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Characterization of Well-Posedness of Piecewise-Linear Systems

Jun-ichi Imura and Arjan van der Schaft

***Abstract*—One of the basic issues in the study of hybrid systems is the well-posedness (existence and uniqueness of solutions) problem of discontinuous dynamical systems. The paper addresses this problem for a class of piecewise-linear discontinuous systems under the definition of solutions of Carathéodory. The concepts of jump solutions or of sliding modes are not considered here. In this sense, the problem to be discussed is one of the most basic problems in the study of well-posedness for discontinuous dynamical systems. First, we derive necessary and sufficient conditions for bimodal systems to be well-posed, in terms of an analysis based on lexicographic inequalities and the smooth continuation property of solutions. Next, its extensions to the multi-modal case are discussed. As an application to switching control, in the case that two state feedback gains are switched according to a criterion depending on the state, we give a characterization of all admissible state feedback gains for which the closed loop system remains well-posed.**

***Index Terms*—Discontinuous systems, hybrid systems, lexicographic inequalities, piecewise-linear systems, well-posedness.**

I. Introduction

VARIOUS approaches to modeling, analysis, and control synthesis of hybrid systems have been developed within the computer science community and the systems and control community, from different points of view (see, e.g., [1]–[6]). In the computer science community, as an extension of finite automata, several models of hybrid systems such as timed automata [7] and hybrid automata [8] have been proposed and some results on verification of their models have been obtained. In the control community, from the dynamical systems and control point of view, models of hybrid systems have been proposed (see e.g. [9], [10]), and several properties such as stability and controllability have been discussed; see [11] and [12] for controllability of switched systems and integrator hybrid systems, respectively, [13] and [14] for stability of general hybrid systems, and [15]–[17] for stability of piecewise-linear systems. One of the main concerns in these researches is how to define and analyze various kinds of properties of hybrid systems with discontinuous changes of vector fields and jumps of solutions (i.e., autonomous switchings and autonomous jumps in the terminology of [10]). However, there are still few results on the basic problem of uniqueness of solutions of piecewise-linear discontinuous systems, while the existing standard theory of discontinuous dynamical systems is not quite satisfactory in spite of the fact that it is crucial for various developments of hybrid systems.

On the other hand, as an approach to modeling of hybrid systems, there is a new attempt in [18] and [19] to generalize in a natural manner dynamical properties of physical systems with jump phenomena which occur between unconstrained motion and constrained motion, such as the collision of a mass to an inelastic wall, so as to develop a framework modeling a class of hybrid systems. This framework is called the complementarity modeling (the corresponding system is called the complementarity system), which can describe several kinds of hybrid systems including electrical network with diodes and relay type systems as well as mechanical systems with unilateral constraints. Such an approach provides a natural and intuitive interpretation of jump phenomena in hybrid systems and makes the analysis relatively easier. In fact, as the first result of the analysis in this line, several algebraic and checkable conditions for well-posedness (existence and uniqueness of solutions) of such systems have been derived in [18]–[22]..

When hybrid (discontinuous) systems are considered from the above physical viewpoint, there also exist physical phenomena such as the collision to an elastic wall, which leads to systems with discontinuous vector fields, but not exhibiting jumps. Does there exist a common algebraic structure in the discontinuous vector field of such systems? Can we extend this to a general framework from the mathematical point of view? As far as we know, however, such questions have not been addressed, although an abstract condition can be found in the well-known book by Filippov [23]. When solutions without jumps are considered, there are, roughly speaking, two kinds of definitions of solutions, that is, Carathéodory's definition and Filippov's definition. The latter yields the concept of a sliding mode. In the case of physical systems such as the collision to an elastic wall, on the other hand, the solution belongs to the former, although we need to extend Carathéodory's definition, in a straightforward manner, to the case of discontinuous vector fields.

Besides from the viewpoint of a generalization of such physical systems, there are in addition the following three points we like to stress as a motivation to address the well-posedness problem in the sense of Carathéodory for discontinuous dynamical systems. First, this problem is a most fundamental one in the study of well-posedness for discontinuous dynamical systems. In other words, compared with the well-posedness problem including the concept of jump phenomena or a sliding mode, it

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is closest to the well-posedness problem in continuous dynamical systems. Therefore, as a first step to establish a theory of well-posedness of general hybrid systems, it will be very meaningful to clarify to what extent this basic problem can be analyzed. The second point is that it may be easier to analyze a system without jumps than with jumps. By representing a system with jumps as a limit of a system without jumps, we may obtain more results on the property of hybrid systems with jumps. A similar approach can be found in [24]–[27]. Third, in many examples of hybrid systems of practical interest, the solutions do not necessarily have jumps in the transition from one mode to the other mode, and also it may be desirable from the practical point of view that no sliding mode exists in closed loop control systems because of the resulting chattering behavior.

In this paper, we address the well-posedness problem in the sense of Carathéodory for the class of piecewise-linear discontinuous systems. We mainly concentrate on bimodal systems, and give several necessary and sufficient conditions for those systems to be well-posed, in terms of the analysis based on lexicographic inequalities and the smooth continuation property. Furthermore, some of the results obtained in the bimodal case will be extended to the case of two kinds of multi-modal systems. Finally, as an application of our result, we discuss the well-posedness problem of feedback control systems with two state feedback gains switched according to a criterion depending on the state. Recently, switching control schemes have attracted considerable attention in the control community (see e.g. [28]–[30]). As one of its basic results, we give a characterization of all admissible state feedback gains for which the corresponding closed loop system is well-posed.

The organization of this paper is as follows. In Section II, piecewise-linear discontinuous systems in the bimodal case are described, together with the definition of solutions of Carathéodory. Section III is devoted to some mathematical preliminaries on lexicographic inequalities and smooth continuation. We give our main results on the well-posedness of bimodal systems in Sections IV and V, and some extensions in Section VI. In Section VII, our results are applied to the well-posedness problem in switching control systems. Section VIII presents a brief summary and some topics for future research.

In the sequel, we will use the following notation: for lexicographic inequalities of $x \in \mathcal{R}^n$, if for some i, $x_j = 0$ $(j = 1, 2, \cdots, i-1)$, while $x_i > (<) 0$, we denote it by $x \succ (\prec) 0$. Furthermore, if $x = 0$ or $x \succ (\prec) 0$, we denote it $x \succeq (\preceq) 0$. We use the notation $*$ representing any fixed but unspecified number or matrix. Finally, I_n and $0_{m,n}$ denote the $n \times n$ identity matrix and the $m \times n$ zero matrix, respectively.

II. Piecewise-Linear Discontinuous Systems

In this section, we describe the basic form of bimodal systems to be studied here, and give a definition of well-posedness for these bimodal systems. Next, we discuss the relation between well-posedness and smooth continuation, and give an equivalent representation of bimodal systems, which will be important for further developments.

A. Description of Bimodal System and Definition of its Solution

Consider the system given by

$$\Sigma_O \begin{cases} \text{mode 1: } \dot{x} = Ax, & \text{if } y = Cx \geq 0 \\ \text{mode 2: } \dot{x} = Bx, & \text{if } y = Cx \leq 0 \end{cases} \tag{1}$$

where $x \in \mathcal{R}^n$, $y \in \mathcal{R}$, and A and B are $n \times n$ matrices (in general different). Since the two linear differential equations $\dot{x} = Ax$ and $\dot{x} = Bx$ are coupled by separating the region of $\mathcal{R}^n$ into two subregions, i.e., $y \geq 0$ and $y \leq 0$, the system Σ_O belongs to the class of piecewise-linear systems. Even when we consider the system Σ_O on any neighborhood of the origin, the argument below holds with some modification. However, for brevity, we consider the system to be defined on the whole R^n.

Remark 2.1: For the system Σ_O, it appears that x satisfying $Cx = 0$ is allowed in both modes. However as illustrated by the example (3), even when $x(T)$ satisfies $Cx(T) = 0$ at some T we will see that only one of both modes will be allowed by considering the behavior $x(t)$, $T \leq t \leq T + \varepsilon$, of the solution for small $\varepsilon > 0$.

At first, we define the well-posedness for the system Σ_O. In the system Σ_O, there may be a set (of measure 0) of points of time where the solution $x(t)$ is not differentiable, although we use hereafter $\dot{x}(t)$ in (1) for simplicity of notation. So formally, the system Σ_O is given by its integral form (which is called the Carathéodory equation):

$$x(t) = x_0 + \int_{t_0}^{t} f(x(\tau))\, d\tau \tag{2}$$

where $f(x)$ is the discontinuous vector field given by the right hand side of (1) and $x(t_0) = x_0$. Moreover if t is a time instant at which the system switches from one mode to another mode, t is said to be an event time. A point $\hat{t}$ is called a right (left)-accumulation point of event times [21], if there exists a sequence $\{t_i\}$ of event times such that $t_i < (>) t_{i+1}$ and $\lim_{i\to\infty} t_i = \hat{t} < \infty$. Then a solution of this system is defined as follows.

Definition 2.1: If $x(t)$ satisfies (2) and is absolutely continuous on $[t_0, t_1)$ for some $t_1 > t_0$, and there is no left-accumulation point of event times on $[t_0, t_1)$, then $x(t)$ is said to be a solution of Σ_O on $[t_0, t_1)$ in the sense of Carathéodory for the initial state $x(t_0)$.

The condition of disallowing the existence of left-accumulation points of event times will be used in the proof of Lemma 2.1. On the other hand, a solution with the right-accumulation points of event times, which is called a Zeno trajectory, is allowed in the above definition of solutions. The well-posedness for the system Σ_O is defined as follows (without loss of generality, we set $t_0 = 0$ hereafter).

Definition 2.2: The system Σ_O is said to be well-posed if there exists a unique solution of (1) on $[0, \infty)$ in the sense of Carathéodory for every initial state $x_0 \in \mathcal{R}^n$.

It is well-known that a sufficient condition for a system given by a first-order differential equation to be well-posed is that it satisfies a global Lipschitz condition. When we apply this to the system Σ_O, it follows that a sufficient condition for well-posedness is that there exists a K such that $B = A + KC$. Note that in this case the vector field is necessarily continuous in the state x.

Now, how about the case of discontinuous vector fields? Let us consider the following example shown in Fig. 1. The equations of motion of this system are given by

$$\begin{cases} \text{mode 1: } \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \\ \qquad \text{if } y = [1\ 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \geq 0 \\ \text{mode 2: } \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k & -d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \\ \qquad \text{if } y = [1\ 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq 0. \end{cases} \tag{3}$$

Suppose that the initial state satisfies $x_1(0) = 0$ and $x_2(0) > 0$, for which there may exist a solution in both modes. Then for some $\varepsilon > 0$, $x_1(t) > 0$ in mode 1 and $x_1(t) > 0$ in mode 2 holds for all $t \in (0, \varepsilon)$. Thus in this initial state, only mode 1 is active. In a similar way, only mode 2 is active for the case $x_1(0) = 0$ and $x_2(0) < 0$. Since a solution from the origin is the same in both modes, we see that this system is well-posed (without jumps and sliding modes), although the vector field is discontinuous in x when $d \neq 0$. On the other hand, we can easily find an example which is not well-posed, as shown below:

$$\begin{cases} \text{mode 1: } \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \\ \qquad \text{if } y = [1\ 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \geq 0 \\ \text{mode 2: } \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \\ \qquad \text{if } y = [1\ 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq 0. \end{cases} \tag{4}$$

In fact, if the initial state $x(0)$ satisfies $(x_1(0), x_2(0)) = (0, 1)$, then for some $\varepsilon > 0$, $x_1(t) > 0$ in mode 1 and $x_1(t) < 0$ in mode 2 holds for all $t \in (0, \varepsilon)$. Thus there exist two solutions for this initial state. Moreover, from the initial state $(x_1(0), x_2(0)) = (0, -1)$, there exists no solution because for some $\varepsilon > 0$, $x_1(t) < 0$ in mode 1 and $x_1(t) > 0$ in mode 2 holds for $t \in (0, \varepsilon)$.

Within the type of physical systems as given by (3), there will exist many systems with discontinuous vector fields, but which are well-posed. In the next sections, we will derive a necessary and sufficient condition for the well-posedness of the system Σ_O including such physical systems.

Remark 2.2: For the system (4), if the discontinuity of the vector field is regarded as a kind of relay-type and the solution concept by Filippov is applied, there exists a unique solution from the initial state $(x_1(0), x_2(0)) = (0, -1)$. In fact, the system Σ_O can be rewritten by $\dot{x} = (1/2)(1+u)Ax + (1/2)(1-u)Bx$, using a relay-type input $u = 1$ if $y > 0$, $u = -1$ if $y < 0$, and $u \in [-1, 1]$ if $y = 0$. Thus for $(x_1(0), x_2(0)) = (0, -1)$, there exists a unique solution which is called a sliding mode given by the equivalent control input $u = 0$. Certainly, Filippov's definition is very important from a practical viewpoint as well as from a mathematical viewpoint, and it is of interest to address the well-posedness problem including sliding modes. However, in some discontinuous systems with mechanical ON/OFF-type (not relay-type) switches, a sliding mode is not in general physically feasible, and even when a sliding mode theoretically exists as in the case of sliding mode control, it is desirable to avoid a sliding mode because of chattering phenomena. Thus in this paper, as the first step, we concentrate on the well-posedness problem in the sense of Definition 2.2 (See also Remark 4.2).

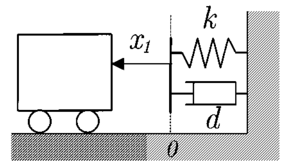

Fig. 1. Collision to an elastic wall.

Remark 2.3: When we consider the case of the collision to an inelastic wall by $d \to \infty$ in the example (3), a jump in the solution will occur. Such a system can be treated within the framework of linear complementarity systems [18]. Thus we conjecture that there exists some relation between linear complementarity systems and systems of the form (1). In other words, there may be a possibility to approximate the complementarity system, i.e., the discontinuous dynamical system with jumps, by a system without jumps given by (1). Some researchers have already studied the relation between the two solutions for a simple physical system as in Fig. 1 (see [27, Ch. 2]), and we plan to return to this issue in a future paper. Note also that the system of the form (1) can be expressed as a bilinear complementarity form [19].

B. Well-Posedness and Smooth Continuation

In this subsection, we will characterize the well-posedness by the concept of smooth continuation, which is defined as follows.

Definition 2.3 [18]: Let $\mathcal{S}$ be a subset of $\mathcal{R}^n$. If for the initial state x_0 there exists an $\varepsilon > 0$ such that $x(t) \in \mathcal{S}$ for all $t \in [0, \varepsilon]$, then we say that the system has the smooth continuation property at x_0 with respect to $\mathcal{S}$, or that smooth continuation is possible from x_0 with respect to $\mathcal{S}$. Moreover, if from all $x_0 \in \mathcal{S}$ smooth continuation is possible with respect to $\mathcal{S}$, then the system is said to have the smooth continuation property with respect to $\mathcal{S}$.

We have the following result.

Lemma 2.1: The following statements are equivalent.

i) The system Σ_O is well-posed.
ii) For the system Σ_O, from every initial state $x_0 \in \mathcal{R}^n$, smooth continuation is possible in only one of the two modes, in other words, with respect to either one of $\{x \in \mathcal{R}^n | Cx \geq 0\}$ or $\{x \in \mathcal{R}^n | Cx \leq 0\}$, except for the case that the solutions in both modes are the same in some time interval.

Proof: i)→ii). Since no finite left-accumulation point of event times exists by Definition 2.1, if a unique local solution $x(t)$ exists at x_0, smooth continuation exists in only one of the two modes, as long as the two solutions in both modes are not the same. This implies ii).

ii)→i). Since ii) implies that there exists a local unique solution from every initial state, we can make a successively connected solution. Then the solution $x(t)$ in (2) is given by

$$x(t) = e^{S_i(t-t_i)} e^{S_{i-1}(t_i - t_{i-1})} \cdots e^{S_0 t_1} x(0)$$

for all $t \in [t_i, t_{i+1})$, where $i \in \{0, 1, 2, \cdots\}$ is the switching number, t_j is an event time ($t_0 = 0$), and $S_j = A$ or B ($j = 0, 1, 2, \cdots, i$). Since there exists a positive real number a such that $\max\{\|e^{At}\|, \|e^{Bt}\|\} \le e^{at}$ for all $t \ge 0$, it follows that $\|x(t)\| \le e^{at}\|x(0)\|$ for all $t \in [t_i, t_{i+1})$ and all $i \in \{0, 1, 2, \cdots\}$. Note here that there exists a unique solution for all $t \ge t_\infty$ even when $t_\infty < \infty$, i.e., a finite right-accumulation point of event times exists. In fact, $\{x(t_i)\}$ has a well-defined limit at t_∞ since $x(t)$ is uniformly continuous on (t_0, t_∞) because of $x(t) = e^{S_i(t-t_i)} x(t_i)$, and from the state $x(t_\infty)$ a unique solution exists again. Thus we have $x \in \mathcal{L}_{\infty e}$ (extended $\mathcal{L}_\infty$ space). Furthermore, since $f(x) \in \mathcal{L}_{1e}$ [with $f(x)$ defined by (2)] if $x \in \mathcal{L}_{\infty e}$, it follows from the theory of Lebesgue integrals that the solution given by (2) is absolutely continuous on any interval of $\mathcal{R}$.

Finally, if ii) holds, no finite left-accumulation points of event times in $\{t_i\}$ exists. In fact, if a finite left-accumulation point of event times exists, then $x(t)$ has a well-defined limit at that point, which is similar to the case of the right-accumulation points, and smooth continuation from that state is not possible in any mode. This is inconsistent with ii). Therefore there exists a unique solution $x(t)$ on $[0, \infty)$ for every initial state $x_0 \in \mathcal{R}^n$, which leads to i). □

By this lemma, we only have to focus on whether or not smooth continuation is possible from every initial state in order to show the well-posedness of the system Σ_O.

