9}$$

For many practical systems, bounded disturbances or parameter uncertainty can be rewritten in this form. In Chen et al. (1997), it is shown that given the quadratic function

$$V_f(x) = x'Px, \tag{10}$$

where $P$ is a positive-definite solution of the matrix Riccati inequality

$$A'P + PA - \frac{1}{4}P\left(B'B - \frac{1}{\gamma^2}GG'\right)P + H'H \leq -pI$$

with $p$ a fixed positive constant, $A = \partial a(0)/\partial x$, $B = b(0)$, $G = g(0)$, $H = \partial h(0)/\partial x$ there exists a region $\Omega$ (defined as the interior of a suitable level set of $V_f$) that is invariant for the uncertain system with control law

$$u = \kappa(x) = -\frac{1}{2}b(x)'V_f(x)'_x \tag{11}$$

and such that along the trajectories of the closed-loop system

$$\frac{\partial}{\partial t} V_f(x(t)) + \|z(t)\|^2 - \gamma^2 \|d(t)\|^2 \leq -\varepsilon \|x(t)\|^2 \quad \forall x(t) \in \Omega,$$

$$(12)$$

where $\varepsilon$ is a positive constant. For further details on the computation of the set $\Omega$ see Chen et al. (1997) and Michalska and Mayne (1993) where a numerical procedure is given for a similar problem.

Owing to this result, we are now in the position to introduce a RH control scheme that solves the nonlinear sub-optimal $H_\infty$ problem.

**Theorem 2.** *Consider the* RH *control scheme based on the solution of a* FHODG *with* $V_f(x) = x'Px$ *given by* (10) *and* $X_f = \Omega$ *where* $\Omega$ *is the interior of a suitable level set of* $V_f(x)$ *such that, along the trajectories of the closed-loop system* (1), (2), (11), *condition* (12) *is satisfied. Suppose that the uncertainty is of the form* (9) *and that Assumption 1 holds, then the control law* $u = \kappa^{RH}(x)$ *solves the state feedback* $H_\infty$ *optimal control problem associated with the cost function* (5).

**Proof.** The proof of this theorem follows by Theorem 1 by showing that if $L_1 \leq L_2$ and $\bar{x} \in \Omega^{RH}(L_1)$ then $V(\bar{x}, L_1) \geq V(\bar{x}, L_2)$

We rewrite the considered functional as

$$V_f(x(\tau + L_2)) + \int_\tau^{\tau + L_2} \|z(t)\|^2 - \gamma^2 \|d(t)\|^2 \, dt$$

$$= V_f(x(\tau + L_2)) - V_f(x(\tau + L_1))$$

$$+ \int_{\tau + L_1}^{\tau + L_2} \|z(t)\|^2 - \gamma^2 \|d(t)\|^2 \, dt$$

$$+ V_f(x(\tau + L_1)) + \int_\tau^{\tau + L_1} \|z(t)\|^2 - \gamma^2 \|d(t)\|^2 \, dt \quad (13)$$

Let us now consider the following feasible state-feedback control law:

$$\tilde{\kappa}(t - \tau, x(t))$$

$$= \begin{cases} \kappa^*(t - \tau, x(t)) & \text{for } t \in [\tau, \tau + L_1], \\ -\frac{1}{2} b(x(t))' V_f(x(t))'_x & \text{for } t \in [\tau + L_1, \tau + L_2], \end{cases} \quad (14)$$

where $\kappa^*(t - \tau, x(t))$ is an optimal solution for the FHODG with horizon $L_1$. Note that $\tilde{u}(t) = \tilde{\kappa}(t - \tau, x(t)) \in \mathscr{U}(\bar{x}, L_2)$. By integrating (12), we obtain for any admissible disturbance

$$V_f(x(\tau + L_2)) + \int_{\tau + L_1}^{\tau + L_2} \|z(t)\|^2 - \gamma^2 \|d(t)\|^2 \, dt$$

$$\leq V_f(x(\tau + L_1)).$$

Therefore, denoting by $\mathscr{N}(\kappa)$ the set of strategy $v$ of the form (9), (13) implies

$$\max_{v \in \mathscr{N}(\tilde{\kappa})} \left[ V_f(x(\tau + L_2)) + \int_\tau^{\tau + L_2} \|z(t)\|^2 - \gamma^2 \|d(t)\|^2 \, dt \right]$$

$$\leq \max_{v \in \mathscr{N}(\tilde{\kappa})} \left[ V_f(x(\tau + L_1)) + \int_\tau^{\tau + L_1} \|z(t)\|^2 - \gamma^2 \|d(t)\|^2 \, dt \right]$$

