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## KINEMATICAL CONSTRAINTS AND ALGEBROIDS

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This note intends to establish a link between the description of dynamics on a Lie algebroid defined by an integrable subbundle of the tangent bundle, and the dynamics associated to the Hamiltonian structure defined by [14] in the case of holonomic constraints.

**Keywords:** Lie algebroids, constrained dynamics.

### 1. Introduction

The study of mechanical systems with constraints has been a topic of continuing interest over the last century. Though the most important and open subject in this field is the nonholonomic case, the tools developed for holonomic systems may be indirectly important and useful in the study of nonholonomic constraints.

The present paper establishes the geometrical structure which defines a Poisson structure on the constrained phase space of a system with kinematical constraints proposed by two of us [14]. The construction in [14] is valid both for holonomic and nonholonomic constraints, and a theorem is presented proving that the almost Poisson structure of the general case is a true Poisson structure (that is satisfying the Jacobi identity) if and only if the constraints are holonomic. This Poisson structure is obtained as a suitable restriction of the canonical structure (defined by the symplectic form of a cotangent bundle) on the submanifold defined by the constraints on that cotangent bundle (see [6]).

The mathematical approach here is inspired by [8] (see also references therein) and combines it with the recently developed Lagrangian and Hamiltonian formalisms for Lie algebroids [15, 9, 7]. The Lie algebroid is the bundle defined by the subbundle of admissible velocities. In order to emphasize the algebroid properties, our notation for the distribution defined by the admissible velocities, denoted by  $\mathcal{D}$ , will be different from the notation for the Lie algebroid itself, which we will denote by  $\mathcal{A}_{\mathcal{D}}$ . Of course,  $\mathcal{A}_{\mathcal{D}}$  is canonically isomorphic to the distribution  $\mathcal{D}$ .

The structure of the paper is as follows. Sections 2 and 3 provide a brief sketch of the Poisson structure proposed in [14], a summary of the definition and main properties of Lie algebroids and the particular example of an integrable subbundle of a tangent bundle. Section 4 contains the main result of the paper: the Hamiltonian dynamics on the dual of the Lie algebroid defined by an integrable subbundle of a tangent bundle is equivalent to the Hamiltonian dynamics defined by van der Schaft and Maschke. Finally, in Section 5 we recall, for completeness, Martínez's definition of a symplectic structure on the Lie algebroid which provides the Poisson algebra we are dealing with. We also establish the relation between the Poisson tensor of the constrained phase space and that of the dual of the Lie algebroid.

## 2. Van der Schaft–Maschke construction for constrained dynamics

The main idea of the construction is that it is possible to define a dynamical description of a constrained system without making explicit use of the constraint equations, just by defining a suitable Poisson structure adopted to the constrained phase space.

Let us consider a dynamical system defined on a given manifold  $M$  and let us assume that it is constrained by a set of conditions on its velocities:

$$A^T(q)\dot{q} = 0, \quad q \in M, \quad \dot{q} \in T_qM,$$

where we consider  $A^T$  as a  $p \times n$  matrix of rank  $p < n$  everywhere ( $n$  is the dimension of  $M$ ) defined on  $T_qM$ , or in more intrinsic terms, as a set of  $p$  independent 1-forms on  $M$ .

From the geometrical point of view, this means that the set of admissible velocities defines a subbundle  $\mathcal{D}$  of the tangent bundle  $TM$ . In the present paper, we only consider the case where  $\mathcal{D}$  is an integrable subbundle of  $TM$ , though the original paper [14] is devoted to the general case of nonintegrable subbundles.

We want to define a Hamiltonian dynamics for this system, and hence we must define the corresponding submanifold of  $T^*M$ , endowed with a Poisson structure. We restrict ourselves to the case of regular Hamiltonians, for simplicity. The definition was given in [14], from the geometric point of view, by specifying the submanifold of  $T^*M$  defined by Legendre transformation of the integrable subbundle defined by the admissible velocities. This definition depends on the Hamiltonian of the system, and not only on the constraints. In any case, we define the constrained submanifold  $\mathcal{X}_c$  on  $T^*M$  as follows.

**DEFINITION 2.1.** The *constrained phase space*  $\mathcal{X}_c \subset T^*M$  is defined as the following set of points of the cotangent bundle:

$$\mathcal{X}_c = \left\{ (q, p) \in T^*M : A(q) \frac{\partial H(q, p)}{\partial p} = 0 \right\}. \quad (1)$$

Throughout we require that the Hamiltonian satisfies the regularity property,

$$\det \left( A^T \frac{\partial^2 H}{\partial p^2} A \right) \neq 0, \tag{2}$$

which implies that the Legendre transformation of any admissible velocity is never contained in the annihilator of the distribution  $\mathcal{D}$ .

An interesting property of the constrained system is that we can provide a system of coordinates for  $\mathcal{X}_c$  which does not depend of the Hamiltonian, contrary to the manifold itself, and hence is the same coordinate system for any Hamiltonian. The set of coordinates is obtained from a mapping from the vector bundle defined by the admissible velocities and the distribution on  $M$  corresponding to the admissible vector fields. In order to understand better the Lie algebroid issues of the following sections, we consider this vector bundle as a separate bundle, not as a subbundle of  $TM$ , and we denote it as  $\mathcal{A}_{\mathcal{D}}$ . In Section 3 we present its Lie algebroid structure. Thus, we define  $S : \mathcal{A}_{\mathcal{D}} \rightarrow \mathcal{D} \subset TM$ , of maximal rank on  $\mathcal{D}$ , satisfying

$$A^T \circ S = 0.$$

We will see how this mapping coincides with the anchor mapping defined in the next section. If we take the dual  $S^*$  of this mapping we can define a set of coordinates for  $\mathcal{X}_c$ ,

$$\tilde{p}^{(1)} = S^*(q)p, \quad p \in T_q^*M.$$

NOTE 2.1. Later on it will become clear why these coordinates are the same for all possible regular Hamiltonians we define. The main idea is that the Legendre transformation of the subbundle of integrable velocities defines a choice of a representative in a certain quotient bundle (i.e., it defines an element of the fiber of this bundle, by choosing an element of the manifold with respect to which we factor out). These coordinates will be defined for this quotient bundle, and any particular representation of the fibers (i.e., the choice of any regular Hamiltonian to define the Legendre transformation) will be admissible for them.

