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# An Intrinsic Hamiltonian Formulation of the Dynamics of LC-Circuits

B. M. Maschke, *Member. IEEE*, A. J. van der Schaft, *Member. IEEE*, and P. C. Breedveld *Member. IEEE*

**Abstract**—First, the dynamics of LC-circuits are formulated as a Hamiltonian system defined with respect to a Poisson bracket which may be degenerate, i.e., nonsymplectic. This Poisson bracket is deduced from the network graph of the circuit and captures the dynamic invariants due to Kirchhoff's laws. Second, the antisymmetric relations defining the Poisson bracket are realized as a physical network using the gyrator element and partially dualizing the network graph constraints. From the network realization of the Poisson bracket, the reduced standard Hamiltonian system as well as the realization of the embedding standard Hamiltonian system are deduced.

## I. INTRODUCTION

THE STUDY of the dynamic equations of electrical circuits—in particular, weakly damped circuits—is often carried out on their lossless part, i.e., by neglecting all resistive effects. This had led to a surprisingly large variety of Lagrangian or Hamiltonian formulations [1]–[4], which differ in the choice of variables and generating functions (i.e., the Lagrangian or Hamiltonian functions) and in the underlying geometric structure (Riemannian or symplectic) of the state-space. However these formulations are dissatisfying in the sense that they depend on strong assumptions on the constitutive relations of the multiports and that the variables and structure are not easily interpretable in terms of the original description of the electrical circuit: a set of charges and flux linkages describing the electrical and magnetic energies in the circuit and the interconnection constraints expressed in Kirchhoff's laws. For instance the dynamics of LC-circuits is described in terms of a constrained Lagrangian system with generalized coordinates being fictitious flux linkages associated with capacitors and fictitious charges with the inductors [2]. A similar Lagrangian formulation was proposed in terms of the more natural variables being capacitors' charges and inductors' flux linkages, which however still relies on the existence of some co-energy functions, that is on the assumption on the invertibility of the constitutive relations of the inductors and capacitors [4]. In the Hamiltonian formulation proposed in [1], the variables are some linear combinations of the charges and flux linkages in the circuit. In the present paper we propose an intrinsic Hamiltonian formulation of the

dynamics of LC-circuits which is directly related with the charges and flux linkages in the circuit and the interconnection constraints expressed in Kirchhoff's laws.

The first main result of the present paper is the direct formulation of the dynamics of LC-circuits as a Hamiltonian system defined with respect to a Poisson bracket which may be degenerate, i.e., nonsymplectic. This Poisson bracket is uniquely determined by the network graph of the circuit and takes account of the dynamic invariants defined by Kirchhoff's laws. The state variables are simply the capacitors' charges and the inductors' fluxes.

The second general result is that the antisymmetric relations defining the Poisson bracket may be realized by a physical network using the gyrator element [5], [6]. But this physical network is obtained by partial dualization of the network graph of the circuit and requires a more general notation: the bond graph [7], [8]. In this way the dynamics of LC-circuits may be related to the dynamics of more general conservative systems [9]–[11].

In Section II, we recall the definition of Hamiltonian systems defined with respect to general Poisson brackets [12], [13], possibly degenerate. It is recalled that the degeneracy of the Poisson bracket is related to the existence of invariant functions, different from the energy function, and to the reduction of standard Hamiltonian systems.

In Section III, the dynamic equations of the capacitors' charges and the inductors' flux linkages are formulated as a Hamiltonian system, generated by the total energy of the circuit with respect to a Poisson bracket, uniquely defined by the network graph. As a consequence the state-variables are directly interpretable and the order of the Hamiltonian systems is equal to the (possibly odd) order of the circuit.

Section IV gives a network realization of the antisymmetric relations associated with the Poisson bracket using the gyrator element [5], [6]. The bond graph representation is used in order to ensure a graphical representation of the dual of any network graph. Finally, the bond graph plays also a key role for the network realization of the embedding standard Hamiltonian system, by enabling to represent the set of symmetry variables conjugated to the invariants of the system as port variables of a network.

## II. NONSTANDARD HAMILTONIAN SYSTEMS

This section recalls very briefly the definition of general Poisson manifolds and the Hamiltonian dynamic systems defined on it. For a detailed treatment of this subject the reader may consult [12]–[14].