The maximum on the right-hand side over all $v \in \mathscr{N}(\tilde{\kappa})$ restricted to $[\tau, \tau + L_1]$ is equal to the maximum over all function $v \in \mathscr{N}(\kappa^*)$ restricted to $[\tau, \tau + L_1]$. Hence, this maximum is nothing but $V(\bar{x}, L_1)$, and then

$$V(\bar{x}, L_2) = \min_{\kappa \in \mathscr{K}} \max_{v \in \mathscr{N}(\kappa)} J(\bar{x}, \kappa, v, L_2)$$

$$\leq \max_{v \in \mathscr{N}(\tilde{\kappa})} J(\bar{x}, \tilde{\kappa}, v, L_2) \leq \max_{v \in \mathscr{N}(\tilde{\kappa})} J(\bar{x}, \tilde{\kappa}, v, L_1)$$

$$= \max_{v \in \mathscr{N}(\kappa^*)} J(\bar{x}, \kappa, v, L_1) = V(\bar{x}, L_1)$$

as required.

Note that differently than with the scheme proposed in Chen et al. (1997) it is easily shown that $\Omega^{RH}(L_2) \supseteq \Omega^{RH}(L_1) \supseteq \Omega, \forall L_2 \geq L_1$.

## 4. Robustness analysis

The main engineering importance of this inverse optimality result is the possibility to achieve robustness of the closed-loop system. In the previous section it is shown that the input–output map of the closed-loop system (1)–(2)–(4) has a finite $L_2$-gain. In this section, uncertainties that can be tolerated at the input without loosing of stability are considered. In particular, we will show that the nonlinear $H_\infty$ control guarantees the same stability margins in the face of input uncertainty as the $H_2$ optimal control law with an additional robustness margin.

We first introduce some definitions.

**Definition 3** (*Sepulchre et al.*, 1996). Consider the following system (C):

$$\dot{x} = f(x, u),$$

$$y = h(x, u) \quad (15)$$

with $u \in U$, $x \in X$ and $y \in \mathbb{R}^m$, where $X$ and $U$ are connected subsets of $\mathbb{R}^n$, respectively $\mathbb{R}^m$, containing the origin. Assume that associated with the system $C$ is a function $s: U \times \mathbb{R}^m \to \mathbb{R}^+$ called the supply rate, which is locally integrable for every admissible time-function $u$ with $u(t) \in U$. We say that the system $C$ is dissipative in $X$ with supply rate $s(u, y)$ if there exists a function $S(x)$, $S(0) = 0$, $S(x) \geq 0$, such that for all $x \in X$

$$S(x(T)) - S(x(0)) \leq \int_0^T s(u(t), y(t)) \, dt \quad (16)$$

for all $u \in U$ and all $T \geq 0$ such that $x(t) \in X$ is the solution of (15) for all $t \in [0, T]$. The function $S$ is called a storage function.

**Definition 4.** System $C$ is said to be passive if it is dissipative with supply rate $s(u, y) = u'y$.

Consider now the input uncertainty $\Delta_1$ reported in Fig. 1, that is a type of uncertainty that cannot be represented by uncertainty (9). In the nominal case $\Delta_1$ is identity, and the feedback loop consists of the (nominal) nonlinear plant in the feedback loop with the nominal control $u = \kappa(x)$. This uncertainty is a common physical situation, in particular, when simplified models of actuators are used for design. For more details on this type of uncertainty see Sepulchre et al. (1996). In the following, we will show that the nonlinear $H_\infty$ control guarantees the same stability margins in the face of input uncertainty as the $H_2$ optimal control law with an additional robustness margin.