With this set of coordinates for  $\mathcal{X}_c$  we can proceed to define a Poisson structure on it, as a suitable restriction of the natural Poisson structure on  $T^*M$  which includes the constraints. The process can be summarized as follows:

- We complement the set of coordinates above to define a set of coordinates adopted to the constrained phase space (see the following sections for a geometrical version).

$$\tilde{p}^{(1)} = S^*(q)p, \quad \tilde{p}^{(2)} = A^T(q)p, \quad p \in T_q^*M.$$

This set of local coordinates  $\{q, \tilde{p}^{(1)}, \tilde{p}^{(2)}\}$  defines a system which will be not, in general, canonical. In any case it is a valid coordinate system on the total phase space, and particularly useful since it is adopted to the constraints.

- We take the natural Poisson structure on  $T^*M$ , and we write it in the coordinates above:

$$\mathcal{P}_{T^*M} = \begin{pmatrix} \{q^i, q^j\} & \{q^i, p_j\} \\ \{p_i, q^j\} & \{p_i, p_j\} \end{pmatrix} \rightarrow \begin{pmatrix} \{q^i, q^j\} & \{q^i, \tilde{p}_j\} \\ \{\tilde{p}_i, q^j\} & \{\tilde{p}_i, \tilde{p}_j\} \end{pmatrix}. \tag{3}$$

- Finally, we define the restriction of this Poisson bracket to the constrained phase space, just by removing the set of brackets of the  $\tilde{p}^{(2)}$  coordinates:

$$\mathcal{P}_c = \begin{pmatrix} \{q^i, q^j\} & \{q^i, \tilde{p}_\alpha^{(1)}\} \\ \{\tilde{p}_\alpha^{(1)}, q^j\} & \{\tilde{p}_\alpha^{(1)}, \tilde{p}_\beta^{(1)}\} \end{pmatrix}. \tag{4}$$

We have used Greek indices for denoting the coordinates of the fibers of the constrained phase space, as we will do in the case of algebroids, in order to make the relation between the two formalisms more evident. The following step in [14] is to define a restricted Hamiltonian  $H_c$  on  $\mathcal{X}_c$ , which allows us to define a Hamiltonian system on the constrained phase space which does not include the constraints explicitly, and which matches with the dynamical behaviour of the original system. The Hamiltonian  $H_c$  will be simply the restriction of  $H$  to the manifold  $\mathcal{X}_c$ . This follows by noting that (1) is equivalent to

$$\frac{\partial H}{\partial p^{(2)}} = 0.$$

### 3. The concept of Lie algebroids

#### 3.1. Generalities

The concept of Lie algebroids has been used in the last fifty years in the algebraic geometric framework, under different names (see [2–4, 10, 11, 13]); but the first proper definition, from the point of view of differential geometry, is due to Pradines [12].

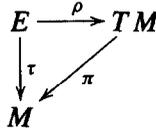
DEFINITION 3.1. A *Lie algebroid* on a manifold  $M$  is a vector bundle  $E \rightarrow M$ , in whose space of sections we define a Lie algebra structure  $(\Gamma(E), [\cdot, \cdot]_E)$ , and a mapping  $\rho : E \rightarrow TM$  which is a homomorphism for this structure in relation with the natural Lie algebra structure of the set of vector fields  $(X(M), [\cdot, \cdot]_{TM})$ . We have therefore

$$\rho([\eta, \xi]_E) = [\rho(\eta), \rho(\xi)]_{TM}, \quad \forall \eta, \xi \in \Gamma(E),$$

and

$$[\eta, f\xi]_E = [\eta, \xi]_E + (\rho(\eta)f)\xi, \quad \forall \eta, \xi \in \Gamma(E), \quad \forall f \in C^\infty(M).$$

The definition may be summarized in the following diagram



For simplicity we often omit the subscripts  $E$  and  $TM$  of the commutator.

If we take coordinates  $\{x^i\}$  in the base manifold and a basis of sections  $\{e_\alpha\}$  in the bundle, we can consider  $\{(x^i, \lambda^\alpha)\}$  to be the coordinates for  $E$ , with respect to which we write the anchor mapping as

$$\rho(e_\alpha) = \rho_\alpha^i \frac{\partial}{\partial x^i},$$

and the Lie bracket in this base as

$$[e_\alpha, e_\beta] = C_{\alpha\beta}^\gamma e_\gamma.$$

The main idea we have to keep in mind is that Lie algebroid is a geometrical object very similar to the tangent bundle. The sections of the bundle  $E$  play the role of vector fields. The other basic objects of differential calculus on  $TM$  can be defined for  $E$  as well:

- An analogue of 1-forms are sections of the dual bundle  $E^* \rightarrow M$ . This definition allows us to consider an action of sections of  $E$  on sections of  $E^*$  as the analogue of the inner action of vector fields on differential forms. We denote it by  $i_\sigma$  for  $\sigma \in \Gamma E$ .
- An analogue of  $p$ -forms is also easy to define: we take simply the sections of the bundle  $(E^*)^{\wedge p} \rightarrow M$ .
- Finally, basic in our construction is the definition of the exterior derivative  $d$ , as the operator which connects the analogue of  $p$ -forms with the analogue of  $(p + 1)$ -forms. We define it as we do in the usual case, first the action on functions, and later the action on higher-order forms:

– For functions,  $d : C^\infty(M) \rightarrow \Lambda^1(E)$  such that

$$\langle df, a \rangle = \rho(a)f, \quad \forall f \in C^\infty(M), \quad a \in \Gamma(E).$$

– For  $p$ -forms we take a direct analogue of the usual definition  $d : \Lambda^p(E) \rightarrow \Lambda^{p+1}(E)$ ,

$$\begin{aligned}
 d\theta(\sigma_1, \dots, \sigma_{p+1}) &= \sum_{i=1}^{p+1} (-1)^{i+1} \rho(\sigma_i)\theta(\sigma_1, \dots, \hat{\sigma}_i, \dots, \sigma_{p+1}) \\
 &\quad + \sum_{i < j} (-1)^{i+j} \theta([\sigma_i, \sigma_j], \sigma_1, \dots, \hat{\sigma}_i, \dots, \hat{\sigma}_j, \dots, \sigma_{p+1}),
 \end{aligned}$$

where by  $\hat{\sigma}_i$  we mean that the corresponding section is omitted.

We may also define an analogue for the Lie derivative using the Cartan formula

$$L_\sigma = i_\sigma \circ d + d \circ i_\sigma.$$

One of the most interesting properties which can also be extended to the Lie algebroid framework, is that, as in the case of a cotangent bundle, the dual of any Lie algebroid is a Poisson manifold.

**THEOREM 3.1.** *For any Lie algebroid  $E$  the dual bundle  $E^*$  is a Poisson manifold.*

*Proof:* See [15] (direct construction), or [9]. □

If we take a basis of sections of  $E^*$  as the dual of the basis  $\{e_\alpha\}$  of sections of  $E$ , and denote the corresponding coordinates as  $(x^i, \mu_\alpha)$ , the Poisson bracket above turns out to be

$$\{x^i, x^j\} = 0, \quad \{x^i, \mu_\alpha\} = \rho_\alpha^i, \quad \{\mu_\alpha, \mu_\beta\} = C_{\alpha\beta}^\gamma \mu_\gamma. \tag{5}$$

### 3.2. The algebroid structure of the integrable subbundle

Let us consider a differentiable manifold  $M$  and an integrable subbundle  $\mathcal{D} \subset TM$ , i.e., a bundle whose sections define a subalgebra in  $X, [\cdot, \cdot]$ . We consider this subbundle as a separate object, and we denote it as  $\mathcal{A}_\mathcal{D}$ .

The algebroid components are defined as follows:

- The vector bundle is the bundle  $\mathcal{A}_\mathcal{D}$ , and its base is the base manifold  $M$ .
- The Lie algebra structure on the sections is inherited from the natural one defined on the tangent bundle. The fact that the subbundle is integrable means that we have a subalgebra of the algebra of vector fields, and hence on  $\mathcal{A}_\mathcal{D}$  we have a relation of the type

$$[e_\alpha, e_\beta] = C_{\alpha\beta}^\gamma e_\gamma, \tag{6}$$

where  $\{e_\mu\}$  defines a basis of sections of  $\mathcal{A}_\mathcal{D}$ , and  $C_{\alpha\beta}^\gamma$  are the structure functions defining the Lie algebra structure on this basis.

- Finally, the anchor mapping is defined as the natural inclusion of the elements of  $\mathcal{A}_\mathcal{D}$  in  $\mathcal{D} \subset TM$ ,

$$\rho_0 : \mathcal{A}_\mathcal{D} \hookrightarrow TM. \tag{7}$$

Regarding the dual bundle  $\mathcal{A}_\mathcal{D}^*$ , we have a new vector bundle, which is not a Lie algebroid but which is a Poisson manifold as we saw above, and will be the natural framework for a Hamiltonian dynamical description. Note that the construction of  $\mathcal{A}_\mathcal{D}^*$  is purely geometrical and no dynamical considerations are taken into account. Another important property of the construction is the canonical mapping from  $T^*M$  to  $\mathcal{A}_\mathcal{D}^*$  defined by the dual of the anchor mapping,

$$\rho^0 : T^*M \rightarrow \mathcal{A}_\mathcal{D}^*.$$

As we will see later, very important in our construction will be the necessity of defining an inverse mapping of  $\rho^0$ , i.e., an application of the sections of  $\mathcal{A}_{\mathcal{D}}^*$  as sections of the cotangent bundle.

NOTE 3.1. The Poisson structure above was used by Weinstein [15] for the definition of his version of Lagrangian mechanics in Lie algebroids, by defining the Legendre transformation and the energy function corresponding to the given Lagrangian (a function on the algebroid); then he employed the Poisson structure to define a kind of Hamiltonian dynamics description, which was mapped back to the algebroid.

#### 4. Equivalence of the Hamiltonian dynamics in both approaches

The main result of this paper is summarized in the title of this section. We prove this result in two steps:

- First we construct a Poisson morphism between the manifolds  $\mathcal{X}_c$  and  $\mathcal{A}_{\mathcal{D}}^*$  with the Poisson structures introduced in the previous sections.
- The second step will be equivalence of Hamiltonians. We start with a constrained Hamiltonian system defined on  $T^*M$  and follow the van der Schaft–Maschke construction for defining the constrained Hamiltonian dynamics on  $\mathcal{X}_c$ . This Hamiltonian function will be later transferred to  $\mathcal{A}_{\mathcal{D}}^*$  by using the Legendre transformation and the anchor mapping in order to define a Hamiltonian system on this last manifold. We also prove that both dynamics are equivalent.

