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Mathematical Structures in the Network Representation of Energy-Conserving Physical Systems

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Abstract

It is shown that network modelling of energy-conserving physical systems naturally leads to the consideration of (nonlinear) implicit generalized Hamiltonian systems. Behavioral systems theory may be invoked to formulate and analyze the system-theoretic properties of these systems.

1 Generalized Hamiltonian modelling

Most of the current modelling approaches of physical systems (e.g. multi-body systems) are based on some sort of *network representation*, where the physical system under consideration is seen as the *interconnection* of a possible large number of simple sub-systems (the elementary building blocks). This way of modelling has several advantages. The knowledge about sub-systems can be stored in libraries, and is re-usable for later occasions. Because of the modularity the modelling process can be performed in a "recursive" manner, first neglecting certain effects and gradually refining the model by adding other sub-systems. Further, the approach is suited to general control design where the overall behavior of the system is sought to be improved by the addition of other sub-systems (controllers).

In our previous work [2], [3], [4], [5], [6], [7], [8], [9] we have mainly concentrated on *energy-conserving* physical systems, where we have argued that the basic dynamic building blocks are of the form

$$\begin{cases} \dot{x} &= J(x)\frac{\partial H}{\partial x}(x) + g(x)f \\ e &= g^T(x)\frac{\partial H}{\partial x}(x) \end{cases} \tag{1}$$

Here $x = (x_1, \cdots, x_n)$ denotes the vector of (independent) energy variables, coordinatizing the state space manifold $\mathcal{X}$, $H(x_1, \cdots, x_n)$ is the total stored energy of the sub-system, with $\frac{\partial H}{\partial x}(x)$ denoting the transposed gradient vector of H, and the $n \times n$ structure matrix

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$J(x)$ (the modulated gyrator in bond graph terminology) is associated with the network topology of the sub-system. Since the internal interconnections are all assumed to be energy-conserving we have the important property

$$J(x) = -J^T(x), \quad \text{for all } x. \tag{2}$$

Finally, the columns $g_j(x), j = 1, \cdots, m$, of the matrix $g(x)$ denote the (state modulated) transformers describing the influence of the external *flow* sources $f_j, j = 1, \cdots, m$. The components e_j of e are the corresponding conjugated *efforts*.

Because of (2) we immediately obtain the energy-balance

$$\frac{d}{dt}H = e^T f, \tag{3}$$

expressing that the increase in energy equals the externally supplied *power* ($e_j^T f_j$ is the power of the j-th source). Thus (1) describes an energy-conserving physical system with *internal* variables $x_1, \cdots, x_n$ (associated with energy storage) and *terminal* or *port* variables $f_1, \cdots, f_m, e_1, \cdots, e_m$ (associated with power).

In [6], [9] the system (1) was called a port-controlled generalized *Hamiltonian* system, because of the following reason. Given the skew-symmetric matrix $J(x)$ we may define a skew-symmetric *bracket* operation on the real functions on $\mathcal{X}$ (the space of energy variables x; generally a manifold) as

$$\{F, G\}(x) = \left[\frac{\partial F}{\partial x}(x)\right]^T J(x)\frac{\partial G}{\partial x}(x), \ F, G : \mathcal{X} \rightarrow \Re \tag{4}$$

In many cases of interest (e.g., if $J(x)$ is *constant*) this bracket satisfies the *Jacobi-identity*

$$\{F, \{G, H\}\}+\{G, \{H, F\}\}+\{H, \{F, G\}\} = 0, \ \forall F, G, H \tag{5}$$

If (and only if!) the Jacobi-identity holds, then, by a generalized form of Darboux's theorem, we may find in the neighborhood of every point $x_0 \in \mathcal{X}$ where $J(x)$ has constant rank, coordinates $q_1, \cdots, q_k, p_1, \cdots, p_k, r_1, \cdots, r_\ell$ for $\mathcal{X}$ in which $J(x)$ takes the form

$$J(x) = \begin{bmatrix} 0 & I_k & 0 \\ -I_k & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{6}$$

In such coordinates (called "*canonical*") the internal dynamics $\dot{x} = J(x)\frac{\partial H}{\partial x}(x)$ takes the form

$$
\begin{array}{rclcrcl}
\dot{q}_1 & = & \frac{\partial H}{\partial p_1}(q,p,r), & & \dot{r}_1 & = & 0 \\
& \vdots & & & & \vdots & \\
\dot{q}_k & = & \frac{\partial H}{\partial p_k}(q,p,r), & & \dot{r}_\ell & = & 0 \\
\dot{p}_1 & = & -\frac{\partial H}{\partial q_1}(q,p,r) & & & & \\
& \vdots & & & & & \\
\dot{p}_k & = & -\frac{\partial H}{\partial q_k}(q,p,r) & & & &
\end{array}
\tag{7}
$$

which are almost the standard Hamiltonian equations of motion except for the appearance of the conserved quantities $r_1, \cdots, r_\ell$.

