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J-Inner–Outer Factorization, J-Spectral Factorization, and Robust Control for Nonlinear Systems

Joseph A. Ball, *Member, IEEE*, and Arjan J. van der Schaft, *Member, IEEE*

***Abstract*—The problem of expressing a given nonlinear state-space system as the cascade connection of a lossless system and a stable, minimum-phase system (inner–outer factorization) is solved for the case of a stable system having state-space equations affine in the inputs. The solution is given in terms of the stabilizing solution of a certain Hamilton–Jacobi equation. The stable, minimum-phase factor is obtained as the solution of an associated nonlinear spectral factorization problem. As an application, one can arrive at the solution of the nonlinear H_∞-control problem for the disturbance feedforward case.**

I. Introduction

It is well known that inner–outer [i.e., lossless (stable, minimum-phase)] and spectral factorization plays a prominent role in H_∞-control theory (see [7], [14], [15], [18], [19], and [21] for the linear case and [9] for the nonlinear case) as well as in chemical process control (see [13] and [30]). Actually the H_∞-control application calls more generally for J-inner–outer factorization, where J is an indefinite signature matrix. While computation of both inner–outer and spectral factorization in terms of state-space realizations is well developed in the linear case, the issues are not so well understood in the nonlinear setting. In particular, the nonlinear inner–outer factorization problem was formulated and studied in [2]–[4] for discrete time systems. Specifically, the inner–outer factorization problem for a stable, invertible, discrete-time system was solved in [4] by constructing an invertible, lossless system having a given zero dynamics in terms of the solution of a certain Hamilton–Jacobi equation. Here we recover a version of the same result, but for stable, invertible, continuous-time systems having state-space equations which are affine in the inputs via a different approach. More precisely, we formulate and solve (again in terms of the solution of a Hamilton–Jacobi equation) a nonlinear spectral factorization problem; this leads directly to the computation of the outer factor in an inner–outer factorization of a given stable plant. Thus we obtain the inner–outer factorization by first solving for the outer factor, rather than in the reverse order as is done in [4]. The solution of the nonlinear spectral factorization problem is modeled on the approach in [10] for the linear case and also has contact with the theory of adjoint systems and input–output Hamiltonian systems developed in [12].

Section II introduces the problem in precise form (both in an input–output and state-space setting) and presents the solution. Section III presents as an application an alternative derivation of the Hamilton–Jacobi equation associated with a system having L_2-gain equal to, at most, a prescribed level γ. Section IV presents the application to nonlinear H_∞-control theory, and Section V discusses the difficulties of our approach for the unstable case. In a companion paper [27], we plan to discuss the special features of the definite case (where J is the identity matrix), the case of noninvertible plants, and the applications to chemical process control. Finally we mention that an announcement of these results appears in [25] and [26].

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II. J-Inner–Outer Factorization

A. Preliminaries

We consider a smooth (C^∞) nonlinear state-space system which is affine in the inputs u

$$\Sigma: \begin{cases} \dot{x} = a(x) + b(x)u, & u \in \boldsymbol{R}^m \\ y = c(x) + d(x)u, & y \in \boldsymbol{R}^p \end{cases} \tag{1}$$

with outputs y, where $x = (x_1, \cdots, x_n)$ are local coordinates for a state-space manifold M. Throughout, we assume the existence of an equilibrium x_0, i.e., $a(x_0) = 0$. Without loss of generality we also assume that $c(x_0) = 0$. Furthermore we make the following standing assumption:

$$x_0 = 0 \text{ is a globally asymptotically stable equilibrium point of } \dot{x} = a(x). \tag{A1}$$

(We remark on the unstable case in Section V.) Because of Assumption (A1), we may assume that the state-space manifold M is equal to R^n, and after a suitable coordinate shift, we may take the equilibrium point x_0 to be equal to $0 \in R^n$. In the sequel we will always abbreviate "globally asymptotically stable" to "asymptotically stable." Assume also that we are given an $m \times m$ signature matrix j and a $p \times p$ signature matrix J. In general we say that a square matrix $\mathcal{J}$ is a signature matrix if $\mathcal{J} = \mathcal{J}^* = \mathcal{J}^{-1}$, for example, $\mathcal{J} = \begin{bmatrix} I_{n_+} & 0 \\ 0 & -I_{n_-} \end{bmatrix}$. The problem of (j, J)-inner–outer factorization of Σ consists of constructing (if possible) a (j, J)-lossless system Θ (the (j, J)-inner factor) and an asymptotically stable minimum

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phase system R (the outer factor), both of the same form as Σ, such that

$$\Sigma = \Theta \circ R. \tag{2}$$

By this we mean that for every initial condition of Σ, there exist initial conditions of Θ and R such that the input–output map of Σ equals the input–output map of the series interconnection of R followed by Θ (for the respective initial conditions). Let us recall (see [29]) that a nonlinear system Σ is called lossless with respect to the supply rate $\frac{1}{2}u^T ju - \frac{1}{2}y^T Jy$, if there exists a function $V(x) \geq 0$ (the storage function) such that

$$V(x(t_1)) - V(x(t_0)) = \tfrac{1}{2}\int_{t_0}^{t_1} (u(t)^T ju(t) - y(t)^T Jy(t))\, dt$$
$$V(0) = 0 \tag{3}$$

for all $t_0 \leq t_1$ and $u(\cdot)$, with $x(t_1)$ denoting the state at time t_1 resulting from initial state $x(t_0)$ at time t_0 and input $u(\cdot)$ on the time interval $[t_0, t_1]$. If V is differentiable, then (3) can be equivalently expressed as

$$\begin{aligned} &V_x(x)[a(x) + b(x)u] \\ &\quad = \tfrac{1}{2}u^T ju - \tfrac{1}{2}[c(x) + d(x)u]^T J[c(x) + d(x)u] \\ &V(0) = 0 \end{aligned}$$

for all x, u, or equivalently, as the system of equations

$$\begin{aligned} V_x(x)a(x) + \tfrac{1}{2}c^T(x)Jc(x) &= 0, \qquad V(0) = 0 \\ V_x(x)b(x) + c^T(x)Jd(x) &= 0 \\ d^T(x)Jd(x) &= j \end{aligned} \tag{4}$$

for all x. Here $V_x(x)$ denotes the row vector of partial derivatives $[\partial V/\partial x_1(x) \cdots \partial V/\partial x_n(x)]$.

In this paper we consider minimum phase systems only for the case of causally invertible systems, i.e., a system of the form of (1) with $d(x)$ an invertible matrix for all x; the factorization problem in the singular case will be discussed in a separate paper [27]. In general, if R is a smooth state-space system

$$R\colon \begin{cases} \dot{x} = \overline{a}(x) + \overline{b}(x)u, & u \in \boldsymbol{R}^m \\ \overline{y} = \overline{c}(x) + \overline{d}(x)u, & \overline{y} \in \boldsymbol{R}^m \end{cases} \tag{5}$$

with $\overline{d}(x)$ an invertible $m \times m$ matrix for all $x \in \boldsymbol{R}^n$, then the inverse of R (the system with the same set of trajectories $(x(t), u(t), y(t))$ but with $\overline{y}(t)$ appearing as the input variable and $u(t)$ as the output variable) is given explicitly as

$$R^{-1}\colon \begin{cases} \dot{x} = [\overline{a}(x) - \overline{b}(x)\overline{d}(x)^{-1}\overline{c}(x)] + \overline{b}(x)\overline{d}(x)^{-1}\overline{y} \\ u = -\overline{d}(x)^{-1}\overline{c}(x) + \overline{d}(x)^{-1}\overline{y}. \end{cases} \tag{6}$$

We say that a system R, as in (5), is minimum phase if the dynamics of the inverse system is Lyapunov stable, i.e., if zero is a stable equilibrium point in the sense of Lyapunov for the system of differential equations

$$\dot{x} = \overline{a}(x) - \overline{b}(x)\overline{d}(x)^{-1}\overline{c}(x).$$

If the inverse system dynamics is asymptotically stable, then we say that R is strictly minimum phase. If, in addition, R itself is asymptotically stable, then we say that R is strictly outer. In the linear case, (strictly) outer corresponds to all poles of the transfer function being in the open left-half plane and all zeros in the closed (open) left-half plane, while (strictly) minimum phase corresponds to all transmission zeros being in the closed (open) left-half plane.

B. Reduction of (j, J)-Inner–Outer to j-Spectral Factorization: The Input–Output Level

The goal of this section is to obtain a (j, J)-inner–outer factorization for a system of the form of (1) under the assumption that the $m \times m$ matrix function $d^T(x)Jd(x)$ has a factorization

$$d^T(x)Jd(x) = \overline{d}^T(x)j\overline{d}(x) \tag{A2}$$

where $x \to \overline{d}(x)$ is a smooth, $m \times m$ matrix function with invertible values for all x. Note that in case $p = m, j = J$, and $d(x)$ is invertible for all x, we can take simply $\overline{d}(x) = d(x)$. If $p > m$ and $d(x)^T d(x)$ is invertible for all x with constant signature (i.e., the number of positive and negative eigenvalues) equal to the signature of j, then a Morse theory argument implies that such a factorization exists. We also impose in this section Assumption (A1) that $x_0 = 0$ is an asymptotically stable equilibrium point for the uncontrolled system $\dot{x} = a(x)$.

To motivate our approach to the nonlinear (j, J)-inner–outer factorization problem, we recall the approach through j-spectral factorization for the linear case. We start with a transfer function G. A property of (j, J)-inner functions is that $\Theta(-s)^T J\Theta(s) = j$. Hence, if G has a (j, J)-inner–outer factorization $G = \Theta R$, then

$$\begin{aligned} G(-s)^T JG(s) &= R(-s)^T \Theta(-s)^T J\Theta(s)R(s) \\ &= R(-s)^T jR(s). \end{aligned}$$

Conversely, if we produce an outer function R such that

$$G(-s)^T JG(s) = R(-s)^T jR(s)$$

and set $\Theta = GR^{-1}$, then Θ has (j, J)-isometric values on the imaginary line. If G has a (j, J)-inner–outer factorization, then Θ also has (j, J)-contractive values on the right-half plane and $G = \Theta R$ is a (j, J)-inner-outer factorization for G. Our goal in this section is to show that a similar analysis holds in the nonlinear case at the input–output level; for the discrete-time case with $j = J = I$, see [3].

In this subsection we consider maps on $L_2^m(\boldsymbol{R}^+)$ and ignore state-space representations. A map $T_\Theta\colon L_2^m(\boldsymbol{R}^+) \to L_2^p(\boldsymbol{R}^+)$ is said to be input–output (j, J)-conservative if

$$\langle JT_\Theta(u), T_\Theta(u)\rangle_{L_2^p} = \langle ju, u\rangle_{L_2^m} \tag{7}$$

for all $u \in L_2^m(\boldsymbol{R}^+)$. We say that T_Θ is input–output (j, J)-lossless (or (j, J)-lossless) if in addition

$$\langle JP_\tau T_\Theta(u), P_\tau T_\Theta(u)\rangle_{L_2^m} \leq \langle jP_\tau u, P_\tau u\rangle_{L_2^p} \tag{8}$$

for all $u \in L_2^n$ and all real τ. Here P_τ is the truncation operator

$$P_\tau f(t) = \begin{cases} f(t), & \text{for } t \leq \tau \\ 0, & \text{for } t > \tau. \end{cases}$$

This input–output notion of (j, J)-inner is closely related, but not equivalent, to the state-space notion introduced in Section II-A (see [35]).