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**Definition 1:** Let  $M$  be a smooth (i.e.,  $C^\infty$ ) manifold and let  $C^\infty(M)$  denote the set of the smooth real functions on  $M$ . A *Poisson bracket* on  $M$  is a bilinear map from  $C^\infty(M) \times C^\infty(M)$  into  $C^\infty(M)$ , denoted as:

$$(F, G) \mapsto \{F, G\} \in C^\infty(M), \quad F, G \in C^\infty(M)$$

which verifies the following properties:

$$\text{—skew-symmetry:} \quad \{F, G\} = -\{G, F\} \quad (1)$$

$$\text{—Jacobi identity:} \quad \{F, \{G, H\}\} + \{G, \{H, F\}\} + \{H, \{F, G\}\} = 0 \quad (2)$$

$$\text{—Leibniz rule:} \quad \{F, G \cdot H\} = \{F, G\} \cdot H + G \cdot \{F, H\}. \quad (3)$$

**Definition 2:** A *Poisson manifold* is a smooth manifold  $M$  endowed with a Poisson bracket.

**Proposition—Definition 3:** Let  $M$  be a Poisson manifold with Poisson bracket  $\{, \}$ . Then any function  $H \in C^\infty(M)$  defines a mapping:

$$X_H(\mathbf{x}): C^\infty(M) \rightarrow \mathbb{R}$$

$$\text{such that: } X_H(F)(\mathbf{x}) = \{F, H\}(\mathbf{x}), \quad \forall F \in C^\infty(M). \quad (4)$$

Because of (3)  $X_H$  is a vector field on  $M$ , called the *Hamiltonian vector-field* with respect to  $H$  and the Poisson bracket.

It follows that the Hamiltonian function  $H$  is a conserved quantity for the Hamiltonian vector field  $X_H$ . Indeed one obtains from (1):

$$X_H(H)(\mathbf{x}) = \{H, H\}(\mathbf{x}) = -\{H, H\}(\mathbf{x}) = 0. \quad (5)$$

**Definition 4:** The *structure matrix*  $J(\mathbf{x}) = (J_{kl}(\mathbf{x}))_{k,l=1,\dots,m}$  associated with the Poisson bracket  $\{, \}$  and coordinate functions  $x_1, \dots, x_m$  is defined by:

$$J_{kl}(\mathbf{x}) = \{x_k, x_l\}; \quad k, l \in \{1, \dots, m\}. \quad (6)$$

In local coordinates  $x_1, \dots, x_m$  the Poisson bracket of two smooth real functions  $F$  and  $G$  may now be expressed in terms of the structure matrix as follows:

$$\{F, G\}(\mathbf{x}) = \sum_{k,l=1}^m \frac{\partial F}{\partial x_k} J_{kl}(\mathbf{x}) \frac{\partial G}{\partial x_l}(\mathbf{x}). \quad (7)$$

which makes clear that  $\{F, G\}(\mathbf{x})$  only depends on the differentials of  $F$  and  $G$  in  $\mathbf{x}$ .

It follows from (1), respectively (2), that the structure matrix satisfies the two following conditions:

$$\text{skew-symmetry:} \quad J_{kl}(\mathbf{x}) = -J_{lk}(\mathbf{x}) \quad k, l = 1, \dots, m \quad (8)$$

$$\text{Jacobi identities:} \quad \sum_{i=1}^m \left( J_{ij}(\mathbf{x}) \frac{\partial J_{ik}}{\partial x_l}(\mathbf{x}) + J_{li}(\mathbf{x}) \frac{\partial J_{kj}}{\partial x_l}(\mathbf{x}) + J_{lk}(\mathbf{x}) \frac{\partial J_{ji}}{\partial x_l}(\mathbf{x}) \right) = 0$$

$$i, j, k = 1, \dots, m. \quad (9)$$

Conversely if a matrix  $J$  with coefficients in  $C^\infty(M)$  satisfies (8) and (9), then it defines locally a Poisson bracket according to (7). It may be noted that, in consequence, any

constant skew-symmetric ( $m \times m$ ) matrix defines a Poisson bracket on  $\mathbb{R}^m$ .