In a mechanical context the simplest examples of systems (1) include the "*generalized spring element*"

$$
\begin{array}{rcll}
\dot{q} & = & f & ,q,f,e \in \Re^3 \\
e & = & \frac{\partial H}{\partial q}(q) &
\end{array}
\tag{8}
$$

with $q \in \Re^3$ the configuration coordinates, $f \in \Re^3$ the velocity, and $e \in \Re^3$ the force delivered by the spring with potential energy $H(q)$. Comparing with (1) we see that $J \equiv 0$, while g is the 3×3 identity matrix. Another basic example are the equations of a point mass in $\Re^3$

$$
\begin{array}{rcll}
\dot{p} & = & f & ,p,f,e \in \Re^3 \\
e & = & \frac{1}{m}p &
\end{array}
\tag{9}
$$

where p is the vector of momenta, $H(p) = \frac{1}{2m} \parallel p \parallel^2$ is the kinetic energy, m is the mass, f is the vector of external forces, and e denotes the velocity of the point mass.

A more involved basic example concerns the equations of a rigid body with fixed center of mass

$$
\begin{bmatrix} \dot{p}_x \\ \dot{p}_y \\ \dot{p}_z \end{bmatrix} = \begin{bmatrix} 0 & -p_z & p_y \\ p_z & 0 & -p_x \\ -p_y & p_x & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial p_x}(p) \\ \frac{\partial H}{\partial p_y}(p) \\ \frac{\partial H}{\partial p_z}(p) \end{bmatrix} + g(p)f
$$

$$
e = g^T(p)\frac{\partial H}{\partial p}(p)
\tag{10}
$$

where $p = (p_x \; p_y \; p_z)$ is the vector of body angular momenta, $H(p) = \frac{1}{2}\left(\frac{p_x^2}{I_x} + \frac{p_y^2}{I_y} + \frac{p_z^2}{I_z}\right)$ is the kinetic energy, f are external torques, and e the corresponding angular velocities. In this case we encounter a non-trivial J, which in fact is determined by the geometry of the underlying matrix group $SO(3)$. (The equations for a rigid body with moving center of mass are similarly obtained by taking instead of $SO(3)$ the special Euclidean group $SE(3)$.) In the above example, the terminal variables f and e are the torques, respectively, the velocities in a body-fixed frame. The values in an inertial frame are obtained by a transformation depending on

the configuration of the rigid body, see e.g. [5].

In addition to these dynamic generalized Hamiltonian sub-systems we may also find *static* Hamiltonian sub-systems in the network representation. In the mechanical context the most well-known examples are kinematic constraints

$$
A^T(q)v = 0,
\tag{11}
$$

with $v \in \Re^n$ the vector of generalized velocities and $A(q)$ some $n \times k$ matrix of rank k. The resulting constraint forces are by d'Alembert of the form

$$
F = A(q)\lambda,
\tag{12}
$$

where the Lagrange multipliers $\lambda \in \Re^k$ are determined by the requirement that the constraints (11) need to be satisfied for all time.

The overall physical system is now obtained by interconnecting the various port-controlled generalized Hamiltonian systems using Kirchhoff's laws (with respect to generalized forces and generalized velocities, or to voltages and currents), resulting in a *mixed* set of differential and algebraic equations. Since the overall system is a power-conserving interconnection of generalized Hamiltonian systems one would expect that this resulting mixed set of differential and algebraic equations is again *Hamiltonian* in some sense. Indeed, it can be seen that it is an *implicit* generalized Hamiltonian system, as defined in [7], [8]. The key concept in the definition of an implicit Hamiltonian system is the notion of a generalized *Dirac structure*, introduced in [10], [11].