A map T_R on $L_2^m(\boldsymbol{R}^+)$ is input–output stable if T_R maps $L_2^m(\boldsymbol{R}^+)$ into itself and is input–output outer if both T_R and its inverse T_R^{-1} are input–output stable. We assume throughout that all mappings on $L_2(\boldsymbol{R}^+)$-spaces are without bias in the sense that the image of zero is zero. In addition we say that the map T_R is causal if

$$P_\tau \circ T_R = P_\tau \circ T_R \circ P_\tau$$

for all real τ and that T_R is bicausal if both T_R and $(T_R)^{-1}$ are causal. We will discuss the distinction between internal stability and input–output stability in more detail in Section IV in connection with the H_∞-control problem; for more information, see [28, Chapter 6].

Now let T_Σ be a mapping from $L_2^m(\boldsymbol{R}^+)$ into $L_2^p(\boldsymbol{R}^+)$ with $T_\Sigma(0) = 0$. The (j, J)-inner–outer factorization problem (at the input-output level) is to produce mappings T_Θ and T_R which are (j, J)-inner and outer, respectively, in the input–output sense, such that $T_\Sigma = T_\Theta \circ T_R$. We assume that all maps are Frechet differentiable (see, for example, [31] for a discussion of the notion of Frechet derivative). For T_Σ a mapping as above, the Frechet derivative of T_Σ at the point $u \in L_2^m(\boldsymbol{R}^+)$, denoted by $DT_\Sigma(u)$, then is a linear mapping from $L_2^m(\boldsymbol{R}^+)$ into $L_2^p(\boldsymbol{R}^+)$ and hence has a transpose, denoted by $[DT_\Sigma(u)]^T$, with respect to the L_2 inner products. The following result provides an extension of the connection between (j, J)-inner–outer factorization and j-spectral factorization to the nonlinear setting.

Theorem 1: Suppose that T_Σ: $L_2^m(\boldsymbol{R}^+) \to L_2^p(\boldsymbol{R}^+)$ and T_R: $L_2^m(\boldsymbol{R}^+) \to L_2^m(\boldsymbol{R}^+)$ are Frechet differentiable mappings without bias such that T_R is outer. Set $T_\Theta = T_\Sigma \circ T_R^{-1}$. Then $T_\Sigma = T_\Theta \circ T_R^{-1}$ is a (j, J)-conservative-outer factorization of T_Σ if and only if

$$[DT_\Sigma(u)]^T \circ JT_\Sigma(u) = [DT_R(u)]^T \circ jT_R(u) \tag{9}$$

for all $u \in L_2^m(\boldsymbol{R}^+)$.

Moreover, if T_Σ has a (j, J)-inner–outer factorization (in the input–output sense) and if (9) holds for a certain bicausal outer T_R, then $T_\Theta = T_\Sigma \circ T_R^{-1}$ is (j, J)-inner and $T_\Sigma = T_\Theta \circ T_R$ is a (j, J)-inner–outer factorization of T_Σ.

The proof requires a preliminary fact of independent interest.

Proposition 1: Suppose T_Θ: $L_2^m(\boldsymbol{R}^+) \to L_2^p(\boldsymbol{R}^+)$ is (j, J)-conservative in the input–output sense. Then

$$[DT_\Theta(u)]^T \circ JT_\Theta(u) = ju \tag{10}$$

for all $u \in \boldsymbol{R}^m$.

Proof: Differentiation of (7) with respect to u in the direction $h \in L_2^m(\boldsymbol{R}^+)$ yields

$$\langle [DT_\Theta(u)]^T \circ JT_\Theta(u), h\rangle = \langle ju, h\rangle$$

from which (10) follows from the arbitrariness of h. Conversely, note that

$$\begin{aligned}
&\langle ju, u\rangle - \langle JT_\Theta(u), T_\Theta(u)\rangle \\
&\quad = \int_0^1 \frac{d}{dt}\{\langle j(tu), tu\rangle - \langle JT_\Theta(tu), T_\Theta(tu)\rangle\}\, dt \\
&\quad = 2\left(\int_0^1 \{\langle j(tu), u\rangle - \langle [DT_\Theta(tu)]^T \circ JT_\Theta(tu), u\rangle\}\, dt\right)
\end{aligned}$$

whence (7) follows from (10).

Proof of Theorem 1: If $T_\Sigma = T_\Theta T_R$ and T_Θ is (j, J)-conservative, then Proposition 1 (with $T_R(u)$ in place of u) gives us

$$\begin{aligned}
&[DT_\Sigma(u)]^T \circ JT_\Sigma(u) \\
&\quad = [DT_R(u)]^T \circ [DT_\Theta(T_R(u))]^T \circ JT_\Theta(T_R(u)) \\
&\quad = [DT_R(u)]^T \circ jT_R(u)
\end{aligned}$$

and (9) follows. Conversely, assume (9) and set $T_\Theta = T_\Sigma \circ T_R^{-1}$. Then

$$\begin{aligned}
&[DT_\Theta(u)]^T \circ JT_\Theta(u) \\
&\quad = [DT_R(T_R^{-1}(u))]^{-T} \circ [DT_\Sigma(T_R^{-1}(u))]^T \\
&\qquad \circ JT_\Sigma(T_R^{-1}(u)) \\
&\quad = [DT_R(T_R^{-1}(u))]^{-T} \circ [DT_R(T_R^{-1}(u))]^T \\
&\qquad \circ jT_R(T_R^{-1}(u)) \\
&\quad = ju
\end{aligned}$$

where the second line follows from (9) with $T_R^{-1}(u)$ in place of u. From the other direction of Proposition 1, we conclude that Θ is (j, J)-conservative.

Suppose now that T_Σ has a (j, J)-inner–outer factorization $T_\Sigma = T_{\Theta'} \circ T_{R'}$ and that $T_\Sigma = T_\Theta \circ T_R$ is a (j, J)-conservative-outer factorization as in the first part of the proof. Then we have the identity

$$T_\Theta = T_{\Theta'} \circ (T_{R'} \circ T_R^{-1}).$$

Since $T_{R'} \circ T_R^{-1}$ is bicausal and outer, the (j, J)-inner property of $T_{\Theta'}$ carries over to the (j, J)-conservative T_Θ as well. Hence the j-spectral factorization approach actually leads to a (j, J)-inner–outer factorization whenever such exists, as desired.

C. Inner–Outer and Spectral Factorization for State-Space Systems

The goal of this section is to implement the j-spectral factorization problem (9) described in Theorem 1 (where T_Σ is known and T_R is to be found) in terms of state-space representations. We assume that T_Σ is the input–output map associated with a system Σ given by (1), and we seek an outer system R given by state-space equations of the form of (5) so that (9) holds. To achieve this we must discuss how to obtain a state-space realization for $[DT_\Sigma]^T \circ JT_\Sigma$ and from this a realization for T_R so that (9) holds. The analysis parallels the approach to spectral factorization in [10].

To obtain a state-space realization of $[DT_\Sigma]^T \circ JT_\Sigma$, we first consider the Hamiltonian extension of Σ [where Σ is given by (1)] introduced in [12], namely

$$\begin{aligned}\dot{x} &= a(x) + b(x)u \\ \dot{p} &= -\left[\frac{\partial a}{\partial x}(x) + \frac{\partial b}{\partial x}(x)u\right]^T p - \frac{\partial^T c}{\partial x}(x)u_a \\ &\quad - u^T \frac{\partial^T d}{\partial x}(x)u_a, \qquad u_a \in \boldsymbol{R}^p \\ y &= c(x) + d(x)u \\ y_a &= b^T(x)p + d^T(x)u_a, \qquad y_a \in \boldsymbol{R}^m \end{aligned} \tag{11}$$

which is a Hamiltonian system with state-space T^*M [the cotangent bundle of the state manifold M with local coordinates (x,p)] having inputs (u,u_a) and outputs (y,y_a). Imposing the interconnection law $u_a = Jy$ on (11) leads to the Hamiltonian system

$$[D\Sigma]^T \circ J\Sigma\colon \begin{cases} \dot{x} = \dfrac{\partial H}{\partial p}(x,p,u) \\ \dot{p} = -\dfrac{\partial H}{\partial x}(x,p,u) \\ y_a = \dfrac{\partial H}{\partial u}(x,p,u) \end{cases} \tag{12}$$

with state-space T^*M, inputs $u \in \boldsymbol{R}^m$, outputs $y_a \in \boldsymbol{R}^m$, and the Hamiltonian function

$$\begin{aligned} H(x,p,u) &= p^T(a(x) + b(x)u) + \tfrac{1}{2}(c(x) + d(x)u)^T \\ &\quad \cdot J(c(x) + d(x)u). \end{aligned} \tag{13}$$

If we impose the initial condition $(x,p) = (0,0)$ and consider the input–output map $T_{[D\Sigma]^T \circ J\Sigma}$ mapping u to y_a, it is not difficult to see that

$$T_{[D\Sigma]^T \circ J\Sigma}(u) = [DT_\Sigma(u)]^T \circ JT_\Sigma(u).$$

In particular, for a linear system Σ given by state-space equations

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

$[D\Sigma]^T \circ J\Sigma$ is nothing other than the series interconnection of $J \circ \Sigma$ and the adjoint system Σ^T having state-space equations

$$\begin{aligned} \dot{p} &= -A^T p - C^T u_a \\ y_a &= B^T p + D^T u_a \end{aligned}$$

and has transfer matrix $G^T(-s)JG(s)$, where $G(s) = C(sI - A)^{-j}B + D$. The nonlinear case is somewhat more complicated. The action of the left factor $[D\Sigma]^T$ in the composition $[D\Sigma]^T \circ J\Sigma$ which produces the output y_a requires not only the output $u_a = y$ of the right factor $J\Sigma$ but also the state vector x of the right factor at each time t.

Our goal now is to produce an invertible outer system R as in (6) so that

$$[D\Sigma]^T \circ J\Sigma = [DR]^T \circ jR. \tag{14}$$

As we no longer insist on zero initial conditions for the state vectors in the state-space equations, this problem is actually somewhat more general than that discussed in Section II-B. Also, as was mentioned in Section II-B, the input–output notions of "(j,J)-inner" and of "outer" in general are not equivalent to the corresponding state-space notions. Here we shall implement the procedure outlined in Theorem 1 in terms of state-space representations. Rather than applying Theorem 1, we simply check directly for the "(j,J)-inner" and "outer" properties in the state-space sense of our solution. Thus Theorem 1 serves more as motivation rather than a logically integral part of the analysis.

