Modelling switching power converters as complementarity systems

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Abstract— **Switched complementarity models of linear circuits with ideal diodes and/or ideal switches allow one to study well-posedness and stability issues for these circuits by employing the complementarity problems of the mathematical programming. In this paper, we demonstrate that other types of typical electronic switching elements such as transistors and thyristors can also be treated in the framework of switched complementarity systems. By employing complementarity methods, we establish well-posedness results for a fairly large class of switched circuits that encompass power converters.**

I. Introduction

Switched Complementarity Systems (SCS) have been recently proposed as a framework for modelling switched electrical networks that contain ideal diodes and/or ideal switches [1], [2]. As shown there, SCS framework has certain advantages in capturing the features of such networks in comparison to, for instance, hybrid automaton framework. Indeed, hybrid automaton models for those networks get too complicated as the number of switching elements increases whereas SCSs provide a compact way of describing the specific multimodal features. Power converters are among the most practical examples of multimodal electrical/electronic systems. Typically, a power converter contains linear circuit elements such as resistors, inductors, capacitors, gyrators, and transformers as well as switching elements such as diodes, transistor, and thyristors.

The main goals of the current paper are i) to demonstrate that electronic switches such as transistors and thyristors can also be put into SCS framework, and ii) to establish a well-posedness/stability theory for power converters by using this framework.

The structure of the paper is as follows. In the following section, we introduce the notational conventions. Section III discusses the behaviors of various electronic switches and explains how to model them within the complementarity framework. In Section IV, we show that the power converters can be modelled as the so-called switched complementarity systems. Well-posedness and stability of these systems are discussed in Section V and Section VI, respectively. Section VII closes the paper with the conclusions. For sake of simplicity we skip the proofs which are reported in [3].

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II. Notational Conventions

The following notational conventions will be in force.

Sets. The set of all real numbers is denoted by $\mathbb{R}$, complex numbers by $\mathbb{C}$. The notation $\mathbb{R}_-$ ($\mathbb{R}_+$) denotes the nonpositive (nonnegative) real numbers. The set of all locally square-integrable Lebesgue-measurable functions is denoted by $\mathcal{L}_2^{\mathrm{loc}}$.

A set $\mathcal{C} \subseteq \mathbb{R}^\ell$ is said to be a *cone* if $v \in \mathcal{C}$ implies that $\alpha v \in \mathcal{C}$ for all $\alpha \geqslant 0$. For any nonempty set $\mathcal{Q} \subseteq \mathbb{R}^\ell$, we define the *dual cone* as the set $\{w \in \mathbb{R}^\ell \mid w^T v \geqslant 0 \text{ for all } v \in \mathcal{Q}\}$. It will be denoted by $\mathcal{Q}^\star$.

Matrices. The set of $n \times m$ matrices is denoted by $\mathbb{R}^{n\times m}$. Transpose of a matrix A is denoted by A^T. If the matrices A and B have the same number of columns then $\mathrm{col}(A, B)$ stands for the matrix obtained by stacking A over B. A square matrix $M \in \mathbb{R}^{m\times m}$ is said to be *nonnegative definite* if $x^T M x \geqslant 0$ for all $x \in \mathbb{R}^m$. It is said to be *positive definite* if it is nonnegative definite and $x^T M x = 0$ implies that $x = 0$.

Other. The symbols $\vee$ and $\wedge$ denote the logical connectives 'or' and 'and', respectively. We write $x \perp y$ if the vectors x and y are orthogonal. For a function f, $f(T-)$ and $f(T+)$ denote $\lim_{t\uparrow T} f(t)$ and $\lim_{t\downarrow T} f(t)$, respectively.

III. Classification of Electronic Switches

Most power electronics converters consist of a combination of typical electronic switches such as diodes, thyristors, bipolar junction transistors, insulated gate bipolar transistors, and metal-oxide-semiconductor field-effect transistors with linear electrical components such as resistors, inductors, and capacitors. The characteristics of the electronic switches are typically idealized by neglecting the voltage drop in the conducting phase and the leakage current in the blocking phase [4], [5]. These equivalent models allow one to simply obtain hybrid or averaged models of the converters, which are suitable for simulation and control design purposes. The characteristics and behaviors of idealized electronic switches and their complementarity classification are summarized below.