A generalized Dirac structure $\mathcal{D}$ on an n-dimensional manifold $\mathcal{X}$ is given by specifying for every $x \in \mathcal{X}$ an n-dimensional subspace

$$
\mathcal{D}(x) \subset T_x\mathcal{X} \times T_x^*\mathcal{X},
\tag{13}
$$

depending smoothly on x, with the property that for all $(X, \alpha) \in T_x\mathcal{X} \times T_x^*\mathcal{X}$

$$
(X, \alpha) \in \mathcal{D}(x) \Longrightarrow \alpha^T X = 0
\tag{14}
$$

Let now $H : \mathcal{X} \to \Re$ be given. Then the (autonomous) *implicit generalized Hamiltonian system* on $\mathcal{X}$ corresponding to the generalized Dirac structure $\mathcal{D}$ on $\mathcal{X}$ and the Hamiltonian H is given by the specification (see [7], [8])

$$
\left(\dot{x}, \frac{\partial H}{\partial x}(x)\right) \in \mathcal{D}(x), \quad \text{for all } x \in \mathcal{X}
\tag{15}
$$

Note that in general the specification (15) puts *constraints* on the state space $\mathcal{X}$, since in general there will not exist for every $x \in \mathcal{X}$ a tangent vector $\dot{x} \in T_x\mathcal{X}$ such that (15) is satisfied. If on the other hand the subspace $D(x)$ for every x can be parametrized by the

co-tangent vectors α, that is, there exists an $n \times n$ matrix $J(x)$ such that

$$D(x) = \{(X, \alpha) \mid X = J(x)\alpha\} \qquad (16)$$

then by (14) $J(x) = -J^T(x)$ for every x, and (15) reduces to the *explicit* generalized Hamiltonian system

$$\dot{x} = J(x)\frac{\partial H}{\partial x}(x) \qquad (17)$$

Now let us consider k port-controlled Hamiltonian systems as in (1), i.e., for $i = 1, \cdots, k$

$$\dot{x}_i = J_i(x_i)\frac{\partial H_i}{\partial x_i}(x_i) + g_i(x_i)f_i$$

$$e_i = g_i^T(x_i)\frac{\partial H_i}{\partial x_i}(x_i) \qquad (18)$$

$$x_i \in \mathcal{X}_i, \quad f_i \in \mathcal{F}_i = \Re^{m_i}, \quad e_i \in \mathcal{E}_i = \Re^{m_i}$$

with $\mathcal{X}_i$ an n_i-dimensional state space, and consider a general power-conserving interconnection of these systems given by an $(m_1 + \cdots + m_k)$-dimensional subspace (possibly parametrized by $x_1, \ldots, x_k$)

$$I(x_1, \cdots, x_k) \subset \mathcal{F}_1 \times \cdots \times \mathcal{F}_k \times \mathcal{E}_1 \times \cdots \mathcal{E}_k \qquad (19)$$

with the property

$$(f_1, \cdots, f_k, e_1, \cdots, e_k) \in I(x_1, \cdots, x_k) \Rightarrow \sum_{i=1}^{k} e_i^T f_i = 0 \qquad (20)$$

Then the resulting interconnected system is an implicit generalized Hamiltonian system with state space $\mathcal{X} := \mathcal{X}_1 \times \cdots \times \mathcal{X}_k$, Hamiltonian (total energy) $H(x_1, \cdots, x_k) := H_1(x_1) + \cdots + H_k(x_k)$, and generalized Dirac structure D on $\mathcal{X}_1 \times \cdots \times \mathcal{X}_k$ defined as

$$(X, \alpha) = (X_1, \cdots, X_k, \alpha_1, \cdots, \alpha_k) \in D(x_1, \cdots, x_k)$$

$$\Longleftrightarrow \begin{cases} X_i = J_i(x_i)\alpha_i + g_i(x_i)f_i \\ e_i = g_i^T(x_i)\alpha_i, \qquad i = 1, \cdots, k \\ (f_1, \cdots, f_k, e_1, \cdots, e_k) \in I(x_1, \cdots, x_k) \end{cases} \qquad (21)$$

Indeed, this defines a Dirac structure since by direct computation

$$\sum_{i=1}^{k} \alpha_i^T X_i = \sum_{i=1}^{k} \alpha_i^T (J_i(x_i)\alpha_i + g_i(x_i)f_i) \qquad (22)$$

$$= \sum_{i=1}^{k} \alpha_i^T g_i(x_i)f_i = \sum_{i=1}^{k} e_i^T f_i = 0,$$

by (20), while it can be checked that the dimension of $D(x_1, \cdots, x_k)$ equals $n_1 + \cdots + n_k$.