If R is an outer system as in (5) (with state manifold denoted by $\overline{M}$), then the same development as above gives that $[DR]^T \circ jR$ has the form

$$\begin{aligned} \dot{\overline{x}} &= \frac{\partial H_R}{\partial p}(\overline{x},\overline{p},u) \\ \dot{\overline{p}} &= -\frac{\partial H_R}{\partial x}(\overline{x},\overline{p},u) \\ y_a &= \frac{\partial H_R}{\partial u}(\overline{x},\overline{p},u) \end{aligned} \tag{15}$$

where

$$\begin{aligned} H_R(\overline{x},\overline{p},u) &= \overline{p}^T(\overline{a}(\overline{x}) + \overline{b}(\overline{x})u) + \tfrac{1}{2}(\overline{c}(\overline{x}) \\ &\quad + \overline{d}(\overline{x})u)^T j(\overline{c}(\overline{x}) + \overline{d}(\overline{x})u). \end{aligned} \tag{16}$$

We first show that the antistable invariant manifold of $[DR]^T \circ jR$ for $u = 0$ (i.e., the set of points $(\overline{x},\overline{p})$ converging to $(\overline{x},\overline{p}) = (0,0)$ as $t \to -\infty$ under the Hamiltonian flow) has the simple form

$$\{(\overline{x},\overline{p}) \in T^*\overline{M}\colon \overline{x} = 0\}. \tag{AS}$$

Indeed, from (16) we see that

$$\frac{\partial H_R}{\partial p}(0,\overline{p},0) = \overline{p}^T\overline{a}\,(0) + \frac{1}{2}\overline{c}^T(0)j\overline{c}(0) = 0$$

for all $\overline{p}$. From (15) it follows that the manifold (AS) is invariant for (15) restricted to $u = 0$. Moreover, from (16) we have that

$$-\frac{\partial H_R}{\partial x}(0,\overline{p},0) = -\frac{\partial^T \overline{a}}{\partial x}(0)\overline{p}.$$

By assumption $x \to \overline{a}(x)$ is an asymptotically stable vector field, and hence the matrix $-(\partial^T\overline{a}/\partial x)(0)$ is antistable (i.e., all its eigenvalues are in the closed right-half plane). Hence the Hamiltonian flow (15) (with $u = 0$) restricted to (AS) is antistable. By dimension count, it follows that (AS) is the antistable invariant manifold for (15) with $u = 0$.

Second, the inverse system is easily computed since by the assumption $\overline{d}(x)$ is invertible; the result is

$$([DR]^T \circ jR)^{-1}\colon \begin{cases} \dot{\overline{x}} = \dfrac{\partial H_R^\times}{\partial p}(\overline{x},\overline{p},y_a) \\ \dot{\overline{p}} = -\dfrac{\partial H_R^\times}{\partial x}(\overline{x},\overline{p},y_a) \\ u = -\dfrac{\partial H_R^\times}{\partial y_a}(\overline{x},\overline{p},y_a) \end{cases} \tag{17}$$

with state-space $T^*\overline{M}$, inputs y_a and outputs u, where the inverse Hamiltonian $H_R^\times(\overline{x},\overline{p},y_a)$ is obtained as the Legendre

transform of $H_R(\overline{x},\overline{p},u)$ with respect to u and y_a, i.e.,

$$H_R^\times(\overline{x},\overline{p},y_a) = H_R(\overline{x},\overline{p},u) - u^T y_a, \quad u \text{ satisfying } \frac{\partial H_R}{\partial u}(\overline{x},\overline{p},u) = y_a. \tag{18}$$

(Note the somewhat unusual sign convention.) Direct computation using (16) gives

$$\begin{aligned} H_R^\times(\overline{x},\overline{p},y_a) = {} & \overline{p}^T(\overline{a}(\overline{x}) - \overline{b}(\overline{x})\overline{d}^{-1}(\overline{x})\overline{c}(\overline{x})) \\ & - \tfrac{1}{2}\overline{p}^T\overline{b}(\overline{x})\overline{d}^{-1}(\overline{x})j\overline{d}^{-T}(\overline{x})\overline{b}^T(\overline{x})\overline{p} \\ & + (\overline{p}^T\overline{b}(\overline{x}) + \overline{c}^T(\overline{x})j\overline{d}(\overline{x}))\overline{d}^{-1}(\overline{x})j\overline{d}^{-T}(\overline{x})y_a \\ & - \tfrac{1}{2}y_a\overline{d}^{-1}(\overline{x})j\overline{d}^{-T}(\overline{x})y_a. \end{aligned} \tag{19}$$

We note from (19) that

$$-\frac{\partial H_R^\times}{\partial \overline{x}}(\overline{x},0,0) = 0.$$

From (17) we see that the other coordinate space

$$\{(\overline{x},\overline{p}) \in T^*\overline{M}\colon \overline{p} = 0\} \tag{S}$$

is an invariant manifold of $([DR]^T \circ jR)^{-1}$ with $y_a = 0$. Moreover, from (17) and (19) we see that the restriction of the dynamics of $([DR]^T \circ jR)^{-1}$ with $y_a = 0$ to this manifold is given by the vector field

$$\frac{\partial H_R^\times}{\partial \overline{p}}(\overline{x},0,0) = \overline{a}^\times(\overline{x})$$

where

$$\overline{a}^\times(\overline{x}) := \overline{a}(\overline{x}) - \overline{b}(\overline{x})\overline{d}^{-1}(\overline{x})\overline{c}(\overline{x}).$$

If R is (strictly) outer, then $\overline{a}^\times(\overline{x})$ is Lyapunov (asymptotically) stable. By dimension count we conclude that the manifold (S) is the stable invariant manifold for the Hamiltonian flow associated with the inverse system $([DR]^T \circ jR)^{-1}$ with $y_a = 0$. To factor $[D\Sigma]^T \circ J\Sigma$ as $[DR]^T \circ jR$, the idea now is to find a canonical change of coordinates $(x,p) \to (\overline{x},\overline{p})$ so that the system $[D\Sigma]^T \circ J\Sigma$ given by (12) and (13) expressed in the new coordinates will have the form of (15) and (16); from (16) we can then read off the desired coefficients $\overline{a}(\overline{x}), \overline{b}(\overline{x}), \overline{c}(\overline{x}), \overline{d}(\overline{x})$ for the desired outer factor. It is crucial that the change of coordinates be canonical to preserve the Hamiltonian structure (see e.g., [1]).

First note from Assumption (A2) the fact that $d^T(x)Jd(x)$ is invertible guarantees the causal invertibility of the system $[D\Sigma]^T \circ J\Sigma$. Indeed, analogously to the procedure for $[DR]^T \circ jR$, the inverse system $([D\Sigma]^T \circ J\Sigma)^{-1}$ is the Hamiltonian system

$$([D\Sigma]^T \circ J\Sigma)^{-1} \begin{cases} \dot{x} = \dfrac{\partial H^\times}{\partial p}(x,p,y_a) \\ \dot{p} = -\dfrac{\partial H^\times}{\partial x}(x,p,y_a) \\ u = -\dfrac{\partial H^\times}{\partial y_a}(x,p,y_a) \end{cases} \tag{20}$$

state-space T^*M, inputs y_a, and outputs u, where the inverse Hamiltonian $H^\times$ is the Legendre transform of $H(x,p,u)$ with respect to u and y_a, i.e.,

$$H^\times(x,p,y_a) = H(x,p,u) - u^T y_a \quad u \text{ satisfying } \frac{\partial H}{\partial u}(x,p,u) = y_a. \tag{21}$$

Explicitly, this works out to be

$$\begin{aligned} H^\times(x,p,y_a) = {} & p^T(a(x) - b(x)E^{-1}(x)d^T(x)Jc(x)) \\ & + \tfrac{1}{2}c^T(x)(J - Jd(x)E^{-1}(x)d^T(x)J)c(x) \\ & - \tfrac{1}{2}p^T b(x)E^{-1}(x)b^T(x)p \\ & + (p^T b(x) + c^T(x)Jd(x))E^{-1}(x)y_a \\ & - \tfrac{1}{2}y_a^T E^{-1}(x)y_a \end{aligned} \tag{22}$$

where $E(x) = d^T(x)Jd(x)$.

The desired canonical change of coordinates $(x,p) \to (\overline{x},\overline{p})$ must be such that the antistable invariant manifold for the Hamiltonian flow induced by H (with $u = 0$) is given by $\{(\overline{x},\overline{p}) \in T^*M\colon \overline{x} = 0\}$ and is such that the manifold $\{(\overline{x},\overline{p}) \in T^*M\colon \overline{p} = 0\}$ is a Lyapunov (or asymptotically, in the strict outer case) stable invariant manifold of $H^\times$ (with $y_a = 0$). Note next that by the same calculation as was done above for H_R, our standing Assumption (A1) that a is asymptotically stable implies that the antistable manifold for H (with $u = 0$) is $\{(x,p) \in T^*M\colon x = 0\}$. To compute a Lagrangian invariant manifold for $([D\Sigma]^T \circ J\Sigma)^{-1}$ with $y_a = 0$, we consider the Hamilton–Jacobi equation $H^\times(x, P_x^T(x), 0) = 0$ for some smooth function $P(x)$ with $P(0) = 0$, i.e.,

$$\begin{aligned} & P_x(x)[a(x) - b(x)E^{-1}(x)d^T(x)Jc(x)] \\ & \quad + \tfrac{1}{2}c^T(x)[J - Jd(x)E^{-1}(x)d^T(x)J]c(x) \\ & \quad - \tfrac{1}{2}P_x(x)b(x)E^{-1}(x)b^T(x)P_x^T(x) = 0, \qquad P(0) = 0 \end{aligned} \tag{23}$$

with the stability side condition

$$\begin{aligned} & \frac{\partial H^\times}{\partial p}(x, P_x^T(x), 0) \\ & \quad = a(x) - b(x)E^{-1}(x)[d^T(x)Jc(x) + b^T(x)P_x^T(x)] \\ & \quad \text{is Lyapunov stable.} \end{aligned} \tag{24}$$

Suppose that there exists a smooth solution P to (23) and (24). Then it is well known (see [22]) that the submanifold of T^*M given by

$$\{(x,p)\colon p = P_x^T(x)\} \tag{25}$$

is an invariant manifold for $([D\Sigma]^T \circ J\Sigma)^{-1}$ with $y_a = 0$, on which the dynamics coordinated by x are given by the Lyapunov stable vector field given in (24). This leads to the canonical transformation $(x,p) \to (x,\overline{p})$ with

$$p = \overline{p} + P_x^T(x).$$

Clearly, in these new coordinates, $\{(x,\overline{p})\colon x = 0\}$ remains the antistable invariant manifold for $[D\Sigma]^T \circ J\Sigma$ with $u = 0$, and the invariant manifold (25) for $([D\Sigma]^T \circ J\Sigma)^{-1}$ with

$y_a = 0$ is given simply as $\{(x,\overline{p}): \overline{p} = 0\}$. The next step is to compute the Hamiltonian $H(x,p,u)$ in these new coordinates

$$\begin{aligned}\overline{H}(x,\overline{p},u) &= H(x,\overline{p} + P_x^T(x),u)\\ &= \overline{p}^T(a(x)+b(x)u) + \tfrac{1}{2}(\overline{c}(x)\\ &\quad + \overline{d}(x)u)^T j(\overline{c}(x)+\overline{d}(x)u)\end{aligned} \tag{26}$$

where $\overline{d}(x)$ is chosen as in (A2)

$$d^T(x)Jd(x) = \overline{d}^T(x)j\overline{d}(x), \qquad \overline{d}(x) \text{ of size } m \times m \tag{27}$$

and

$$\overline{c}(x) = \overline{d}(x)E^{-1}(x)[d^T(x)Jc(x) + b^T(x)P_x^T(x)]. \tag{28}$$

Comparison of (26) and (16) now gives that $[D\Sigma]^T \circ J\Sigma = [DR]^T \circ jR$ where R is the new system defined by

$$R\colon \begin{cases}\dot{x} = a(x) + b(x)u\\ \overline{y} = \overline{c}(x) + \overline{d}(x)u\end{cases} \tag{29}$$

with $\overline{c}$ and $\overline{d}$ as in (28) and (27), respectively. Clearly, from (A1) R is asymptotically stable. The inverse dynamics for R is given by

$$\begin{aligned}&a(x) - b(x)\overline{d}^{-1}(x)\overline{c}(x)\\ &\quad = a(x) - b(x)E^{-1}(x)[d^T(x)Jc(x) + b^T(x)P_x^T(x)]\end{aligned}$$

which is Lyapunov stable by (24). Thus R is also minimum phase and so is an outer factor of Σ.