Remark 2.4: Note that ii) in Lemma 2.1 is equivalent to the following statement:

iii) for every initial state $x_0 \in \mathcal{R}^n$, there exists an $\varepsilon > 0$ such that a unique solution $x(t)$ of Σ_O exists on $[0, \varepsilon)$, under the assumption that there are no left-accumulation points of event times.

In this sense, there is a slight difference between the smooth continuation property and the existence of local unique solutions.

Remark 2.5: After Section V, we will consider other types of discontinuous systems such as multi-modal systems. For all these systems, Definition 2.1 and 2.2 can be straightforwardly extended and Lemma 2.1 and 2.2 also hold for these systems.

C. Equivalent Representation of the Bimodal System Σ_O

For the system Σ_O, define the following row-full rank matrices:

$$T_A \triangleq \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{h-1} \end{bmatrix}, \quad T_B \triangleq \begin{bmatrix} C \\ CB \\ \vdots \\ CB^{k-1} \end{bmatrix} \tag{6}$$

where h and k are the observability indexes of the pairs (C, A) and (C, B), respectively . In addition, let $\mathcal{S}_A^+$, $\mathcal{S}_A^-$, $\mathcal{S}_B^+$, and $\mathcal{S}_B^-$ be sets defined by

$$\mathcal{S}_N^+ \triangleq \{x \in \mathcal{R}^n | T_N x \succeq 0\}, \quad \mathcal{S}_N^- \triangleq \{x \in \mathcal{R}^n | T_N x \preceq 0\} \tag{7}$$

for $N = A, B$, using the lexicographic inequalities defined in the end of Section I. Then noting that $T_A x = [y, \dot{y}, \cdots, y^{(h-1)}]^T$ for the system $\dot{x} = Ax$ and $T_B x = [y, \dot{y}, \cdots, y^{(k-1)}]^T$ for the system $\dot{x} = Bx$, we introduce the system given by

$$\Sigma_{AB} \begin{cases} \text{mode 1: } \dot{x} = Ax, & \text{if } x \in \mathcal{S}_A^+ \\ \text{mode 2: } \dot{x} = Bx, & \text{if } x \in \mathcal{S}_B^-. \end{cases} \tag{8}$$

We call T_A and T_B the rule (or observability) matrices of the system Σ_{AB}. The well-posedness for the system Σ_{AB} is defined similar to Definition 2.2. The following result shows that the system Σ_O is well-posed if and only if the system Σ_{AB} is well-posed.

Lemma 2.3: The system Σ_{AB} is equivalent to the original system Σ_O, i.e., both systems have the same solutions.

Proof: If a solution $x(t)$ of Σ_O exists in mode 1 on some time interval, i.e., for some $\varepsilon > 0$, $y(t) = Cx(t) \ge 0$ on $[\tau, \tau + \varepsilon)$ for $\dot{x} = Ax$, then a solution satisfies either one of $y(t) > 0$, $y(t) = 0$ and $\dot{y}(t) > 0$, $y(t) = \dot{y}(t) = 0$ and $\ddot{y}(t) > 0, \cdots$ $y(t) = \dot{y}(t) = \cdots = y^{(h-2)}(t) = 0$ and $y^{(h-1)}(t) \ge 0$ for each $t \in [\tau, \tau + \varepsilon)$. Using the lexicographic inequality notations, this implies $T_A x(t) \succeq 0$ holds on $[\tau, \tau + \varepsilon)$. Conversely, suppose that a solution $x(t)$ of $\dot{x} = Ax$ satisfies $T_A x(t) \succeq 0$ on some time interval. Then if $T_A x(t) \succ 0$ on that interval, $y(t) \ge 0$ holds on that interval. On the other hand, if $T_A x(T) = 0$ for some T, the definition of the observability index implies that $y(t) \equiv 0$ for $t \ge T$. The case of $\dot{x} = Bx$ is similar. Thus a solution in modes 1 and 2 of Σ_O is equivalent to a solution in modes 1 and 2 of Σ_{AB}, respectively, which implies that both systems have the same solutions. □

This lemma implies that the solution does not exist in mode 1 from the initial state $x(0)$ satisfying, for example, $y(0) = 0$ and $\dot{y}(0) < 0$, although it is included in mode 1 at $t = 0$. In other words, we only have to consider the regions $\mathcal{S}_A^+$ and $\mathcal{S}_B^-$ which express the sets of all the initial states from which smooth continuation is possible in mode 1 and 2, respectively.

III. Preliminaries on Lexicographic Inequalities and Smooth Continuation

In this section, as a preparation, we give mathematical preliminaries on lexicographic inequalities and smooth continuation for solutions of linear systems. Most of the results obtained in this section will play a central role in the study of well-posedness in the next sections.

First of all, we define the following classes of matrices, which is used throughout the paper.

Definition 3.1: Let $\mathcal{L}^n$ be the set of $n \times n$ lower-triangular matrices. In addition, let $\mathcal{L}_+^n$ be the set of elements in $\mathcal{L}^n$ with all diagonal elements positive.

Definition 3.2: Let $\mathcal{G}_0^n$ be the set defined by

$$\mathcal{G}_0^n \triangleq \left\{ \Gamma \in \mathcal{R}^{n\times n} \middle| \Gamma = \begin{bmatrix} * & \gamma_{12} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & \gamma_{n-1,n} \\ * & \cdots & \cdots & \cdots & * \end{bmatrix}, \gamma_{i,i+1} \geq 0,\ i = 1, 2, \cdots, n-1 \right\}$$

where γ_{ij} is the (i, j) element of the matrix Γ. In addition, let $\mathcal{G}_+^n$ be the set of elements in $\mathcal{G}_0^n$ with all the $(i, i+1)$ elements $\gamma_{i,i+1}$ positive.

A. Lemmas on Lexicographic Inequalities

First we give some lemmas on lexicographic inequalities. Throughout this subsection, x will be a vector in $\mathcal{R}^n$.

Lemma 3.1: Let T be an $m \times n$ real matrix with rank $T =$ rank $T_1 = r$, where $T = [T_1^T\ T_2^T]^T$ and $T_1 \in \mathcal{R}^{r\times n}$. Then $Tx \succeq (\preceq) 0$ if and only if $T_1 x \succeq (\preceq) 0$.

Proof: $Tx \succeq 0$ is equivalent to $T_1 x \succ 0$, or $T_1 x = 0$ and $T_2 x \succeq 0$. Hence, $Tx \succeq 0$ implies $T_1 x \succeq 0$. Conversely, consider $T_1 x = 0$. Then rank $T =$ rank $T_1 = r$ yields $T_2 x = 0$. Thus $T_1 x \succeq 0$ implies $Tx \succeq 0$. The case of $Tx \preceq 0 \leftrightarrow T_1 x \preceq 0$ is similar. □

This lemma shows that the row full-rank submatrix T_1 of T is enough for representing the relation of the lexicographic inequality. Thus the following result is obtained: let T be an $m \times n$ matrix and let $\bar{t}_i^T$ be the ith row vector of T. Let also $T_i \triangleq [\bar{t}_1\ \bar{t}_2\ \cdots \bar{t}_i]^T$. Suppose that rank $T_i =$ rank $T_{i+1} = i$. Then from Lemma 3.1, we can use, in place of T, $\tilde{T} = [\bar{t}_1\ \cdots\ \bar{t}_i\ \bar{t}_{i+2}\ \cdots\ \bar{t}_m]^T$ which is obtained by removing the $i+1$th row vector $\bar{t}_{i+1}^T$ from T. Hence we can assume without loss of generality that T is row-full rank, whenever we consider $Tx \succeq (\preceq) 0$.

The following lemma shows that the set $\mathcal{L}_+^n$ characterizes the coordinate transformations preserving the lexicographic inequality relation.

Lemma 3.2: Let T be an $n \times n$ real matrix. Then $x \succeq (\preceq) 0 \leftrightarrow Tx \succeq (\preceq) 0$ if and only if $T \in \mathcal{L}_+^n$.

Proof: ($\leftarrow$) Obvious. ($\rightarrow$) First, we will prove that if $x \succeq 0 \leftrightarrow Tx \succeq 0$ holds, then T is nonsingular. So assume that T is singular and rank $T = m < n$. Then from Lemma 3.1, there exists a $T_1 \in \mathcal{R}^{m\times n}$ with rank $T_1 = m$ such that $Tx \succeq 0 \leftrightarrow T_1 x \succeq 0$. So we consider $x \succeq 0 \leftrightarrow T_1 x \succeq 0$. Let T_2 be an $(n-m) \times n$ matrix such that $\tilde{T} \triangleq [T_1^T\ T_2^T]^T$ is nonsingular, and let $z \triangleq [\bar{z}_1^T\ \bar{z}_2^T]^T$ where $\bar{z}_i = T_i x$. Then $x = \tilde{T}^{-1} z = M_1 \bar{z}_1 + M_2 \bar{z}_2$ where $[M_1\ M_2] = \tilde{T}^{-1}$. When $\bar{z}_1 = 0$ and $\bar{z}_2$ is any vector, we obtain $x = M_2 \bar{z}_2$. In addition, since rank $M_2 = n-m$, there exists a $z_2 \in \mathcal{R}^{n-m}$ such that $x \prec 0$. This is inconsistent with the condition that $T_1 x \succeq 0 \rightarrow x \succeq 0$. Hence, T is nonsingular.

Now we define the new coordinates $z = [z_1, z_2, \cdots, z_n]^T \triangleq Tx$. Denote the (i,j)th element of T by t_{ij}. Suppose that, for $k \in \{1, 2, \cdots, n\}$, $x_i = 0$ $(i = 1, 2, \cdots, k-1)$, $x_k > 0$, and x_j $(j = k+1, k+2, \cdots, n)$ are arbitrary. We prove the assertion for $\succeq$ by induction. First, let us consider $k = 1$. From

$$z_1 = t_{11} x_1 + t_{12} x_2 + \cdots + t_{1n} x_n,$$

we have $t_{1i} = 0$ $(i = 2, 3, \cdots, n)$ because $z_1 \geq 0$ and x_i $(i = 2, 3, \cdots, n)$ are arbitrary. Furthermore, if $t_{11} < 0$, then $z_1 < 0$ for $x_1 > 0$, and if $t_{11} = 0$, then T is singular. Hence we conclude $t_{11} > 0$. Next assume that, for $k = k_* \in \{1, 2, \cdots, n-1\}$, $t_{ii} > 0$ $(i = 1, 2, \cdots, k_*)$, and $t_{ij} = 0$ $(i = 1, 2, \cdots, k_*,\ j = i+1, i+2, \cdots, n)$. Under this inductive assumption, let us consider $k = k_* + 1$. From $x_1 = \cdots = x_{k_*} = 0$, it follows that

$$z_{k_*+1} = t_{k_*+1, k_*+1} x_{k_*+1} + t_{k_*+1, k_*+2} x_{k_*+2} + \cdots + t_{k_*+1, n} x_n.$$

Thus noting that $z_i = 0$ $(i = 1, 2, \cdots, k_*)$, we have $t_{k_*+1, i} = 0$ $(i = k_*+2, \cdots, n)$ since $z_{k_*+1} \geq 0$ and x_i $(i = k_* + 2, \cdots, n)$ are arbitrary. In addition, similarly to the case $k = 1$, it is verified that $t_{k_*+1, k_*+1} > 0$. The proof of the assertion for $\preceq$ is similar. □

While Lemma 3.2 is concerned with the nonsingular matrices case, the following result treats the singular matrix case.

Lemma 3.3: Let T and S be $l \times n$ and $m \times n$ real matrices with rank $T = l$, rank $S = m$, and $l \geq m$, respectively. Then the following statements are equivalent.

i) $Sx \succeq (\preceq) 0$ for all x satisfying $Tx \succeq (\preceq) 0$.
ii) $S = [M\ 0]T$ for some $M \in \mathcal{L}_+^m$.

Proof: i)→ii). Let Q be any $(n-l) \times n$ matrix such that $[T^T\ Q^T]^T (\triangleq \tilde{T})$ is nonsingular. We denote the new coordinates by $z \triangleq [z_1^T\ z_2^T]^T$, where $z_1 = Tx$ and $z_2 = Qx$. Then i) is equivalent to that $Nz \succeq 0$ for all $z_1 \succeq 0$, where $N \triangleq S\tilde{T}^{-1}$. Let N_1 and N_2 be $m \times l$ and $m \times (n-l)$ matrices, respectively, satisfying $N = [N_1\ N_2]$. When $z_1 \succeq 0$ and z_2 is arbitrary, $N_2 = 0$ is necessary for $Nz \succeq 0$. Thus i) is equivalent to the condition that $N_1 z_1 \succeq 0$ for all $z_1 \succeq 0$. Similarly to the proof of Lemma 3.2 and noting rank $S = m$, we can prove that $N_1 = [M\ 0]$ for some $M \in \mathcal{L}_+^m$. Hence it follows that $S = N\tilde{T} = N_1 T = [M\ 0]T$.

ii)→i). If $Tx \succeq 0$, then $[I_m\ 0]Tx \succeq 0$, which implies that $[M\ 0]Tx \succeq 0$ because $M \in \mathcal{L}_+^m$. Hence ii) provides $Sx \succeq 0$. The proof of the case with $\preceq$ is similar. □

As can be easily seen from the proof, it is noted that even if i) is replaced by i)' $Sx \succeq (\preceq) 0$ for all x satisfying $Tx \succ (\prec) 0$, or i)' $Sx \succ (\prec) 0$ for all x satisfying $Tx \succ (\prec) 0$, Lemma 3.3 still holds. This fact will be used in the proof of Lemma 3.4 below.

Moreover, when we describe the singular case in terms of a form corresponding to Lemma 3.2, the following corollary is obtained from Lemma 3.3.

Corollary 3.1: Let T and S be $l \times n$ and $m \times n$ real matrices with rank $T = l$, rank $S = m$, and $l \geq m$, respectively. Then the following statements are equivalent.

i) $Sx \succeq (\preceq) 0 \leftrightarrow Tx \succeq (\preceq) 0$.
ii) $l = m$ and $S = MT$ for some $M \in \mathcal{L}_+^m$.

Proof: i)→ii). We can prove rank $T =$ rank S in a similar way to the first part of the proof in Lemma 3.2. The latter part in ii) follows from Lemma 3.3. Conversely, it follows from ii) that $Sx \succeq (\preceq) 0 \leftrightarrow MTx \succeq (\preceq) 0 \leftrightarrow Tx \succeq (\preceq) 0$, which implies i). □

From the definition of the lexicographic inequality, it follows that for any nonsingular $n \times n$ matrix T we have the properties:

$$\{x \in \mathcal{R}^n | Tx \succeq 0\} \bigcup \{x \in \mathcal{R}^n | Tx \preceq 0\} = \mathcal{R}^n,$$
$$\{x \in \mathcal{R}^n | Tx \succeq 0\} \bigcap \{x \in \mathcal{R}^n | Tx \preceq 0\} = \{0\}.$$

The following lemma generalizes this property.

Lemma 3.4: Let T and S be $l \times n$ and $m \times n$ real matrices with rank $T = l$, rank $S = m$, and $l \geq m$. Let also T_1 and T_2 be $m \times n$ and $(l-m) \times n$ real matrices with rank $T_1 = m$ and $T = [T_1^T \ T_2^T]^T$. Then the following statements are equivalent.

i) $\{x \in \mathcal{R}^n | Tx \succeq 0\} \bigcup \{x \in \mathcal{R}^n | Sx \preceq 0\} = \mathcal{R}^n$.
ii) $\{x \in \mathcal{R}^n | Tx \succeq 0\} \bigcap \{x \in \mathcal{R}^n | Sx \preceq 0\} = \{x \in \mathcal{R}^n | T_1 x = 0, T_2 x \succeq 0\}$.
iii) $S = [M \ 0]T$ for some $M \in \mathcal{L}_+^m$.

Proof: Since the complement of $\{x \in \mathcal{R}^n | Tx \succeq 0\}$ in $\mathcal{R}^n$ is $\{x \in \mathcal{R}^n | Tx \prec 0\}$, i) is equivalent to i)' $Sx \preceq 0$ for all x satisfying $Tx \prec 0$. From remarks in Lemma 3.3, it follows that i)'↔iii). Next, we have

$$\begin{aligned}
\text{ii)} &\leftrightarrow \{x \in \mathcal{R}^n | Tx \prec 0\} \bigcup \{x \in \mathcal{R}^n | Sx \succ 0\} \\
&= \mathcal{R}^n / \{x \in \mathcal{R}^n | T_1 x = 0, T_2 x \succeq 0\} \\
&\rightarrow \{x \in \mathcal{R}^n | Tx \succeq 0\} \subseteq \{x \in \mathcal{R}^n | Sx \succ 0\} \\
&\quad \bigcup \{x \in \mathcal{R}^n | T_1 x = 0, T_2 x \succeq 0\} \\
&\rightarrow Sx \succ 0 \text{ for all } T_1 x \succ 0.
\end{aligned}$$

Thus from remarks in Lemma 3.3, ii)→iii) follows. On the other hand, since $Sx \preceq 0$ is equivalent to $T_1 x \preceq 0$, iii)→ii) holds. □

Remark 3.1: The sets defined by lexicographic inequalities such as $\{x \in \mathcal{R}^n | Tx \succeq 0\}$ in Lemma 3.4 are in general neither open nor closed, contrary to what might be suggested by the notation.

B. Characterization of Smooth Continuation Property

In this subsection, we discuss when the system $\dot{x} = Ax$ has smooth continuation property with respect to $x \succeq (\preceq) 0$.

The following result show that the set $\mathcal{G}_0^n$, which is defined in Definition 3.2, characterizes the smooth continuation property of linear systems.