Note that Kirchhoff's laws for the flows $f_1, \cdots, f_k$ and efforts $e_1, \cdots, e_k$, as well as static Hamiltonian sub-systems as given in (11), (12), are a special case

of a power-conserving interconnection I. (Also any transformer-type of interconnection will define a power-conserving interconnection I.) Thus we have indeed shown that any power-conserving interconnection of port-controlled generalized Hamiltonian systems (1) defines an *implicit* generalized Hamiltonian system (15).

Usually the following *integrability condition* is imposed on the generalized Dirac structure:

$$< L_{X_1}\alpha_2, X_3 > + < L_{X_2}\alpha_3, X_1 > + < L_{X_3}\alpha_1, X_2 > = 0 \qquad (23)$$

for all pairs of vector fields and one-forms (X_1, α_1), $(X_2, \alpha_2), (X_2, \alpha_3)$ which are elements of $\mathcal{D}(x)$ for all $x \in \mathcal{X}$. If this is satisfied, then D is called a (true) Dirac structure [10], [11]. When the generalized Dirac structure is given as in (16), then (23) is nothing else then the Jacobi-identity (5). Similarly to this case one can show ([10], [11]) that if and only if the integrability condition (23) is satisfied one may find local (canonical) coordinates $q_1, \cdots, q_k, p_1, \cdots, p_k, r_1, \cdots, r_\ell, s_1, \cdots, s_q$ in the neighborhood of any point $x_0 \in \mathcal{X}$ where the dimensions of the distribution $D(x) \cap (T_x\mathcal{X} \times 0)$ and the co-distribution $D(x) \cap (0 \times T_x^*\mathcal{X})$ are constant, such that the implicit Hamiltonian system (15) is given as

$$\begin{array}{llll}
\dot{q}_1 & = & \frac{\partial H}{\partial p_1}, & \dot{r}_1 & = & 0, \\
& \vdots & & & \vdots & \\
\dot{q}_k & = & \frac{\partial H}{\partial p_k}, & \dot{r}_\ell & = & 0, \\
\dot{p}_1 & = & -\frac{\partial H}{\partial q_1}, & 0 & = & \frac{\partial H}{\partial s_1}, \\
& \vdots & & & \vdots & \\
\dot{p}_k & = & -\frac{\partial H}{\partial q_k}, & 0 & = & \frac{\partial H}{\partial s_q},
\end{array} \qquad (24)$$

Comparing (24) with (7) we see that while (7) explicitates the conserved quantities, the equations (24) also make explicit the algebraic *constraints*

$$\begin{array}{lll}
0 & = & \frac{\partial H}{\partial s_1}(q, p, r, s) \\
& \vdots & \\
0 & = & \frac{\partial H}{\partial s_q}(q, p, r, s)
\end{array} \qquad (25)$$

If H is non-degenerate in the energy-variables $s_1, \cdots, s_q$, that is

$$\text{rank}\left[\frac{\partial^2 H}{\partial s_i \partial s_j}\right] = q, \qquad (26)$$

then by the Implicit function theorem one may locally express the variables $s_1, \cdots, s_q$ as functions of q, p, r, i.e. $s_i = s_i(q, p, r)$, $i = 1, \cdots, q$. Defining the *constrained* Hamiltonian

$$H_c(q, p, r) := H(q, p, r, s(q, p, r)) \qquad (27)$$

it follows that (24) reduces to the same format as (7)

$$\begin{aligned}
\dot{q}_1 &= \frac{\partial H_c}{\partial p_1} & \dot{r}_1 &= 0 \\
&\vdots & &\vdots \\
\dot{q}_k &= \frac{\partial H_c}{\partial p_k} & \dot{r}_\ell &= 0 \\
\dot{p}_1 &= -\frac{\partial H_c}{\partial q_1} & & \\
&\vdots & & \\
\dot{p}_k &= -\frac{\partial H_c}{\partial q_k} & &
\end{aligned} \tag{28}$$