**Definition 5** (Sepulchre et al., 1996). System $C$ is said to be output feedback passive (OFP) if it is dissipative with respect to $s(u, y) = u'y - \rho y'y$ for some $\rho \in \mathbb{R}$.

This means that the input–output system $C$ has an excess or shortage of passivity, that depends on the sign of $\rho$, characterized by the fact that it is rendered passive by the output feedback transformation $u = \rho y + v$. The possibility of achieving passivity of interconnected systems with excess or shortage of passivity motivates the interest in OFP systems. The relevance of OFP property in the study of stability margin and the equivalence between OFP property and the *disk margin* property are reviewed in the recent monograph by Sepulchre et al. (1996).

**Theorem 3.** *If there exists a $C^1$, positive-semidefinite function $V(x)$ such that*

$$0 = h(x)'h(x) + V_x(x)a(x)$$
$$- \frac{1}{4}V_x(x)\left[b(x)b(x)' - \frac{1}{\gamma^2}g(x)g(x)'\right]V_x(x)', \quad (17)$$

*and*

$$\kappa(x) = -\frac{1}{2}b(x)'V_x(x)',$$

$$v(x) = \frac{1}{2\gamma^2}\, g(x)'V_x(x)'$$

*then the system*

$$\dot{x} = a(x) + b(x)u + g(x)d, \quad (18)$$
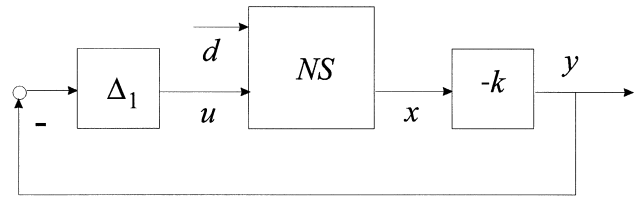
$$y = -\kappa(x) \quad (19)$$



Fig. 1. Nonlinear feedback loop with the control law $k(x)$, input uncertainty $\Delta_1$ and disturbance $d$.

*with input $u$ and output $y$ is* OFP, *with $\rho = -\frac{1}{2}$, with a $C^1$ storage function $S(x)$ for every $d$ such that*

$$\gamma^2(\|d\|^2 - \|v(x) - d\|^2) \leq \|h(x)\|^2. \quad (20)$$

**Proof.** Let $S(x) = \frac{1}{2}V(x)$, then

$$S_x(x)b(x) = -\kappa(x)',$$

$$S_x(x)g(x) = \gamma^2 v(x)',$$

$$S_x(x)a(x) = -\frac{1}{2}h(x)'h(x) + \frac{1}{2}\kappa(x)'\kappa(x) - \frac{\gamma^2}{2}v(x)'v(x).$$

Then along the solutions of system (18),

$$\frac{\partial}{\partial t}S(x) = S_x(x)[a(x) + b(x)u + g(x)d]$$

$$= -\frac{1}{2}h(x)'h(x) + \frac{1}{2}\kappa(x)'\kappa(x)$$

$$\quad - \frac{\gamma^2}{2}v(x)'v(x) - \kappa(x)'u + \gamma^2 v(x)'d$$

$$= -\frac{1}{2}h(x)'h(x) + \frac{1}{2}\kappa(x)'\kappa(x)$$

$$\quad - \frac{\gamma^2}{2}v(x)'v(x) - \kappa(x)'u + \gamma^2 v(x)'d$$

$$\quad + \frac{\gamma^2}{2}d'd - \frac{\gamma^2}{2}d'd$$

$$= \frac{1}{2}\kappa(x)'\kappa(x) - \kappa(x)'u$$

$$\quad + \frac{\gamma^2}{2}(\|d\|^2 - \|v(x) - d\|^2) - \frac{1}{2}\|h(x)\|^2 \quad (21)$$

and then for all $d$ such that $\gamma^2(\|d\|^2 - \|v(x) - d\|^2) \leq \|h(x)\|^2$,

$$\frac{\partial}{\partial t}S(x) \leq \frac{1}{2}\kappa(x)'\kappa(x) - \kappa(x)'u = \frac{1}{2}y'y + y'u$$

and so system (18), (19), with input $u$ and output $y$, is OFP with $\rho = -\frac{1}{2}$. $\square$

This means that the input–output system (18), (19) has a shortage of passivity characterized by the fact that it is

rendered passive by the output feedback transformation $u = -\frac{1}{2}y + v$. In the linear case, this shortage of passivity translates into the fact that the Nyquist plot of the input–output transfer function (18), (19) does not enter the circle of radius one and centered at $(-1,0)$ (Sepulchre et al., 1996). To guarantee the stability of the feedback interconnection in Fig. 1, the shortage of passivity of (18), (19) must be compensated for by a sufficient excess of passivity of the uncertainty $\Delta_1$.