Lemma 3.5: For the system $\dot{x} = Ax$, the following statements are equivalent.

i) The system has the smooth continuation property with respect to $x \succeq (\preceq) 0$.
ii) $A \in \mathcal{G}_0^n$.
iii) There exists a matrix $T \in \mathcal{L}_+^n$ such that

$$TAT^{-1} = \begin{bmatrix} \tilde{A}_{11} & 0 & \cdots & 0 \\ \tilde{A}_{21} & \tilde{A}_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \tilde{A}_{p1} & \cdots & \tilde{A}_{p,p-1} & \tilde{A}_{pp} \end{bmatrix} \tag{9}$$

where

$$\tilde{A}_{ii} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ * & \cdots & \cdots & \cdots & * \end{bmatrix} \in \mathcal{R}^{n_i \times n_i},$$

$$\tilde{A}_{ij} = \begin{bmatrix} * & \cdots & * \\ \vdots & & \vdots \\ * & \cdots & * \end{bmatrix} \in \mathcal{R}^{n_i \times n_j}, \quad \text{for } i > j,$$

and $n = n_1 + n_2 + \cdots + n_p$ $(p \in \{1, 2, \cdots, n\})$.

Proof: i)→ii). Suppose that, for $k \in \{2, \cdots, n\}$, $x_i(0) = 0$ $(i = 1, 2, \cdots, k-1)$, $x_k(0) > 0$, and x_j $(j = k+1, k+2, \cdots, n)$ take any values. We prove the assertion by induction. First, consider $k = 2$. Let a_{ij} be the (i, j)th element of A. So from

$$x_1(t) = t\{a_{12}x_2(0) + a_{13}x_3(0) + \cdots + a_{1n}x_n(0)\} + o(t^2),$$

it follows that $a_{1j} = 0$ $(j = 3, 4, \cdots, n)$. In fact, if $a_{1j} \neq 0$ for some $j \in [3, 4, \cdots, n]$, then there exists an $\varepsilon > 0$ such that $x_1(t) < 0$ for all $t \in [0, \varepsilon]$ at some $x_j(0)$, which is inconsistent with the condition i). In addition, since $x_2(0) > 0$, no smooth continuation is possible if $a_{12} < 0$. Hence we have $a_{12} \geq 0$.

Next assume that, for $k = k_* \in \{2, 3, \cdots, n-1\}$, $a_{i,i+1} \geq 0$ $(i = 1, 2, \cdots, l)$, $a_{l,l+1} = 0$, $a_{i,i+1} > 0$ $(i = l+1, l+2, \cdots, k_*-1)$, and $a_{ij} = 0$ $(i = 1, 2, \cdots, k_*-1, \ j = i+2, i+3, \cdots, n)$. Under this assumption, let us consider $k = k_* + 1$. By inductive calculations, noting that $x_i(t) \equiv 0$ $(i = 1, 2, \cdots, l)$, it is verified that

$$\begin{aligned}
x_{l+1}(t) = \frac{t^{k_*-l}}{(k_*-l)!} \Bigg\{ &\prod_{i=l+1}^{k_*} a_{i,i+1} x_{k_*+1}(0) \\
&+ \prod_{i=l+1}^{k_*-1} a_{i,i+1} a_{k_*,k_*+2} x_{k_*+2}(0) + \cdots \\
&+ \prod_{i=l+1}^{k_*-1} a_{i,i+1} a_{k_*,n} x_n(0) \Bigg\} + o(t^{k_*-l+1}).
\end{aligned}$$

where $\prod_{i=l}^{m} a_{i,i+1} = 1$ for $l > m$. From this, it follows that $a_{k_*,k_*+1} \geq 0$ and $a_{k_*,j} = 0$ $(j = k_*+2, \cdots, n)$. Thus by induction, ii) holds.

ii)→iii). Suppose that, for $i = k_j$, $a_{i,i+1} = 0$ $(j = 1, 2, \cdots, s;\ s \leq n-1)$, and for the other i, $a_{i,i+1} > 0$. Set $k_0 = 0$ and $k_{s+1} = n$. Let us consider the coordinate transformation $z = [z_1, z_2, \cdots, z_n]^T \triangleq Tx$ given by

$$\begin{aligned}
z_{k_j+1} &\triangleq x_{k_j+1}, \\
z_{k_j+l} &\triangleq \sum_{i_{(k_j+1)}=1}^{k_j+2} \sum_{i_{(k_j+2)}=1}^{i_{(k_j+1)}+1} \cdots \sum_{i_{(k_j+l-1)}=1}^{i_{(k_j+l-2)}+1} \\
&\quad \cdot a_{k_j+1, i_{(k_j+1)}} a_{i_{(k_j+1)}, i_{(k_j+2)}} \cdots \\
&\quad a_{i_{(k_j+l-2)}, i_{(k_j+l-1)}} x_{i_{(k_j+l-1)}}, \\
&\qquad l = 2, \cdots, k_{j+1} - k_j,\ j = 0, 1, \cdots, s.
\end{aligned} \tag{10}$$

where $i_{k_j} = k_j + 1$ (note that $s = 0$ implies that all elements $a_{i,i+1}$ are positive). The matrix T is given by

$$T = \begin{bmatrix} T_{11} & 0 & \cdots & 0 \\ T_{21} & T_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ T_{s+1,1} & \cdots & T_{s+1,s} & T_{s+1,s+1} \end{bmatrix} \tag{11}$$

where

$$T_{ii} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ * & a_{k_{(i-1)}+1,\,k_{(i-1)}+2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ * & \cdots & * & \prod_{j=1}^{k_i - k_{i-1} - 1} a_{k_{(i-1)}+j,\,k_{(i-1)}+j+1} \end{bmatrix} \in \mathcal{R}^{(k_i - k_{(i-1)}) \times (k_i - k_{(i-1)})},$$

$$T_{ij} = \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ * & \cdots & \cdots & * \\ \vdots & & & \vdots \\ * & \cdots & \cdots & * \end{bmatrix} \in \mathcal{R}^{(k_i - k_{(i-1)}) \times (k_j - k_{(j-1)})}, \quad \text{for } i > j.$$

Thus from $a_{i,i+1} > 0$ for all $i \in \{1, 2, \cdots, n\}$ except for $i = k_j$, we conclude $T \in \mathcal{L}_+^n$. Furthermore, by direct computation, it is verified that TAT^{-1} satisfies (9).

iii)→i). The case $x(0) = 0$ is trivial. So we consider the case $x(0) \succ 0$. Denote the new coordinates by $z = [z_1, z_2, \cdots, z_n]^T \triangleq Tx$. From Lemma 3.2, $T \in \mathcal{L}_+^n$ implies that $x \succ 0 \leftrightarrow z \succ 0$. Let $\overline{z}_k$ $(k = 1, 2, \cdots, p)$ be defined by

$$\overline{z}_k \triangleq \begin{bmatrix} z_{(\sum_{i=1}^{k-1} n_i)+1} \\ \vdots \\ z_{(\sum_{i=1}^{k} n_i)} \end{bmatrix}.$$

where $\overline{z}_1 = [z_1, z_2, \cdots, z_{n_1}]^T$ for $k = 1$.

Note that $x(0) \succ 0$, namely $z(0) \succ 0$, is equivalent to $\overline{z}_i(0) = 0$ $(i = 1, 2, \cdots, k-1)$ and $\overline{z}_k(0) \succ 0$ for all $k \in \{1, 2, \cdots, p\}$. So from the structure of the A-matrix of the system, for each $k \in \{1, 2, \cdots, p\}$, there exists an $\varepsilon > 0$ such that

$$\begin{cases} \overline{z}_i(t) = 0, & i = 1, 2, \cdots, k-1 \\ \overline{z}_k(t) \succ 0 \end{cases}, \quad \forall\, t \in [0, \varepsilon]$$

which implies that $x(t) \succ 0$ for all $t \in [0, \varepsilon]$. The case $x(0) \prec 0$ is proven in the same way. □

From Lemma 3.5, it turns out that, by the coordinate transformation given in (11), any linear system with the smooth continuation property is transformed into a system whose A-matrix is given by (9). In addition, the equivalence between ii) and iii) suggests that all the coordinates transformations given by elements in $\mathcal{L}_+^n$ preserve the smooth continuation property of the linear system. This is shown in the following lemma.

Lemma 3.6: Let M be a matrix in $\mathcal{L}_+^n$ and Γ be a matrix in $\mathcal{G}_0^n(\mathcal{G}_+^n)$. Then $M\Gamma M^{-1} \in \mathcal{G}_0^n(\mathcal{G}_+^n)$.

Proof: Let M_k and Γ_k $(k = 1, 2, \cdots, n-1)$ be $k \times k$ matrices with $M_k \in \mathcal{L}_+^k$ and $\Gamma_k \in \mathcal{G}_0^k$. When $k = 1$, $M_1\Gamma_1 M_1^{-1} \in \mathcal{G}_0^1$ is obvious. Assume that $M_k \Gamma_k M_k^{-1} \in \mathcal{G}_0^k$ for any $k \in \{1, 2, \cdots, n-1\}$. Under this assumption, let us show $M_{k+1}\Gamma_{k+1}M_{k+1}^{-1} \in \mathcal{G}_0^{k+1}$. Denote the (i, j)th element of M_{k+1} and Γ_{k+1} by m_{ij} and γ_{ij}, respectively. After some calculations, we have

$$M_{k+1}\Gamma_{k+1}M_{k+1}^{-1} = \left[\begin{array}{c|c} M_k\Gamma_k M_k^{-1} + S & \eta \\ \hline * \cdots \cdots * & * \end{array}\right]$$

where

$$S = \left[\begin{array}{c} 0_{k-1,\,k} \\ \hline * \cdots * \end{array}\right], \quad \eta = \left[\begin{array}{c} 0_{k-1,\,1} \\ \hline \dfrac{m_{kk}}{m_{k+1,\,k+1}}\gamma_{k,\,k+1} \end{array}\right]$$

Thus from $M_k\Gamma_k M_k^{-1} \in \mathcal{G}_0^k$, $m_{k,k} > 0$, $m_{k+1,k+1} > 0$, and $\gamma_{k,k+1} > 0$, it follows that $M_{k+1}\Gamma_{k+1}M_{k+1}^{-1} \in \mathcal{G}_0^{k+1}$. By induction, we conclude $M\Gamma M^{-1} \in \mathcal{G}_0^n$. The proof in the case of $\mathcal{G}_+^n$ is similar. □

There is another type of the smooth continuation property with respect to $x \succeq (\preceq) 0$, where ε in Definition 2.3 is independent of the initial state $x(0)$. In other words, if there exists a positive constant ε such that $x(t) \succeq (\preceq) 0$ for all $x(0)$ satisfying $x(0) \succeq (\preceq) 0$ and all $t \in [0, \varepsilon]$, we call this the uniform smooth continuation property with respect to $x \succeq (\preceq) 0$. The following lemma characterizes this property.

Corollary 3.2: For the system $\dot{x} = Ax$, the following statements are equivalent.

i) The system has the uniform smooth continuation property with respect to $x \succeq (\preceq) 0$.
ii) There exists a positive constant ε such that $e^{At} \in \mathcal{L}_+^n$ for all $t \in [0, \varepsilon]$.
iii) $x(t) \succeq (\preceq) 0$ for all $x(0)$ satisfying $x(0) \succeq (\preceq) 0$ and all $t \in [0, \infty)$.
iv) $e^{At} \in \mathcal{L}_+^n$ for all $t \in [0, \infty)$.
v) $A \in \mathcal{L}^n$.

Proof: Since $x(t) = e^{At}x(0)$, i)↔ii), and iii)↔iv) are straightforward from Lemma 3.2. We prove iv)→ii)→v)→iv). First, iv)→ii) is trivial. Next, ii)→v). Note that e^{At} is a one-parameter subgroup in $\mathcal{L}_+^n$ around $t = 0$. Thus the tangent vector at $t = 0$ is A. On the other hand, the tangent space $T_e\mathcal{L}_+^n$ at the identity matrix is $\mathcal{L}^n$. Hence $A \in \mathcal{L}^n$. Finally, v)→iv). If $A \in \mathcal{L}^n$, simple calculations show

$$e^{At} = \begin{bmatrix} e^{a_{11}t} & 0 & \cdots & 0 \\ * & e^{a_{22}t} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ * & \cdots & * & e^{a_{nn}t} \end{bmatrix}, \qquad A = [a_{ij}]$$

which implies iv). □

Obviously, the uniform smooth continuation property implies the smooth continuation property, but the converse is not true. Corollary 3.2 asserts that the uniform smooth continuation property in the local sense [i.e., i)] is equivalent to the global one [i.e., iii)] in the case of linear systems. Thus the sets $\{x \in \mathcal{R}^n | x \succeq 0\}$ and $\{x \in \mathcal{R}^n | x \preceq 0\}$ are invariant subsets of $\mathcal{R}^n$ with respect to the dynamics $\dot{x} = Ax$ with the uniform smooth continuation property i).

IV. Characterization of Well-Posedness of Bimodal Systems

In this section, we discuss the well-posedness of Σ_O given by (1), or equivalently of Σ_{AB} given by (8). First, we give a result in the case that both pairs (C, A) and (C, B) are observable. This will clarify a fundamental issue in the algebraic structure for well-posed bimodal systems. Next, the unobservable case is treated as a generalization of the observable case.

A. Observable Case

In this subsection, we assume that the pairs (C, A) and (C, B) are observable, that is, T_A and T_B are nonsingular, where

$$T_A \triangleq \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}, \quad T_B \triangleq \begin{bmatrix} C \\ CB \\ \vdots \\ CB^{n-1} \end{bmatrix}. \tag{12}$$

In addition, we consider the following two systems:

$$\Sigma_A \begin{cases} \text{mode 1: } \dot{x} = Ax, & \text{if } x \in \mathcal{S}_A^+ \\ \text{mode 2: } \dot{x} = Bx, & \text{if } x \in \mathcal{S}_A^- \end{cases} \tag{13}$$

$$\Sigma_B \begin{cases} \text{mode 1: } \dot{x} = Ax, & \text{if } x \in \mathcal{S}_B^+ \\ \text{mode 2: } \dot{x} = Bx, & \text{if } x \in \mathcal{S}_B^- \end{cases} \tag{14}$$

where $\mathcal{S}_N^+$ and $\mathcal{S}_N^-$ $(N = A, B)$ are given by (7). Utilizing the fact that $\mathcal{S}_A^+ \bigcup \mathcal{S}_A^- = \mathcal{R}^n$, the system Σ_A is given by the rule matrix T_A only. The system Σ_B is defined by the rule matrix T_B in the same way.

Now the main idea to characterize the well-posedness of Σ_{AB} is as follows. First, note that $\mathcal{S}_A^+$ and $\mathcal{S}_B^-$ express sets of all initial states from which smooth continuation is possible in mode 1 and mode 2, respectively. Next, if the solutions in both modes are the same on some time interval, they must satisfy $y(t) = 0$ on that interval, and so such a solution is only the origin $x(t) = 0$ under the assumption of observability. Thus from Lemma 2.1, we can see that the system Σ_{AB} is well-posed if and only if $\mathcal{S}_A^+ \bigcup \mathcal{S}_B^- = \mathcal{R}^n$ and $\mathcal{S}_A^+ \bigcap \mathcal{S}_B^- = \{0\}$.

On the other hand, from Lemma 3.4, it follows that $\mathcal{S}_A^+ \bigcup \mathcal{S}_B^- = \mathcal{R}^n$ is equivalent to $\mathcal{S}_A^+ \bigcap \mathcal{S}_B^- = \{0\}$, and also is equivalent to $T_B T_A^{-1} \in \mathcal{L}_+^n$. Thus we conclude that either one of these conditions holds if and only if the system Σ_{AB} is well-posed.

Moreover, we will derive another type of condition by using the relation $T_B T_A^{-1} \in \mathcal{L}_+^n$. Since $T_B T_A^{-1} \in \mathcal{L}_+^n$ implies $\mathcal{S}_A^- = \mathcal{S}_B^-$ from Lemma 3.2, if Σ_{AB} is well-posed, then Σ_A is also well-posed. Moreover, in the new coordinates $z \triangleq T_A x$, the system Σ_A is described by

$$\tilde{\Sigma}_A \begin{cases} \text{mode 1: } \dot{z} = T_A A T_A^{-1} z, & \text{if } z \succeq 0 \\ \text{mode 2: } \dot{z} = T_A B T_A^{-1} z, & \text{if } z \preceq 0. \end{cases} \tag{15}$$

Then the well-posedness of $\tilde{\Sigma}_A$ implies that smooth continuation is possible in each mode. Thus by Lemma 3.5, $T_A B T_A^{-1} \in \mathcal{G}_0^n$ must hold. Note also that $T_A A T_A^{-1} \in \mathcal{G}_0^n$ is automatically satisfied. More strictly, as seen in the proof below, we can prove that $T_A B T_A^{-1} \in \mathcal{G}_+^n$ and that this is also a sufficient condition for the system Σ_{AB} to be well-posed.

Thus we come to the first main result on the well-posedness.

Theorem 4.1: Suppose that both pairs (C, A) and (C, B) are observable. Then the following statements are equivalent.

i) Σ_O (or equivalently Σ_{AB}) is well-posed.
ii) Σ_A is well-posed.
iii) Σ_B is well-posed.
iv) $\mathcal{S}_A^+ \bigcup \mathcal{S}_B^- = \mathcal{R}^n$.
v) $\mathcal{S}_A^+ \bigcap \mathcal{S}_B^- = \{0\}$.
vi) $T_B T_A^{-1} \in \mathcal{L}_+^n$.
vii) $T_A B T_A^{-1} \in \mathcal{G}_+^n$.
viii) $T_B A T_B^{-1} \in \mathcal{G}_+^n$.