From (4) it follows that in the local coordinates:  $x_1, \dots, x_m$ , the Hamiltonian vector field  $X_H(\mathbf{x})$  is given by the following vector:

$$\begin{bmatrix} X_H^1(\mathbf{x}) \\ \vdots \\ X_H^m(\mathbf{x}) \end{bmatrix} = J(\mathbf{x}) \begin{bmatrix} \frac{\partial H}{\partial x_1}(\mathbf{x}) \\ \vdots \\ \frac{\partial H}{\partial x_m}(\mathbf{x}) \end{bmatrix} \quad (10)$$

and thus the dynamical equations of motion determined by the Hamiltonian vector field  $X_H$  are given in local coordinates by:

$$\begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_m \end{bmatrix} = J(\mathbf{x}) \begin{bmatrix} \frac{\partial H}{\partial x_1}(\mathbf{x}) \\ \vdots \\ \frac{\partial H}{\partial x_m}(\mathbf{x}) \end{bmatrix}. \quad (11)$$

These equations may also be interpreted being a local matrix representation of a bundle map from the cotangent bundle  $T^*M$  to the tangent bundle  $TM$  mapping the differential of a Hamiltonian function  $H$  to its Hamiltonian vector field  $X_H(\mathbf{x})$ . The structure matrix  $J(\mathbf{x})$  of the Poisson bracket is the representation of this map in local coordinates.

**Proposition—Definition 5:** The *rank of a Poisson bracket*  $\{, \}$  at a point  $x \in M$  is defined as the rank of the structure matrix at this point (which may be shown to be independent of the choice of local coordinates).

By the skew-symmetry of the Poisson bracket (8), the rank of a Poisson bracket at any point is even. A Poisson bracket on  $M$  whose rank is equal everywhere to the dimension of  $M$  is called *nondegenerate*, and in this case the Poisson manifold  $M$  is called a *symplectic manifold*. Thus the dimension of a symplectic manifold is necessarily even.

The definition of canonical coordinates as used for *standard* Hamiltonian systems (i.e., Hamiltonian vector fields on a symplectic manifold) may be generalized to Poisson brackets with the aid of the following proposition (see e.g., [12]–[14]).

**Proposition 1:** Let  $M$  be an  $m$ -dimensional Poisson manifold. Suppose the Poisson bracket has constant rank  $2n$  in a neighborhood of a point  $x_0 \in M$ . Then locally around  $x_0$  one can find coordinates  $(q, p, r) = (q_1, \dots, q_n, p_1, \dots, p_n, r_1, \dots, r_l)$  where:  $(2n + l) = m$ , which satisfy:

$$\begin{aligned} \{q_i, p_j\} &= \delta_{ij} \\ \{q_i, q_j\} &= \{p_i, p_j\} = 0, \\ \{q_i, r_{i'}\} &= \{p_i, r_{i'}\} = \{r_{j'}, r_{i'}\} = 0 \\ & i, j = 1, \dots, n, \quad i', j' = 1, \dots, l, \end{aligned} \quad (12)$$

or equivalently, the  $m \times m$  structure matrix  $J(q, p, r)$  is given as follows:

$$J(q, p, r) = \begin{bmatrix} 0_n & I_n & 0_{n \times l} \\ -I_n & 0_n & 0_{n \times l} \\ 0_{l \times n} & 0_{l \times n} & 0_{l \times l} \end{bmatrix} \quad (13)$$

**Definition 6:** The coordinates  $(q, p, r)$  satisfying (12) are called *canonical coordinates* of the Poisson manifold with Poisson bracket  $\{, \}$ .

*Remark:* As noted before, any skew-symmetric constant  $(m \times m)$  matrix defines a Poisson bracket on  $\mathbb{R}^m$ . In this particular case, Proposition 1 reduces to the well-known fact from linear algebra that there exist linear (and thus *global!*) coordinates for  $\mathbb{R}^m$  in which  $J$  takes the form (13).

Let  $M$  be an  $m$ -dimensional Poisson manifold with the Poisson bracket of constant rank  $2n$  in a neighborhood of a point  $x_0 \in M$ , and canonical coordinates  $(q, p, r)$ , then it follows from (11) and (13) that every Hamiltonian vector field  $X_H$  has the following expression in the coordinates  $(q, p, r)$ :

$$\begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i}(q, p, r) \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i}(q, p, r) \quad i = 1, \dots, n \\ \dot{r}_j &= 0 \quad j = 1, \dots, l. \end{aligned} \quad (14)$$

If  $M$  is a symplectic manifold then  $2n = m = \dim M$  and  $l = 0$ , and (14) reduces to the standard Hamiltonian equations.

*Definition 7:* Let  $M$  be a Poisson manifold. The *distinguished* or *Casimir functions* are those smooth functions  $F \in C^\infty(M)$  which satisfy:

$$\{F, G\} = 0, \quad \forall G \in C^\infty(M) \quad (15)$$

(or equivalently, since  $\{F, G\} = -X_F(G)$ , all those functions  $F$  such that  $X_F = 0$ ).