A. Ideal switches (IS)

Let $S_{IS} \in \{\text{ON}, \text{OFF}\}$ denote the "discrete state" of the switch, v_{IS} the voltage across the switch, and i_{IS} the current through the switch. The behavior of an ideal switch can be given by

$$\begin{aligned}&(S_{IS} = \text{ON} \wedge v_{IS} = 0 \wedge i_{IS} \in \mathbb{R})\\ &\qquad \vee (S_{IS} = \text{OFF} \wedge v_{IS} \in \mathbb{R} \wedge i_{IS} = 0).\end{aligned} \tag{1}$$

Figure 1 depicts the voltage-current characteristic of an ideal switch.

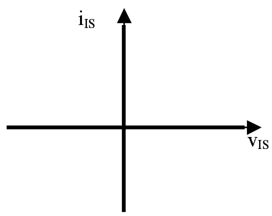

Fig. 1. Voltage-current characteristic of an ideal switch.

B. Ideal diodes (D)

Let v_D the voltage across the diode, and i_D the current through the diode. The behavior of an ideal diode can be given by

$$(v_D = 0 \wedge i_D \in \mathbb{R}_+) \vee (v_D \in \mathbb{R}_+ \wedge i_D = 0). \tag{2}$$

Figure 2 depicts the voltage-current characteristic of an ideal diode.

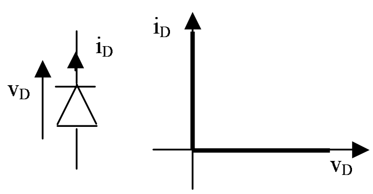

Fig. 2. Ideal diode and its voltage-current characteristic.

C. Electronic switches (ES)

Let $S_{ES} \in \{\text{ON}, \text{OFF}\}$ denote the "discrete state" of the electronic switch, v_{ES} the voltage across the switch, and i_{ES} the current through the switch. The behavior of an ideal electronic switch can be given by

$$\begin{aligned}&(S_{ES} = \text{ON} \wedge v_{ES} = 0 \wedge i_{ES} \in \mathbb{R}_+)\\ &\qquad \vee (S_{ES} = \text{OFF} \wedge v_{ES} \in \mathbb{R}_- \wedge i_{ES} = 0).\end{aligned} \tag{3}$$

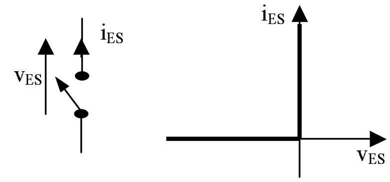

Fig. 3. Electronic switch and the corresponding characteristic.

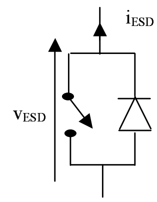

Fig. 4. Parallel connection of an electronic switch and a diode.

Figure 3 depicts the voltage-current characteristic of an electronic switch. Another interesting and widely used switch consists of a parallel connection of an ES with a D. Figure 4 depicts this type of switch which will be denoted by ESD. If $S_{ESD} \in \{\text{ON}, \text{OFF}\}$ denotes the "discrete state" of the switch, v_{ESD} the voltage across the switch, and i_{ESD} the current through the switch, then the behavior of the ESD can be given by

$$\begin{aligned}&(S_{ESD} = \text{ON} \wedge v_{ESD} = 0 \wedge i_{ES} \in \mathbb{R})\\ &\quad \vee (S_{ESD} = \text{OFF} \wedge v_{ESD} = 0 \wedge i_{ESD} \in \mathbb{R}_+)\\ &\quad \vee (S_{ESD} = \text{OFF} \wedge v_{ESD} = \mathbb{R}_+ \wedge i_{ESD} = 0).\end{aligned} \tag{4}$$

D. Thyristors (T)

Let $S_T \in \{\text{ON}, \text{OFF}\}$ denote the "discrete state" of the thyristor, v_T the voltage across the switch, and i_T the current through the switch. The behavior of an ideal thyristor can be given by

$$\begin{aligned}&(S_T = \text{ON} \wedge v_T = 0 \wedge i_T \in \mathbb{R}_+)\\ &\qquad \vee (S_T = \text{ON} \wedge v_T \in \mathbb{R}_+ \wedge i_T = 0)\\ &\qquad \vee (S_T = \text{OFF} \wedge v_T \in \mathbb{R} \wedge i_T = 0).\end{aligned} \tag{5}$$

Figure 5 depicts the voltage-current characteristic of an ideal thyristor.