Proof: We have already proven i)↔iv)↔v)↔vi) and vi)→ii). So let us prove ii)→vii)→vi). ii)→vii). We have shown in (15) $T_A B T_A^{-1} \in \mathcal{G}_0^n$. Furthermore, letting γ_{ij} be the (i, j)th element of $\Gamma \triangleq T_A B T_A^{-1}$, and noting that $C T_A^{-1} = [1\ 0\ \cdots\ 0]$, we obtain

$$\begin{cases} CB &= CT_A^{-1}\Gamma T_A &= [*\ \gamma_{12}\ 0\ \cdots\cdots\ 0]T_A \\ CB^2 &= CT_A^{-1}\Gamma^2 T_A &= [*\ *\ \gamma_{12}\gamma_{23}\ 0\ \cdots\ 0]T_A \\ \quad\vdots & & \quad\vdots \\ CB^{n-1} &= CT_A^{-1}\Gamma^{n-1} T_A &= \left[*\ \cdots\cdots\ *\ \prod_{i=1}^{n-1}\gamma_{i,i+1}\right]T_A. \end{cases} \tag{16}$$

From these calculations, it follows that

$$T_B = L T_A \tag{17}$$

where

$$L \triangleq \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ * & \gamma_{12} & \ddots & & \vdots \\ \vdots & \ddots & \gamma_{12}\gamma_{23} & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ * & \cdots & \cdots & * & \prod_{i=1}^{n-1}\gamma_{i,i+1} \end{bmatrix}. \tag{18}$$

This implies that all elements $\gamma_{i,i+1}$ are positive, since T_A and T_B are nonsingular. Hence $T_A B T_A^{-1} \in \mathcal{G}_+$. vii)→vi). In a similar way to (16), we obtain the equation (17) from vii). Since $L \in \mathcal{L}_+^n$, vi) holds.

The proof of vi)→iii)→viii)→vi) is similar. □

Remark 4.1: From Theorem 4.1, it turns out that the well-posedness property of the bimodal system Σ_{AB} with both (C, A) and (C, B) observable is characterized by either one of the following two properties: a) the preservation property of the lexicographic inequality relation between two rule matrices T_A and T_B, which is characterized by the set $\mathcal{L}_+^n$, and b) the smooth continuation property which is characterized by the set $\mathcal{G}_+^n$ (or $\mathcal{G}_0^n$). The former corresponds to iv), v), or vi) in Theorem 4.1, and the latter to vii) or viii). Note also that the well-posedness property of Σ_{AB} can be given by the equivalence between Σ_{AB}, Σ_A, and Σ_B. Furthermore, from vii), it follows that a parametrization of all matrices B for which Σ_{AB} is well-posed is given by the form $B = T_A^{-1}\Gamma T_A$ for any $\Gamma \in \mathcal{G}_+^n$.

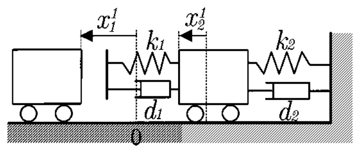

Fig. 2. Elastic collision between two objects.

Remark 4.2: When the well-posedness condition in Theorem 4.1 is not satisfied, there is still some possibility that the system is well-posed with sliding modes, if we allow the existence of sliding modes. However, such a situation is not possible under the assumption of observability. In fact, whenever the well-posedness condition in Theorem 4.1 is not satisfied, $\mathcal{S}_A^+ \bigcap \mathcal{S}_B^- \neq \{0\}$ holds, which implies that there exists two different solutions from the initial state $x_0 \in \mathcal{S}_A^+ \bigcap \mathcal{S}_B^-$. We here use the fact that if $Cx(t) = 0$ is satisfied on some time interval, then $x(t) \equiv 0$ by observability.

Example 4.1: Consider the physical system in Fig. 2. The equations of motion of this system are given by

$$\text{mode 1:} \begin{cases} \dot{x}^1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x^1, \\ \dot{x}^2 = \begin{bmatrix} 0 & 1 \\ -k_2 & -d_2 \end{bmatrix} x^2, \quad \text{if } y = [1\ 0\ {-1}\ 0]x \leq 0 \end{cases}$$

$$\text{mode 2:} \begin{cases} \begin{bmatrix} \dot{x}^1 \\ \dot{x}^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -k_1 & -d_1 & k_1 & d_1 \\ 0 & 0 & 0 & 1 \\ k_1 & d_1 & -k_1-k_2 & -d_1-d_2 \end{bmatrix} \\ \quad \cdot \begin{bmatrix} x^1 \\ x^2 \end{bmatrix}, \quad \text{if } y = [1\ 0\ {-1}\ 0]x \leq 0 \end{cases}$$

where $x = [(x^1)^T\ (x^2)^T]^T = [x_1^1\ x_2^1\ x_1^2\ x_2^2]^T$. These provide

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -k_2 & -d_2 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -k_1 & -d_1 & k_1 & d_1 \\ 0 & 0 & 0 & 1 \\ k_1 & d_1 & -k_1-k_2 & -d_1-d_2 \end{bmatrix},$$

$$C = [1\ 0\ -1\ 0].$$

Simple calculations show that the pair (C, A) is observable if and only if $k_2 \neq 0$, and also the pair (C, B) is observable if and only if $k_2 \neq 0$. Thus we here assume $k_2 \neq 0$.

From the equations at the bottom of the page it follows that

$$T_B T_A^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ * & * & 1 & 0 \\ * & * & * & 1 \end{bmatrix}$$

which belongs to the set $\mathcal{L}_+^4$. Hence the system is well-posed. We also have

$$T_A B T_A^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ * & * & 1 & 0 \\ * & * & * & 1 \\ * & * & * & * \end{bmatrix}$$

which belongs to the set $\mathcal{G}_+^4$.

B. Unobservable Case

The following result is concerned with the case that both pairs (C, A) and (C, B) are unobservable.

Theorem 4.2: Denote the observability indexes of the pairs (C, A) and (C, B) by m_A and m_B, respectively. Then the following statements are equivalent.

i) Σ_O (or equivalently Σ_{AB}) is well-posed.

ii) The following conditions are satisfied.
   a) $m_A = m_B$.
   b) $T_B = M T_A$ for some $M \in \mathcal{L}_+^{m_A}$.
   c) $(A - B)x = 0$ for all $x \in \text{Ker } T_A$.

iii) The following conditions are satisfied.
   a) $m_A = m_B$.
   b) $T_A B = \Gamma T_A$ for some $\Gamma \in \mathcal{G}_+^{m_A}$.
   c) $(A - B)x = 0$ for all $x \in \text{Ker } T_A$.

Let us compare Theorem 4.2 with Theorem 4.1, which deals with the observable case. If $m_A = m_B = n$, ii) and iii) in Theorem 4.2 generalize vi) and vii) in Theorem 4.1, respectively. However, in the unobservable case (i.e., $m_A < n$ and $m_B < n$), additional conditions ii)a) and ii)c) [or iii)a) and iii)c)] are required. The former condition implies that the dimension of the unobservability subspace in both modes must be the same for the well-posedness. For example, if $m_A > m_B$, then for the initial state in some subset of the $(m_A - m_B)$-dimensional unobservability subspace smooth continuation is possible in both modes and the two different solutions exist, which implies that the system is not well-posed. The latter condition, on the other hand, implies that the solutions in the unobservability subspace Ker $T_A (= \text{Ker } T_B)$ must be the same in both modes.

$$T_A = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & k_2 & d_2 \\ 0 & 0 & -k_2 d_2 & k_2 - d_2^2 \end{bmatrix}$$

$$T_B = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -2k_1 & -2d_1 & 2k_1 + k_2 & 2d_1 + d_2 \\ (4d_1 + d_2)k_1 & -2k_1 + (4d_1 + d_2)d_1 & -(4d_1 + d_2)k_1 - (2d_1 + d_2)k_2 & (2k_1 + k_2) - 4d_1^2 - 3d_1 d_2 - d_2^2 \end{bmatrix}$$

Since this theorem is a special case of Theorem 5.1 in the next section, the proof will follow from that of Theorem 5.1 (see Remark 5.1).

Remark 4.3: From Theorem 4.2, it follows that whenever the pair (C, A) is observable and the pair (C, B) is unobservable, the system Σ_O is not well-posed. However, if the number of the criterions which specify admissible regions of the state in each mode, i.e., the dimension of y in (1), is more than one, then the situation is different. The details will be given in Theorem 5.1 and Example 5.1 in the next section.

Remark 4.4: The conditions in Theorem 4.2 can be checked as follows. First, check the condition iii)a). If it is not satisfied, we conclude that the system is not well-posed. Otherwise, check b) and c) in iii). So pick any matrix $\tilde{T}_A$ such that $T \triangleq [T_A^T \ \tilde{T}_A^T]^T$ is nonsingular. Then we can show that b) and c) are equivalent to

$$[I_{m_B} \ 0_{m_B, n-m_B}]TBT^{-1}\begin{bmatrix} I_{m_B} \\ 0_{n-m_B, m_B}\end{bmatrix} \in \mathcal{G}_+^{m_B},$$
$$[I_{m_B} \ 0_{m_B, n-m_B}]TBT^{-1}\begin{bmatrix} 0_{m_B, n-m_B} \\ I_{n-m_B}\end{bmatrix} = 0, \tag{19}$$

and

$$[0_{n-m_B, m_B} \ I_{n-m_B}]T(A-B)T^{-1}\begin{bmatrix} 0_{m_B, n-m_B} \\ I_{n-m_B}\end{bmatrix} = 0. \tag{20}$$

Thus if these conditions are satisfied, we conclude that the system is well-posed. Otherwise, we conclude that the system is not well-posed. Note here that we only have to check the condition for some $\tilde{T}_A$, since the well-posedness does not depend on the choice of $\tilde{T}_A$.

Furthermore, for this class of systems, we can show that if the system Σ_O is well-posed, then the time-reversed system below is well-posed:

$$\Sigma_O^- \begin{cases} \text{mode 1:} \dot{x} = -Ax, & \text{if } y = Cx \geq 0 \\ \text{mode 2: } \dot{x} = -Bx, & \text{if } y = Cx \leq 0. \end{cases}$$

Theorem 4.3: For the system Σ_O given by (1), the following statements are equivalent.

i) Σ_O is well-posed.

ii) Σ_O^- is well-posed.

Proof: We prove i)→ii). Let m_A and m_B be the observability indexes of the pairs (C, A) and (C, B), respectively. Let also T_A^- and T_B^- given by (6) with $-A$ and $-B$ instead of A and B, and with $h = m_A$ and $k = m_B$, respectively. Note that $m_A = m_B$ because of i). Then since there exists $M \in \mathcal{L}_{m_A}^+$ such that $T_B = MT_A$, we have $T_B^- = E_{m_A}T_B = E_{m_A}ME_{m_A}T_A^-$ where

$$E_{m_A} = \begin{bmatrix} 1 & 0 & \dots & \dots & 0 \\ 0 & -1 & 0 & \dots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & (-1)^{m_A-2} & 0 \\ 0 & \dots & \dots & 0 & (-1)^{m_A-1} \end{bmatrix}$$

Thus we will show $E_{m_A}ME_{m_A} \in \mathcal{L}_+^{m_A}$. From simple calculations, the (i, i)th element of $E_{m_A}ME_{m_A}$ is given by $(-1)^{2(i-1)}m_{i,i} > 0$, where $m_{i,i}$ is the (i, i)th element of M. So noting that both E_{m_A} and M are lower-triangular, $E_{m_A}ME_{m_A} \in \mathcal{L}_+^{m_A}$ holds. Similarly for ii)→ i). □

Example 4.2: Consider the system in Example 4.1 again. Assume that $k_2 = 0$ and $d_2 \neq 0$. Then since

$$T_A = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & d_2 \end{bmatrix},$$
$$T_B = \begin{bmatrix} C \\ CB \\ CB^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -2k_1 & -2d_1 & 2k_1 & 2d_1+d_2 \end{bmatrix},$$

we have $m_A = 3$ and $m_B = 3$. Thus iii)a) in Theorem 4.2 is satisfied. Letting $\tilde{T}_A \triangleq [0\ 0\ 1\ 0]$, we have

$$TAT^{-1} = \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -d_2 & 0 \\ \hline 0 & 0 & 1/d_2 & 0 \end{array}\right],$$
$$TBT^{-1} = \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ -2k_1 & -2d_1 & 1 & 0 \\ k_1 d_2 & d_1 d_2 & -d_2 & 0 \\ \hline 0 & 0 & 1/d_2 & 0 \end{array}\right].$$

Using (19) and (20) in Remark 4.4, we can show that b) and c) in iii) are satisfied. Therefore, the system is well-posed.

V. Well-Posedness of Bimodal Systems with Multiple Criteria

In this section, we treat bimodal systems given by multiple criteria.

A. Description of Bimodal Systems with Multiple Criteria

Let us start with the following example:

$$\Sigma_{AB} \begin{cases} \text{mode 1: } \dot{x} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} x, & \text{if } x \succeq 0 \\ \text{mode 2: } \dot{x} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} x, & \text{if } x \preceq 0. \end{cases} \tag{21}$$

Since smooth continuation in each mode is possible, that is, both A-matrices belong to $\mathcal{G}_0^2$, this system is well-posed. Then let us consider what is the original system Σ_O of this Σ_{AB}. So from mode 1, we can see that $C = [1\ \ 0]$. However, in this case, $T_A = I_2$ and $T_B = [1\ \ 0]$, and so (C, A) is observable but (C, B) is not observable. This implies that the system of the form (1) given by $C = [1\ \ 0]$ is not equivalent to the system Σ_{AB}, and so is not the original system of Σ_{AB}.

How can this well-posed bimodal system be characterized by our framework? In fact, the original system for Σ_{AB} in (21) is given in terms of two criteria $Cx \geq (\leq) 0$ and $\overline{C}x \geq (\leq) 0$ where $C = [1\ 0]$ and $\overline{C} = [0\ 1]$ as follows.

$$\Sigma_O \begin{cases} \text{mode 1: } \dot{x} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} x, & \text{if } Cx \geq 0 \\ \text{mode 2: } \dot{x} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} x, & \text{if } \begin{bmatrix} C \\ \overline{C} \end{bmatrix} x \preceq 0. \end{cases} \tag{22}$$

In this section, we will generalize this example to consider the following bimodal system:

$$\Sigma_O \begin{cases} \text{mode 1: } \dot{x} = Ax, & \text{if } Cx \succeq 0 \\ \text{mode 2: } \dot{x} = Bx, & \text{if } Dx \preceq 0 \end{cases} \tag{23}$$

where

$$C = \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_p \end{bmatrix} \in \mathcal{R}^{p\times n}, \quad D = \begin{bmatrix} D_1 \\ D_2 \\ \vdots \\ D_s \end{bmatrix} \in \mathcal{R}^{s\times n},$$

and C_i^T and D_j^T are n-dimensional vectors. In this definition, note that it is at least required for well-posedness that $\{x \in \mathcal{R}^n | Cx \succeq 0\} \bigcup \{x \in \mathcal{R}^n | Dx \preceq 0\} = \mathcal{R}^n$.

First, we give an equivalent representation to the above system, as in Section II. So we introduce the following rule matrices:

$$T_A \triangleq \begin{bmatrix} T_{A1} \\ T_{A2} \\ \vdots \\ T_{Ap} \end{bmatrix} \in \mathcal{R}^{m_A\times n}, \quad T_B \triangleq \begin{bmatrix} T_{B1} \\ T_{B2} \\ \vdots \\ T_{Bs} \end{bmatrix} \in \mathcal{R}^{m_B\times n} \tag{24}$$

where

$$T_{Ai} \triangleq \begin{bmatrix} C_i \\ C_iA \\ \vdots \\ C_iA^{h_i-1} \end{bmatrix} \in \mathcal{R}^{h_i\times n}, \qquad i = 1, 2, \cdots, p,$$

$$T_{Bi} \triangleq \begin{bmatrix} D_i \\ D_iB \\ \vdots \\ D_iB^{k_i-1} \end{bmatrix} \in \mathcal{R}^{k_i\times n}, \qquad i = 1, 2, \cdots, s,$$

and each h_i $(i = 1, 2, \cdots, p)$ is the maximum value of the rank such that $[T_{A1}^T\ T_{A2}^T\ \cdots T_{Ai}^T]^T$ has a row-full rank. Similarly for k_i. Note that $\sum_{i=1}^p h_i = m_A$ and $\sum_{i=1}^s k_i = m_B$, and then rank $T_A = m_A$ and rank $T_B = m_B$.

Using these rule matrixes, we consider the system given by

$$\Sigma_{AB} \begin{cases} \text{mode 1: } \dot{x} = Ax, & \text{if } x \in \mathcal{S}_A^+ \\ \text{mode 2: } \dot{x} = Bx, & \text{if } x \in \mathcal{S}_B^- \end{cases} \tag{25}$$

where $\mathcal{S}_N^+$ and $\mathcal{S}_N^-$ $(N = A, B)$ is defined by (7), where T_A and T_B are given by (24). Then, similar to Lemma 2.3, we can prove that the system Σ_{AB} is equivalent to the original system Σ_O. Therefore, we focus on the well-posedness of Σ_{AB}.

B. Well-Posedness Conditions

We consider the general case that both pairs are not necessarily observable. Let $\mathcal{T}_A$ be the set of $(n - m_A) \times n$ matrices such that $T \triangleq [T_A^T\ \tilde{T}_A^T]^T$ is nonsingular, that is,

$$\mathcal{T}_A \triangleq \left\{ \tilde{T}_A \in \mathcal{R}^{(n-m_A)\times n} \mid T \text{ is nonsingular} \right\}. \tag{26}$$

Let also $\mathcal{T}_B$ be defined in the same way.