The next step is to compute Θ as $\Theta = \Sigma \circ R^{-1}$. From (29) we see that R^{-1} is given by

$$R^{-1}\colon \begin{cases}\dot{\tilde{x}} = a(\tilde{x}) - b(\tilde{x})\overline{d}^{-1}(\tilde{x})\overline{c}(\tilde{x}) + b(\tilde{x})\overline{d}^{-1}(\tilde{x})\overline{y}\\ u = -\overline{d}^{-1}(\tilde{x})\overline{c}(\tilde{x}) + \overline{d}(\tilde{x})^{-1}\overline{y}.\end{cases} \tag{30}$$

Combine with (1) for Σ and use the interconnection law that u in (1) is equal to u in (30) to get

$$\Sigma \circ R^{-1}\colon \begin{cases}\dot{x} = a(x) - b(x)\overline{d}^{-1}(x)\overline{c}(\tilde{x}) + b(x)\overline{d}^{-1}(\tilde{x})\overline{y}\\ \dot{\tilde{x}} = a(\tilde{x}) - b(\tilde{x})\overline{d}^{-1}(\tilde{x})\overline{c}(\tilde{x}) + b(\tilde{x})\overline{d}^{-1}(\tilde{x})\overline{y}\\ y = c(x) - d(x)\overline{d}^{-1}(\tilde{x})\overline{c}(\tilde{x}) + d(x)\overline{d}(\tilde{x})^{-1}\overline{y}.\end{cases} \tag{31}$$

Note that if the initial state $(x(0),\tilde{x}(0))$ is on the diagonal $x(0) = \tilde{x}(0)$, then the state vector remains on the diagonal for all $t > 0$ $(x(t) = \tilde{x}(t))$, and the reduced system Θ (with state vector (x,x) parameterized simply by x) is given by

$$\Theta\colon \begin{cases}\dot{x} = a(x) - b(x)\overline{d}^{-1}(x)\overline{c}(x) + b(x)\overline{d}^{-1}(x)\overline{y}\\ y = c(x) - d(x)\overline{d}^{-1}(x)\overline{c}(x) + d(x)\overline{d}^{-1}(x)\overline{y}.\end{cases} \tag{32}$$

It is not difficult to check, by using (23) satisfied by P together with (27) and (28) for $\overline{d}$ and $\overline{c}$, that the system of equations

$$\begin{aligned}&P_x(x)[a(x) - b(x)\overline{d}^{-1}(x)\overline{c}(x)] + \tfrac{1}{2}[c(x) - d(x)\overline{d}^{-1}(x)\overline{c}(x)]^T\\ &\qquad \cdot J[c(x) - d(x)\overline{d}^{-1}(x)\overline{c}(x)] = 0\\ &P_x(x)b(x)\overline{d}^{-1}(x) + [c(x) - d(x)\overline{d}^{-1}(x)\overline{c}(x)]^T\\ &\qquad \cdot Jd(x)\overline{d}^{-1}(x) = 0\\ &[d(x)\overline{d}^{-1}(x)]^T J[d(x)\overline{d}^{-1}(x)] = j\end{aligned} \tag{33}$$

holds for all x, and hence P is a storage function for Θ with supply rate $\frac{1}{2}\overline{y}^T j\overline{y} - \frac{1}{2}y^T Jy$ if $P(x) \geq 0$. One also easily checks that $\Sigma = \Theta \circ R$ in the sense that given any initial state x_0 for Σ, there is a choice of initial state for $\Theta \circ R$ (namely, (x_0,x_0)) so that the input–output map for Σ is the same as the input–output map associated with $\Theta \circ R$. The following summarizes our discussion.

Theorem 2: Assume that (A1) and (A2) hold. Suppose that there exists a solution $P \geq 0$ to (23) and (24). Then a (j,J)-inner–outer factorization of Σ is $\Theta \circ R$ with R and Θ defined by (29) and (32), respectively.

Remark: In the context of H_∞ control, it is important to consider the larger system $\Sigma \circ R^{-1}$ defined by (31) rather than Θ. While it is clear that $\Sigma = (\Sigma \circ R^{-1}) \circ R$ at the input–output level, there is no obvious storage function for $\Sigma \circ R^{-1}$, and hence it is not clear if $\Sigma \circ R^{-1}$ is (j,J)-inner. We shall return to this topic in Section IV.

Let us now consider the case where the inverse system $([D\Sigma]^T \circ J\Sigma)^{-1}$ for $y_a = 0$ does not possess any dynamics corresponding to purely imaginary eigenvalues, that is, where the linearization of $([D\Sigma]^T \circ J\Sigma)$ does not have purely imaginary transmission zeros. In this situation the stable invariant manifold for the Hamiltonian flow induced by $H^\times$ (with $y_a = 0$) is automatically Lagrangian (see [22]), and one can state a local version of Theorem 2 as follows. Consider the linearization at (0, 0) of (21) for $y_a = 0$. This gives rise to a linear Hamiltonian system

$$\begin{aligned}&\begin{bmatrix}\dot{x}\\ \dot{p}\end{bmatrix} = \begin{bmatrix}A & -R\\ -Q & -A^T\end{bmatrix}\begin{bmatrix}x\\ p\end{bmatrix}\\ &A = \frac{\partial a}{\partial x}(0) - b(0)E^{-1}(0)d^T(0)\frac{\partial c}{\partial x}(0)\\ &R = b(0)E^{-1}(0)b^T(0)\\ &Q = \frac{\partial^T c}{\partial x}(0)[J - Jd(0)E^{-1}(0)d^T(0)J]\frac{\partial^T c}{\partial x}(0).\end{aligned} \tag{34}$$

Assume also that

i) The Hamiltonian matrix in (34) does not have purely imaginary eigenvalues; and
ii) The generalized eigenspace corresponding to eigenvalues in the open left-half complex plane for the Hamiltonian matrix in (34) has the form span $\begin{bmatrix}I\\ X\end{bmatrix}$ for some positive semidefinite matrix X.

Condition ii) is equivalent to: there exists a positive semidefinite solution $X \geq 0$ to the algebraic Riccati equation

$$XA + A^T X - XRX + Q = 0 \tag{35}$$

with stability side condition

$$A - RX \text{ is asymptotically stable.} \tag{36}$$

(Here A, R, Q are as in (34).) Then we have the following result. For guidelines to the details of the proof, we refer to [23].

Theorem 3: Assume (A1) and (A2) hold. Assume in addition that conditions i) and ii) above are satisfied. Then there exists a neighborhood $W \subset M$ of $x_0 = 0$ and a smooth function $P\colon W \to \boldsymbol{R}$ with $P(x) \geq 0$ for $x \in W$ which is the unique solution of (23) together with the strengthened local

version of (24)

$$a(x) - b(x)E^{-1}(x)[d^T(x)Jc(x) + b^T(x)P_x^T(x)] \text{ is asymptotically stable for } x \in W. \tag{37}$$

If we define R and Θ as in (29) and (32), respectively, for $x \in W$, then $\Sigma = \Theta \circ R$ is a (j, J)-inner–outer factorization of Σ on W, where the outer factor R is strictly minimum phase on W.

Furthermore, if the function P: $W \to \boldsymbol{R}$ can be extended to a global solution P: $M \to \boldsymbol{R}$ with $a(x) - b(x)E^{-1}(x)[d^T(x)Jc(x) + b^T(x)P_x^T(x)]$ globally asymptotically stable and with the nonnegativity constraint $P(x) \geq 0$ holding for all $x \in M$, then Θ and R are globally defined inner and outer factors with R strictly minimum phase on M.

Remark: In the case where $J = I_p$ and $j = I_m$, one can use inertia-type theorems to guarantee a unique local solution of (23) and (24). We will discuss this case in more detail in [27].

III. Application to L_2-Gain

As an application of the theory of nonlinear spectral factorization developed in Section II, we give here a new interpretation of the state-space characterization of systems having finite gain at most equal to a tolerance level γ.

Let us consider as in Section II a smooth nonlinear state-space system Σ which is affine in the inputs

$$\Sigma\colon \begin{cases} \dot{x} = a(x) + b(x)u, & u \in \boldsymbol{R}^m \\ y = c(x) + d(x)u, & y \in \boldsymbol{R}^p. \end{cases} \tag{38}$$

We assume that

$$a(x) \text{ is asymptotically stable} \tag{A1}$$

and that

$$\gamma^2 I_m - d^T(x)d(x) = \overline{d}^T(x)\overline{d}(x) \tag{A2}$$

for a smooth, everywhere invertible $m \times m$ matrix function $\overline{d}(x)$, where γ is a fixed positive number. As usual, we also assume that $a(0) = 0, c(0) = 0$. Assume first that the associated input–output map T_Σ: $u \to y$ (defined with the state initialized to the equilibrium point $x(0) = 0$) is a well-defined mapping from $L_{2,e}^m(\boldsymbol{R}^+)$ into $L_{2,e}^p(\boldsymbol{R}^+)$ (where $L_{2,e}^n(\boldsymbol{R}^+)$, in general, consists of $\boldsymbol{R}^n$-vector functions defined on $\boldsymbol{R}^+ = [0, \infty)$ which are square-integrable in norm over each finite interval). Then we say that T_Σ has finite gain less than or equal to γ if

$$\|T_\Sigma(u)\|_2 \leq \gamma \|u\|_2, \qquad u \in L_2^m(\boldsymbol{R}^+). \tag{39}$$

By the causality of T_Σ, this in turn is equivalent to

$$\int_0^{t_1} \|y(t)\|^2\, dt \leq \gamma^2 \int_0^{t_1} \|u(t)\|^2\, dt$$

for all $t_1 > 0$ whenever $y = T_\Sigma(u)$.

A closely related state-space idea is that there exists a storage function $V(x)$ for Σ with respect to the supply rate $\frac{1}{2}\gamma^2\|u\|^2 - \frac{1}{2}\|y\|^2$ (in the sense of dissipative systems rather than of lossless systems as in Section II—see [29]), i.e., that there exists a function $V(x) \geq 0$ on M so that

$$V(x(t_1)) - V(x(t_0)) \leq \tfrac{1}{2}\int_{t_0}^{t_1} (\gamma^2\|u(t)\|^2 - \|y(t)\|^2)\, dt \qquad V(0) = 0 \tag{40}$$

for all $t_0 \leq t_1$ and $u(\cdot)$ with $x(t_1)$ denoting the state at time t_1 resulting from initial state $x(t_0)$ at time t_0 and input $u(\cdot)$ on the time interval $[t_0, t_1]$. In particular, if we take $t_0 = 0, x(t_0) = 0$, we recover (39) as a consequence of (40). If the system is reachable, conversely one can construct a function $V(x) \geq 0$ satisfying (40) as a consequence of (39). If V is differentiable, then (40) can be equivalently expressed as

$$V_x(x)[a(x) + b(x)u] + \tfrac{1}{2}[c(x) + d(x)u]^T[c(x) + d(x)u] - \tfrac{1}{2}\gamma^2 u^T u \leq 0, \qquad V(0) = 0 \tag{41}$$

for all x and u. As the expression on the left is quadratic in u, we can compute its maximum over u explicitly. Then (41) has an equivalent formulation

$$\begin{aligned} &V_x(x)[a(x) + b(x)(\gamma^2 I_m - d^T(x)d(x))^{-1}d^T(x)c(x)] \\ &\quad + \tfrac{1}{2}c^T(x)(I_p - \gamma^{-2}d(x)d^T(x))^{-1}c(x) \\ &\quad + \tfrac{1}{2}V_x(x)b(x)(\gamma^2 I_m - d^T(x)d(x))^{-1}b^T(x)V_x(x) \leq 0 \\ &\quad V(0) = 0 \end{aligned} \tag{42}$$

for all x.