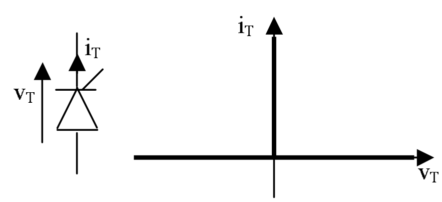

Fig. 5. Thyristor and the corresponding characteristic.

E. Complementarity classification

In [2], it is shown that the IS and D type of switching elements can be put into the so-called cone complementarity framework. Now, define the cones

$$\mathcal{K}_{-1} = \{0\} \quad \mathcal{K}_0 = \mathbb{R}_+ \quad \mathcal{K}_1 = \mathbb{R}. \tag{6}$$

Note that the corresponding dual cones are given by

$$\mathcal{K}_{-1}^{\star} = \mathbb{R} \quad \mathcal{K}_{0}^{\star} = \mathbb{R}_{+} \quad \mathcal{K}_{1}^{\star} = \{0\}. \tag{7}$$

It can be verified that the relations (1) and (2) can be rewritten as

$$\mathcal{K}_{\pi_{IS}} \ni v_{IS} \perp i_{IS} \in \mathcal{K}_{\pi_{IS}}^{\star} \tag{8}$$
$$\mathcal{K}_{\pi_{D}} \ni v_{D} \perp i_{D} \in \mathcal{K}_{\pi_{D}}^{\star} \tag{9}$$

where $\pi_{IS} = -1$ if $S_{IS} =$ ON, $\pi_{IS} = 1$ if $S_{IS} =$ OFF, and $\pi_D = 0$.

One of the novelties of this paper is to show that ESD and T type of switches can be reformulated in terms of cone complementarity variables. Indeed, the relations (4) and (5) can be given in terms of cone complementarity variables as follows:

$$\mathcal{K}_{\pi_{ESD}} \ni v_{ESD} \perp i_{ESD} \in \mathcal{K}_{\pi_{ESD}}^{\star} \tag{10}$$
$$\mathcal{K}_{\pi_{T}} \ni v_{T} \perp i_{T} \in \mathcal{K}_{\pi_{T}}^{\star} \tag{11}$$

Here, $\pi_{ESD} = -1$ if $S_{ESD} =$ ON, $\pi_{ESD} = 0$ if $S_{ESD} =$ OFF, $\pi_T = 0$ if $S_T =$ ON, and $\pi_T = -1$ if $S_T =$ OFF.

The ES type switches, however, cannot be treated in the cone complementarity framework. It turns out that the cone complementarity framework can be generalized in such a way that one can deal with ES type switches in this generalized framework. For the sake of brevity, we will confine ourselves with the cone complementarity framework and exclude ES type switches from our treatment.

IV. COMPLEMENTARITY MODEL OF POWER CONVERTERS

Consider a power converter that consists of linear RLCTG[1] elements, voltage/current sources, and switches of type D, IS, ESD, and T. By extracting the switches, we get an m-port electrical circuit where m is the number of switches. Under suitable conditions (for instance when the network does not contain all-capacitor loops or nodes with the only elements incident being inductors, see [6, Ch. 4] for more details), this m-port circuit, which consists of RLCTG elements and external sources, is described by the state-space system

$$\dot{x}(t) = Ax(t) + Bz(t) + Eu(t) \tag{12a}$$
$$w(t) = Cx(t) + Dz(t) + Fu(t) \tag{12b}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^k$, and $(z, w) \in \mathbb{R}^{m+m}$. Here x denotes the state (typically the voltage across the capacitors and the currents through the inductors), u denotes the external sources, and (z_i, w_i) denotes either the voltage-current of the current-voltage pairs of the ith port.