Theorem 5.1: Suppose that the rank of T_A and T_B given by (24) are m_A and m_B, respectively, and $m_A \geq m_B$. Then the following statements are equivalent.

i) Σ_O (or equivalently Σ_{AB}) is well-posed.

ii) The following conditions are satisfied.
 a) rank $[T_{A1}^T\ T_{A2}^T\ \cdots T_{Ai}^T]^T = m_B$ for some $i \in \{1, 2, \cdots, p\}$.
 b) $T_B = [M\ \ 0_{m_B, m_A-m_B}]T_A$ for some $M \in \mathcal{L}_+^{m_B}$.
 c) $(A - B)x = 0$ for all $x \in$ Ker T_B.

iii) The following conditions are satisfied.
 a) rank $[T_{A1}^T\ T_{A2}^T\ \cdots T_{Ai}^T]^T = m_B$ for some $i \in \{1, 2, \cdots, p\}$.
 b) $[I_{m_B}\ \ 0_{m_B, m_A-m_B}]T_AB = \Gamma[I_{m_B}\ \ 0_{m_B, m_A-m_B}]T_A$ for some $\Gamma \in \mathcal{G}_0^{m_B}$.
 c) $D_i = [\underbrace{*\cdots*}_{\bar{k}_i}a\ \ 0\ \ \cdots\ \ 0]T_A$ for every $i \in \{1, 2, \cdots, s\}$, where $\bar{k}_i = k_1 + k_2 + \cdots + k_{i-1}$, $k_0 = 0$, and $a > 0$.
 d) $(A - B)x = 0$ for all $x \in$ Ker T_B.

Proof: i)→ii). From i), it follows that $\mathcal{S}_A^+ \bigcup \mathcal{S}_B^- = \mathcal{R}^n$, which implies by Lemma 3.4 that T_A and T_B satisfy $T_B = [M\ \ 0]T_A$ for some $M \in \mathcal{L}_+^{m_B}$. In addition, let two new coordinates be defined by $z = [z_1^T\ \ z_2^T]^T \triangleq Tx$ and $w = [w_1^T\ \ w_2^T]^T \triangleq \hat{T}x$, where $T \triangleq [T_A^T\ \ \tilde{T}_A^T]^T$ and $\hat{T} \triangleq [T_B^T\ \tilde{T}_B^T]^T$ for any $\tilde{T}_A \in \mathcal{T}_A$ and any $\tilde{T}_B \in \mathcal{T}_B$. Then Σ_{AB} is transformed into

$$\tilde{\Sigma}_{AB} \begin{cases} \text{mode 1: } \dot{z} = TAT^{-1}z, & \text{if } z_1 \succeq 0 \\ \text{mode 2: } \dot{w} = \hat{T}B\hat{T}^{-1}w, & \text{if } w_1 \preceq 0. \end{cases} \tag{27}$$

Here TAT^{-1} and $\hat{T}B\hat{T}^{-1}$ are given by

$$TAT^{-1} = \left[\begin{array}{c|c} \tilde{A}_{11} & 0_{m_A, n-m_A} \\ \hline * & * \end{array}\right], \qquad \tilde{A}_{11} \in \mathcal{G}_0^{m_A}, \tag{28}$$

$$\hat{T}B\hat{T}^{-1} = \left[\begin{array}{c|c} \tilde{B}_{11} & 0_{m_B, n-m_B} \\ \hline * & * \end{array}\right], \qquad \tilde{B}_{11} \in \mathcal{G}_0^{m_B}. \tag{29}$$

Let z_1 be denoted by $z_1 \triangleq [z_{11}^T\ z_{12}^T]^T$ where $z_{11} \in \mathcal{R}^{m_B}$ and $z_{12} \in \mathcal{R}^{m_A-m_B}$. So let us consider the case of $z_{11}(0) = 0$ and $z_{12}(0) \succ 0$, which also implies $w_1(0) = 0$ because $T_B = [M\ \ 0]T_A$. From (28) and (29), smooth continuation in each mode is possible from this state, and the solution in mode 2 is in the $n - m_B$ dimensional unobservable invariant subspace with $w_1(t) \equiv 0$, namely, Ker T_B. Thus due to uniqueness of the solution, the solution in mode 1 must satisfy $z_{11}(t) = 0$ as far as $z_{12} \succeq 0$ holds. Hence a) follows from this. Furthermore, the vector fields in both modes must be the same on Ker $T_B \cap \{z \in \mathcal{R}^n | z_{12} \succeq 0\}$. From the property of linear systems, this implies that $Ax = Bx$ for all $x \in$ Ker T_B.

ii)→iii). We only have to show b) and c) in iii). It follows from ii)b) that

$$\begin{aligned} [I_{m_B}\ 0]T_AB &= M^{-1}T_BB = M^{-1}\tilde{B}_{11}T_B \\ &= M^{-1}\tilde{B}_{11}M[I_{m_B}\ 0]T_A \end{aligned} \tag{30}$$

where $\tilde{B}_{11}$ is the same as (29). From Lemma 3.6, this implies $\Gamma \triangleq M^{-1}\tilde{B}_{11}M \in \mathcal{G}_0^{m_B}$, namely, iii)b). Moreover, letting m_{ij}

be the (i, j) element of M in ii)b), the relation $T_B = [M\ 0]T_A$ implies that, for $i \in \{1, 2, \cdots, s\}$,

$$D_i = \left[\underbrace{* \cdots *}_{\overline{k}_i}\ m_{\overline{k}_i+1, \overline{k}_i+1}\ 0\ \cdots 0\right] T_A.$$

Since $m_{\overline{k}_i+1, \overline{k}_i+1} > 0$, we have iii)c).

iii)→i). First, we show $T_B = [M\ 0]T_A$ for some $M \in \mathcal{L}_+^{m_B}$. From b) and c) in iii), it follows that

$$\begin{aligned} D_1 B &= a[1\ 0\ \cdots 0][I_{m_B}\ 0]T_A B \\ &= a[1\ 0\ \cdots 0]\Gamma[I_{m_B}\ 0]T_A \\ &= a[*\ \gamma_{12}\ 0\ \cdots\ 0][I_{m_B}\ 0]T_A \\ &= a[*\ \gamma_{12}\ 0\ \cdots\ 0]T_A. \end{aligned}$$

Thus by calculating similarly $D_1B^2, \cdots, D_1B^{k_1-1}, D_2B, \cdots, D_2B^{k_2-1}, \cdots$, and $D_sB^{k_s-1}$, we can derive $T_B = [M\ \ 0]T_A$ for some $M \in \mathcal{L}_+^{m_B}$. In addition, since $[M\ \ 0]T_Ax \preceq 0 \leftrightarrow [I_{m_B}\ 0]T_Ax \preceq 0$, Σ_{AB} is equivalent to

$$\Sigma_A \begin{cases} \text{mode 1: } \dot{x} = Ax, & \text{if } T_Ax \succeq 0 \\ \text{mode 2: } \dot{x} = Bx, & \text{if } [I_{m_B}\ 0]T_Ax \preceq 0. \end{cases} \tag{31}$$

In the new coordinates $z \triangleq [z_1^T\quad z_2^T]^T = Tx$, where $T \triangleq [T_A^T\ \tilde{T}_A^T]^T$ for any $\tilde{T}_A \in \mathcal{T}_A$, Σ_A is transformed into

$$\tilde{\Sigma}_A \begin{cases} \text{mode 1: } \dot{z} = TAT^{-1}z, & \text{if } z_1 \succeq 0 \\ \text{mode 2: } \dot{z} = TBT^{-1}z, & \text{if } [I_{m_B}\ 0]z_1 \preceq 0. \end{cases} \tag{32}$$

Note here that TAT^{-1} is given by (28). On the other hand, it follows from b) that, in mode 2,

$$\dot{z}_{11} = [I_{m_B}\ 0]T_ABT^{-1}z = \Gamma[I_{m_B}\ 0]z_1 = \Gamma z_{11}$$

where z_{11} is the m_B-dimensional vector defined by $z_1 = [z_{11}^T\ z_{12}^T]^T$. Thus, smooth continuation in each mode is possible. Furthermore, from a) and d), which mean that the vector fields in both modes are the same on Ker T_B, i.e., the invariant subspace given by $z_{11}(0) = 0$, it follows that the solutions in both modes are the same when $z_{11}(0) = 0$ and $z_{12}(0) \succ 0$. Therefore, Σ_{AB} is well-posed. □

Compared with Theorem 4.2, ii)a) or iii)a) in Theorem 5.1 implies that the dimension of the invariant subspace KerT_B in mode 2 must be the same as either of the dimension of the invariant subspaces given by Ker $[T_{A_1}^T\ T_{A_2}^T\ \cdots T_{A_i}^T]^T$ $(i = 1, 2, \cdots, p)$ in mode 1. By this condition and ii)c) or iii)d), when solutions exist in both modes, they are necessarily the same. iii)c) comes from the relation between T_A and T_B on the k_ith row in ii)b).

Remark 5.1: When $p = 1$ and $s = 1$, Theorem 5.1 is reduced to Theorem 4.2, although $\mathcal{G}_0^{m_B}$ is replaced by $\mathcal{G}_+^{m_B}$ in iii)b). In the proof of Theorem 5.1, the condition iii)b) in Theorem 4.2 comes from the fact that $\tilde{B}_{11}$ in (30) is given by

$$\tilde{B}_{11} = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \ldots & 0 & 1 \\ * & \ldots & \ldots & \ldots & * \end{bmatrix} \in \mathcal{G}_+^{m_B}.$$

Remark 5.2: When in Theorem 5.1 we consider the case that the pairs (C, A) and (D, B) are observable (i.e., $m_A = m_B = n$), the condition ii) is reduced into $T_BT_A^{-1} \in \mathcal{L}_+^n$, and the condition iii) is reduced into $T_ABT_A^{-1} \in \mathcal{G}_0^n$ and iii)c).

Remark 5.3: The conditions in Theorem 5.1 can be checked as described in Remark 4.4. Namely, the conditions iii)b) and d) are replaced by (19) with $\mathcal{G}_0^{m_B}$ instead of $\mathcal{G}_+^{m_B}$, and (20).

Remark 5.4: In terms of $z = Tx$ and $w = \hat{T}x$ given in the proof of Theorem 5.1, where

$$\tilde{T}_B = \left[\frac{[0_{m_A-m_B, m_B}\ I_{m_A-m_B}]T_A}{\tilde{T}_A}\right],$$

every well-posed bimodal system given by (23), or equivalently (25), with rank $T_A = m_A$ and rank $T_B = m_B (\leq m_A)$ can be transformed into the following canonical form:

$$\tilde{\Sigma}_O \begin{cases} \text{mode 1: } \dot{z} = \left[\begin{array}{c|c} \tilde{A}_{11} & 0_{m_A, n-m_A} \\ \hline \tilde{A}_{21} & \tilde{A}_{22} \end{array}\right] z, \\ \quad \text{if } [I_{m_A}\ 0_{m_A, n-m_A}]z \succeq 0 \\ \text{mode 2: } \dot{w} = \left[\begin{array}{c|c} \tilde{B}_{11} & 0_{m_B, n-m_B} \\ \hline * & \tilde{B}_{22} \end{array}\right] w, \\ \quad \text{if } [I_{m_B}\ 0_{m_B, n-m_B}]w \preceq 0 \end{cases}$$

where $[I_{m_B}\ 0_{m_B, n-m_B}]w = [M\ 0_{m_B, n-m_B}]z$ for some $M \in \mathcal{L}_+^{m_B}$, $\Sigma_{i=1}^l h_i = m_B$ for some $l \in \{1, 2, \cdots, p\}$,

$$\tilde{B}_{22} = [0_{n-m_B, m_B}\ \ I_{n-m_B}] \left[\begin{array}{c|c} \tilde{A}_{11} & 0_{m_A, n-m_A} \\ \hline \tilde{A}_{21} & \tilde{A}_{22} \end{array}\right] \cdot \begin{bmatrix} 0_{m_B, n-m_B} \\ I_{n-m_B} \end{bmatrix},$$

and

$$\tilde{A}_{11} = \begin{bmatrix} \hat{A}_{11} & 0 & \ldots & 0 \\ \hat{A}_{21} & \hat{A}_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \hat{A}_{p1} & \ldots & \hat{A}_{p,p-1} & \hat{A}_{pp} \end{bmatrix} \in \mathcal{R}^{m_A \times m_A},$$

$$\tilde{B}_{11} = \begin{bmatrix} \hat{B}_{11} & 0 & \ldots & 0 \\ \hat{B}_{21} & \hat{B}_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \hat{B}_{s1} & \ldots & \hat{B}_{s,s-1} & \hat{B}_{ss} \end{bmatrix} \in \mathcal{R}^{m_B \times m_B},$$

$$\hat{A}_{ii} = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \ldots & 0 & 1 \\ * & \ldots & \ldots & \ldots & * \end{bmatrix} \in \mathcal{R}^{h_i \times h_i},$$

$$\hat{A}_{ij} = \begin{bmatrix} 0 & \ldots & \ldots & 0 \\ \vdots & & & \vdots \\ 0 & \ldots & \ldots & 0 \\ * & \ldots & \ldots & * \end{bmatrix} \in \mathcal{R}^{h_i \times h_j}, \qquad \text{for } i > j,$$

$$\hat{B}_{ii} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & 0 & 1 \\ * & \dots & \dots & \dots & * \end{bmatrix} \in \mathcal{R}^{k_i \times k_i},$$

$$\hat{B}_{ij} = \begin{bmatrix} 0 & \dots & \dots & 0 \\ \vdots & & & \vdots \\ 0 & \dots & \dots & 0 \\ * & \dots & \dots & * \end{bmatrix} \in \mathcal{R}^{k_i \times k_j}, \qquad \text{for } i > j.$$

From Lemma 3.5, we see that $\tilde{A}_{11}$ and $\tilde{B}_{11}$ are the same as the form (9) of all A-matrices for which the system has the smooth continuation property with respect to $x \succeq 0$.

Example 5.1: Let us check the well-posedness of the following simple example:

$$\Sigma_O \begin{cases} \text{mode 1: } \dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x, & \text{if } Cx \succeq 0 \\ \text{mode 2: } \dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix} x, & \text{if } Dx \leq 0 \end{cases}$$

where

$$C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad D = D_1 = [1\ 0\ 0].$$

Then we obtain $m_A = 3$ and $m_B = 2$ from

$$T_A = \begin{bmatrix} C_1 \\ C_1 A \\ C_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$T_B = \begin{bmatrix} D_1 \\ D_1 B \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Thus ii)a) is satisfied. From $T_B = [I_2\ 0]T_A$, we obtain ii)b). In addition, noting $T_A A T_A^{-1} = A$ and $T_A B T_A^{-1} = B$, c) is satisfied. Therefore, this system is well-posed, although (C, A) is observable and (D, B) is not observable.

VI. Extensions to Multi-Modal Cases

In this section, we extend several results for the case of bimodal systems given by (1) to the case of multi-modal systems with multiple criteria and multi-modal systems based on affine-type inequalities. We only discuss the observable case, as a first step to investigate to what extent our framework can be generalized, although the unobservable case may be extended in a similar way.

A. Multi-Modal Systems with Multiple Criteria

We here consider multi-modal systems with multiple criteria. For any matrix $C = [C_1^T\ C_2^T\ \cdots\ C_r^T]^T \in \mathcal{R}^{r \times n}$ where $r \leq n$, let the criterion vector be $y = [y_1\ y_2\ \cdots\ y_r]^T = Cx$. We assume throughout that there exists no constant k such that $C_i = kC_j$ for each $i, j \in \{1, 2, \cdots, r\}$. Let $\mathcal{I} \subset \{1, 2, \cdots, r\}$ be the index set satisfying $y_i \geq 0$ for $i \in \mathcal{I}$ and $y_i \leq 0$ for $i \notin \mathcal{I}$. The index set $\mathcal{I}$ represents the mode (location) of the system. Note that there are 2^r possible choices for the index set $\mathcal{I}$, and so there exist 2^r modes. Moreover, let $\mathcal{C}_I$ be a subset of R^n defined by

$$\mathcal{C}_I \triangleq \{\, x \in \mathcal{R}^n | y_i \geq 0 \text{ for } i \in \mathcal{I}, \quad y_i \leq 0 \text{ for } i \notin \mathcal{I}\}. \quad (33)$$

By numbering the index sets $\mathcal{I}$ from 1 to 2^r, we use the number $i \in \{1, 2, \cdots, 2^r\}$ in place of $\mathcal{I}$ to express the mode.

Then we consider the original 2^r-modal system Σ_O given by

$$\Sigma_O \begin{cases} \text{mode } 1: & \dot{x} = A_1 x, & \text{if } x \in \mathcal{C}_1 \\ \text{mode } 2: & \dot{x} = A_2 x, & \text{if } x \in \mathcal{C}_2 \\ \quad\vdots & \quad\vdots & \quad\vdots \\ \text{mode } 2^r: & \dot{x} = A_{2^r} x, & \text{if } x \in \mathcal{C}_{2^r} \end{cases} \quad (34)$$

where $x \in \mathcal{R}^n$. For example, for $r = 2$, we have the 4 modal system given by

$$\begin{cases} \text{mode 1: } \dot{x} = A_1 x,\ x \in \mathcal{C}_1 = \{x \in R^n | y_1 \geq 0,\ y_2 \geq 0\} \\ \text{mode 2: } \dot{x} = A_2 x,\ x \in \mathcal{C}_2 = \{x \in R^n | y_1 \geq 0,\ y_2 \leq 0\} \\ \text{mode 3: } \dot{x} = A_3 x,\ x \in \mathcal{C}_3 = \{x \in R^n | y_1 \leq 0,\ y_2 \geq 0\} \\ \text{mode 4: } \dot{x} = A_4 x,\ x \in \mathcal{C}_4 = \{x \in R^n | y_1 \leq 0,\ y_2 \leq 0\}. \end{cases} \quad (35)$$

In addition, we assume that every pair (C_i, A_k) $(i = 1, 2, \cdots, r;\ k = 1, 2, \cdots, 2^r)$ is observable. So the rule matrices

$$T_{A_k}^i \triangleq \begin{bmatrix} C_i \\ C_i A_k \\ \vdots \\ C_i A_k^{n-1} \end{bmatrix} \in \mathcal{R}^{n \times n}$$

are all nonsingular. So let $\mathcal{S}_I$ be a subset of $\mathcal{R}^n$ defined by

$$\mathcal{S}_I \triangleq \{x \in \mathcal{R}^n | T_{A_I}^i x \succeq 0 \text{ for } i \in \mathcal{I}, \quad T_{A_I}^i x \preceq 0 \text{ for } i \notin \mathcal{I}\}$$

Using the sets $\mathcal{S}_I$, we also define the 2^r-modal system Σ_{A_0} as follows:

$$\Sigma_{A_0} \begin{cases} \text{mode 1:} & \dot{x} = A_1 x, & \text{if } x \in \mathcal{S}_1 \\ \text{mode } 2: & \dot{x} = A_2 x, & \text{if } x \in \mathcal{S}_2 \\ \quad\vdots & \quad\vdots & \quad\vdots \\ \text{mode } 2^r: & \dot{x} = A_{2^r} x, & \text{if } x \in \mathcal{S}_{2^r}. \end{cases} \quad (36)$$

For a vector $x, y \in \mathcal{R}^n$, the notation $x \geq y$ expresses $x_i \geq y_i$ for all i. Similarly for the other notation $\leq$, $>$, and $<$. For a closed convex polyhedral cone $\mathcal{C} \triangleq \{x \in \mathcal{R}^n | Fx \geq 0\}$ where F is an $m \times n$ real matrix, let int $\mathcal{C}$ be the interior of $\mathcal{C}$ and let $\partial\mathcal{C}$ be the boundary of $\mathcal{C}$.