An alternative derivation of (41) from the point of view of spectral factorization can be done as follows. An obvious sufficient condition for (39) to hold is that there exists a smooth system R

$$R\colon \begin{cases} \dot{x} = \overline{a}(x) + \overline{b}(x)u \\ \overline{y} = \overline{c}(x) + \overline{d}(x)u \end{cases} \tag{43}$$

such that

$$\tfrac{1}{2}(\gamma^2\|u\|^2 - \|T_\Sigma(u)\|^2) = \tfrac{1}{2}\|T_R(u)\|^2 \tag{44}$$

for all u in $L_2^m(\boldsymbol{R}^+)$. Frechet-differentiation of (44) with respect to u then gives

$$\gamma^2 u - [DT_\Sigma(u)]^T \circ T_\Sigma(u) = [DT_R(u)]^T \circ T_R(u). \tag{45}$$

Conversely, (45), together with initial conditions $T_R(0) = 0, T_\Sigma(0) = 0$, implies (44) by a straightforward integration along the line segment $\{tu\colon 0 \leq t \leq 1\} \subset L_2^m(\boldsymbol{R}^+)$.

Thus to show that T_Σ has gain at most γ, it suffices to produce a system R so that (45) is satisfied. But a state-space realization of the map $u \to \gamma^2 u - [DT_\Sigma(u)]^T \circ T_\Sigma(u)$, computed in the form $u \to \gamma^2 u + [DT_\Sigma(u)]^T \circ (-T_\Sigma(u))$, is

$$\gamma^2 I_m - [D\Sigma]^T \circ \Sigma \begin{cases} \dot{x} = \dfrac{\partial \tilde{H}}{\partial p}(x, p, u) \\ \dot{p} = -\dfrac{\partial \tilde{H}}{\partial x}(x, p, u) \\ y_a = \dfrac{\partial H}{\partial u}(x, p, u) \end{cases} \tag{46}$$

with state-space T^*M, inputs $u \in \boldsymbol{R}^m$, and outputs $y_a \in \boldsymbol{R}^m$ where the Hamiltonian function $\tilde{H}(x,p,u)$ is given by

$$\tilde{H}(x,p,u) = p^T(a(x)+b(x)u) - \tfrac{1}{2}(c(x)+d(x)u)^T (c(x)+d(x)u) + \tfrac{1}{2}\gamma^2 u^T u. \tag{47}$$

(This amounts to the Hamiltonian extension (11) of Σ with connection law $u_a = -y$ in series connection with the memoryless system $u \to \gamma^2 u$.) Due to Assumption (A2), the inverse system is easily computed; it is again a Hamiltonian system

$$(\gamma^2 I_m - [D\Sigma]^T \circ \Sigma)^{-1} \begin{cases} \dot{x} = \dfrac{\partial \tilde{H}^\times}{\partial p}(x,p,y_a) \\ \dot{p} = -\dfrac{\partial \tilde{H}^\times}{\partial x}(x,p,y_a) \\ u = -\dfrac{\partial \tilde{H}^\times}{\partial y_a}(x,p,y_a) \end{cases} \tag{48}$$

with the Hamiltonian equal to the Legendre transform of $H(x,p,u)$ with respect to u and y_a

$$\tilde{H}^\times(x,p,y_a) = \tilde{H}(x,p,u) - u^T y_a, \quad u \text{ satisfying } y_a = \frac{\partial \tilde{H}}{\partial u}(x,p,u). \tag{49}$$

From (47), $\tilde{H}^\times(x,p,y_a)$ works out explicitly to be

$$\begin{aligned} \tilde{H}^\times(x,p,y_a) = {} & p^T[a(x)+b(x)(\gamma^2 - d^T(x)d(x))^{-1} \\ & \cdot d^T(x)c(x)] - \tfrac{1}{2}p^T b(x) \\ & \cdot (\gamma^2 I_m - d^T(x)d(x))^{-1} b^T(x)p \\ & - \tfrac{1}{2}c^T(x)(I_p - \gamma^{-2}d(x)d^T(x))^{-1}c(x) \\ & + p^T b(x)(\gamma^2 I_m - d^T(x)d(x))^{-1}y_a \\ & + c^T(x)d(x)(\gamma^2 I_m - d^T(x)d(x))^{-1}y_a \\ & - \tfrac{1}{2}y_a^T(\gamma^2 I_m - d^T(x)d(x))^{-1}y_a. \end{aligned} \tag{50}$$

To factor $\gamma^2 I_m - [D\Sigma]^T \circ \Sigma$ as $[DR]^T \circ R$ with R stable, we know from experience in Section II that we need to compute the antistable invariant manifold of $\gamma^2 I_m - [D\Sigma]^T \circ \Sigma$ with $u = 0$ and some invariant manifold of $(\gamma^2 I_m - [D\Sigma]^T \circ \Sigma)^{-1}$ with $y_a = 0$. The former is simply $\{(x,p)\colon x = 0\}$ due to Assumption (A1). As for the latter, we consider the Hamilton–Jacobi equation

$$\tilde{H}^\times(x, -V_x^T(x), 0) = 0, \qquad V(0) = 0. \tag{51}$$

This is the same as (42) but with an equality rather than inequality. We introduce the change of coordinates $p = \overline{p} - V_x^T(x)$ and compute the original Hamiltonian in terms of these new coordinates. The result is

$$\begin{aligned} \overline{\tilde{H}}(x,\overline{p},u) &= \overline{p}^T[a(x)+b(x)u] - \tfrac{1}{2}[c(x)+d(x)u]^T \\ &\quad \cdot [c(x)+d(x)u] + \tfrac{1}{2}\gamma^2 u^T u - V_x(x) \\ &\quad \cdot [a(x)+b(x)u] \\ &= \overline{p}^T[a(x)+b(x)u] + \tfrac{1}{2}[\overline{c}(x)+\overline{d}(x)u]^T \\ &\quad \cdot [\overline{c}(x)+\overline{d}(x)u] \end{aligned} \tag{52}$$

with

$$\overline{d}^T(x)\overline{d}(x) = \gamma^2 I_m - d^T(x)d(x) \tag{53}$$

as in (A2), and

$$\overline{c}(x) = -\overline{d}^{-T}(x)(d^T(x)c(x) + b^T(x)V_x^T(x)) \tag{54}$$

where we used (51). By comparison with (19) (where now $j = I_m$), we see that $\gamma^2 I_m - [D\Sigma]^T \circ \Sigma = [DR]^T \circ R$ with R given by

$$R\colon \begin{cases} \dot{x} = a(x) + b(x)u \\ \overline{y} = \overline{c}(x) + \overline{d}(x)u \end{cases} \tag{55}$$

with $\overline{c}$ and $\overline{d}$ as in (54) and (53), respectively. We summarize the discussion as follows. The result is an alternate interpretation for the role of (42) for a system having gain $\leq \gamma$. One can also verify the result directly by using a standard completion-of-squares argument based on (51).

Theorem 4: Suppose that Σ is a system satisfying Assumptions (A1) and (A2), and suppose that $V(x) \geq 0$ is a smooth solution of (42) with equality. Then the input–output map T_Σ of Σ (defined with $x(0) = 0$) satisfies

$$\gamma^2\|u\|^2 - \|T_\Sigma(u)\|^2 = \|T_R(u)\|^2$$

where R is (55) with $\overline{c}(x)$ and $\overline{d}(x)$ given by (53) and (54), and hence T_Σ has finite gain at most γ.

IV. Nonlinear H_∞-Control

Consider a nonlinear system $P\colon \left[\begin{smallmatrix} w \\ u \end{smallmatrix}\right] \to \left[\begin{smallmatrix} z \\ y \end{smallmatrix}\right]$ with state-space equations

$$P\colon \begin{cases} \dot{x} = A(x) + B_1(x)w + B_2(x)u \\ z = C_1(x) + D_{12}(x)u \\ y = C_2(x) + D_{21}(x)w. \end{cases} \tag{56}$$

Here w is a reference and/or disturbance signal, u is the control signal, z is an error signal, and y is a measurement signal with values in $\boldsymbol{R}^{n_w}, \boldsymbol{R}^{n_u}, \boldsymbol{R}^{n_z}$, and $\boldsymbol{R}^{n_y}$, respectively. The H_∞-problem is to design a dynamic compensator $K\colon y \to u$ with state-space equations

$$\begin{aligned} \dot{x}_K &= A_K(x_K) + B_K(x_K)y \\ u &= C_K(x_K) + D_K(x_K)y \end{aligned} \tag{57}$$

so that

i) System (56) and (57) is "internally stable," and
ii) The closed-loop input–output map (with $x(0) = 0, x_K(0) = 0$) has L_2-gain at most γ

$$\|z\|_2 \leq \gamma\|w\|_2)$$

for all $w \in L_2^{n_w}(\boldsymbol{R}^+)$.

Here we assume that $A(0) = 0$, $C_1(0) = 0$, $C_2(0) = 0$, $A_K(0) = 0$, $C_K(0) = 0$ and all functions $A, B_1, B_2, C_1, D_{12}, C_2, D_{21}$ are smooth.