So $\Delta_1$ *represents* some dynamic uncertainty of the form

$$u = -\Delta_1 y \tag{22}$$

that can be tolerated at the input if the system is OFP with $\rho = -\frac{1}{2}$ (Sepulchre et al., 1996). This class of uncertainties includes static sector nonlinearity $u = \phi(\cdot)$ in the sector $(\frac{1}{2}, \infty)$, that is, $\frac{1}{2}\tau'\tau < \tau'\phi(\tau) < \infty$ for all $\tau$ in $\mathbb{R}^m$, for which a similar result is derived in van der Schaft (1993), but also all the linear dynamic uncertainties whose Nyquist plot lie to the right of the vertical line with abscissa $\frac{1}{2}$.

In the formulation of Theorem 3, the extra constraint (20) imposed on the disturbance $d$ is rather artificial, just to enforce that the correct inequality is satisfied. We therefore consider the following class of disturbances that is more restrictive than the previous one but it is more understandable. Suppose that the disturbance $d$ is given by

$$d = \Delta_2 h(x), \tag{23}$$

where $\Delta_2$ is an arbitrary nonlinear system having a finite $L_2$-*gain* $< 1/\gamma$. Then the robustness of the closed-loop system (18), (19), (23), given in Fig. 2, is stated in the following corollary.

**Corollary 4.** *If there exists a $C^1$, positive-semidefinite function $V(x)$ such that* (17) *holds then the input–output map of the closed-loop system* (18), (19), (22), *with input $d$ and output $h(x)$, has $L_2$-gain $\leq \gamma$. Moreover, in view of the small-gain theorem, the closed-loop system* (18), (19), (22), (23), *reported in Fig. 2, will be closed-loop stable (see Definition 1.2.5 in van der Schaft, 1996) for all perturbations $\Delta_1$ and $\Delta_2$ of the form* (22) *and* (23) *with $\Delta_1$ such that can be tolerated at the input of an OFP system with $\rho = -\frac{1}{2}$ and $\Delta_2$ having an $L_2$-gain smaller than $1/\gamma$.*

**Proof.** From (21) and (22), with $\Delta_1$ such that can be tolerated at the input of an OFP system with $\rho = -\frac{1}{2}$, it immediately follows that the closed-loop system (18), (19), (22), with input $d$ and output $h(x)$ satisfies

$$\frac{\partial}{\partial t}S(x) \leq \frac{\gamma^2}{2}(\|d\|^2 - \|v(x) - d\|^2) - \frac{1}{2}\|h(x)\|^2$$
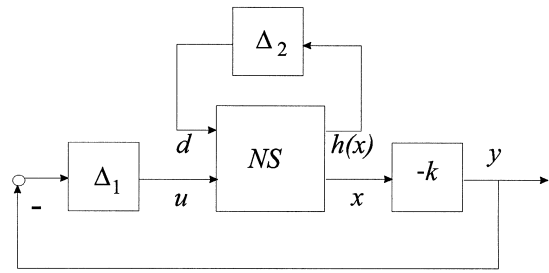
$$\leq \frac{1}{2}(\gamma^2\|d\|^2 - \|h(x)\|^2)$$



Fig. 2. Nonlinear feedback loop with the control law $k(x)$ and uncertainties $\Delta_1$ and $\Delta_2$.

and then it is dissipative with a supply rate $s(d, y) = \gamma^2\|d\|^2 - \|y\|^2$ that means that the input–output map of the closed-loop system (18), (19), (22) has $L_2$-*gain* $\leq \gamma$ (van der Schaft, 1996). Moreover, from the small gain theorem it follows that the closed-loop system (18), (19), (22), (23) will be stable for all perturbation $\Delta_1$ and $\Delta_2$ of the form (22) and (23) with $\Delta_1$ such that can be tolerated at the input of an OFP system with $\rho = -\frac{1}{2}$ and $\Delta_2$ having an $L_2$-*gain* smaller than $1/\gamma$.