As it is well-known from circuit theory, the matrix quadruple (A, B, C, D) is not arbitrary but has a certain structure. Indeed, the circuit should satisfy passivity property when there is no external input present.

[1] Here R stands for resistor, L for inductor, C for capacitor, G for gyrator, and T for transformer

Definition IV.1 [7] A linear system $\Sigma(A, B, C, D)$ given by

$$\dot{x}(t) = Ax(t) + Bz(t) \tag{13a}$$
$$w(t) = Cx(t) + Dz(t) \tag{13b}$$

is called *passive*, or *dissipative* with respect to the supply rate $z^T w$, if there exists a nonnegative function $V : \mathbb{R}^n \to \mathbb{R}_+$ such that for all $t_0 \leqslant t_1$ and all trajectories (u, x, y) of the system (13) the following inequality holds:

$$V(x(t_0)) + \int_{t_0}^{t_1} z^T(t) w(t) dt \geqslant V(x(t_1)).$$

If exists the function V is called a *storage function*.

The following proposition is one of the classical results of systems and control theory.

Proposition IV.2 [7] *Consider a system $\Sigma(A, B, C, D)$ in which (A, B, C) is a minimal representation. The following statements are equivalent.*

- *$\Sigma(A, B, C, D)$ is passive.*
- *The transfer matrix $G(s) := D + C(sI - A)^{-1}B$ is* positive real*, i.e., $x^*[G(\lambda) + G^*(\lambda)]x \geqslant 0$ for all complex vectors x and all $\lambda \in \mathbb{C}$ such that Re$(\lambda) > 0$ and λ is not an eigenvalue of A.*
- *The matrix inequalities*
$$\begin{bmatrix} A^T K + KA & KB - C^T \\ B^T K - C & -(D + D^T) \end{bmatrix} \leqslant 0 \tag{14}$$
and $K = K^T \geqslant 0$ have a solution K.

Moreover, in case $\Sigma(A, B, C, D)$ is passive, all solutions K to the linear matrix inequalities (14) *are positive definite and K is a solution to* (14) *if and only if $V(x) = \frac{1}{2} x^T K x$ defines a storage function of the system $\Sigma(A, B, C, D)$.*

At this point, we bring switches into the picture. Consider the ith port, i.e. ith switch. The pair (z_i, w_i) corresponds to either voltage-current or current-voltage pair of the ith switch as mentioned above. In view of the relations (8), (9), (10) and (11) one has

$$\mathcal{K}_{\pi_i(t)} \ni z_i(t) \perp w_i(t) \in \mathcal{K}_{\pi_i(t)}^{\star} \tag{15}$$

by choosing the switching function π_i accordingly to the type of the ith switch.

Consequently, the overall power converter can be represented by a switched complementarity system of the form

$$\dot{x}(t) = Ax(t) + Bz(t) + Eu(t) \tag{16a}$$
$$w(t) = Cx(t) + Dz(t) + Fu(t) \tag{16b}$$
$$\mathcal{C}_{\pi(t)} \ni z(t) \perp w(t) \in \mathcal{C}_{\pi(t)}^{\star} \tag{16c}$$

where $(z, x, w) \in \mathbb{R}^{m+n+m}$, $\pi : \mathbb{R}_+ \to \{-1, 0, 1\}^m$ is the *switching function*, and

$$\mathcal{C}_{\pi(t)} = \mathcal{K}_{\pi_1(t)} \times \mathcal{K}_{\pi_2(t)} \times \cdots \times \mathcal{K}_{\pi_m(t)} \tag{17}$$

with

$$\mathcal{K}_{-1} = \{0\} \quad \mathcal{K}_0 = \{\mathbb{R}_+\} \quad \mathcal{K}_1 = \{\mathbb{R}\}. \tag{18}$$

An interesting illustrative example is represented by the so-called full bridge power converter represented in Figure 6. In such converter the elements indicated as switches in the figure characterize the type of power converter:

- if the switches are D and u is a sinusoidal voltage the circuit is an AC/DC converter;
- if the switches are T the circuit is a controlled AC/DC converter in which the average value of the output voltage can be controlled by selecting the conduction of the thyristors;
- if the switches are ESD than the full bridge converter can operate either as a DC/DC converter, as a DC/AC converter or as an AC/DC converter depending on the fact that the input voltage u is constant or sinusoidal and on the modulation imposed to the switches.