Then the following result is a natural extension to that for bimodal systems.

Theorem 6.1: Suppose that every pair (C_i, A_j) $(i = 1, 2, \cdots, r;\ j = 1, 2, \cdots, 2^r)$ is observable. Then the following statements are equivalent.

i) Σ_O is well-posed.
ii) Σ_{A_0} is well-posed.
iii) $\bigcup_{j=1}^{2^r} \mathcal{S}_j = \mathcal{R}^n$ and $\mathcal{S}_j \bigcap \mathcal{S}_k = \{0\}$ for all $j, k(\neq j) \in \{1, 2, \cdots, 2^r\}$.

Proof: i)$\leftrightarrow$ii) can be proven in the same way as Lemma 2.3. i)$\rightarrow$iii). Since $\mathcal{S}_i$ is a set of all initial states from which smooth continuation is possible in mode i, it follows that $\bigcup_{j=1}^{2^r} \mathcal{S}_j = \mathcal{R}^n$. In order to prove the latter part of iii), we assume that there exists some j and $k(\neq j)$ such that

$\mathcal{S}_j \bigcap \mathcal{S}_k \neq \{0\}$ and $\mathcal{S}_j \bigcap \mathcal{S}_k \neq \emptyset$. So let $x_*(\neq 0)$ be an element of $\mathcal{S}_j \bigcap \mathcal{S}_k$. Then for some $\varepsilon > 0$, the solution in mode j from the initial state x_* satisfies $x(t) \in \operatorname{int} \mathcal{C}_j$ for all $t \in (0, \varepsilon]$, while the solution in mode k from x_* satisfy $x(t) \in \operatorname{int} \mathcal{C}_k$ because of observability. This implies that the solution is not unique, which is in contradiction with ii). Hence the latter part of iii) holds. iii)→i) follows from the multi-modal version of Lemma 2.1. □

From Theorem 6.1, it turns out that the well-posedness of Σ_O is characterized by condition iii). When is condition iii) satisfied? It seems difficult to interpret condition iii) in terms of some simple algebraic relation between the matrices $T^i_{A_j}$ as in the case of bimodal systems. However, we give below an algorithm to check condition iii).

First, the following simple lemma is useful for the algorithm.

Lemma 6.1: Let $\mathcal{S}$ be a set defined by $\{x \in \mathcal{R}^n | T_i x \succeq 0, \; i = 1, 2, \cdots, r\}$ where T_i is an $n \times n$ real matrix, and let $\mathcal{C}$ be a set defined by $\{x \in \mathcal{R}^n | Cx = 0\}$ where C is a $1 \times n$ real matrix. Then there exist $(n-1) \times (n-1)$ matrices $\overline{T}_i$ $(i = 1, 2, \cdots, r)$ such that

$$\mathcal{S} \bigcap \mathcal{C} = \{z \in \mathcal{R}^{n-1} | \overline{T}_i z \succeq 0, \; i = 1, 2, \cdots, r\} \tag{37}$$

Proof: In the new coordinates $\overline{z} = [w \; z^T]^T = Mx$ where $M = [C^T \; \tilde{T}^T]^T$ is nonsingular for some $\tilde{T} \in \mathcal{R}^{n-1 \times n}$, and $w = Cx$ and $z = \tilde{T}x$, we have $T_i x = T_i M^{-1} \overline{z} \succeq 0$. So when $w = 0$, this yields

$$T_i M^{-1} \begin{bmatrix} 0_{1, n-1} \\ I_{n-1} \end{bmatrix} \tilde{T} x \succeq 0 \tag{38}$$

Then by applying Lemma 3.1 to (38), we can derive an $(n-1) \times (n-1)$ matrix $\overline{T}_i$ in (37). □

In order to clarify the idea of the algorithm, let us first discuss the necessity of condition iii) in Theorem 6.1.

Suppose that condition iii) in Theorem 6.1 holds. Then we have

$$\bigcup_{i=1}^{2^r} \mathcal{C}_i = \mathcal{R}^n, \; \mathcal{C}_j \bigcap \mathcal{C}_k = \{0\} \text{ or } = \partial\mathcal{C}_j \bigcap \partial\mathcal{C}_k, \quad \forall\, j, k \in \{1, 2, \cdots, 2^r\} \tag{39}$$

where $\mathcal{C}_i$ is given by (33). Next, let us consider a necessary condition for condition iii) with respect to the set of x satisfying $\mathcal{C}_j \bigcap \mathcal{C}_k = \partial\mathcal{C}_j \bigcap \partial\mathcal{C}_k$, which is given by $\bigcup_{\alpha_1=1}^{r} \{x \in \mathcal{R}^n | C_{\alpha_1} x = 0\}$. So for each $\alpha_1 \in \{1, 2, \cdots, r\}$, we consider the set defined by

$$\mathcal{S}^{(1)}_{i, \alpha_1} = \mathcal{S}_i \bigcap \{x \in \mathcal{R}^n \mid C_{\alpha_1} x = 0\}, \qquad i = 1, 2, \cdots, 2^r \tag{40}$$

Note here that, from Lemma 6.1, $\mathcal{S}^{(1)}_{i, \alpha_1}$ is a set in $\mathcal{R}^{n-1}$ which is expressed by

$$\mathcal{S}^{(1)}_{i, \alpha_1} = \{z \in \mathcal{R}^{n-1} | T^{(1)}_{i, \alpha_1, j} z \succeq 0, \quad j = 1, 2, \cdots, r\}$$

where $T^{(1)}_{i, \alpha_1, j}$ is an $(n-1) \times (n-1)$ matrix. Concerning $\mathcal{S}^{(1)}_{i, \alpha_1}$, a necessary condition for condition iii) is that for each $\alpha_1 \in \{1, 2, \cdots, r\}$

$$\bigcup_{i=1}^{2^r} \mathcal{S}^{(1)}_{i, \alpha_1} = \mathcal{R}^{n-1}, \quad \mathcal{S}^{(1)}_{j, \alpha_1} \bigcap \mathcal{S}^{(1)}_{k, \alpha_1} = \{0\}, \quad \forall\, j, k(\neq j) \in \{1, 2, \cdots, 2^r\}. \tag{41}$$

Noting that the condition (41) has a similar form to that of condition iii), we will repeat the above discussion for (41). So let $C^{(1)}_{i, \alpha_1, j}$ be the first row vector of the matrix $T^{(1)}_{i, \alpha_1, j}$, and let $\mathcal{C}^{(1)}_{i, \alpha_1}$ be defined by

$$\mathcal{C}^{(1)}_{i, \alpha_1} = \{z \in \mathcal{R}^{n-1} | C^{(1)}_{i, \alpha_1, j} z \geq 0, \quad j = 1, 2, \cdots, r\} \tag{42}$$

Then if (41) holds for each $\alpha_1 \in \{1, 2, \cdots, r\}$, the following relation on the first row of the lexicographic inequalities must hold for each $\alpha_1 \in \{1, 2, \cdots, r\}$:

$$\bigcup_{i=1}^{2^r} \mathcal{C}^{(1)}_{i, \alpha_1} = \mathcal{R}^{n-1}, \quad \mathcal{C}^{(1)}_{j, \alpha_1} \bigcap \mathcal{C}^{(1)}_{k, \alpha_1} = \{0\} \text{ or } = \partial\mathcal{C}^{(1)}_{j, \alpha_1} \bigcap \partial\mathcal{C}^{(1)}_{k, \alpha_1}, \quad \forall\, j, k(\neq j) \in \{1, 2, \cdots, 2^r\}. \tag{43}$$

Next let us consider a necessary condition for (41) with respect to the set of z satisfying $\mathcal{C}^{(1)}_{j, \alpha_1} \bigcap \mathcal{C}^{(1)}_{k, \alpha_1} = \partial\mathcal{C}^{(1)}_{j, \alpha_1} \bigcap \partial\mathcal{C}^{(1)}_{k, \alpha_1}$. Note that this set is included in the union of the sets given by $\{z \in \mathcal{R}^{n-1} | C^{(1)}_{(\alpha_1, \alpha_2)} z = 0\}$ for all $\alpha_2 \in \{1, 2, \cdots, r_{\alpha_1}\}$, where $C^{(1)}_{(\alpha_1, \alpha_2)}$ is given by $C^{(1)}_{i, \alpha_1, j}$ for some (i, α_1, j), and r_{α_1} is some finite number. So if we define the set

$$\mathcal{S}^{(2)}_{i, (\alpha_1, \alpha_2)} = \mathcal{S}^{(1)}_{i, \alpha_1} \bigcap \left\{ z \in \mathcal{R}^{n-1} | C^{(1)}_{(\alpha_1, \alpha_2)} z = 0 \right\}, \tag{44}$$

a necessary condition for (41) is given by for each $\alpha_2 \in \{1, 2, \cdots, r_{\alpha_1}\}$

$$\bigcup_{i=1}^{2^r} \mathcal{S}^{(2)}_{i, (\alpha_1, \alpha_2)} = \mathcal{R}^{n-2}, \quad \mathcal{S}^{(2)}_{j, (\alpha_1, \alpha_2)} \bigcap \mathcal{S}^{(2)}_{k, (\alpha_1, \alpha_2)} = \{0\}, \quad \forall\, j, k(\neq j) \in \{1, 2, \cdots, 2^r\} \tag{45}$$

and also concerning the first row of the lexicographic inequalities in $\mathcal{S}^{(2)}_{i, (\alpha_1, \alpha_2)}$, (45) implies

$$\bigcup_{i=1}^{2^r} \mathcal{C}^{(2)}_{i, (\alpha_1, \alpha_2)} = \mathcal{R}^{n-2}, \quad \mathcal{C}^{(2)}_{j, (\alpha_1, \alpha_2)} \bigcap \mathcal{C}^{(2)}_{k, (\alpha_1, \alpha_2)} = \{0\} \text{ or } = \partial\mathcal{C}^{(2)}_{j, (\alpha_1, \alpha_2)} \bigcap \partial\mathcal{C}^{(2)}_{k, (\alpha_1, \alpha_2)}, \quad \forall\, j, k(\neq j) \in \{1, 2, \cdots, 2^r\} \tag{46}$$

where $\mathcal{C}^{(2)}_{i, (\alpha_1, \alpha_2)}$ is defined in a similar way to (42).

Thus in a similar way, for $h \in \{1, 2, \cdots, n-1\}$, we define the set $\mathcal{S}^{(h)}_{i, \overline{\alpha}_h}$ as

$$\mathcal{S}^{(h)}_{i, \overline{\alpha}_h} = \mathcal{S}^{(h-1)}_{i, \overline{\alpha}_{h-1}} \bigcap \left\{ z \in \mathcal{R}^{n-h+1} | C^{(h-1)}_{\overline{\alpha}_h} z = 0 \right\} \tag{47}$$

where $\overline{\alpha}_h$ implies $(\alpha_1, \alpha_2, \cdots, \alpha_h)$, $\alpha_s \in \{1, 2, \cdots, r_{\overline{\alpha}_{s-1}}\}$ $(s = 1, 2, \cdots, h)$ with $r_{\overline{\alpha}_0} = r$, $\mathcal{S}^{(0)}_{i, \overline{\alpha}_0} = \mathcal{S}_i$, and $C^{(0)}_{\overline{\alpha}_1} =$

C_{α_1}. Note that $\mathcal{S}^{(h)}_{i,\overline{\alpha}_h}$ is expressed using some $T^{(h)}_{i,\overline{\alpha}_h,j}$ $(j = 1, 2, \cdots, r)$ (see Lemma 6.1). Furthermore, let $C^{(h)}_{i,\overline{\alpha}_h,j}$ be the first row vector of the matrix $T^{(h)}_{i,\overline{\alpha}_h,j}$, and let $\mathcal{C}^{(h)}_{i,\overline{\alpha}_h}$ be defined by

$$\mathcal{C}^{(h)}_{i,\overline{\alpha}_h} = \left\{ z \in \mathcal{R}^{n-h} | C^{(h)}_{i,\overline{\alpha}_h,j} z \geq 0, \quad j = 1, 2, \cdots, r \right\}.$$

Then we can show that for each $\alpha_s \in \{1, 2, \cdots, r_{\overline{\alpha}_{s-1}}\}$ $(s = 1, 2, \cdots, h)$, the following relation must hold:

$$\begin{aligned} &\bigcup_{i=1}^{2^r} \mathcal{C}^{(h)}_{i,\overline{\alpha}_h} = \mathcal{R}^{n-h}, \\ &\mathcal{C}^{(h)}_{j,\overline{\alpha}_h} \bigcap \mathcal{C}^{(h)}_{k,\overline{\alpha}_h} = \begin{cases} \{0\} \text{ or } \partial\mathcal{C}^{(h)}_{j,\overline{\alpha}_h} \bigcap \partial\mathcal{C}^{(h)}_{k,\overline{\alpha}_h}, \\ \quad \text{if } h = 1, 2, \cdots, n-2 \\ \{0\}, \\ \quad \text{if } h = n-1, \end{cases} \\ &\forall j, k (\neq j) \in \{1, 2, \cdots, 2^r\}. \end{aligned} \tag{48}$$

From the converse argument of the above one, we see that if (48) holds for each $h \in \{1, 2, \cdots, n-1\}$ and each $\alpha_s \in \{1, 2, \cdots, r_{\overline{\alpha}_{s-1}}\}$ $(s = 1, 2, \cdots, h)$, then condition iii) in Theorem 6.1 holds.

Next, let us show how to check (48). The first condition of (48) is equivalent to

$$\begin{aligned} &\bigcap_{i=1}^{2^r} \left\{ z \in \mathcal{R}^{n-h} | C^{(h)}_{i,\overline{\alpha}_h,j_i} z < 0 \right\} = \emptyset, \\ &\forall\, j_1, j_2, \cdots, j_{2^r} \in \{1, 2, \cdots, r\}. \end{aligned} \tag{49}$$

So letting

$$F^{(h)}_{\overline{\alpha}_h,\overline{j}_{2^r}} = \begin{bmatrix} C^{(h)}_{1,\overline{\alpha}_h,j_1} \\ C^{(h)}_{2,\overline{\alpha}_h,j_2} \\ \vdots \\ C^{(h)}_{2^r,\overline{\alpha}_h,j_{2^r}} \end{bmatrix}$$

where $\overline{j}_{2^r}$ implies $(j_1, j_2, \cdots, j_{2^r})$, (49) is rewritten by

$$\begin{aligned} &\left\{ z \in \mathcal{R}^{n-h} | F^{(h)}_{\overline{\alpha}_h,\overline{j}_{2^r}} z < 0 \right\} = \emptyset, \\ &\forall\, j_1, j_2, \cdots, j_{2^r} \in \{1, 2, \cdots, r\} \end{aligned} \tag{50}$$

Thus we only have to solve the feasibility problem of the form $F^{(h)}_{\overline{\alpha}_h,\overline{j}_{2^r}} z < 0$. An answer of this kind of problem is given, for example, by solving the following linear programming: min λ subject to $F^{(h)}_{\overline{\alpha}_h,\overline{j}_{2^r}} z \leq \lambda e$ or min λ subject to $F^{(h)}_{\overline{\alpha}_h,\overline{j}_{2^r}} z \leq \lambda e$ and $-e \leq z \leq e$, where e is some vector with all elements positive. Letting λ_* be an optimal solution, if $\lambda_* = 0$, then the set $\{z \in \mathcal{R}^{n-h} | F^{(h)}_{\overline{\alpha}_h,\overline{j}_{2^r}} z < 0\}$ is empty, and if $\lambda_* < 0$, then it is not an empty set.

Concerning the second condition of (48), on the other hand, the following lemma is obtained.

Lemma 6.2: Let $\mathcal{C}_i$ be a set defined by $\mathcal{C}_i \triangleq \{x \in \mathcal{R}^n | F_i x \geq 0\}$ $(i = 1, 2)$ where F_i is an $m_i \times n$ real matrix. Then the following statements are equivalent.

i) $\mathcal{C}_1 \bigcap \mathcal{C}_2 = \{0\}$ or $= \partial\mathcal{C}_1 \bigcap \partial\mathcal{C}_2$.

ii) $\text{int}\,\mathcal{C}_1 \bigcap \text{int}\,\mathcal{C}_2 = \emptyset$, i.e., $\{x \in \mathcal{R}^n | F_1 x > 0\} \bigcap \{x \in \mathcal{R}^n | F_2 x > 0\} = \emptyset$.