## 5. Conclusion

In this paper, we have shown that, under regularity assumptions, a RH nonlinear $H_\infty$ control law, which is based on a finite horizon optimal control problem, is inverse optimal with respect to a modified infinite horizon $H_\infty$ problem if increasing the horizon causes a decrease of the optimal cost function. This inverse optimal result has been obtained by showing that the value function of the finite horizon problem is solution of a stationary HJI equation and, even if it has been obtained without considering any constraints, it gives some important guidelines to achieve RH control schemes that solve the $H_\infty$ problem. Beyond the standard robustness results of the $H_\infty$ problem, also robustness properties of the $H_\infty$ control law in face of dynamic input uncertainties are analyzed.

## References

Chen, H., & Allgöwer, F. (1998). A quasi-infinite horizon nonlinear model predictive control scheme with guaranteed stability. *Automatica*, 34, 1205–1217.

Chen, H., Scherer, C. W., & Allgöwer, F. (1997). A game theoretical approach to nonlinear robust receding horizon control of constrained systems. *American Control Conference '97*.

De Nicolao, G., & Bitmead, R. R. (1997). Fake Riccati equations for stable receding–horizon control. *European Control Conference '97*, Bruxelles, 1–4 July.

De Nicolao, G., Magni, L., & Scattolini, R. (1996). On the robustness of receding–horizon control with terminal constraints. *IEEE Transactions on Automatic Control*, 41, 451–453.

De Nicolao, G., Magni, L., & Scattolini, R. (1997). Stabilizing predictive control of nonlinear ARX models. *Automatica*, 33, 1691–1697.

De Nicolao, G., Magni, L., & Scattolini, R. (1998). Stabilizing receding–horizon control of nonlinear time-varying systems. *IEEE Transactions on Automatic Control, AC-43*, 1030–1036.

Glad, S. T. (1987). Robustness of nonlinear state feedback — a survey. *Automatica, 23*, 425–435.

Isidori, A., & Astolfi, A. (1992). Disturbance attenuation and $H_\infty$ control via measurement feedback in nonlinear systems. *IEEE Transactions on Automatic Control, 37*, 1283–1293.

Isidori, A., & Lin, W. (1997). Global inverse $L2$-gain state feedback design for a class of nonlinear systems. *Proceedings of the 36th IEEE CDC*, San Diego (pp. 2831–2836).

Jacobson, D. H. (1977). *Extensions of linear-quadratic control, optimization and matrix theory*. New York: Academic Press.

Keerthi, S. S., & Gilbert, E. G. (1988). Optimal, infinite-horizon feedback laws for a general class of constrained discrete-time systems. *Journal of Optimization Theory and Applications, 57*, 265–293.

Lall, S., & Glover, K. (1994). A game theoretic approach to moving horizon control. D. W. Clarke (Ed.), *Advances in model-based predictive control* (pp. 131–144). Oxford: Oxford University Press.

Magni, L., & Sepulchre, R. (1997). Stability margins of nonlinear receding horizon control via inverse optimality. *System and Control Letters, 32*, 241–245.

Mayne, D. Q., & Michalska, H. (1990). Receding horizon control of nonlinear systems. *IEEE Transaction on Automatic Control, 35*, 814–824.

Michalska, H., & Mayne, D. Q. (1993). Robust receding horizon control of constrained nonlinear systems. *IEEE Transactions on Automatic Control, 38*, 1512–1516.

Parisini, T., & Zoppoli, R. (1995). A receding–horizon regulator for nonlinear systems and a neural approximation. *Automatica, 31*, 1443–1451.

Poubelle, M. A., Bitmead, R. R., & Gevers, M. (1988). Fake algebraic Riccati technique and stability. *IEEE Transactions on Automatic Control, 33*, 379–381.