which is an explicit Hamiltonian dynamics on the constrained state space $\mathcal{X}_c = \{(q, p, r, s) \mid \frac{\partial H}{\partial s_i}(q, p, r, s) = 0, \; i = 1, \cdots, q\}$. Also note that while under the assumption (26) the variables $s_1, \cdots, s_q$ together with the Hamiltonian H define a (constraint) submanifold $\mathcal{X}_c$ of $\mathcal{X}$, dually the level sets of the variables $r_1, \cdots, r_\ell$ define a *foliation* of $\mathcal{X}$. Both the constraint submanifold $\mathcal{X}_c$ and the foliation are *invariant* for the Hamiltonian dynamics. It should be noted, though, that there are cases of interest where the generalized Dirac structure does *not* satisfy the integrability condition (23). In fact, the generalized Dirac structure resulting from kinematic constraints (11) satisfies (23) if and only if the kinematic constraints are *holonomic*, i.e., can be integrated to geometric constraints ([3], [1]). Furthermore, also if the integrability condition (23) is satisfied the actual *construction* of the canonical coordinates q_i, p_i, r_i, s_i, may be very involved, and preferably should be avoided.

2 Representations of Dirac structures and implicit Hamiltonian systems

There are different ways of representing generalized Dirac structures, which each have their own advantages, and which naturally come up in different modelling approaches. The most general representation is as follows [7]. Let D be a generalized Dirac structure on an n-dimensional manifold $\mathcal{X}$ with local coordinates $(x_1, \cdots, x_n)$. Then locally there exist $n \times n$ matrices $E(x)$ and $F(x)$, depending smoothly on x, such that locally

$$D(x) = \{(X, \alpha) \mid F(x)X = E(x)\alpha\}$$

$$\text{rank } [F(x) \vdots - E(x)] = n \tag{29}$$

$$E(x)F^T(x) + F(x)E^T(x) = 0$$

(Conversely, any D defined as in (29) is a generalized Dirac structure). Given a Hamiltonian $H : \mathcal{X} \to \Re$ the corresponding implicit Hamiltonian system is thus given as

$$F(x)\dot{x} = E(x)\frac{\partial H}{\partial x}(x) \tag{30}$$

which is in the usual implicit (or descriptor) form. This representation naturally comes up in the Hamiltonian description of electrical LC-circuits [4]. Note that two distributions and two co-distributions are intrinsically attached to any generalized Dirac structure.

$$\begin{aligned}
G_0(x) &= \{X \in T_x\mathcal{X} \mid (X, 0) \in D(x)\} \\
&= D(x) \cap (T_x\mathcal{X} \times 0)
\end{aligned}$$

$$\begin{aligned}
P_0(x) &= \{\alpha \in T_x^*\mathcal{X} \mid (0, \alpha) \in D(x)\} \\
&= D(x) \cap (0 \times T_x^*\mathcal{X})
\end{aligned}$$

$$G_1(x) = \{X \in T_x\mathcal{X} \mid \exists \, \alpha \text{ s.t } (X, \alpha) \in D(x)\}$$

$$P_1(x) = \{\alpha \in T_x^*\mathcal{X} \mid \exists \, X \text{ s.t. } (X, \alpha) \in D(x)\} \tag{31}$$

Given the representation (29) for D it immediately follows that (note that ker $[F(x) \vdots - E(x)] = \text{im} \begin{bmatrix} E^T(x) \\ -F^T(x) \end{bmatrix}$)

$$G_0(x) = \text{ker } F(x), \quad P_0(x) = \text{ker } E(x)$$

$$G_1(x) = \text{im } E^T(x), \quad P_1(x) = \text{im } F^T(x) \tag{32}$$

The distribution G_1 describes the set of admissible flows $\dot{x}$. In particular, if G_1 is constant-dimensional (or equivalently, rank $E(x)$ is constant) and *involutive* then there are $(n - \dim G_1)$ independent conserved quantities for (30). Dually the co-distribution P_1 describes the set of algebraic constraints of (30), i.e.

$$\frac{\partial H}{\partial x}(x) \in P_1(x) \tag{33}$$

(Note that the integrability condition (16) implies involutivity of G_1 and P_1, see [10], [11], [1]).