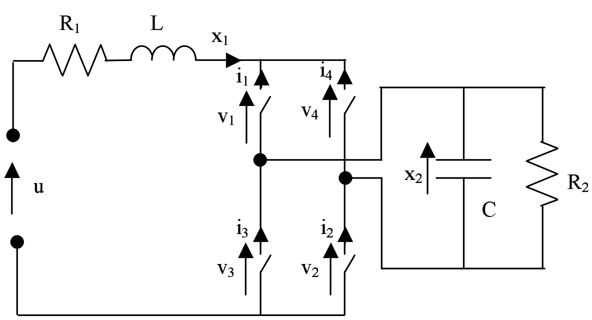

Fig. 6. Full bridge power converter.

By considering as state variables the current in the inductance and the voltage on the capacitor, the full bridge can be modelled by the linear time invariant state space equations:

$$\begin{aligned} L\dot{x}_1(t) &= -R_1 x_1(t) + u(t) - v_1(t) - v_3(t), \\ C\dot{x}_2(t) &= -\frac{1}{R_2} x_2(t) - i_2(t) + i_4(t). \end{aligned}$$

Moreover one can also write that

$$\begin{aligned} x_1(t) &= -i_1(t) - i_4(t) = -i_2(t) - i_3(t) \\ x_2(t) &= -v_2(t) + v_3(t) = -v_1(t) + v_4(t) \end{aligned}$$

By choosing $z_1 = v_1$, $z_2 = v_3$, $z_3 = i_2$, $z_4 = i_4$ and $w_1 = i_1$, $w_2 = i_3$, $w_3 = v_2$, $w_4 = v_4$, the full bridge model can be represented in the form (16) with

$$A = \begin{bmatrix} -\frac{R_1}{L} & 0 \\ 0 & -\frac{1}{R_2 C} \end{bmatrix} \tag{19}$$

$$B = \begin{bmatrix} -\frac{1}{L} & -\frac{1}{L} & 0 & 0 \\ 0 & 0 & -\frac{1}{C} & \frac{1}{C} \end{bmatrix} \tag{20}$$

$$E = \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} \quad C = \begin{bmatrix} -1 & 0 \\ -1 & 0 \\ 0 & -1 \\ 0 & 1 \end{bmatrix} \tag{21}$$

$$D = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad F = 0. \tag{22}$$

V. Well-posedness

In this section, our aim is to study well-posedness (in the sense of existence and uniqueness of solutions) of the switched complementarity systems (16).

Let us consider only the constant switching functions for the moment, i.e. $\pi(t) = \bar{\pi} \in \{-1, 0, 1\}^m$ for all t. Define $\bar{\mathcal{C}} := \mathcal{C}_{\bar{\pi}}$. Then, we have the following cone complementarity system

$$\dot{x}(t) = Ax(t) + Bz(t) + Eu(t) \tag{23a}$$

$$w(t) = Cx(t) + Dz(t) + Fu(t) \tag{23b}$$

$$\bar{\mathcal{C}} \ni z(t) \perp w(t) \in \bar{\mathcal{C}}^\star. \tag{23c}$$

Some nomenclature is in order. Let $\mathcal{C} \subseteq \mathbb{R}^m$ be a cone. Given an m vector q and an $m \times m$ matrix M, the linear cone complementarity problem LCCP($\mathcal{C}, q, M$) is to find an m vector z such that

$$z \in \mathcal{C} \tag{24a}$$

$$w := q + Mz \in \mathcal{C}^\star \tag{24b}$$

$$z^T w = 0. \tag{24c}$$

If such a vector z exists, we say that z *solves* (is a *solution of*) LCCP($\mathcal{C}, q, M$). The following proposition is an immediate consequence of [2, Theorem II.7].