Thus the Casimir functions correspond to the kernel of the map:  $F \mapsto X_F$  from  $C^\infty(M)$  modulo  $\mathbb{R}$  to the vector fields on  $M$  given by (4). In local terms  $F$  is a Casimir function if  $dF(x)$  is in the kernel of the map  $J(x): T_x^*M \mapsto T_xM$  for every  $x$  in  $M$ . Furthermore, under the assumptions of Proposition 1, the Casimir functions are all functions depending only on  $r_1, \dots, r_l$  where  $(q, p, r)$  are canonical coordinates.

Proposition 1 has some interesting consequences concerning the local *reduction* of the generalized Hamiltonian (14) to *lower-dimensional standard* Hamiltonian equations, as well as concerning the local *embedding* of (14) into *higher-dimensional standard* Hamiltonian equations, see also [11]. Indeed, by the local projection  $\pi: \mathbb{R}^{2n+l} \rightarrow \mathbb{R}^{2n}$  defined as:

$$(q_1, \dots, q_n, p_1, \dots, p_n, r_1, \dots, r_l) \mapsto (q_1, \dots, q_n, p_1, \dots, p_n)$$

$\mathbb{R}^{2n}$  inherits the Poisson bracket defined by (12) in  $\mathbb{R}^{2n+l}$  by leaving out  $r_1, \dots, r_l$ . In fact this Poisson bracket on  $\mathbb{R}^{2n}$  is *nondegenerate* and the dynamics (14) projects locally to the standard Hamiltonian equations on  $\mathbb{R}^{2n}$ :

$$\begin{aligned} \dot{q}_i &= \frac{\partial H_r}{\partial p_i}(q, p) \\ \dot{p}_i &= -\frac{\partial H_r}{\partial q_i}(q, p) \quad i = 1, \dots, n \end{aligned} \quad (16)$$

with the Hamiltonian function:  $H_r(q, p) = H(q, p, r)$  parameterized by  $r$ . On the other hand, consider locally the embed-

ding space  $\mathbb{R}^{2n+2l}$  with linear coordinates  $(q_1, \dots, q_n, p_1, \dots, p_n, r_1, \dots, r_l, s_1, \dots, s_l)$  and nondegenerate Poisson bracket defined by (12) and the additional relations:

$$\begin{aligned} \{s_{i'}, r_{j'}\} &= \delta_{i'j'}, \quad i', j' = 1, \dots, l \\ \{q_i, s_{i'}\} &= \{p_i, s_{i'}\} = \{s_{j'}, s_{i'}\} = 0 \\ i &= 1, \dots, n, \quad i', j' = 1, \dots, l. \end{aligned} \quad (17)$$

Then (14) can be locally embedded into the standard Hamiltonian equations on  $\mathbb{R}^{2n+2l}$  given as:

$$\begin{aligned} \dot{q}_i &= \frac{\partial H_e}{\partial p_i}(q, p, r, s) \\ \dot{p}_i &= -\frac{\partial H_e}{\partial q_i}(q, p, r, s) \quad i = 1, \dots, n \\ \dot{s}_i &= \frac{\partial H_e}{\partial r_i}(q, p, r, s) \\ \dot{r}_i &= -\frac{\partial H_e}{\partial s_i}(q, p, r, s) \quad i = 1, \dots, l \end{aligned} \quad (18)$$

with Hamiltonian:  $H_e(q, p, r, s) = H(q, p, r)$ , i.e., not depending on  $s$ .

Furthermore, we note that, conversely, the transition from the standard Hamiltonian system (18)  $\mathbb{R}^{2n+2l}$  to the standard Hamiltonian system (16), via the nonstandard Hamiltonian system (14) can be interpreted as the *canonical reduction* of order caused by the *symmetry* of the Hamiltonian  $H_e$  with respect to translations in the  $s$ -coordinates, see e.g., [13] for an introduction into this subject. From this point of view the Casimir functions can be seen as the *conserved quantities* corresponding to the infinitesimal symmetries:  $\frac{\partial}{\partial s_1}, \dots, \frac{\partial}{\partial s_l}$ .

### III. LC-CIRCUIT DYNAMICS

This section recalls first the definition and assumptions on the LC-circuits considered in this paper. Then the dynamic equations are formulated in “natural” coordinates, i.e., in terms of the energy variables of the elements (charges of the capacitors and flux linkages of the inductors), as a Hamiltonian system defined on a Poisson manifold. The Poisson bracket is shown to be determined by the constraint relations, i.e., Kirchhoff’s laws induced by the network graph, among the inductors’ voltages and among the capacitors’ currents. Finally this formulation is compared with the standard Hamiltonian equations proposed in [1].