##### 4.1. Geometry and Legendre transformation

First of all we must clarify the geometrical structure of the manifold  $\mathcal{X}_c$ . As we mentioned in the previous sections, it is defined as the Legendre transform of the subbundle  $\mathcal{D}$ . If we denote by  $A$  the set of 1-forms which defines the constraints (semibasic 1-forms if we consider them as forms defined on the cotangent bundle), then

$$\mathcal{X}_c = \{(q, p) \in T^*M \mid i_{X_H} A = 0\} \quad (8)$$

is obviously transformed into the subbundle  $\mathcal{D}$  by the Legendre transformation.

Another interesting condition required in [14] is that the Legendre transformation does not map any element in the bundle of admissible velocities into the annihilator of  $\mathcal{D}$  in  $T^*M$  (see [14] for discussion). This is the meaning of the regularity condition (2) which appears in [14], and it will be very important in the following sections.

It is remarkable that, due to the locality of Legendre transformation, this relation does not depend critically on the global Hamiltonian  $H \in C^\infty(T^*M)$  but only on its restriction  $H_c \in C^\infty(\mathcal{X}_c)$ . This will be useful later, since we need to consider only the restriction of the Hamiltonian, which is a function on  $\mathcal{X}_c$ , and therefore is mapped to an energy function defined only on  $\mathcal{D}$ , and which can be transferred trivially to  $\mathcal{A}_{\mathcal{D}}$ . This property allows us to define a Legendre transformation between

$\mathcal{A}_{\mathcal{D}}$  and  $\mathcal{A}_{\mathcal{D}}^*$  identical to the Legendre transformation we used between  $TM$  and  $T^*M$ .

LEMMA 4.1. *The Legendre transformation constructed with  $H_c$  maps also  $\mathcal{X}_c$  on  $\rho_0(\mathcal{A}_{\mathcal{D}}) = \mathcal{D}$ .*

*Proof:* It is obvious that the Legendre transformation and the Hamiltonian define an element of the tangent bundle:

$$\begin{aligned} \mathcal{F}H : T^*M &\rightarrow TM, \\ \mathcal{F}H(q, w) &= (q, \xi^w(H)) = (q, \langle dH, \xi^w \rangle), \quad q \in M, \quad w \in T_q^*M, \end{aligned}$$

where  $\xi^w$  is the vertical vector in  $T_{(q,w)}(T^*M)$  canonically identified with element  $w \in T_q^*M$ . Since we know that the image of elements of  $\mathcal{X}_c$  belongs to the sub-bundle of integrable velocities, the Legendre transformation will depend only on the values of the Hamiltonian restricted to  $\mathcal{X}_c$ , since we know from the properties of exterior derivative (see for instance [1] ) that

$$(dH)_{\mathcal{X}_c} = d(H_{\mathcal{X}_c}) = dH_c. \quad \square$$

Finally, we would like to characterize carefully the structure of  $\mathcal{X}_c$  in an intrinsic way. For the sake of simplicity we only consider a particular case.

ASSUMPTION 4.1. Throughout the rest of the paper we will consider Hamiltonians of the mechanical type (i.e., quadratic in momenta) and regular (i.e.,  $\det \frac{\partial^2 H}{\partial p^2} \neq 0$ ). This implies that the manifold  $\mathcal{X}_c$  is a vector subbundle of  $T^*M$ , as can be seen easily from the definition.

The most important feature will be the existence of a quotient vector bundle defined as

$$\mathcal{P} = \frac{T^*M}{\mathcal{P}^{(2)}},$$

where  $\mathcal{P}^{(2)}$  is the annihilator of the distribution  $\mathcal{D}$ .  $\mathcal{P}$  is defined with the same base as  $T^*M$ , but with a fiber at the point  $p \in M$  which is the quotient vector space defined by  $T_p^*M$  and  $\mathcal{P}_p^{(2)}$ .

Given an element  $P \in T^*M$ , the choice of a representative in the class above as a fiber of  $\mathcal{P}^{(2)}$ , defines a different bundle where we take only one element of each class. The quotient bundle becomes then a vector bundle (i.e. fibres become vector spaces). It is easy to see that, for quadratic Hamiltonians, the Legendre transform of  $\mathcal{D}$  provides such a choice.

PROPOSITION 4.1. *The Legendre transformation of  $\mathcal{D}$  (i.e.  $\mathcal{X}_c$ ) defines a representative for the quotient bundle  $\mathcal{P}$ .*

*Proof:* We know that the bundle of admissible velocities is transferred isomorphically onto the bundle  $\mathcal{X}_c$  included in  $T^*M$ . This implies that the transformation of  $p$  independent vector fields in  $\mathcal{D}$  defines  $p$  independent elements on  $T^*M$ . We

can take these elements of  $T^*M$  as representatives of the classes of the quotient bundle  $\mathcal{P}$  since this ensures that the intersection of  $\mathcal{X}_c$  with  $\mathcal{P}$  is transversal, i.e., the images of two independent vector fields in  $\mathcal{D}$  cannot belong to the same class (otherwise their difference, which is also an admissible vector field, would belong to the annihilator, contradicting the assumption). As a result we can write that, locally, the bundle  $T^*M$  is the sum of  $\mathcal{X}_c$  and the annihilator  $\mathcal{P}^{(2)}$ .  $\square$

These results allow us to understand geometrically the construction in [14] of the choice of the coordinate system  $\{p^1, p^{(2)}\}$  that we saw above. The set of coordinates  $p^{(1)}$  is natural for the quotient bundle  $\mathcal{P}$ , which (as we will see later) has a one-to-one relation with the dual of the Lie algebroid. Of course, any particular bundle obtained by choosing a representative in the class can be parametrized by this set of coordinates. Because of the previous result, it can also be used, together with any set of coordinates for the annihilator, as a coordinate system for the cotangent bundle  $T^*M$  once a representative for the quotient bundle has been chosen. The Legendre transformation of the bundle of admissible velocities provides such a choice, and therefore justifies geometrically the nature of the set of coordinates used in [14].