Proof: i)→ii) is trivial. ii)→i). We only have to show that if ii) holds, then there also exist no elements in the intersection of the boundary of a closed convex polyhedral cone and the interior of another cone. Let f_{11} be the 1st row vector of F_1 and let $\overline{F}_1$ be the matrix such that $F_1 = [f_{11}^T\ \overline{F}_1^T]^T$. Then we will show $\mathcal{N} = \emptyset$ where

$$\mathcal{N} \triangleq \{x \in \mathcal{R}^n | f_{11} x = 0, \quad \overline{F}_1 x > 0\} \bigcap \text{int}\,\mathcal{C}_2.$$

Assume $\mathcal{N} \neq \emptyset$, and let x_* be an element of $\mathcal{N}$. Note that an element of $\mathcal{C}_1$ can be expressed by $x = \sum_{i=1}^{m_1} \alpha_i u_i + \{\text{an element of Ker } F_1\}$, where $\alpha_i \geq 0$ and $F_1 u_i = e_i$ (the ith element of e_i is 1 and the others are 0). So x_* is expressed by $x_* = \sum_{i=2}^{m_1} \alpha_i u_i + \{\text{an element of Ker } F_1\}$ where $\alpha_i > 0$. Now for $\tilde{x}_* = x_* + \varepsilon u_1$ where $\varepsilon > 0$ is sufficiently small, we have $\tilde{x}_* \in \text{int}\,\mathcal{C}_1 \bigcap \text{int}\,\mathcal{C}_2$, which implies that ii) is not true. Hence, it follows that if ii) is true, then $\mathcal{N} = \emptyset$. For any other boundary of $\mathcal{C}_i$, similar discussion holds. This completes the proof. □

Thus by Lemma 6.2, the second condition of (48) can be also checked using, e.g., the linear programming.

Based on the above discussion, an algorithm for checking condition iii) is given as follows.

Step 1: Set $\overline{h} = 0$.

Step 2: Set $h = \overline{h} + 1$. For each $\alpha_s \in \{1, 2, \cdots, r_{\overline{\alpha}_{s-1}}\}$ $(s = 1, 2, \cdots, h)$ and each $i \in \{1, 2, \cdots, 2^r\}$ derive $\mathcal{S}^{(h)}_{i,\overline{\alpha}_h}$ and $\mathcal{C}^{(h)}_{i,\overline{\alpha}_h}$.

Step 3: Check whether (48) is true or not for each $\alpha_s \in \{1, 2, \cdots, r_{\overline{\alpha}_{s-1}}\}$ $(s = 1, 2, \cdots, h)$. If it is true for all cases, then go to Step 2 if $h < n - 1$, or we conclude that condition iii) holds if $h = n - 1$. Otherwise, we conclude that condition iii) is not satisfied.

Since (39) is always satisfied, the statement on (39) is omitted in the above algorithm. The proposed algorithm includes some redundant calculations, so it will have to be refined from the viewpoint of its computational complexity. However, the algorithm is meaningful in the sense that it provides one of approaches to determine systematically the well-posedness in the sense of Carathéodory of any multi-modal piecewise-linear system (34).

Finally, we give a simple example to illustrate the idea of the proposed algorithm.

Example 6.1: Consider the 4-modal system of (35) where

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix},$$

$$A_4 = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix}$$

and, $C_1 = [1\ 0\ 0]$ and $C_2 = [0\ 1\ 0]$. Then we have

$$T^1_{A_1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad T^2_{A_1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix},$$
$$T^1_{A_2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & -1 & -1 \end{bmatrix}, \quad T^2_{A_2} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix},$$
$$T^1_{A_3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad T^2_{A_3} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix},$$
$$T^1_{A_4} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & -1 & 1 \end{bmatrix}, \quad T^2_{A_4} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & -1 \\ -1 & 1 & 0 \end{bmatrix}.$$

Step 2 ($h = 1$): For $\alpha_1 = 1$, we have in $\mathcal{S}^{(1)}_{i,1}$ $(i = 1, 2, 3, 4)$

$$T^{(1)}_{1,1,1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad T^{(1)}_{1,1,2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$
$$T^{(1)}_{2,1,1} = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}, \quad T^{(1)}_{2,1,2} = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix},$$
$$T^{(1)}_{3,1,1} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad T^{(1)}_{3,1,2} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$
$$T^{(1)}_{4,1,1} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}, \quad T^{(1)}_{4,1,2} = \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix}.$$

and in $\mathcal{C}^{(1)}_{i,1}$ $(i = 1, 2, 3, 4)$

$$C^{(1)}_{1,1,1} = [1\ \ 0], \quad C^{(1)}_{1,1,2} = [1\ \ 0],$$
$$C^{(1)}_{2,1,1} = [-1\ \ 0], \quad C^{(1)}_{2,1,2} = [-1\ \ 0],$$
$$C^{(1)}_{3,1,1} = [-1\ \ 0], \quad C^{(1)}_{3,1,2} = [1\ \ 0],$$
$$C^{(1)}_{4,1,1} = [1\ \ 0], \quad C^{(1)}_{4,1,2} = [-1\ \ 0].$$

For $\alpha_1 = 2$, $\mathcal{S}^{(1)}_{i,2}$ and $\mathcal{C}^{(1)}_{i,2}$ $(i = 1, 2, 3, 4)$ are obtained similarly. **Step 3** ($h = 1$): It can be easily verified that (48) holds for $h = 1$. **Step 2** ($h = 2$): For $\alpha_1 = 1$, we obtain $r_{\alpha_1} = 1$, and $C^{(1)}_{(1,1)} = [1 \quad 0]$ for $\alpha_2 = 1$. Then we have in $\mathcal{S}^{(2)}_{i,(1,1)}$ $(= \mathcal{C}^{(2)}_{i,(1,1)})$ $(i = 1, 2, 3, 4)$

$$T^{(2)}_{1,(1,1),1} = C^{(2)}_{1,(1,1),1} = 1, \quad T^{(2)}_{1,(1,1),2} = C^{(2)}_{1,(1,1),2} = 1,$$
$$T^{(2)}_{2,(1,1),1} = C^{(2)}_{2,(1,1),1} = -1, \quad T^{(2)}_{2,(1,1),2} = C^{(2)}_{2,(1,1),2} = -1,$$
$$T^{(2)}_{3,(1,1),1} = C^{(2)}_{3,(1,1),1} = 1, \quad T^{(2)}_{3,(1,1),2} = C^{(2)}_{3,(1,1),2} = -1,$$
$$T^{(2)}_{4,(1,1),1} = C^{(2)}_{4,(1,1),1} = -1, \quad T^{(2)}_{4,(1,1),2} = C^{(2)}_{4,(1,1),2} = 1.$$

On the other hand, for $\alpha_1 = 2$, we obtain $r_{\alpha_1} = 2$, and $C^{(1)}_{(2,1)} = [1 \quad 0]$ for $\alpha_2 = 1$ and $C^{(1)}_{(2,2)} = [0 \quad 1]$ for $\alpha_2 = 2$. Then similarly we can derive $\mathcal{S}^{(2)}_{i,(\alpha_1,\alpha_2)}$ (=$\mathcal{C}^{(2)}_{i,(\alpha_1,\alpha_2)}$) for $(\alpha_1, \alpha_2) = (2, 1), (2, 2)$. **Step 3** ($h = 2$): It is verified that (48) is satisfied for $h = 2$ and for every $(\alpha_1, \alpha_2) = (1, 1), (2, 1), (2, 2)$. Thus we conclude that condition iii) of Theorem 6.1 is satisfied for this system, and so the system is well-posed.

B. Multi-Modal Systems with Affine Inequalities

We here start with the bimodal system given by

$$\Sigma_O(\alpha) \begin{cases} \text{mode 1: } \dot{x} = Ax, & \text{if } Cx \geq \alpha \\ \text{mode 2: } \dot{x} = Bx, & \text{if } Cx \leq \alpha \end{cases} \tag{51}$$

where $x \in \mathcal{R}^n$, $C \in \mathcal{R}^{1\times n}$, and $\alpha \in R$ is any given constant. Note that the inequality constraint is affine. This system is equivalent to the following system:

$$\Sigma_{AB}(\alpha) \begin{cases} \text{mode 1: } \dot{x} = Ax, & \text{if } x \in \overline{\mathcal{S}}^+_A(\alpha) \\ \text{mode 2: } \dot{x} = Bx, & \text{if } x \in \overline{\mathcal{S}}^-_B(\alpha) \end{cases} \tag{52}$$

where

$$\overline{\mathcal{S}}^+_A = \{x \in \mathcal{R}^n | \overline{T}_A x \succeq \overline{\alpha}\}, \quad \overline{\mathcal{S}}^-_B = \{x \in \mathcal{R}^n | \overline{T}_B x \preceq \overline{\alpha}\},$$
$$\overline{T}_A = \begin{bmatrix} T_A \\ CA^n \end{bmatrix}, \quad \overline{T}_B = \begin{bmatrix} T_B \\ CB^n \end{bmatrix}$$

and $\overline{\alpha} = [\alpha\ 0\ \cdots\ 0]^T \in \mathcal{R}^{n+1}$, and T_A and T_B are defined by (12). In fact, if $Cx(t) \geq \alpha$ in $\Sigma_O(\alpha)$ on some time interval, then $\overline{T}_A x(t) \succeq \overline{\alpha}$ on that interval. Conversely, if $\overline{T}_A x(t) \succ \overline{\alpha}$ on some time interval, then $Cx(t) \geq \alpha$ on that interval, and if $\overline{T}_A x(t) = \overline{\alpha}$, then $Cx \equiv \alpha$. The same argument holds in mode 2. Thus each mode in $\Sigma_O(\alpha)$ is identified with each mode in $\Sigma_{AB}(\alpha)$. Denote by $(M)_{(i,j)}$ the (i, j) element of a matrix M.

Theorem 6.2: Suppose that both pairs (C, A) and (C, B) are observable. Then for any given constant $\alpha \in \mathcal{R}$, the following statements are equivalent.

i) $\Sigma_O(\alpha)$ is well-posed.
ii) $\Sigma_{AB}(\alpha)$ is well-posed.
iii) The following conditions are satisfied.
  a) $\Sigma_{AB}(0)$ is well-posed.
  b) $(T_B T_A^{-1})_{(1,j)} = 0$, $j = 2, 3, \cdots, n$.
  c) (c1) $(T_A A T_A^{-1})_{(n,1)}\alpha \geq 0$ and $(T_B B T_B^{-1})_{(n,1)}\alpha > 0$, or (c2) $(T_A A T_A^{-1})_{(n,1)}\alpha < 0$ and $(T_B B T_B^{-1})_{(n,1)}\alpha \leq 0$, or (c3) $(T_A A T_A^{-1})_{(n,1)}\alpha = 0$ and $(T_B B T_B^{-1})_{(n,1)}\alpha = 0$.

Proof: i)↔ii) has already been proven. ii)→iii). In the two new coordinates $z \triangleq T_A x - \tilde{\alpha}$ and $w \triangleq T_B x - \tilde{\alpha}$, where $\tilde{\alpha} = [\alpha\ 0\ \cdots 0]^T \in \mathcal{R}^n$, $\Sigma_{AB}(\alpha)$ is described by

$$\tilde{\Sigma}_{AB}(\alpha) \begin{cases} \text{mode 1: } \dot{z} = T_A A T_A^{-1} z + T_A A T_A^{-1}\tilde{\alpha}, \\ \qquad\qquad \text{if } \begin{bmatrix} z \\ CA^n T_A^{-1}(z + \tilde{\alpha}) \end{bmatrix} \succeq 0 \\ \text{mode 2: } \dot{w} = T_B B T_B^{-1} w + T_B B T_B^{-1}\tilde{\alpha}, \\ \qquad\qquad \text{if } \begin{bmatrix} w \\ CB^n T_B^{-1}(w + \tilde{\alpha}) \end{bmatrix} \preceq 0. \end{cases} \tag{53}$$

So from ii) it follows that $\{x \in \mathcal{R}^n | z \succ 0\} \bigcap \{x \in \mathcal{R}^n | w \prec 0\} = \emptyset$, which implies that $w \succeq 0$ for all $z \succ 0$. Since $w = T_B T_A^{-1} z + (T_B T_A^{-1} - I)\tilde{\alpha}$, this means that $(T_B T_A^{-1} - I)\tilde{\alpha} \succeq 0$. On the other hand, from ii) it follows that $w \preceq 0$ for all $z \prec 0$, which means that $(T_B T_A^{-1} - I)\tilde{\alpha} \preceq 0$. Therefore, we have

$(T_B T_A^{-1} - I)\tilde{\alpha} = 0$, which leads to iii)b). In addition, from Lemma 3.3, we have $T_B = MT_A$ for some $M \in \mathcal{L}_+^n$.

Now let us consider $z(0) = 0$, where both modes may be admissible because of $w = Mz$. For mode 1, if $CA^n T_A^{-1}\tilde{\alpha} \geq (<)0$, smooth continuation is (not) possible, while for mode 2, if $CB^n T_B^{-1}\tilde{\alpha} \leq (>)0$, smooth continuation is (not) possible. From Lemma 2.1, smooth continuation in both modes is not possible at the same time, except for the origin $z(t) = w(t) = 0$. Hence iii)c) holds.

iii)→ii). Consider (53), where z and w are defined above. iii)a) and b) imply $w = Mz$ for some $M \in \mathcal{L}_+^n$. Thus in each case of $z(0) \succ 0$ and $w(0) \prec 0$, smooth continuation in only one of the two modes is possible. In addition, when $z(0) = 0$, iii)c) guarantees smooth continuation in only one of the two modes or $z(t) = w(t) \equiv 0$. From Lemma 2.1, this implies ii). □

This theorem asserts that the well-posedness of $\Sigma_O(\alpha)$ for all $\alpha \in \mathcal{R}$ is characterized by that of $\Sigma_O(0)$, provided that iii)c) holds. In iii)c), (c1) implies that, whenever $z(0) = 0$, smooth continuation in mode 1 is possible, while not in mode 2. (c2) implies the converse situation of (c1). In addition, (c3) corresponds to the case that smooth continuation in both modes is possible and their solutions are the same.

Remark 6.1: Let $\overline{A}$, $\overline{C}$, and $\overline{x}$ be defined by

$$\overline{A} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \quad \overline{C} = [C \quad -1], \quad \overline{x} = \begin{bmatrix} x \\ \eta \end{bmatrix}$$

Then the system $\Sigma_O(\alpha)$ is rewritten as

$$\Sigma_O(\alpha) \begin{cases} \text{mode 1: } \dot{\overline{x}} = \overline{A}\overline{x}, & \text{if } \overline{C}\overline{x} \geq 0 \\ \text{mode 2: } \dot{\overline{x}} = \overline{B}\overline{x}, & \text{if } \overline{C}\overline{x} \leq 0 \end{cases}$$

which has the same form as the system Σ_O of equations (1). Thus an alternative approach to derive a well-posedness condition of $\Sigma_O(\alpha)$ will be to directly apply the results derived in the previous sections. However, it is noted that the proof based on this approach is not straightforward (although possible), since we have to take into account the following points: an additional condition $\eta(0) = \alpha$ is required in this case, and also a pair $(\overline{C}, \overline{A})$ may not be observable even when the pair (C, A) is observable.

Remark 6.2: In the case of polyhedral sets, instead of the constraints sets given by affine inequalities, some extension may be possible by considering the intersection of sets such as $\overline{\mathcal{S}}_A^+(\alpha)$. Furthermore, the case such as $y = h(x) \leq 0$ or ≥ 0 may be discussed. These extensions are topics for further research.

Based on the above result, we consider the well-posedness of the following r-modal system:

$$\Sigma_O(\alpha_1, \alpha_2, \cdots, \alpha_{r-1}) \begin{cases} \text{mode 1:} & \dot{x} = A_1 x, & \text{if } x \in \mathcal{C}_1 \\ \text{mode 2:} & \dot{x} = A_2 x, & \text{if } x \in \mathcal{C}_2 \\ \quad\vdots & \quad\vdots & \quad\vdots \\ \text{mode } r: & \dot{x} = A_r x, & \text{if } x \in \mathcal{C}_r \end{cases} \tag{54}$$

where $x \in \mathcal{R}^n$, $\alpha_1 > \alpha_2 > \cdots > \alpha_{r-1}$ are any real numbers, and

$$\begin{aligned} \mathcal{C}_1 &= \{x \in \mathcal{R}^n | Cx \geq \alpha_1\}, \\ \mathcal{C}_i &= \{x \in \mathcal{R}^n | \alpha_{i-1} \geq Cx \geq \alpha_i\}, \quad i \in \{2, \cdots, r-1\}, \\ \mathcal{C}_r &= \{x \in \mathcal{R}^n | \alpha_{r-1} \geq Cx\}, \end{aligned}$$

and $C \in \mathcal{R}^{1\times n}$. Let us also introduce the bimodal system given by

$$\Sigma_O(A_i, A_{i+1}, \alpha_i) \begin{cases} \text{mode } i\text{: } \dot{x} = A_i x, \\ \quad \text{if } x \in \{x \in \mathcal{R}^n | Cx \geq \alpha_i\} \\ \text{mode } i+1\text{: } \dot{x} = A_{i+1} x, \\ \quad \text{if } x \in \{x \in \mathcal{R}^n | Cx \leq \alpha_i\} \end{cases} \tag{55}$$

for $i \in \{1, 2, \cdots, r-1\}$. Then noting that we only have to focus on smooth continuation from the initial state x satisfying $Cx = \alpha_i$ $(i = 1, 2, \cdots, r-1)$ to show the well-posedness of the system $\Sigma_O(\alpha_1, \alpha_2, \cdots, \alpha_{r-1})$, the following fact will be straightforwardly obtained.