An LC-circuit is composed of a set of multiport inductors and capacitors interconnected through their ports by a graph  $\Gamma$  called *network graph* [15] or *port connection graph* [16]. The capacitor and inductor elements are defined by their constitutive relation, whereas the network graph defines the relations among their port variables arising from Kirchhoff’s laws.

*Definition 8:* An  $n$ -port capacitor (respectively inductor) is defined by a set of energy variables, the charge:  $q \in \mathbb{R}^n$  (resp. the flux linkage:  $\phi \in \mathbb{R}^n$ ), an energy function:  $E_C(q) \in C^\infty(\mathbb{R}^n)$  (resp.  $E_L(\phi) \in C^\infty(\mathbb{R}^n)$ ) and two sets of port variables: the current  $i_C \in \mathbb{R}^n$  (resp.  $i_L \in \mathbb{R}^n$ ) and the voltage

$v_C \in \mathbb{R}^n$  (resp.  $v_L \in \mathbb{R}^n$ ), related by the constitutive relations: elements:

$$i_C = \left( \frac{dq_i}{dt} \right)_{i=1, \dots, n} \quad \text{and} \quad v_C = \left( \frac{\partial E_C}{\partial q_i} \right)_{i=1, \dots, n} \quad (19)$$

respectively for the inductor:

$$v_L = \left( \frac{d\phi_i}{dt} \right)_{i=1, \dots, n} \quad \text{and} \quad i_L = \left( \frac{\partial E_L}{\partial \phi_i} \right)_{i=1, \dots, n} \quad (20)$$

It may be noted that, as the capacitors' and inductors' constitutive relations are derived from the stored energy function, they satisfy Maxwell's reciprocity equations—resulting from energy conservation, hence they define reciprocal multiports.

For the sake of simplicity, we shall group in the sequel all the capacitors of the circuit into one  $n_C$ -port with the energy-function  $E_C(q)$  equal to the sum of the energy-functions of the capacitors and all the inductors into one  $n_L$ -port with the energy-function  $E_L(\phi)$  equal to the sum of the energy-functions of the inductors.

*Definition 9:* The network graph is defined as an oriented graph whose edges correspond one-to-one to the ports of the capacitors and inductors and the orientation corresponds to the sign convention of the voltage variables (and the opposite sign convention of the current variables).

The network graph describes the connection constraints among the port variables of the elements due to Kirchhoff's laws which may be formulated in the following generalized form [15].

*Proposition 2: Kirchhoff's voltage law.* The sum of voltages along any cycle (or loop) in the network graph vanishes.

*Proposition 3: Kirchhoff's current law.* The sum of currents along any cocycle (or cutset) in the network graph vanishes.

Moreover, for the sake of simplicity, we shall make the following assumptions on the electrical circuit:

- (H1) The network graph is connected.
- (H2) The  $n_C$  ports of the capacitors correspond to a tree, denoted by  $\mathcal{C}$ , in the network graph.
- (H2) The  $n_L$  ports of the inductors correspond to a cotree, denoted by  $\mathcal{L}$ , in the network graph, complementary to  $\mathcal{C}$ .

The assumption (H1) says that the circuit consists of one part. The assumptions (H2) and (H3) say that there is no capacitor loop or inductor cutset, i.e., that the topological constraints induced by the circuit do not constrain the space of the energy variables  $(q, \phi)$  to a proper subset of  $\mathbb{R}^{n_C} \times \mathbb{R}^{n_L}$  which may be chosen as the state-space of the system. If this is not the case, the results presented here remain valid by replacing the space of energy-variables  $\mathbb{R}^{n_C} \times \mathbb{R}^{n_L}$  by some proper subspace  $M$  of it [9].