Proposition V.1 *Suppose that M is nonnegative definite. Let $\mathcal{Q}_M := \{z \mid z \in \bar{\mathcal{C}},\ Mz \in \bar{\mathcal{C}}^\star,\ \text{and}\ z^T Mz = 0\}$. Then, the following statements are equivalent.*

1) *LCCP($\bar{\mathcal{C}}, q, M$) is solvable.*
2) $q \in \mathcal{Q}_M^\star$.

With these preparations, we are in a position to formulate a well-posedness result for systems (23).

Theorem V.2 *Consider the cone complementarity system* (23). *Suppose that $\Sigma(A, B, C, D)$ is passive and the triple (A, B, C) is minimal. Let*

$$\mathcal{Q}_D := \{z \mid z \in \bar{\mathcal{C}},\ Dz \in \bar{\mathcal{C}}^\star,\ \text{and}\ z^T Dz = 0\}.$$

Then, the following statements are equivalent.

1) *For a given real-analytic input u and an initial state x_0, there exist an absolutely continuous state trajectory x with $x(0) = x_0$ and a pair $(z, w) \in \mathcal{L}_2^{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^{m+m})$ such that* (23) *are satisfied for almost all $t \geqslant 0$.*
2) $Cx_0 + Fu(0) \in \mathcal{Q}_D^\star$.

Moreover, if such a triple (z, x, w) exists then the state trajectory x and $\mathrm{col}(B, D + D^T)z$ are unique.

The uniqueness of $\mathrm{col}(B, D+D^T)z$ instead of that of the whole vector z which one might expect, has a nice physical interpretation from a power electronics point of view. For instance, by considering the full bridge example reported in the previous section, it is simple to see that $\mathrm{col}(B, D+D^T)$ is not full column rank. This corresponds to the fact that during the so called free wheeling operating conditions (all four switches are conducting) the current sharing among the switches is not unique. It should be noticed that most power electronic converters present such kind of operating conditions during their ordinary behavior.

Above theorem considers only a *constant* switching function. Next step is to extend this theorem to arbitrary switching functions and more general inputs functions. Arbitrary switching and/or discontinuous inputs introduce the possibility of discontinuous state trajectories. An immediate question is how to define a jump in the state that is triggered by a switching and/or discontinuity of the input. A natural way of introducing a jump rule can be obtained via the stored energy in the system. To formalize this idea, let

$$\mathcal{Q}_D(\mathcal{C}) := \{z \mid z \in \mathcal{C},\, Dz \in \mathcal{C}^\star, \text{ and } z^T Dz = 0\}.$$

Suppose that T is an isolated switching time or a discontinuity point of the input u. Consider the minimization problem

$$\text{minimize } \tfrac{1}{2}(\bar{x} - x(T-))^T K(\bar{x} - x(T-)) \tag{25a}$$
$$\text{subject to } C\bar{x} + Fu(T+) \in \mathcal{Q}_D^\star(\mathcal{C}_{\pi(T+)}) \tag{25b}$$

where $\xi \mapsto \frac{1}{2}\xi^T K\xi$ is a storage function for the system $\Sigma(A, B, C, D)$. Passivity of the system allows us to prove that the above minimization problem has a unique solution for any $x(T-)$, $u(T+)$, and $\pi(T+)$. Note that the condition (25b) implies that there exists a solution for the initial state $\bar{x}$ and the input $t \mapsto u(t - T+)$ after the time instant T due to Theorem V.2. Our jump rule defines the solution $\bar{x}$ of the above problem as the state at $T+$. Note that $\bar{x}$ is the closest state (in the metric defined by the storage function) to the state before the jump among the states for which there exists a solution after the jump.

In the sequel of the paper, we consider switching functions $\pi : \mathbb{R}_+ \to \{-1, 0, 1\}^m$ that have only isolated discontinuities. Such switching functions will be called *admissible switching function*. The set Γ_f is defined as the union of the discontinuity points of a function f and zero. We say that an input u is *admissible* if it is real-analytic for all $0 < t \notin \Gamma_u$ and Γ_u is a set of isolated points.