If the co-distribution P_1 has constant dimension (or equivalently $F(x)$ has constant rank), then there is another interesting representation of the generalized Dirac structure. Indeed, since $F(x)$ has constant rank we may locally always transform the equations $F(x)X = E(x)\alpha$ into the form

$$\begin{bmatrix} F_1(x) \\ 0 \end{bmatrix} X = \begin{bmatrix} E_1(x) \\ E_2(x) \end{bmatrix} \alpha \tag{34}$$

where $F_1(x)$ is surjective. Furthermore it follows that ker $F_1(x) = \text{im } E_2^T(x)$. One derives (see [1] for details) that there exists a skew-symmetric matrix $J(x)$ satisfying $F_1(x)J(x) = E_1(x)$. Thus the equations (34) can be rewritten as

$$\begin{aligned}
X - J(x)\alpha &\in \text{ker } F_1(x) = \text{im } E_2^T(x) \\
0 &= E_2(x)\alpha
\end{aligned} \tag{35}$$

or equivalently, defining the constant rank matrix $g(x) := E_2^T(x)$

$$\begin{aligned}
X &= J(x)\alpha + g(x)\lambda \\
0 &= g^T(x)\alpha
\end{aligned} \tag{36}$$

where λ are Lagrange multipliers. The corresponding implicit generalized Hamiltonian system is therefore given as

$$\begin{aligned}
\dot{x} &= J(x)\frac{\partial H}{\partial x}(x) + g(x)\lambda \\
0 &= g^T(x)\frac{\partial H}{\partial x}(x)
\end{aligned} \quad (37)$$

which can be interpreted as a port-controlled Hamiltonian system (1), where the efforts e are set to zero. An appealing example of this representation is formed by a classical mechanical system $\dot{q} = \frac{\partial H}{\partial p}(q,p)$, $\dot{p} = -\frac{\partial H}{\partial q}(q,p)$ with kinematic constraints $A^T(q)v = 0$ ($v = \frac{\partial H}{\partial p}(q,p)$ being the generalized velocity), which can be written as

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}\begin{bmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{bmatrix}(q,p) + \begin{bmatrix} 0 \\ A(q) \end{bmatrix}\lambda$$

$$0 = A^T(q)\frac{\partial H}{\partial p}(q,p)$$

$$(38)$$

The Lagrange multipliers λ have the interpretation of being *constraint forces*. It can be shown that the generalized Dirac structure in this case satisfies the integrability condition (16) if and only if the constraints are *holonomic* ([3], [1]).

Dually, a third representation of Dirac structures can be obtained if the distribution G_1 is constant-dimensional (or equivalently the matrix $E(x)$ has constant rank). In this case the implicit Hamiltonian system (15) can be written as

$$\begin{aligned}
\frac{\partial H}{\partial x} &= \omega(x)\dot{x} + p^T(x)\lambda \\
0 &= p(x)\dot{x}
\end{aligned} \quad (39)$$

with $\omega(x) = -\omega^T(x)$ and $p(x)$ of constant rank ([1]). From a bond-graph point of view this corresponds to writing all elements in *derivative* causality, instead of in *integral* causality as in (37). The characterizations of the integrability condition (23) in these three different representations are detailed in [1].

3 Implicit port-controlled generalized Hamiltonian systems

The overall picture is however not yet complete. Indeed, we have shown that a power-conserving interconnection of port-controlled generalized Hamiltonian systems leads to an implicit generalized Hamiltonian system, with respect to a generalized Dirac structure resulting from the structure matrices J_i together with the definition of the interconnection, and a Hamiltonian which is the sum of the Hamiltonians of all the subsystems. However we cannot yet treat implicit generalized Hamiltonian systems *with* external ports. Moreover, if we start from an implicit generalized Hamil-

tonian system (15), and we make a division into subsystems different from the one we started with, then it is likely that (some of) the sub-systems will not have anymore the structure of a port-controlled generalized Hamiltonian system (1), but instead (at least, if the modularization is power-conserving) will be some sort of *implicit* port-controlled generalized Hamiltonian system.