Theorem 6.3: The multi-modal system $\Sigma_O(\alpha_1, \alpha_2, \cdots, \alpha_{r-1})$ is well-posed if and only if the bimodal system $\Sigma_O(A_i, A_{i+1}, \alpha_i)$ is well-posed for all $i \in \{1, 2, \cdots, r-1\}$.

Using Theorem 6.3, we can determine whether the multi-modal system $\Sigma_O(\alpha_1, \alpha_2, \cdots, \alpha_{r-1})$ is well-posed or not, as shown in the example below.

Example 6.2: Consider the physical system in Fig. 3. Assume that $k_1 = 0$, $\alpha_1 = 0$, and $\alpha_2 = -1$. Then the dynamics of the system is given by

$$\Sigma_O(0, -1) \begin{cases} \text{mode 1: } \dot{x} = A_1 x, \\ \qquad\qquad \text{if } x \in \{x \in \mathcal{R}^n | Cx \geq 0\} \\ \text{mode 2: } \dot{x} = A_2 x, \\ \qquad\qquad \text{if } x \in \{x \in \mathcal{R}^n | 0 \geq Cx \geq -1\} \\ \text{mode 3: } \dot{x} = A_3 x, \\ \qquad\qquad \text{if } x \in \{x \in \mathcal{R}^n | -1 \geq Cx\} \end{cases}$$

where $x = [x_1, \ x_2]^T$, $C = [1 \quad 0]$, and

$$A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ 0 & -d_1 \end{bmatrix},$$
$$A_3 = \begin{bmatrix} 0 & 1 \\ -k_2 & -d_1 - d_2 \end{bmatrix}.$$

Then for $\Sigma_O(A_i, A_{i+1}, \alpha_i)$ $(i = 1, 2)$ we obtain $T_{A_1} = T_{A_2} = T_{A_3} = I_2$ where T_{A_i} $(i = 1, 2, 3)$ is the rule matrix. Thus for $i = 1$, since $\alpha_1 = 0$ and $T_{A_1} T_{A_2}^{-1} \in \mathcal{L}_+^2$, $\Sigma_O(A_1, A_2, 0)$ is well-posed. For $i = 2$, on the other hand, $T_{A_2} T_{A_3}^{-1} \in \mathcal{L}_+^2$ implies iii)a) in Theorem 6.2. In addition, we have $(T_{A_2} A_2 T_{A_2}^{-1})_{(2,1)}\alpha_2 = 0$ and $(T_{A_3} A_3 T_{A_3}^{-1})_{(2,1)}\alpha_2 = k_2$, which implies that iii)b) holds for any $k_2 \geq 0$. Thus $\Sigma_O(A_2, A_3, -1)$ is also well-posed for any $k_2 \geq 0$. Hence from Theorem 6.3, the 3-modal system $\Sigma_O(0, -1)$ is well-posed for any $k_2 \geq 0$. From this, it turns out that the well-posedness of $\Sigma_O(A_2, A_3, -1)$, which expresses the system with the discontinuity between $A_2 x$ and $A_3 x$, depends on the values of k_2, on the other hand, does not depend on the values of d_1 and d_2. Furthermore, if $k_1 \neq 0$, the dynamics in mode 3 is given by the affine form. An extension of the well-posedness condition to the affine form is seen in [31].

Remark 6.3: Consider the system

$$\Sigma_O \begin{cases} \text{mode 1: } \dot{x} = Ax, & \text{if } |Cx| \geq \alpha, \\ \text{mode 2: } \dot{x} = Bx, & \text{if } |Cx| \leq \alpha, \end{cases} \quad \alpha > 0 \tag{56}$$

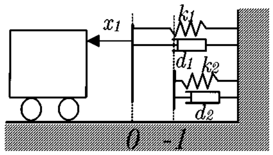

Fig. 3. 3-modal system.

which may appear as the closed loop system resulting from the use of switching controllers. From Theorem 6.3, we can show that this system is well-posed if and only if the bimodal system

$$\Sigma_O(A, B, \alpha)\begin{cases}\text{mode 1: } \dot{x} = Ax, & \text{if } Cx \geq \alpha,\\ \text{mode 2: } \dot{x} = Bx, & \text{if } Cx \leq \alpha\end{cases} \tag{57}$$

is well-posed. Thus the well-posedness problem for the system given by (56) is reduced to that for the system given by (57).

VII. Application to Well-Posedness Problem in Control Switching

The well-posedness conditions as obtained in the previous sections can be applied to several issues in hybrid systems theory. Especially, by combining a stability condition of piecewise-linear systems by Johansson and Rantzer [16] with our result, we can determine stability of those systems where the existence of a unique solution without sliding modes is guaranteed.

As another application, we discuss in this section a well-posedness problem of switching control systems where the state feedback gains are switched according to a criterion depending on the state.

Consider the stabilization problem for the control system given by

$$\dot{x} = Ax + Bu, \quad u = \begin{cases}K_1 x, & \text{if } Cx \geq 0,\\ K_2 x, & \text{if } Cx \leq 0\end{cases} \tag{58}$$

where $x \in \mathcal{R}^n$, $u \in \mathcal{R}^m$, $C \in \mathcal{R}^{1\times n}$, and K_1 and K_2 are feedback gains. Consider a simple example given by

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u$$

and $K_1 = [k_1\ k_2]$, $K_2 = [\overline{k}_1\ \overline{k}_2]$, and $C = [c_1\ c_2]$. Then letting T_{A+BK_1} and T_{A+BK_2} be the rule matrices (i.e., the observability matrices) for the pairs $(C, A+BK_1)$ and $(C, A+BK_2)$, and assuming that these matrices are nonsingular, we obtain

$$T_{A+BK_2}T_{A+BK_1}^{-1} = \begin{bmatrix}1 & 0\\ * & a\end{bmatrix}$$

where $a \triangleq (c_1(c_1+c_2\overline{k}_2) - c_2^2\overline{k}_1/c_1(c_1+c_2k_2) - c_2^2 k_1)$. Thus from Theorem 4.2, we conclude that the closed loop system is well-posed if and only if $a > 0$. This example shows that even if each controller stabilizes each system in the usual sense, the total system is not necessarily well-posed. For example, consider the case of $c_1 = 1$, $c_2 = 1$, $k_1 = -1$, $k_2 = -3$, $\overline{k}_1 = -1$ and $\overline{k}_2 = -1$. Then $A + BK_1$ and $A + BK_2$ are stable, but $a < 0$. Note that such a case is not rare and the stability in the usual sense for each mode does not automatically provide the well-posedness of the closed loop system.

As shown in the above example, for any given closed loop system, the well-posedness can be determined by checking the corresponding conditions derived in the previous sections. Moreover, we can give an parametrization of all feedback gains which guarantee the well-posedness of the closed loop systems in question. Such a parametrization provides a clear structure in the parameter space of all admissible feedback gains in the study of stabilizability with well-posedness, and also will be useful to find a feedback gain which stabilizes the system with well-posedness, using the numerical methods such as the LMI techniques.

For the closed loop system with two modes given by (58), letting $K \triangleq K_2 - K_1$ and denoting $A + BK_1$ by A again, we have

$$\Sigma_O\begin{cases}\text{mode 1: } \dot{x} = Ax, & \text{if } y = Cx \geq 0\\ \text{mode 2: } \dot{x} = (A+BK)x, & \text{if } y = Cx \leq 0.\end{cases} \tag{59}$$

For the single-input control system (59), we will use the information on the relative degree of the pair (C, A, B), which expresses at what stage the effect of $u = Kx$, which leads to the discontinuity of vector fields, on the output $y = Cx$ appears.

Theorem 7.1: Assume that the pair (C, A) is observable and the relative degree for the triple (C, A, B) is $p(\leq n)$ (i.e., $CB = CAB = \cdots = CA^{p-2}B = 0$ and $CA^{p-1}B \neq 0$). Then the following statements are equivalent.

i) The system Σ_O is well-posed.
ii) $K^T \in \text{span}\{C^T, (CA)^T, \cdots, (CA^{p-1})^T\} + \{\xi \in \mathcal{R}^n | \xi = \gamma(CA^p)^T, \gamma CA^{p-1}B > -1\}$.

Proof: i)→ii). From Theorem 4.2, i) implies that $(C, A+BK)$ is observable. Thus from Theorem 4.1, there exists an $M \in \mathcal{L}_+^n$ such that $T_{A+BK} = MT_A$, where T_{A+BK} and T_A are the observability matrices for the pairs $(C, A+BK)$ and (C, A), respectively. Noting that $C(A+BK)^i = CA^i$ $(i = 0, 1, \cdots, p-1)$ and $C(A+BK)^p = CA^p + CA^{p-1}BK$, we obtain

$$CA^p + CA^{p-1}BK = m_{p+1,1}C + m_{p+1,2}CA + \cdots + m_{p+1,p}CA^{p-1} + m_{p+1,p+1}CA^p$$

where $m_{p+1,i}$ is the $(p+1, i)$ element of M, and $m_{p+1,p+1} > 0$. This implies that $K = \sum_{i=1}^{p} \overline{m}_{p+1,i}CA^{i-1} + \gamma CA^p$ where $\overline{m}_{p+1,i} = m_{p+1,i}/CA^{p-1}B$ and $\gamma = (m_{p+1,p+1} - 1)/CA^{p-1}B$. From $m_{p+1,p+1} > 0$, ii) follows.

ii)→i). Let $\mu \triangleq CA^{p-1}B$ and let K be given by $K = \sum_{i=1}^{p+1} \kappa_i CA^{i-1}$ where κ_i $(i = 1, 2, \cdots, p)$ are any values and $\kappa_{p+1}\mu > -1$. Then simple calculations show that there exists a matrix $M \in \mathcal{L}_+^n$ such that $T_{A+BK} = MT_A$ Furthermore since M is nonsingular, the pair $(C, A+BK)$ is observable. Hence by Theorem 4.1, Σ_O is well-posed. □

Remark 7.1: It follows from Theorem 7.1 that for $p = n$ the closed loop system is well-posed for any K. Note also that the case $K = \kappa_1 C$ corresponds to the vector field of the closed loop system being Lipschitz continuous.

Remark 7.2: Theorem 7.1 can be extended to the multi-input case. If the relative degrees for all inputs are different from each other, the extension is straightforward. On the other hand, if some relative degrees are the same, the condition for well-posedness becomes more complicated. Furthermore, Theorem 7.1 can be extended to the case of affine inequalities as given below:

$$\Sigma_O(A,\, A+BK,\, \alpha) \begin{cases} \text{mode 1: } \dot{x} = Ax, & \text{if } Cx \geq \alpha \\ \text{mode 2: } \dot{x} = (A+BK)x, & \text{if } Cx \leq \alpha. \end{cases}$$

VIII. Conclusion

We have discussed the well-posedness problem in the sense of Carathéodory for a class of piecewise-linear discontinuous systems, and we have derived necessary and sufficient conditions for those systems to be well-posed. The obtained results are based on the lexicographic inequality relation and the smooth continuation property. As an application to switching control problems, we have given a necessary and sufficient condition for two state feedback gains, which are switched according to a criterion depending on the state, to maintain the well-posedness property of the closed loop system.

There are several open problems on well-posedness of discontinuous systems to be addressed in the future. We will have to discuss well-posedness of multi-modal systems in the unobservable case as an extension of Section VI. In addition, extensions to the case of nonlinear systems should be addressed. It will be also interesting to discuss some relations with the well-posedness of complementarity systems as mentioned in Remark 2.3. Finally, basic results derived here such as the smooth continuation property may be useful to solve well-posedness problems arising in the framework of hybrid automata as exposed e.g., in [8].

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References

[1] R. L. Grossman, A. Nerode, A. P. Ravn, and H. Rischel, Eds., *Hybrid Systems*. ser. Lecture Notes in Computer Science 736. New York, NY: Springer-Verlag, 1993.

[2] P. Antsaklis, W. Kohn, A. Nerode, and S. Sastry, Eds., *Hybrid Systems II*. ser. Lecture Notes in Computer Science 999. New York, NY: Springer-Verlag, 1995.

[3] R. Alur, T. A. Henzinger, and E. D. Sontag, Eds., *Hybrid Systems III*. ser. Lecture Notes in Computer Science 1066. New York, NY: Springer-Verlag, 1996.

[4] P. Antsaklis, W. Kohn, A. Nerode, and S. Sastry, Eds., *Hybrid Systems IV*. ser. Lecture Notes in Computer Science 1273. New York, NY: Springer-Verlag, 1997.

[5] O. Maler, Ed., *Hybrid and Real Time Systems*. ser. Lecture Notes in Computer Science 1201. New York, NY: Springer-Verlag, 1997.

[6] "Special issue on hybrid systems," *IEEE Trans. Automat. Contr.*, vol. AC-43, 1998.

[7] R. Alur and D. L. Dill, "A theory of timed automata.," *Theoretical Computer Science*, vol. 126, pp. 183–235, 1994.

[8] R. Alur, C. Courcoubetis, N. Halbwachs, T. A. Henzinger, P.-H. Ho, and X. Nicollin, "The algorithmic analysis of hybrid systems," *Theoretical Computer Science*, vol. 138, pp. 3–34, 1995.

[9] P. J. Antsaklis, J. A. Stiver, and M. Lemmon, "Hybrid system modeling and autonomous control systems," in *Hybrid Systems*, ser. Lect. Notes in Computer Sciences, 736. Berlin: Springer-Verlag, 1993, pp. 366–392.

[10] M. S. Branicky, V. S. Borkar, and S. K. Mitter, "A unified framework for hybrid control," in *Proc. 33rd IEEE Conf. on Decision and Control*, 1994, pp. 4228–4234.

[11] J. Ezzine and A. H. Haddad, "Controllability and observability of hybrid systems," *Int. J. Contr.*, vol. 49, no. 6, pp. 2045–2055, 1989.

[12] M. Tittus and B. Egardt, "Control design for integrator hybrid systems," *IEEE Trans. Automat. Contr.*, vol. 43, pp. 491–500, 1998.

[13] M. S. Branicky, "Multiple Lyapunov functions and other analysis tools for switched and hybrid systems," *IEEE Trans. Automat. Contr.*, vol. 43, pp. 475–482, 1998.

[14] H. Ye, N. Michel, and L. Hou, "Stability theory for hybrid dynamical systems," *IEEE Trans. Automat. Contr.*, vol. 43, pp. 461–474, 1998.

[15] M. A. Wicks, P. Peleties, and R. A. DeCarlo, "Construction of piecewise Lyapunov functions for stabilizing switched systems," in *Proc. 33rd IEEE Conf. on Decision and Control*, 1994, pp. 3492–3497.

[16] M. Johansson and A. Rantzer, "Computation of piecewise quadratic Lyapunov functions for hybrid systems," *IEEE Trans. Automat. Contr.*, vol. AC-43, pp. 555–559, 1998.

[17] A. Hassibi and S. Boyd, "Quadratic stabilization and control of piecewise-linear systems," in *Proc. American Control Conf.*, 1998, pp. 3659–3664.

[18] A. J. van der Schaft and J. M. Schumacher, "The complementary-slackness class of hybrid systems," *Math. Contr., Signals, Syst.*, vol. 9, pp. 266–301, 1996.

[19] ——, "Complementarity modeling of hybrid systems," *IEEE Trans. Automat. Contr.*, vol. AC-43, pp. 483–490, 1998.

[20] Y. J. Lootsma, A. J. van der Schaft, and M. K. Çamlıbel, "Uniqueness of solutions of linear relay systems," *Automatica*, vol. 35, no. 3, pp. 467–478, 1999.

[21] W. P. M. H. Heemels, J. M. Schumacher, and S. Weiland, "Linear complementarity systems," Dept. of Elect. Eng., Eindhoven Univ. of Technology, Internal rep. 97 I/01.

[22] ——, "Complementarity problems in linear complementarity systems," in *Proc. American Control Conf.*, 1998, pp. 706–710.

[23] A. F. Filippov, *Differential Equations With Discontinuous Righthand Sides*. Dordrecht, The Netherlands: Kluwer, 1988.

[24] R. W. Brockett, "Smooth dynamical systems which realize arithmetical and logical operations," in *Three Decades of Mathematical Systems Theory*, H. Nijmeijer and J. M. Schumacher, Eds. Berlin, Germany: Springer, 1989, pp. 19–30.

[25] M. S. Branicky, "Analyzing continuous switching systems: Theory and examples," in *Proc. American Control Conf.*, 1994, pp. 3110–3114.

[26] ——, "Universal computation and other capabilities of hybrid and continuous dynamical systems," *Theoretical Computer Science*, vol. 138, pp. 67–100, 1995.

[27] B. Brogliato, *Nonsmooth Impact Mechanics—Models, Dynamics and Control*, ser. Lect. Notes in Contr. Inform. Sci. 220. Berlin, Germany: Springer-Verlag, 1996.

[28] A. S. Morse, "Supervisory control of families of linear set-point controllers—I," *IEEE Trans. Automat. Contr.*, vol. 41, pp. 1413–1431, 1996.

[29] S. Rangan and K. Poolla, "Robust adaptive stabilizing with multiple H_∞ uncertainty models and switching," in *Proc. American Control Conf.*, 1998, pp. 3659–3664.

[30] D. Nešić, E. Skafidas, I. M. Y. Mareels, and R. J. Evans, "On the use of switched linear controllers for stabilizability of implicit rerursive equations," in *Proc. American Control Conf.*, 1998, pp. 3639–3643.

[31] J. Imura and A. J. van der Schaft, "Well-posedness of a class of dynamically interconnected systems," in *Proc. 38th IEEE Conf. on Decision and Control*, 1999, pp. 3031–3036.

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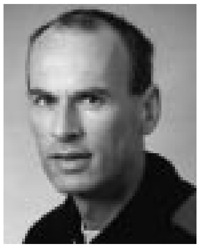

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