Theorem V.3 *Consider the switched complementarity system* (16)*. Suppose that $\Sigma(A, B, C, D)$ is passive and the triple (A, B, C) is minimal. Let K be such that $\xi \mapsto \frac{1}{2}\xi^T K\xi$ is a storage function for the system $\Sigma(A, B, C, D)$. Also let an admissible input u, an initial state x_0, and an admissible switching function π be given. Define $\Gamma = \Gamma_u \cup \Gamma_\pi$. Then, there exist a state trajectory x and a pair $(z, w) \in \mathcal{L}_2^{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^{m+m})$ such that*

1) *$x(0-) = x_0$ and x is absolutely continuous at all points $0 < t \notin \Gamma$.*
2) (23) *are satisfied for almost all $0 < t \notin \Gamma$.*
3) *For times $0 \leqslant t \in \Gamma$, $x(t+)$ is the unique minimum of*

$$\text{minimize } \tfrac{1}{2}(\bar{x} - x(t-))^T K(\bar{x} - x(t-)) \tag{26a}$$
$$\text{subject to } C\bar{x} + Fu(t+) \in \mathcal{Q}_D^\star(\mathcal{C}_{\pi(t+)}). \tag{26b}$$

Moreover, if such a triple (z, x, w) exists then the state trajectory x and $\mathrm{col}(B, D + D^T)z$ are unique.

Most typically, the states of a power converter are the voltages across the capacitors and the currents through the inductors. A jump in the state, therefore, can be modelled by an impulse in the current through a capacitor and an impulse in the voltage across an inductor, respectively. The next lemma characterizes the jumps in the state variable from this perspective.

Lemma V.4 *Let $\bar{x}$ be the solution of the minimization problem* (26)*. Then, there exists a unique $\bar{z} \in \mathcal{Q}(\mathcal{C}_{\pi(t+)})$ such that $\bar{x} = x(t-) + B\bar{z}$.*

As a consequence of the above lemma, one can think of the jump at time instant t as a result of an impulse $\bar{z}\delta_t$ in the z variable. Here, δ_t is the Dirac distribution that is supported at the time instant t. Alternative characterizations of the jump multiplier $\bar{z}$ can be given as follows.

Theorem V.5 *Suppose that $t \in \Gamma$ and $x(t+)$ is the unique solution of the minimization problem* (26)*. Let $\bar{z} \in \mathcal{Q}(\mathcal{C}_{\pi(t+)})$ be such that $\bar{x} = x(t-) + B\bar{z}$. Define $\mathcal{Q} := \mathcal{Q}_D(\mathcal{C}_{\pi(t+)})$. Then the following characterizations can be obtained for $\bar{z}$.*

1) *The jump multiplier $\bar{z}$ is the unique solution to*
$$\mathcal{Q} \ni v \perp C(x(t-) + Bv) \in \mathcal{Q}^\star \tag{27}$$
2) *The cone $\mathcal{Q}$ is equal to pos $N := \{N\lambda \mid \lambda \geqslant 0\}$ and $\mathcal{Q}^\star = \{v \mid N^T v \geqslant 0\}$ for some real matrix N. The re-initialized state $x(t+)$ is equal to $x(t-) + BN\bar{\lambda}$ and $\bar{z} = N\bar{\lambda}$ where $\bar{\lambda}$ is a solution of the following ordinary LCP.*
$$0 \leqslant \lambda \perp (N^T Cx(t-) + N^T CBN\lambda) \geqslant 0. \tag{28}$$
3) *The jump multiplier $\bar{z}$ is the unique minimizer of*
$$\text{minimize } \tfrac{1}{2}(x(t-) + Bv)^T K(x(t-) + Bv) \tag{29}$$
$$\text{subject to } v \in \mathcal{Q} \tag{30}$$

VI. STABILITY

In this section we discuss the stability of switched complementarity systems (16) under a passivity assumption. The Lyapunov stability of hybrid and switched systems in general has already received considerable attention [8]–[13]. From now on, we denote the unique global trajectory for a given switch function π and initial state x_0 of a switched complementarity system by $(u^{\pi,x_0}, x^{\pi,x_0}, y^{\pi,x_0})$. For the study of stability we consider the source-free case.