The precise notion of "internally stable" is taken in two distinct senses (one input–output, the other internal state space) in the literature. These two senses are equivalent in the linear case but not, in general, in the nonlinear case. For the input–output sense of internal stability, we say that (56) and

(57) are internally stable (IO sense) if the enlarged system of equations

$$\begin{aligned} \dot{x} &= A(x) + B_1(x)w + B_2(x)u \\ z &= C_1(x) + D_{12}(x)u \\ y &= C_2(x) + D_{21}(x)w \\ \dot{x}_K &= A_K(x_K) + B_K(x_K)(y + v_2) \\ u &= v_1 + C_K(x_K) + D_K(x_K)(y + v_2) \end{aligned} \tag{58}$$

with both $x(0) = 0$ and $x_K(0) = 0$ as initial conditions, determines a well-defined causal map

$$H\colon \begin{bmatrix} w \\ v_1 \\ v_2 \end{bmatrix} \to \begin{bmatrix} z \\ u \\ y \end{bmatrix}$$

which is stable with finite gain. On the other hand, (56) and (57) are said to be internally stable (internal state-space sense) if the vector field

$$\begin{aligned} (x, x_K) \to (&A(x) + B_2(x)C_K(x_K) + B_2(x)D_K(x_K)C_2(x), \\ &A_K(x) + B_K(x_K)C_2(x))) \end{aligned}$$

associated with the dynamics of the closed-loop system, (56) and (57), is asymptotically stable in the sense that $(x(t), x_K(t)) \to (0, 0)$ for any choice of initialization $(x(0), x_K(0))$, where $(x(t), x_K(t))$ is determined by (56) and (57) with $w = 0$. With extra hypotheses, it is possible to prove input–output stability from internal stability in the state-space sense and conversely. More precisely, if the closed-loop system is globally reachable and uniformly observable, then internal asymptotic stability (state-space sense) follows from input–output stability with finite gain (see [32]); conversely, if the mappings $A(x), B_1(x), B_2(x), C_1(x), C_2(x), D_{12}(x), D_{21}(x), A_K(x_K), B_K(x_K), C_K(x)$ all satisfy a global Lipschitz estimate and (0, 0) is a global, exponentially stable equilibrium point for the closed-loop system, then the closed-loop system is stable with finite gain (see [34]). A simple example of a system which is internally stable (state-space sense) but does not have finite L_2-gain is

$$\begin{aligned} \dot{x} &= -x^3 + u \\ y &= x. \end{aligned}$$

A number of recent papers (see [5], [8], [11], [16], [17], [20], and [22]–[24]) have shown how a solution of the nonlinear H_∞-control problem can be obtained from a smooth solution of a Hamilton–Jacobi equation for the state-feedback case (see [5], [11], [22], and [23]) or (at least locally) from smooth solutions of a coupled pair of Hamilton–Jacobi equations for the measurement feedback case (see [8], [16], [17], [20], and [24]). Our purpose here is to recover this type of result via a (j, J)-inner–outer factorization procedure for the disturbance feedforward case (to be described below).

Additional standard assumptions (called the regular case in the linear theory) are

$$D_{12}^T(x)D_{12}(x) > 0 \text{ for all } x \tag{A3}$$

and

$$D_{21}(x)D_{21}^T(x) > 0 \text{ for all } x. \tag{A4}$$

For the discussion here we shall assume the strengthened form of (A4)

$$D_{21}(x) \text{ is square and invertible for all } x \tag{A4'}$$

(and hence $n_y = n_w$). In this case we can solve the last of (56) for the disturbance w in terms of the measurement y. With this assumption in force, (56) can be rearranged to form a system G having the same trajectories as P but with inputs equal to (u, y) and outputs equal to (z, w)

$$G\colon \begin{cases} \dot{x} = [A(x) - B_1(x)D_{21}^{-1}(x)C_2(x)] \\ \qquad + B_2(x)u + B_1(x)D_{21}^{-1}(x)y \\ z = C_1(x) + D_{12}(x)u \\ \tilde{w} = -\gamma D_{21}(x)^{-1}C_2(x) + \gamma D_{21}(x)^{-1}y \end{cases} \tag{59}$$

where we have set $\tilde{w} = \gamma w$. We shall assume in addition that the vector field $a(x) := A(x) - B_1(x)D_{21}^{-1}(x)C_2(x)$ is asymptotically stable. In this case, the original problem for P is in the disturbance feedforward Case in the classification of [20].

The article [9] relates solutions K of the nonlinear H_∞-control problem for the system P in (56) (with internal stability taken in the input–output sense and where one seeks gain at most equal to γ) to (j, J)-inner–outer factorization $G = \Theta \circ R$ of the system G in (59) $\left(\text{with } J = \begin{bmatrix} I_{n_z} & 0 \\ 0 & -I_{n_w} \end{bmatrix}\right.$ and $\left. j = \begin{bmatrix} I_{n_u} & 0 \\ 0 & -I_{n_y} \end{bmatrix}\right)$. For the results to be valid, R is required to be incrementally outer, i.e., the input–output map T_R associated with R (with zero initial condition on the state vector) is incrementally stable in the sense that

$$\|P_\tau[T_R(u + v) - T_R(u)]\| \le \tilde{\gamma}\|P_\tau v\|)$$

for some $\tilde{\gamma} < \infty$, for all $\tau < \infty$ and $u \in L_{2,e}(\boldsymbol{R}^+), v \in L_2(\boldsymbol{R}^+)$, and similarly for $(T_R)^{-1}$. Then under appropriate conditions one can use $(T_R)^{-1}$ to parameterize many solutions K of the nonlinear H_∞-problem (see [9]). As the present paper is primarily concerned with state-space representations, we do not use these results in the derivation to follow but, rather, analyze directly the validity of the same procedure for giving solutions in the input–output sense.

Our aim here is to use the results of Section II to compute explicitly a state-space realization of a (j, J)-inner–outer factorization of G as in (59) in terms of a state-space realization of the original plant P and to give conditions for compensators K derived in a simple way from the outer factor R to solve the nonlinear H_∞-control problem (for the original plant P) in the internal state-space sense.

Under the assumption that $a(x) = A(x) - B_1(x)\ D_{21}^{-1}(x)C_2(x)$ is asymptotically stable and it is straightforward, by using the results of Section II, to compute a (j, J)-inner–outer factorization $G = \Theta \circ R$ of G given by (59). The result is as follows.

Theorem 5: Let G be as in (59) and assume that $a(x) := A(x) - B_1(x)D_{21}^{-1}(x)C_2(x)$ is an asymptotically stable vector field. Suppose that there exists a smooth solution $P(x)$ of the Hamilton–Jacobi equation

$$\begin{aligned} &P_x(x)[A(x) - B_2(x)E_1^{-1}(x)D_{12}^T(x)C_1(x)] \\ &\quad + \tfrac{1}{2}C_1(x)[I - D_{12}(x)E_1^{-1}(x)D_{12}^T(x)]C_1^T(x) \\ &\quad - \tfrac{1}{2}P_x(x)[B_2(x)E_1^{-1}(x)B_2(x) - \gamma^{-2}B_1(x) \\ &\quad \cdot B_1^T(x)]P_x^T(x) = 0, \qquad P(0) = 0 \end{aligned} \tag{60}$$

with $P(x) \geq 0$ and with stabilizing side condition

$$\begin{aligned} &A(x) - B_2(x)E_1^{-1}(x)D_{12}^T(x)C_1(x) - [B_2(x)E_1^{-1}(x)B_2(x) \\ &\quad - \gamma^{-2}B_1(x)B_1^T(x)]P_x^T(x) \text{ is asymptotically stable} \end{aligned} \tag{61}$$

where

$$E_1(x) = D_{12}^T(x)D_{12}(x). \tag{62a}$$

Assume that $E_1(x)$ in (62a) has a factorization

$$E_1(x) = \overline{d}_1^T(x)\overline{d}_1(x) \tag{62b}$$

for a smooth, square, and invertible function $\overline{d}_1(x)$. Then $G = \Theta \circ R$ is a (j, J)-inner–outer factorization of G where

$$R: \begin{cases} \dot{x} = A(x) - B_1(x)D_{21}^{-1}(x)C_2(x) \\ \qquad + B_2(x)u + B_1(x)D_{21}^{-1}(x)y \\ u' = \overline{d}_1(x)E_1^{-1}(x)(D_{12}^T(x)C_1(x) + B_2^T(x)P_x^T(x)) \\ \qquad + \overline{d}_1(x)u \\ y' = -\gamma D_{21}^{-1}(x)C_2(x) - \gamma^{-1}B_1^T(x)P_x^T(x) \\ \qquad + \gamma D_{21}^{-1}(x)y \end{cases} \tag{63}$$

and

$$\Theta: \begin{cases} \dot{x} = [A(x) - B_2(x)E_1^{-1}(x)(D_{12}^T(x)C_1(x) \\ \qquad + B_2^T(x)P_x^T(x)) + \gamma^{-2}B_1(x)B_1^T(x)P_x^T(x)] \\ \qquad + B_2(x)\overline{d}_1^{-1}(x)u' + \gamma^{-1}B_1(x)y' \\ z = (I - D_{12}(x)E_1^{-1}(x)D_{12}^T(x))C_1(x) \\ \qquad - D_{12}(x)E_1^{-1}(x)B_2^T(x)P_x^T(x) \\ \qquad + D_{12}(x)\overline{d}_1^{-1}(x)u' \\ \tilde{w} = \gamma^{-1}B_1^T(x)P_x^T(x) + y'. \end{cases} \tag{64}$$

From the results of [9] (see [7], [18], [19], and [21] for the linear case), it is known that, under appropriate conditions, compensators $K: y \to u$ solving the H_∞-control problem for the plant P arise in the form

$$\begin{aligned} u &= r_1(H(y'), y') \\ y &= r_2(H(y'), y') \end{aligned} \tag{65}$$

where $(u', y') \to (u, y) = (r_1(u', y'), r_2(u', y'))$ defines the inverse $(T_R)^{-1}$ of the input–output map T_R associated with the outer factor R in the (j, J)-inner–outer factorization $G = \Theta \circ R$, and where H is a free-parameter stable plant. The so-called "central compensator" corresponds to the choice $H = 0$ in (65). Thus the central compensator $K_c: y \to u$ is specified by

$$T_R(K(y), y) = (0, y').$$

By imposing the constraint $u' = 0$ in (63) and solving for $K_c: y \to u$, we obtain the state-space equations for the central compensator K_c. The result is

$$K_c: \begin{cases} \dot{\hat{x}} = A(\hat{x}) - B_1(\hat{x})D_{21}^{-1}(\hat{x})C_2(\hat{x}) \\ \qquad - B_2(\hat{x})E_1^{-1}(\hat{x})D_{12}^T(\hat{x})C_1(\hat{x}) \\ \qquad - B_2(\hat{x})E_1^{-1}(\hat{x})B_2^T(\hat{x})P_x^T(\hat{x}) \\ \qquad + B_1(\hat{x})D_{21}^{-1}(\hat{x})y \\ u = -E_1^{-1}(\hat{x})(D_{12}^T(\hat{x})C_1(\hat{x}) + B_2^T(\hat{x})P_x^T(\hat{x})). \end{cases} \tag{66}$$

We now discuss conditions for this compensator to be a solution of the H_∞-control problem [for the original plant P given by (56)] in the internal state-space sense. If we close the loop [(56) combined with the compensator $K_c: y \to u$ in (66)], we obtain as the closed-loop system

$$T_{c.\ell.}: \begin{cases} \dot{x} = A(x) - B_2(x)E_1^{-1}(\hat{x})(D_{12}^T(\hat{x})C_1(\hat{x}) \\ \qquad + B_2^T(\hat{x})P_x^T(\hat{x})) + B_1(x)w \\ \dot{\hat{x}} = A(\hat{x}) + B_1(\hat{x})D_{21}^{-1}(\hat{x})(C_2(x) - C_2(\hat{x})) \\ \qquad - B_2(\hat{x})E_1^{-1}(\hat{x})(D_{12}^T(\hat{x})C_1(\hat{x}) \\ \qquad + B_2^T(\hat{x})P_x^T(\hat{x})) + B_1(\hat{x})D_{21}(\hat{x})^{-1}D_{21}(x)w \\ z = C_1(x) - D_{12}(x)E_1^{-1}(\hat{x})(D_{12}^T(\hat{x})C_1(\hat{x}) \\ \qquad + B_2^T(\hat{x})P_x^T(\hat{x})). \end{cases} \tag{67}$$

The issue now is to show that the vector field

$$\begin{aligned} (x, \hat{x}) \to &(A(x) - B_2(x)E_1^{-1}(x)(D_{12}^T(\hat{x})C_1(\hat{x}) \\ &+ B_2^T(\hat{x})P_x^T(\hat{x})), A(\hat{x}) + B_1(\hat{x})D_{21}^{-1}(\hat{x}) \\ &\cdot (C_2(x) - C_2(\hat{x})) - B_2(\hat{x})E_1^{-1}(\hat{x}) \\ &\cdot (D_{12}^T(\hat{x})C_1(\hat{x}) + B_2^T(\hat{x})P_x^T(\hat{x}))) \end{aligned} \tag{68}$$

is asymptotically stable. Note that the diagonal (x, x) is invariant under the flow of this vector field. In fact, if the initial condition is taken on the diagonal and an arbitrary input signal w is fed in, then the state $(x(t), \hat{x}(t))$ at time t remains on the diagonal $x(t) = \hat{x}(t)$ for all $t > 0$, i.e., a minimal realization for the input–output map [where states are initialized at $(x(0), \hat{x}(0)) = (0, 0)$] lies on the diagonal.