#### 4.2. Equivalence of geometric structures

The result to prove in this subsection is the existence of a Poisson morphism between the aforementioned manifolds. We will do it in two steps: first we prove that the bundles  $\mathcal{X}_c$  and  $\mathcal{A}_{\mathcal{D}}^*$  are diffeomorphic, and later we will see that the Poisson structures are equivalent.

First of all we recall the well-known result.

LEMMA 4.2. *The restriction of  $\rho^0$  to the bundle  $\mathcal{P}$  defines a bijection from  $\mathcal{P}$  to  $\mathcal{D}^*$ .*

This result allows us to construct the desired mapping between  $\mathcal{A}_{\mathcal{D}}^*$  and  $\mathcal{X}_c$ , but for completeness, and for the definition of the Poisson algebra on the dual of the Lie algebroid, we will construct a full commutative diagram involving both spaces as well as the tangent bundle and the Lie algebroid.

THEOREM 4.1. *The bundles  $\mathcal{D}^*$  and  $\mathcal{X}_c$  are (locally) diffeomorphic manifolds.*

*Proof:* We proceed step by step in the construction of the mapping:

- Take the restriction of the Hamiltonian  $H \in C^\infty(T^*M)$  to the subbundle  $\mathcal{X}_c$  and denote it by  $H_c$ . We have seen that the Legendre transformation defined by this function behaves similarly to the original one, since the Legendre transformation is a local construction. This defines a local diffeomorphism (the system is regular)

$$\Psi : \mathcal{X}_c \rightarrow \rho_0(\mathcal{A}_{\mathcal{D}}).$$

- This relation establishes a diffeomorphism that can be used to map the functions defined on  $\mathcal{X}_c$  to the subbundle of admissible velocities, in particular

the function  $H_c$  itself. It obviously defines an energy function  $E_{\mathcal{D}}$  and a Lagrangian  $L_{\mathcal{D}}$  which belong to  $C^\infty(\mathcal{D})$ .

- Now, it is also obvious that this subbundle in  $TM$  is uniquely identified with  $\mathcal{A}_{\mathcal{D}}$  (since the anchor mapping is injective), and we can define an energy function on  $\mathcal{A}_{\mathcal{D}}$  as

$$E_{\mathcal{A}_{\mathcal{D}}} = \rho_0^*(E_{\mathcal{D}}),$$

as well as the image of the corresponding Lagrangian  $L_{\mathcal{A}_{\mathcal{D}}} = \rho_0^*(L_{\mathcal{D}})$

- This Lagrangian allows us to define a new Legendre transformation, between  $\mathcal{A}_{\mathcal{D}}$  and  $\mathcal{A}_{\mathcal{D}}^*$ , which is, in some sense, the inverse of the transformation we used in the first step. This transformation is again a local diffeomorphism

$$\Phi : \mathcal{A}_{\mathcal{D}} \rightarrow \mathcal{A}_{\mathcal{D}}^*.$$

Now it is trivial to verify that the mapping:

$$\Phi \circ \rho_0^{-1} \circ \Psi : \mathcal{X}_c \rightarrow \mathcal{A}_{\mathcal{D}}^*$$

defines the desired (local) diffeomorphism.

From the geometrical point of view, we have the following commutative diagram:

$$\begin{array}{ccc}
 \mathcal{A}_{\mathcal{D}} & \xrightarrow{\rho_0} & \rho_0(\mathcal{A}_{\mathcal{D}}) \subset TM \\
 \mathcal{F}H_{\mathcal{A}_{\mathcal{D}}^*} \uparrow & \mathcal{F}L_{\mathcal{A}_{\mathcal{D}}} & \uparrow \mathcal{F}H \quad \mathcal{F}L \\
 \mathcal{A}_{\mathcal{D}}^* & \xleftarrow{\rho_0|_{\mathcal{X}_c}} & \mathcal{X}_c \subset T^*M
 \end{array}$$

where it must be taken into account that the commutativity property is satisfied only on the subbundle  $\mathcal{X}_c \subset T^*M$ .

As the Hamiltonian is mechanical (as we have assumed previously), the Legendre transformation is actually a bundle isomorphism, since it is linear on the fibers. The result can also be proven for a more general case, where we do not require the quadratic property, but in such a case the previous diagram cannot be directly defined, since then  $\mathcal{X}_c$  is not a vector subbundle of the cotangent bundle and hence the transversality property defining the relation between  $\mathcal{X}_c$  and  $\mathcal{D}^*$  is no longer valid. □

### 4.3. Equivalence of Hamiltonian systems

The final step of our construction is to establish the equivalence of the corresponding Hamiltonian dynamics. In order to do this, we need to prove that the bundle isomorphism we have constructed above is actually a Poisson morphism when we consider its action on the functions of both spaces:

**THEOREM 4.2.** *The algebras obeying (4) and (5) are equivalent Poisson algebras.*

*Proof:* The main idea is to use the diffeomorphism constructed above to transfer the structure from one bundle to another. Let us take a basis of sections on  $\mathcal{X}_c$ ,

denoted by  $\{E^\alpha\}$ . Since the Legendre transformation is an invertible mapping, we can define a basis of sections for the subbundle  $\rho_0(\mathcal{D})$  as

$$E_\alpha = \mathcal{F}(E^\alpha)$$

which will span the Lie algebra of admissible velocities. This set of sections can be transferred by  $\rho_0^{-1}$  to the Lie algebroid  $\mathcal{D}$  in such a way that the Lie algebra structure is preserved (the anchor mapping is a homomorphism). The mapping allows us also to define a basis of sections on  $\mathcal{D}$  as

$$e_\alpha = \rho_0^{-1}(E_\alpha)$$

(the mapping is well defined since  $E_\alpha$  belongs to the image  $\rho_0(\mathcal{D})$ ). Now, we can use the dual basis of sections of  $\mathcal{D}^*$  for writing the Poisson structure, which will take the form:

$$\begin{aligned} \{x^i, x^j\} &= 0, & \{x^i, \mu_\alpha\} &= \rho_\alpha^i(q), \\ \{\mu_\alpha, x^i\} &= -\rho_\alpha^i(q), & \{\mu_\alpha, \mu_\beta\} &= C_{\alpha\beta}^\gamma \mu_\gamma, \end{aligned}$$

where  $C_{\alpha\beta}^\gamma$  are the structure constants of  $\mathcal{D}$  in the basis  $\{e_\alpha\}$  and  $\mu_\alpha$  are the coordinates of  $\mathcal{D}^*$  with respect to its dual basis  $\{e^\alpha\}$ . But this Poisson structure is equivalent to  $P_c$

$$\begin{aligned} \{q^i, q^j\} &= 0, & \{q^i, p_\alpha^{(1)}\} &= S_\alpha^j(q), \\ \{p_\alpha^{(1)}, q^i\} &= -S_\alpha^i(q), & \{p_\alpha^{(1)}, p_\beta^{(1)}\} &= \langle p, [S^\alpha, S^\beta] \rangle, \end{aligned}$$

once we identify  $S$  with the coordinate expression of the anchor mapping. □

**COROLLARY 4.1.** *The Hamiltonian dynamical systems  $(\mathcal{X}_c, P_c, H_c)$  and  $(\mathcal{D}^*, P_{\mathcal{D}^*}, (\Psi^{-1} \circ \rho_0^{-1} \circ \Phi^{-1})^*(H_c))$  are equivalent.*

## 5. The symplectic structure

### 5.1. Martínez's symplectic structure

In this section we prove, for completeness, that the Poisson structure we saw above comes from a symplectic structure, and that the relation we have seen for the Poisson algebras can be also extended to the level of Poisson tensors in the manifolds  $\mathcal{D}^*$  and  $\mathcal{X}_c$ . The concept of "symplectic" deserves special explanation.

**DEFINITION 5.1.** A *symplectic structure on a Lie algebroid  $E$*  is a symplectic structure  $\omega$  on the vector bundle  $E$ , such that

$$d\omega = 0.$$

Here the exterior derivative must be considered to be the exterior derivative associated to the algebroid, and hence the symplectic form is closed in this new

cohomology (which, in general, does not coincide with the usual de Rham cohomology).

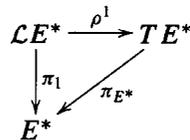
The natural framework for dealing with these objects is the Hamiltonian formalism developed by Martínez [7]. Given the algebroid  $E$ , the main object of this construction is the analogue of the usual  $T(T^*M)$  (the usual Hamiltonian vector field is a section of this bundle). It is reasonable to consider a similar construction to the extension  $\mathcal{L}E$  but taking the dual bundle  $E^*$  as the starting point.

DEFINITION 5.2. We define the *extension of the dual bundle*  $E^*$  as

$$\mathcal{L}E^* = \{(a, b, v) \in E^* \times E \times TE^* \mid \pi(a) = \tau(b), v \in T_a E^*, \rho(b) = T_a \tau(v)\}.$$

It is easy to see that this is a vector bundle over  $E^*$ , and we consider it as a bundle on  $E^*$ . The properties of projections are very similar:

- $\pi_2(a, b, v) = b$  is the analogue of the projection  $T\pi_m : T(TM) \rightarrow TM$  for the usual case.
- The third projection  $\rho_1(a, b, v) = v$  will define an algebroid structure on  $\mathcal{L}E^*$



with the Lie algebra structure on the sections coming from those on  $E$  and  $TE^*$ .

Similarly to the case of  $\mathcal{L}E$  it can be proved that this bundle admits a Lie algebroid structure [7], with  $\rho_1$  as the anchor mapping.

**Basis and coordinates.** We can define the *basis* of sections on  $\mathcal{L}E^*$  considering the usual basis for  $E$  ( $\{e_\alpha\}$ ) and  $E^*$  ( $\{e^\alpha\}$ ), and *coordinates*  $(x_i, \mu_\alpha)$  for  $E^*$

$$\mathcal{X}_\alpha(a) = \left( a, e_\alpha(\tau(a)), \rho_\alpha^i \frac{\partial}{\partial x^i} \right)$$

and

$$\mathcal{P}^\alpha(a) = \left( a, 0, \frac{\partial}{\partial \mu_\alpha} \right).$$

In the set of coordinates corresponding to this set of sections, we can construct the canonical 1-form, which serves as the analogue of the canonical 1-form of the cotangent bundle.

DEFINITION 5.3. Let us define the *canonical 1-form*  $\theta$  as the section of  $(\mathcal{L}E^*)^*$  with the following coordinate expression

$$\theta(x, \mu) = \mu_\alpha \mathcal{X}^\alpha.$$

Similarly to the case of the cotangent bundle, we can construct now the corresponding symplectic 2-form.