Definition VI.1 (Equilibrium point) A state $\bar{x}$ is an equilibrium point of the switched complementarity system (16), if for all admissible switching functions π $x^{\pi,\bar{x}}(t) = \bar{x}$ for almost all $t \geqslant 0$ and all π, i.e. for all solutions starting in $\bar{x}$ the state stays in $\bar{x}$.

Note that in an equilibrium point $\dot{x} = 0$, which leads in a simple way to the following characterization of equilibria of a switched complementarity system.

Lemma VI.2 *A state $\bar{x}$ is an equilibrium point of the switched complementarity system* (16)*, if and only if for all $\bar{\pi} \in \{-1, 0, 1\}^m$ there exist $z^{\bar{\pi}} \in \mathbb{R}^m$ and $w^{\bar{\pi}} \in \mathbb{R}^m$ satisfying*

$$0 = A\bar{x} + Bz^{\bar{\pi}} \tag{31a}$$

$$w^{\bar{\pi}} = C\bar{x} + Dz^{\bar{\pi}} \tag{31b}$$

$$\mathcal{C}_{\bar{\pi}} \ni z^{\bar{\pi}} \perp w^{\bar{\pi}} \in \mathcal{C}^{\star}_{\bar{\pi}}. \tag{31c}$$

From this lemma it follows that $\bar{x} = 0$ is an equilibrium. Note that if A is invertible we get $\bar{x} = -A^{-1}Bz^{\bar{\pi}}$ and

$$\mathcal{C}_{\bar{\pi}} \ni z^{\bar{\pi}} \perp [-CA^{-1}B + D]z^{\bar{\pi}} \in \mathcal{C}^{\star}_{\bar{\pi}}$$

which is a homogeneous LCP over a cone.

Definition VI.3 Let $\bar{x}$ be an equilibrium point of the switched complementarity system (16) and d denote a metric on $\mathbb{R}^n$.

1) $\bar{x}$ is called *stable*, if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $d(x^{\pi,x_0}(t), \bar{x}) < \varepsilon$ for almost all $t \geqslant 0$ whenever $d(x_0, \bar{x}) < \delta$ and π being an admissible switching function.
2) $\bar{x}$ is called asymptotically stable if $\bar{x}$ is stable and there exists an $\eta > 0$ such that $\lim_{t\to\infty} d(x^{\pi,x_0}(t), \bar{x}) = 0$ whenever $d(x_0, \bar{x}) < \delta$ and π being an admissible switching function. By $\lim_{t\to\infty} d(x^{\pi,x_0}(t), \bar{x}) = 0$ we mean that for every $\varepsilon > 0$ there exists a t_ε such that $d(x^{\pi,x_0}(t), \sigma), \bar{x}) < \varepsilon$ whenever $t \geqslant t_\varepsilon$.

Theorem VI.4 *Consider the switched complementarity system* (16)*. Suppose that $\Sigma(A, B, C, D)$ is passive and the triple (A, B, C) is minimal. Let K be such that $\xi \mapsto \frac{1}{2}\xi^T K\xi$ is a storage function for the system $\Sigma(A, B, C, D)$. The system* (16) *has only stable equilibrium points $\bar{x}$. Moreover, if $A^TK + KA < 0$ is invertible $\bar{x} = 0$ is the only equilibrium point, which is asymptotically stable.*

VII. CONCLUSIONS AND FUTURE WORKS

We have shown that the cone complementarity framework can be successfully used to establish well-posedness and stability results for a wide class of power electronics converters. By exploiting the presented complementarity classification of the typical electronic switches, the existing results on switched complementarity systems have been extended to circuits that contain thyristors and switches consisting of a parallel connection of a diode and an electronic switch.

Future work will deal with the analysis in the complementarity framework of circuits which contain also other types of electronic switches and with the stability of power converters in the presence of nonzero external inputs.

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