Implicit port-controlled generalized Hamiltonian systems have been defined in [7] as follows:

Let $\mathcal{X}$ be the n-dimensional manifold of energy variables, and $H : \mathcal{X} \to \Re$ the Hamiltonian (total stored energy). Furthermore let $\mathcal{F}$ be the linear space $\Re^m$ of external flows f, with dual the space $\mathcal{F}^*$ of external efforts e. Then consider a Dirac structure D on $\mathcal{X} \times \mathcal{F}$, only depending on $x \in \mathcal{X}$, that is $D(x)$ is an $(n+m)$-dimensional subspace of $T_x\mathcal{X} \times T_x^*\mathcal{X} \times \mathcal{F} \times \mathcal{F}^*$. The implicit port-controlled generalized Hamiltonian system is now defined by the specification

$$(\dot{x}, \frac{\partial H}{\partial x}(x), f, -e) \in D(x), \text{for all } x \in \mathcal{X} \quad (40)$$

(Here the minus sign in front of the effort e comes from the natural identification $(\alpha, e) \in T_x^*\mathcal{X} \times \mathcal{F}^* \mapsto (\alpha, -e) \in (T_x\mathcal{X} \times \mathcal{F})^*$.) A clear example of such an implicit port-controlled generalized Hamiltonian system is that of a port-controlled generalized Hamiltonian system (1) *with constraints*

$$\begin{aligned}
\dot{x} &= J(x)\frac{\partial H}{\partial x}(x) + g(x)f + k(x)\lambda \\
e &= g^T(x)\frac{\partial H}{\partial x}(x) \\
0 &= k^T(x)\frac{\partial H}{\partial x}(x)
\end{aligned} \quad (41)$$

(for example, an actuated mechanical system with kinematic constraints), as can be seen by rewriting (41) as

$$\begin{bmatrix} \dot{x} \\ f \end{bmatrix} = \begin{bmatrix} J(x) & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \frac{\partial H}{\partial x}(x) \\ -e \end{bmatrix} + \begin{bmatrix} g(x) & k(x) \\ I & 0 \end{bmatrix}\tilde{\lambda} \quad (42)$$

$$0 = \begin{bmatrix} g^T(x) & I \\ k^T(x) & 0 \end{bmatrix}\begin{bmatrix} \frac{\partial H}{\partial x}(x) \\ -e \end{bmatrix}$$

and comparing with (37). The integrability condition (23) for the resulting Dirac structure on $\mathcal{X} \times \mathcal{F}$ has an interesting interpretation, as will be shown in [1].

In [6], [9] it has been shown that the controllability and observability properties of port-controlled generalized Hamiltonian systems are closely connected. For example, it has been shown that a linear port-controlled Hamiltonian system with *non-degenerate* quadratic Hamiltonian $\frac{1}{2}x^TQx$ (that is, $\det Q \neq 0$) is controllable if and only if it is observable. A similar statement can be proved for a *linear* port-controlled

Hamiltonian system with constraints

$$\dot{x} = JQx + Gf + K\lambda$$

$$e = G^T Qx \qquad (43)$$

$$0 = K^T Qx$$

with $J = -J^T$ and Hamiltonian $H(x) = \frac{1}{2}x^T Qx$, $Q = Q^T$, $\det Q \neq 0$. Suppose for simplicity that the constraints are independent, i.e. K is injective. Then following the work of Willems [12] the controllability of the triple (x, f, λ) can be checked by considering the first and third set of equations rewritten as

$$\underbrace{\begin{bmatrix} I\frac{d}{dt} - JQ & -G & -K \\ K^T Q & 0 & 0 \end{bmatrix}}_{R\left(\frac{d}{dt}\right)} \begin{bmatrix} x \\ f \\ \lambda \end{bmatrix} = 0 \qquad (44)$$

and by verifying that the rank of $R(s)$ is *maximal* for every s. Furthermore from the specific form of the equations it is seen that this is equivalent with controllability only of the x-trajectories. On the other hand, observability of the variables (x, λ) from the external variables (f, e) can be verified by rewriting (43) as

$$\underbrace{\begin{bmatrix} I\frac{d}{dt} - JQ & -K \\ G^T Q & 0 \\ K^T Q & 0 \end{bmatrix}}_{P\left(\frac{d}{dt}\right)} \begin{bmatrix} x \\ \lambda \end{bmatrix} = \begin{bmatrix} G & 0 \\ 0 & I \\ 0 & 0 \end{bmatrix} \begin{bmatrix} f \\ e \end{bmatrix} \qquad (45)$$

and by checking that the rank of $P(s)$ is maximal for every s. Furthermore, again from the specific form of the equations it follows that observability of (x, λ) from (f, e) is the same as observability of x from (f, e). Now it is seen that

$$R^T(-s) = \begin{bmatrix} -Q & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} P(s) \begin{bmatrix} -Q^{-1} & 0 \\ 0 & I \end{bmatrix} \qquad (46)$$

and thus controllability is equivalent with observability.

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