DEFINITION 5.4. Let us define a *symplectic 2-form of the algebroid E* in the coordinates above as:

$$\omega = -d\theta = \mathcal{X}^\alpha \wedge \mathcal{P}_\alpha + \frac{1}{2}\mu_\gamma C_{\alpha\beta}^\gamma \mathcal{X}^\alpha \wedge \mathcal{X}^\beta$$

Since the exterior derivative for the algebroid is nilpotent,  $d\omega = 0$ . In the coordinates fixed by the basis above, the symplectic matrix becomes

$$\omega(x, \mu) = \begin{pmatrix} 0 & I \\ -I & \frac{1}{2}\mu_\gamma C_{\alpha\beta}^\gamma \end{pmatrix}.$$

As it happens in the case of normal symplectic manifolds, the symplectic form provides a relation between sections of  $\mathcal{L}E^*$  and sections of  $\mathcal{L}E$ :

$$\sigma \in \mathcal{L}E \rightarrow \mu = \omega(\sigma, \cdot) \in \mathcal{L}E^*.$$

This relation can be extended to the functions on  $E^*$  by using the exterior derivative

$$f \in C^\infty(E^*) \rightarrow \sigma_f \text{ such that } i_{\sigma_f}\omega = df.$$

Finally, this relation allows us to establish a Poisson algebra structure on  $C^\infty(E^*)$  by using the following expression: given  $f, g \in C^\infty(E^*)$ , define

$$\{f, g\} = \omega(\sigma_f, \sigma_g). \tag{9}$$

THEOREM 5.1. *The Poisson structure defined by (9) is precisely the structure proposed by Weinstein for the Lagrangian description.*

*Proof:* This follows from [7]. The main idea is to consider only the set of linear functions on  $E^*$  (i.e. elements of  $E$ ) and compute the corresponding bracket. The only tool we need is the fact that the corresponding sections in  $\mathcal{L}E^*$  for any element in the base coordinates:

$$\sigma_{x^i} = \rho_i^\alpha \mathcal{P}_\alpha, \quad \sigma_{\mu_\alpha} = \mathcal{X}_\alpha - \frac{1}{2}C_{\alpha\beta}^\gamma \mu_\gamma \mathcal{P}_\beta.$$

It is trivial now to verify that the Poisson tensor in these coordinates has the form proposed in [15]. □

The Hamiltonian formalism is now defined completely analogously to the usual case: we take a function  $H \in C^\infty(E^*)$  and we obtain the corresponding dynamical section  $\sigma_H$  on  $\mathcal{L}E^*$  defined by the symplectic equation

$$i_{\sigma_H}\omega = dH.$$

**5.2. The Poisson tensor**

A similar construction can be made in terms of Poisson tensors. In this case, we search for a section of  $\Lambda \in \Gamma(\mathcal{L}E^* \wedge \mathcal{L}E^*)$  such that the Poisson bracket of functions can be written as

$$\{f, g\} = \Lambda(df, dg), \quad \forall f, g \in C^\infty(E^*),$$

where  $d$  is the exterior derivative of the algebroid  $\mathcal{L}E^* \rightarrow E^*$ . Obviously, if such an object exists, it will reduce to the usual Poisson tensor in the usual case. We also consider in this framework the graded Lie algebra of sections

$$\mathcal{T} = \Gamma\left(\bigwedge^{\bullet} \mathcal{L}E^*\right).$$

We can also introduce an analogue of the Schouten bracket, as a bracket operation in the graded Lie algebra  $\mathcal{T}$ , by adopting the usual construction that can be found for instance in [5]. Thus the Schouten bracket of a  $p$  section  $A$  and a  $q$  section  $B$  is defined to be the  $p + q - 1$  section whose action on the analogue of differential forms is

$$i_{[A,B]}\beta = i_A \circ d \circ i_B(\beta) + (-1)^q i_B \circ d \circ i_A(\beta), \quad \beta \in \Gamma\left(\bigwedge^{(p+q-1)} (\mathcal{L}E^*)^*\right).$$

It is quite obvious that this operation defines an inner product on  $\mathcal{T}$ , which has the same properties as the usual Schouten brackets on multivector fields, since the properties of the operators  $i$  and  $d$  are the same.

With this construction, we can adopt also the well-known result for the Poisson tensor.

LEMMA 5.1.  $\Lambda$  is a Poisson tensor on the Lie algebroid  $\mathcal{L}E^*$  if and only if the corresponding Schouten bracket vanishes, i.e.,

$$[\Lambda, \Lambda]\beta = 2i_\Lambda \circ d \circ i_\Lambda(\beta) = 0, \quad \forall \beta \in \bigwedge^3 (\mathcal{L}E^*)^*.$$

*Proof:* Since the objects are linear, it is enough to consider  $\beta$  as an element of the form  $\beta = df \wedge dg \wedge dh$ , where  $f, g, h \in C^\infty(E^*)$ . In this case, if we rewrite the expression above we obtain

$$[\Lambda, \Lambda]\beta = 2i_\Lambda \circ d \circ i_\Lambda(df \wedge dg \wedge dh) = 2(\{f, \{g, h\}\} + \text{cyclic}).$$

Hence, the lemma follows. □

Let us consider now the following section of  $\mathcal{L}E^* \wedge \mathcal{L}E^*$ , written in the basis introduced above,

$$\Lambda(x^i, \mu_\alpha) = \mathcal{X}_\alpha \wedge \mathcal{P}^\alpha + \frac{1}{2} c_{\alpha\beta}^\gamma \mu_\gamma \mathcal{P}^\alpha \wedge \mathcal{P}^\beta. \tag{10}$$

PROPOSITION 5.1. For any two functions in  $E^*$ ,

$$\Lambda(df, dg) = \{f, g\}, \quad \forall f, g \in C^\infty(E^*),$$

where the bracket above is the Poisson bracket (5).