The connection of the solution $P(x)$ of (60) and (61) with stability of the vector field (68) is given by the following result.

Theorem 6: If the solution $P(x)$ of (60) and (61) is proper and positive definite, then the vector field

$$x \to A(x) - B_2(x)E_1^{-1}(x)(D_{12}^T(x)C_1(x) + B_2^T(x)P_x^T(x)) \tag{69}$$

[the diagonal of the vector field (68)] is Lyapunov stable. If in addition the system

$$\begin{aligned} \dot{x} &= A(x) - B_2(x)E_1^{-1}(x)(D_{12}^T(x)C_1(x) \\ &\quad + B_2^T(x)P_x^T(x)) \\ \begin{bmatrix} z \\ w \end{bmatrix} &= \begin{bmatrix} (I - D_{12}(x)E_1^{-1}(x)D_{12}^T(x))C_1(x) \\ - D_{12}(x)E_1^{-1}(x)B_2^T(x)P_x^T(x) \\ \gamma^{-1}B_1^T(x)P_x^T(x) \end{bmatrix} \end{aligned} \tag{70}$$

is detectable, then the vector field (69) is asymptotically stable.

Proof: To abbreviate the notation, write the realization for the system Θ in (64) in the form

$$\Theta: \begin{cases} \dot{x} = A_\Theta(x) + B_{\Theta 1}(x)u' + B_{\Theta 2}(x)y' \\ z = C_{\Theta 1}(x) + D_{12}(x)\bar{d}_1^{-1}(x)u' \\ w = C_{\Theta 2}(x) + y'. \end{cases} \tag{71}$$

We know that Θ is (j, J)-inner with storage function P, and hence [as can also be checked directly from (60) and (64)]

$$P_x(x)A_\Theta(x) + \tfrac{1}{2}C_{\Theta 1}^T(x)C_{\Theta 1}(x) - \tfrac{1}{2}C_{\Theta 2}^T(x)C_{\Theta 2}(x) = 0 \tag{72}$$

$$P_x(x)[B_{\Theta 1}(x)\ B_{\Theta 2}(x)] + [C_{\Theta 1}^T(x)\ -C_{\Theta 2}^T(x)] \cdot \begin{bmatrix} D_{12}(x)\bar{d}_1^{-1}(x) & 0 \\ 0 & I \end{bmatrix} = 0. \tag{73}$$

From (72) and (73), we see that

$$\begin{aligned} &P_x(x)(A_\Theta(x) - B_{\Theta 2}(x)C_{\Theta 2}(x)) \\ &\quad = -\tfrac{1}{2}C_{\Theta 1}^T(x)C_{\Theta 1}(x) - \tfrac{1}{2}C_{\Theta 2}^T(x)C_{\Theta 2}(x). \end{aligned}$$

The assertions of the theorem now follow from standard Lyapunov arguments, upon noting that $A_\Theta(x) - B_{\Theta 2}(x)C_{\Theta 2}(x)$ coincides with the vector field (69) (see [16] and [23]).

The result of Theorem 6 is not satisfactory in that the ultimate goal is to prove the stability of the full vector field (68). For this purpose it is useful to consider the full system $G \circ R^{-1}$ where G is given by (69), and R is given by (63). The resulting system is given by (31) after the appropriate substitutions

$$\begin{aligned} a(x) &= A(x) - B_1(x)D_{21}^{-1}(x)C_2(x) \\ b(x) &= [B_2(x) \quad B_1(x)D_{21}^{-1}(x)] \\ c(x) &= \begin{bmatrix} C_1(x) \\ -\gamma D_{21}^{-1}(x)C_2(x) \end{bmatrix} \\ d(x) &= \begin{bmatrix} D_{12}(x) & 0 \\ 0 & \gamma D_{21}^{-1}(x) \end{bmatrix} \\ \bar{c}(x) &= \begin{bmatrix} \bar{d}_1(x)E_1^{-1}(x)(D_{12}^T(x)C_1(x) + B_2^T(x)P_x^T(x)) \\ -\gamma D_{21}^{-1}(x)C_2(x) - \gamma^{-1}B_1^T(x)P_x^T(x) \end{bmatrix} \\ \bar{d}(x) &= \begin{bmatrix} d_1(x) & 0 \\ 0 & D_{21}^{-1}(x) \end{bmatrix}. \end{aligned} \tag{74}$$

The input-state-output trajectories $\{(w(t), x(t), \hat{x}(t), z(t))\}$ associated with the closed-loop system $T_{c.\ell.}$ given by (67) coincides with the input-state-output trajectories $\{((u'(t), y'(t)), (x(t), \hat{x}(t)), (z(t), w(t))\}$ of the system Θ (64) subject to $u'(t) \equiv 0$ and ignoring $y'(t)$. If we write the system equations for $G \circ R^{-1}$ in the form

$$G \circ R^{-1}: \begin{cases} \dot{\tilde{x}} = A_{\hat{\Theta}}(\tilde{x}) + B_{\hat{\Theta}1}(\tilde{x})u' + B_{\hat{\Theta}2}(\tilde{x})y' \\ z = C_{\hat{\Theta}1}(\tilde{x}) + D_{12}(x)\bar{d}_1^{-1}(\hat{x})u' \\ w = C_{\hat{\Theta}2}(\tilde{x}) + D_{21}^{-1}(x)D_{21}(\hat{x})y' \end{cases} \tag{75}$$

[where $\tilde{x} = (x, \hat{x})$], then the system equations for the closed-loop system $T_{c.\ell.}$ in (67) can be written as

$$T_{c.\ell.}: \begin{cases} \dot{\tilde{x}} = A_{\hat{\Theta}}(\tilde{x}) - B_{\hat{\Theta}2}(\tilde{x})D_{21}^{-1}(\hat{x})D_{21}(x)C_{\hat{\Theta}2}(\tilde{x}) \\ \qquad + B_{\hat{\Theta}2}(\tilde{x})D_{21}^{-1}(\hat{x})D_{21}(x)w \\ z = C_{\hat{\Theta}1}(\tilde{x}). \end{cases} \tag{76}$$

If $G \circ R^{-1}$ is (j, J)-lossless with storage function $S(\tilde{x}) \geq 0$, which is also proper and positive definite, then we can proceed exactly as in the proof of Theorem 5 to show that the state dynamics

$$\tilde{x} \to A_{\hat{\Theta}}(\tilde{x}) - B_{\hat{\Theta}2}(\tilde{x})D_{21}^{-1}(\hat{x})D_{21}(x)C_{\hat{\Theta}2}(\tilde{x}))$$

is Lyapunov stable and is, in fact, asymptotically stable under the appropriate detectability hypothesis. This leads to the following result. While the result is not definitive, it is comparable to those obtained elsewhere in the literature on this problem (see, e.g., [16], [17], [8], and [20]); in fact, the method of analysis which we have presented here has become the standard method for obtaining what stability results there are for this problem.

Theorem 6: Suppose that the system $G \circ R^{-1}$ [equal to (31) with the substitutions (74)] is (j, J)-inner with storage function $S(\tilde{x})$ ($\tilde{x} = (x, \hat{x})$) which is positive definite and proper. Then the vector field (68) is Lyapunov stable. If in addition the system

$$\begin{aligned} \dot{\tilde{x}} &= A_{\hat{\Theta}}(\tilde{x}) \\ \begin{bmatrix} z \\ w \end{bmatrix} &= \begin{bmatrix} C_{\hat{\Theta}1}(\tilde{x}) \\ C_{\hat{\Theta}2}(\tilde{x}) \end{bmatrix} \end{aligned}$$

is detectable, then the vector field (68) is asymptotically stable and the compensator K_c as in (66) is a solution of the H_∞-control problem for the plant P in (56) in the internal state-space sense.

Remark: Note, however, that if $D_{21}(x)$ and/or $D_{12}(x)$ is not constant, then the last of the conditions of (4)

$$\left(\text{with } d(\tilde{x}) = \begin{bmatrix} D_{12}(x)\bar{d}_1^{-1}(\hat{x}) & 0 \\ 0 & D_{21}^{-1}(x)D_{21}(\hat{x}) \end{bmatrix}\right)$$

in general, holds only on the diagonal so $G \circ R^{-1}$ cannot be (j, J)-inner. The result of Theorem 6 requires only the validity of the first two of (4) for $G \circ R^{-1}$. For purposes of verifying that K_c solves the H_∞-control problem, it is sufficient that there exist a smooth, proper, positive definite storage function S with respect to which $G \circ R^{-1}$ is (j, J)-dissipative. This in turn is somewhat stronger than just working directly with the condition that there be a smooth, positive definite, proper storage function $S(\tilde{x})$ with respect to which $T_{c.\ell.}$ is dissipative with respect to the supply rate $w^T w - z^T z$.

A related discussion of the connections between (j, J)-inner–outer factorization and the H_∞-control problem appears in [6].

In this paper we have shown how the disturbance feedforward case of the nonlinear H_∞-control problem can be reduced to a J-inner–outer factorization problem which, in turn, can be reduced to solving a Hamilton–Jacobi equation with stabilizing side condition. Solving such a Hamilton–Jacobi equation in practice may be difficult; typically one cannot solve it explicitly and hence must resort to some approximate or numerical procedure. In [4] and [5], explicit calculations were carried out for the case where the plant is a memoryless system composed with a linear system. Here we offer another example where the presence of extra structure in the original plant enables one to find an explicit solution.

Example: Consider the system

$$\Sigma: \begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x)) \end{cases}$$

where f and h are smooth with $f(0) = 0$ and $h(0) = 0$, u and y are in $\boldsymbol{R}^m$, and x is in the state manifold M. We assume that Σ is a lossless system in the sense that there exists a storage function $H: M \to \boldsymbol{R}$ with $H(0) = 0, H(x) > 0$ for $x \neq 0$, such that we have

$$\frac{d}{dt}\{H(x(t))\} = u(t)^T y(t)$$

along trajectories $(u(t), x(t), y(t))$ of Σ, or equivalently

$$H_x(x)f(x) = 0, \quad H_x(x)g(x) = h^T(x) \tag{77}$$

and that Σ is zero-state detectable, i.e., $\lim_{t\to\infty} x(t) = 0$ whenever $y(t) = 0$ for all $t \geq 0$). The perturbation Σ_p of Σ based on the normalized stable kernel-representation of Σ (see [33]) is now given by

$$P: \begin{cases} \dot{x} = f(x) + g(x)u + g(x)w, \qquad w \in \boldsymbol{R}^m \\ z = \begin{bmatrix} u \\ y \end{bmatrix} \\ y = g^T(x)H_x^T(x) + w. \end{cases} \tag{78}$$

The robust stabilization problem as considered in [33] is to construct a measurement feedback law $K: y \to u$ to minimize the L_2-gain from w to $z = \begin{bmatrix} u \\ y \end{bmatrix}$ for the resulting closed-loop system.