Scokaert, P. O. M., & Mayne, D. Q. (1998). Min–max feedback model predictive control for constrained linear systems. *IEEE Transactions on Automatic Control, 43*, 1136–1142.

Scokaert, P. O. M., Rawlings, J. B., & Meadows, E. S. (1997). Discrete-time stability with perturbations: application to model predictive control. *Automatica, 33*, 463–470.

Sepulchre, R., Jankovic, M., & Kokotovic, P. V. (1996). *Constructive nonlinear control*. Berlin: Springer.

Tadmor, G. (1992). Receding horizon revisited: an easy way to robustly stabilize an LTV system. *System and Control Letters, 18*, 285–294.

van der Schaft, A. J. (1992). $L_2$-Gain analysis of nonlinear systems and nonlinear state feedback $H_\infty$ control. *IEEE Transactions on Automatic Control, 37*, 770–784.

van der Schaft, A. J. (1993). Nonlinear state space $H_\infty$ control theory. In H. L. Trentelman & J. C. Willems (Eds.), Essays on control: perspectives in theory and its applications (pp. 153–190). Basel: Birkhauser.

van der Schaft, A. J. (1996). $L_2$-Gain and passivity techniques in nonlinear control. Berlin: Springer.

Vincent, T. L., & Grantham, W. J. (1997). *Nonlinear and optimal control systems*. New York: Wiley.

**Lalo Magni** was born in Bormio (SO), Italy, in 1971. He received the Laurea Degree (Summa cum laude) in Computer Engineering from the University of Pavia, Italy, in 1994. He got his Ph.D. in Electronic and Computer Engineering in 1998. Currently, he is an Assistant Professor at the University of Pavia. From October 1996 to February 1997 and in March 1998 he was at CESAME, Universitè Catholique de Louvain, Louvain La Neuve (Belgium). From October to November 1997 he was at the University of Twente with the System and Control Group in the Faculty of Applied Mathematics. His current research interests include non-linear control, predictive control, receding–horizon control, robust control, hybrid systems and fault detection.



**Henk Nijmeijer** (M'83, SM'91, F'00) was born in Assen, Netherlands, on March 16, 1955. He received the M.Sc. and Ph.D. degrees from the University of Groningen, Netherlands, in 1979 and 1983, respectively. From 1983 until April 2000 he has been with the Faculty of Mathematical Sciences, University of Twente, Enschede, Netherlands. Since May 2000 he is a Professor of Dynamics and Control at the Faculty of Mechanical Engineering, Eindhoven University of Technology, Eindhoven, Netherlands. His research interests are non-linear control, non-linear systems and non-linear dynamics. He has (co-)authored numerous papers in these areas, and is involved in the editorial board of various journals, including Automatica, the International Journal of Control and the SIAM Journal on Control and Optimization.



**A.J. (Arjan) van der Schaft** was born in Vlaardingen, Netherlands, in 1955. He received the undergraduate and Ph.D. degrees in Mathematics from the University of Groningen, Netherlands, in 1979 and 1983, respectively. In 1982, he joined the Faculty of Mathematical Sciences, University of Twente, Enschede, Netherlands, where he is presently a full-time professor. His research interests include the mathematical modelling of physical and engineering systems and the control of nonlinear and hybrid systems. He has served as Associate Editor for Systems and Control Letters, Journal of Non-linear Science, and the IEEE Transactions on Automatic Control. Currently he is an Associate Editor for the SIAM Journal on Control and Optimization. He is the author of *System Theoretic Descriptions of Physical Systems* (Amsterdam, Netherlands: CWI, 1984) and coauthor of *Variational and Hamiltonian Control Systems* (Berlin, Germany: Springer-Verlag, 1987) and *Nonlinear Dynamical Control Systems* (Berlin, Germany: Springer-Verlag, 1990), as well as the author of $L_2$-*Gain and Passivity Techniques in Nonlinear Control* (London, UK: Springer-Verlag, 1996, second revised and enlarged edition, Springer Communications and Control Engineering Series, 2000) and the coauthor of *An Introduction to Hybrid Dynamical Systems* (London, UK: Springer-Verlag, LNCIS 251, 2000).