*Proof:* This is a straightforward computation, since we know that in the coordinates above

$$df = \rho_\alpha^i \frac{\partial f}{\partial x^i} \mathcal{X}^\alpha + \frac{\partial f}{\partial \mu_\alpha} \mathcal{P}_\alpha.$$

If we take the coordinate functions for  $E^*$ , the calculation is immediate. □

Since we know that this bracket of functions is a well-defined Poisson bracket, we obtain the following corollary.

COROLLARY 5.1.  $\Lambda$  is a Poisson tensor.

### 5.3. Poisson tensors on $\mathcal{X}_c$ and $\mathcal{L}\mathcal{A}_\mathcal{D}^*$

In order to compare the Poisson tensors we have defined on the constrained phase space  $\mathcal{X}_c$  and on  $\mathcal{L}\mathcal{A}_\mathcal{D}^*$ , we will use the mappings we have defined above. In particular, we know that for the bundle  $\mathcal{L}\mathcal{A}_\mathcal{D}^*$ , the anchor mapping  $\rho_2$  establishes a relation

$$\rho_2 : \mathcal{L}\mathcal{A}_\mathcal{D}^* \rightarrow T\mathcal{A}_\mathcal{D}^*,$$

whose image consists of the vector fields on  $\mathcal{A}_\mathcal{D}^*$  along the leaves of the foliation of the anchor mapping  $\rho_0$  (this is the meaning of the constraint in the definition of  $\mathcal{L}\mathcal{A}_\mathcal{D}^*$ ). But this defines the following diagram

$$\begin{array}{ccccc} \mathcal{L}\mathcal{A}_\mathcal{D}^* & \xrightarrow{\rho_2} & T\mathcal{A}_\mathcal{D}^* & \xleftarrow{T\rho^0} & T\mathcal{X}_c \\ \updownarrow & & \updownarrow & & \updownarrow \\ (\mathcal{L}\mathcal{A}_\mathcal{D}^*)^* & \xleftarrow{\rho^2} & T^*\mathcal{A}_\mathcal{D}^* & \xleftarrow{T^*\rho^0} & T^*\mathcal{X}_c. \end{array}$$

These mappings allow us to define, for instance on  $\mathcal{X}_c$ , a Poisson tensor as the image of (10) under the transformations above. We thus define a tensor  $\hat{\Lambda} = (T\rho_0 \circ \rho_2 \circ \eta)_* \Lambda$ , which acts on elements of  $T^*\mathcal{X}_c$  as

$$\hat{\Lambda}(\alpha, \beta) = \Lambda(\hat{\alpha}, \hat{\beta}),$$

where  $\hat{\alpha} = \hat{\Lambda} = (T\rho_0 \circ \rho_2 \circ \eta)^*(\alpha)$  and  $\hat{\beta} = \hat{\Lambda} = (T\rho_0 \circ \rho_2 \circ \eta)^*(\beta)$  for any  $\alpha$  and  $\beta$  in  $T^*\mathcal{X}_c$ .

*A priori* the main difference between the tensors  $\hat{\Lambda}$  and  $\Lambda_c$  is the fact that the first is defined only on the leaves of the foliation defined by the admissible velocities, while we have not yet proved this property for the Poisson structure on the constrained phase space.

LEMMA 5.2. *The Poisson structure  $\Lambda_c$  is constant along the leaves of the foliation.*

*Proof:* The only point to be proved is that the bracket of any function of the base, constant on each leaf ( $g$ ) and any function ( $f$ ) of the quotient bundle (once injected in  $T^*M$  by taking a representative) vanishes. This is quite simple to see in coordinates: in the Poisson tensor expressed in coordinates it is obvious that the bracket

$$\{q, p_\alpha^{(1)}\} = \rho_\alpha$$

is defined only along the foliation, by the image of the mapping  $\rho$ . The corresponding Hamiltonian vector field  $X_f$  corresponding to any function  $f \in C^\infty(\mathcal{X}_c)$  will be given by

$$X_f = \{f, \cdot\} = -\rho_\alpha^i \frac{\partial f}{\partial p_\alpha^{(1)}} \frac{\partial}{\partial x^i} + \rho_\alpha^i \frac{\partial f}{\partial x^i} \frac{\partial}{\partial p_\alpha^{(1)}} + c_{\alpha\beta}^\gamma p_\gamma^{(1)} \frac{\partial f}{\partial p_\alpha^{(1)}} \frac{\partial}{\partial p_\beta^{(1)}}.$$

From this it is immediate that the functions on the base, transversal to the foliation, are Casimir functions for the Poisson algebra.  $\square$

The above result shows how the restriction to forms defined on the leaves of the foliation does not change the bracket, i.e., for any two functions  $f, g \in C^\infty(\mathcal{X}_c)$ ,

$$\Lambda_c(df, dg) = \Lambda_c(df_{\mathcal{F}}, dg_{\mathcal{F}}),$$

where by  $df_{\mathcal{F}}$  we denote the exterior derivative along the leaves. Hence, the transformation

$$\hat{\alpha} \leftrightarrow \alpha$$

with respect to the effect on the Poisson bracket of forms, is a one-to-one relation.

But now, since all the mappings are linear, we can restrict ourselves to the linear functions on both spaces, and then it is trivial to see the following result.

THEOREM 5.2. *The tensors  $\Lambda_c$  and  $\hat{\Lambda}$  are the same Poisson tensor.*

This means that not only Poisson algebras, but also Poisson tensors defined on the constrained phase space and the dual of the Lie algebroid are identical. All these results provide a nice geometrical justification of the construction proposed in [14].

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