This problem can be solved by using the J-spectral factorization approach discussed in this section. With P as in (78), the associated system G in (59) becomes

$$G: \begin{cases} \dot{x} = [f(x) - g(x)g^T(x)H_x^T(x)] \\ \qquad + g(x)u + g(x)y \\ z = \begin{bmatrix} u \\ y \end{bmatrix} \\ \tilde{w} = -\gamma g^T(x)H_x^T(x) + \gamma y. \end{cases} \tag{79}$$

If we now make the substitutions

$$\begin{aligned} a(x) &= f(x) - g(x)g^T(x)H_x^T(x) \quad b(x) = [g(x) \quad g(x)] \\ c(x) &= \begin{bmatrix} 0 \\ 0 \\ -\gamma g^T(x)H_x^T(x) \end{bmatrix} \quad d(x) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & \gamma \end{bmatrix} \\ J &= \begin{bmatrix} I_m & 0 & 0 \\ 0 & I_m & 0 \\ 0 & 0 & -I_m \end{bmatrix} \quad E^{-1}(x) = \begin{bmatrix} 1 & 0 \\ 0 & (1-\gamma^2)^{-1} \end{bmatrix} \end{aligned} \tag{80}$$

then (23) with side condition (24) reduces to

$$\begin{aligned} &P_x(x)[f(x) - (1-\gamma^2)^{-1}g(x)g^T(x)H_x^T(x)] \\ &\quad - \tfrac{1}{2}[(1-\gamma^2)^{-1} + 1]P_x(x)g(x)g^T(x)P_x^T(x) \\ &\quad - \tfrac{1}{2}\gamma^2(1-\gamma^2)^{-1}H_x(x)g(x)g^T(x)H_x^T(x) = 0 \end{aligned} \tag{81}$$

with stability side condition

$$\begin{aligned} &f - (1-\gamma^2)^{-1}gg^TH_x^T - (2-\gamma^2)(1-\gamma^2)^{-1}gg^TP_x^T \\ &\quad \text{is asymptotically stable.} \end{aligned} \tag{81'}$$

This is exactly the Hamilton–Jacobi equation (29) considered in [33]. Equation (81) has solution

$$P(x) = \gamma^2(\gamma^2 - 2)^{-1}H(x)$$

for $\gamma > \sqrt{2}$. The assumption that Σ is zero-state detectable implies that P also meets the stability side condition (81'). Moreover it can be shown that the infimum of the set of γ's for which a solution exists is $\gamma^* = \sqrt{2}$ (see [33]). From (27)–(29) with the substitutions of (80) (or from (63) with the appropriate substitutions) we obtain, for $\gamma > \sqrt{2}$, the equation shown at the bottom of the page with $\bar{d}(x) = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{\gamma^2-1} \end{bmatrix}$ if we take $j = \begin{bmatrix} I_m & 0 \\ 0 & -I_m \end{bmatrix}$. The central compensator is obtained by setting $u' = 0$; this yields

$$K_c: \begin{cases} \dot{\hat{x}} = f(\hat{x}) - g(\hat{x})g^T(\hat{x})H_x^T(\hat{x}) \\ \qquad - \gamma^2(\gamma^2-2)^{-1}g(\hat{x})g^T(\hat{x})H_x^T(\hat{x}) + g(\hat{x})y \\ \quad = [f(\hat{x}) - \gamma^2(\gamma^2-2)^{-1}g(\hat{x})g^T(\hat{x})H_x^T(\hat{x})] \\ \qquad + g(\hat{x})[y - g^T(\hat{x})H_x^T(\hat{x})] \\ u = -\gamma^2(\gamma^2-2)^{-1}g^T(\hat{x})H_x^T(\hat{x}). \end{cases}$$

This is precisely the controller obtained via the certainty equivalence principle in [33]!

Based on the linear theory, the same controller for the general "disturbance feedforward problem" has been proposed in [20], where it has been also shown that this controller solves the suboptimal H_∞-control problem at least locally.

V. The Unstable Case

In this section we discuss the difficulties of our approach to the (j, J)-inner–outer factorization problem in the case where the vector field a in (1) for the system Σ is not stable.

Let us assume that we are given a system Σ with state-space realization as in (1) for which the vector field a is not necessarily stable. Then it is still the case that the system $[D\Sigma]^T \circ J\Sigma$ can be formed and has the Hamiltonian realization as in (12) with Hamiltonian $H(x, p, u)$ given by (13) and that the inverse system $([D\Sigma]^T \circ J\Sigma)^{-1}$ has Hamiltonian realization (20) with Hamiltonian $H^\times$ given by (22). To write the system in the form $[DR]^T \circ jR$ with R outer, the first step in our procedure is to find a new choice of canonical coordinates $(\overline{x}, \overline{p})$ so that the antistable invariant manifold for the H-Hamiltonian flow (with $u = 0$) is given by $\{(\overline{x}, \overline{p}): \overline{x} = 0\}$, and that the stable invariant manifold for the $H^\times$-Hamiltonian flow (with $y_a = 0$) is given by $\{(\overline{x}, \overline{p}): \overline{p} = 0\}$. Indeed it is always possible to find local coordinates $\overline{x}, \overline{p}$ in this way (see [1]). We can still proceed as before to find new canonical coordinates $(\overline{x}, \overline{p})$ so that

$$R: \begin{cases} \dot{x} = [f(x) - g(x)g^T(x)H_x^T(x)] + g(x)u + g(x)y \\ \begin{bmatrix} u' \\ y' \end{bmatrix} = \bar{d}(x)\left(\begin{bmatrix} \gamma^2(\gamma^2-2)^{-1}g^T(x)H_x^T(x) \\ \gamma^2(\gamma^2-2)^{-1}(\gamma^2-1)^{-1}g^T(x)H_x^T(x) \end{bmatrix} + \begin{bmatrix} u \\ y \end{bmatrix}\right) \end{cases}$$

the stable invariant manifold for the $H^\times$-Hamiltonian flow (with $y_a = 0$) is equal to the manifold $\{(\overline{x},\overline{p})\colon \overline{p} = 0\}$ by solving(23) with the stability side condition (24). The remaining difficulty is that the antistable invariant manifold for the H-Hamiltonian flow (with $u = 0$) is not necessarily equal to $\{(\overline{x},\overline{p})\colon \overline{x} = 0\}$, so that the system R defined by (29), in general, is not stable.

To conserve notation, let us assume that we already have this situation at the start. Thus we are given a system Σ as in (1), and Assumption (A2) is replaced by

$$p = m \text{ and } d(x) \text{ is invertible for all } x \tag{A2$'$}$$

and (A1) is replaced by

$$x_0 = 0 \text{ is a globally asymptotically stable equilibrium point of } \dot{x} = a(x) - b(x)d^{-1}(x)c(x). \tag{A1$'$}$$

With these assumptions in place, it is clear that the stable invariant manifold for the $H^\times$-Hamiltonian flow (with $y_a = 0$) is $\{(x,p)\colon p = 0\}$. We seek a new choice of canonical coordinates $(\overline{x},\overline{p})$ so that the stable invariant manifold of the $H^\times$-Hamiltonian flow is still of the form $\{(\overline{x},\overline{p})\colon \overline{p} = 0\}$ but also the antistable invariant manifold for the H-Hamiltonian flow is equal to $\{(\overline{x},\overline{p})\colon \overline{x} = 0\}$.

We assume that the antistable invariant manifold has the form $\mathcal{M} = \{(P_p(p),p)\colon (x,p) \in T^*M \text{ for some } x\}$ for a smooth function $p \to P(p)$. This leads to a Hamilton–Jacobi equation of the form

$$H(P_p^T(p),p,0) = 0)$$

with stability side condition

$$-\frac{\partial H}{\partial x}(P_p^T(p),p,0) \text{ is an antistable vector field.}$$

Explicitly, the Hamilton–Jacobi equation is

$$p^T a(P_p^T(p)) + \tfrac{1}{2}[c(P_p^T(p))]^T jc(P_p^T(p)) = 0 \tag{82}$$

with stability side condition

$$\left[\frac{\partial a}{\partial x}(P_p^T(p))\right]^T p + \left[\frac{\partial c}{\partial x}(P_p(p))\right]^T jc(P_p^T(p)) \text{ is stable.} \tag{83}$$

Assume that we can solve (82) with the side condition (83). Then the new canonical set of coordinates $(\overline{x},p)$ given by $(x,p) = (\overline{x} + P_p^T(p),p)$ has all the required properties and Step 1 of our procedure is complete. The Hamiltonian in the new coordinates $(\overline{x},p)$ becomes

$$\begin{aligned}\overline{H}(\overline{x},p,u) &= H(\overline{x} + P_p^T(p),p,u)\\ &= p^T[a(\overline{x} + P_p^T(p)) + b(\overline{x} + P_p^T(p))u]\\ &\quad + \tfrac{1}{2}[c(\overline{x} + P_p^T(p)) + d(\overline{x} + P_p^T(p))u]^T\\ &\quad \cdot j[c(\overline{x} + P_p^T(p)) + d(\overline{x} + P_p^T(p))u].\end{aligned} \tag{84}$$

The goal now is to identify functions $\overline{a}(\overline{x}),\overline{b}(\overline{x}),\overline{c}(\overline{x}),\overline{d}(\overline{x})$ so that $\overline{H}$ has the form of (16), i.e., so that

$$\begin{aligned}\overline{H}(\overline{x},p,u) &= p^T[\overline{a}(\overline{x}) + \overline{b}(\overline{x})u]\\ &\quad + \tfrac{1}{2}[\overline{c}(\overline{x}) + d(\overline{x})u]^T j[\overline{c}(\overline{x}) + \overline{d}(\overline{x})u].\end{aligned} \tag{85}$$

Note that p appears linearly in (85) while the dependence on p of $\overline{H}(\overline{x},p,u)$ in (84) is a complicated nonlinear dependence. This yields a fundamental obstruction to finding $\overline{a},\overline{b},\overline{c},\overline{d}$ so that (85) matches (84).

The analysis here hinges on the demand that $[D\Sigma]^T \circ J\Sigma$ match with $[DR]^T \circ jR$ at the state-space level. As has been suggested by Helton, conceivably there is an R with a state-space dimension larger than that of Σ for which $[D\Sigma]^T \circ J\Sigma = [DR]^T \circ jR$ at least at the input–output level. This issue of existence of (j,J)-inner–outer factorization in some sense for a unstable system Σ remains a current area of research (see, e.g., [36]).

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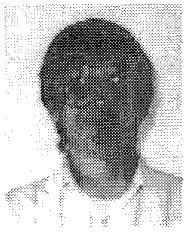

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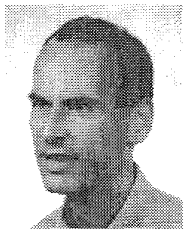

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