Following the notation in [1], the fundamental loop matrix corresponding to the capacitor's tree  $\mathcal{C}$  may be written as:  $B = (I_{n_L \times n_L} B_{LC})$ , and the fundamental cutset matrix corresponding to the complementary inductor's cotree  $\mathcal{L}$  may be written as:  $Q = (-B_{LC}^t I_{n_C \times n_C})$ , where  $B_{LC}$  is an  $n_L \times n_C$  matrix with coefficients in  $\{-1, 0, 1\}$ . Then Kirchhoff's laws imply the two following relations on the port variables of the

$$v_L = -B_{LC} v_C \quad (21)$$

$$i_C = B_{LC}^t i_L. \quad (22)$$

Using the state-variables:  $x = (q, \phi) \in \mathbb{R}^{n_C} \times \mathbb{R}^{n_L}$ , i.e., the energy variables of the inductors and capacitors, and assembling (19)–(22), one obtains the following dynamical equation of the LC-circuit:

$$\dot{x} = JdH(x) \quad (23)$$

where

$$J = \begin{bmatrix} 0_{n_C \times n_C} & B_{LC}^t \\ -B_{LC} & 0_{n_L \times n_L} \end{bmatrix} \quad (24)$$

$$H(x) = E_C(q) + E_L(\phi) \quad (25)$$

and

$$dH(x) = \left( \frac{\partial H}{\partial x_1}, \dots, \frac{\partial H}{\partial x_n} \right)^t. \quad (26)$$

*Proposition 4:* The dynamical equations (23) of the energy-variables (charges in the capacitors and flux linkages in the inductors) of an LC-circuit are a Hamiltonian system with Hamiltonian function defined as the sum of the electrical and magnetic energy functions of the circuit, with respect to the Poisson bracket defined on the manifold  $\mathbb{R}^{n_C} \times \mathbb{R}^{n_L}$  by the structure matrix  $J$  defined by the fundamental loop matrix according to (24).

Indeed the matrix  $J$  is skew-symmetric and constant and hence it verifies the conditions (8), (9) of a structure matrix. Thus it defines a Poisson bracket on the state-space of the energy-variables  $\mathbb{R}^{n_C} \times \mathbb{R}^{n_L}$ . And according to Section II, (23) is the local representation of a Hamiltonian vector field.

Thus the "natural" dynamics associated with an electrical circuit may be directly expressed as a Hamiltonian vector field defined on a Poisson manifold. It is remarkable that using this formalism, the two concepts of energy-storage elements (i.e., the inductors and capacitors) and network graph which may be defined separately, correspond exactly to the distinct objects of the Hamiltonian function, respectively Poisson bracket. Indeed the constitutive relations of the inductors and capacitors are defined by the energy functions defining the Hamiltonian function, while the topological constraints on the port-variables represented by the network graph are fully captured into the structure matrix of the Poisson bracket.

The degeneracy of the Poisson bracket or its structure matrix may be related to the topological constraints induced by the network graph. For this purpose, consider the following partition of the tree  $\mathcal{C}$  and the cotree  $\mathcal{L}$  [17]:

- $\mathcal{C}_1$  a maximal subset of  $\mathcal{C}$  which forms a cotree
- $\mathcal{L}_1$  a maximal subset of  $\mathcal{L}$  which forms a tree
- $\mathcal{C}_2$  the complement of  $\mathcal{C}_1$  in  $\mathcal{C}$
- $\mathcal{L}_2$  the complement of  $\mathcal{L}_1$  in  $\mathcal{L}$ .

It may be shown [17] that:

$$\exists r \in \mathbb{N}; |\mathcal{C}_1| = |\mathcal{L}_1| = r, \quad (27)$$

where  $|\mathcal{X}|$  denotes the number of elements of any set  $\mathcal{X}$  and thus:

$$|\mathcal{C}_2| = n_C - r = s \quad \text{and} \quad |\mathcal{L}_2| = n_L - r = l. \quad (28)$$

According to this partition and to the corresponding partition of the voltage and current variables:  $i^t = (i_{\mathcal{L}_1}^t, i_{\mathcal{L}_2}^t, i_{\mathcal{C}_1}^t, i_{\mathcal{C}_2}^t)$  and  $v^t = (v_{\mathcal{L}_1}^t, v_{\mathcal{L}_2}^t, v_{\mathcal{C}_1}^t, v_{\mathcal{C}_2}^t)$ , the fundamental loop matrix corresponding to the tree  $\mathcal{L}_1 \cup \mathcal{C}_2$  is:

$$\tilde{B} = \begin{bmatrix} \tilde{B}_{11} & 0_{r \times l} & I_r & \tilde{B}_{12} \\ \tilde{B}_{21} & I_l & 0_{l \times r} & 0_{l \times s} \end{bmatrix} \quad (29)$$

and the fundamental cutset matrix corresponding to the cotree  $\mathcal{C}_1 \cup \mathcal{L}_2$  is:

$$\tilde{Q} = \begin{bmatrix} I_r & -\tilde{B}_{21}^t & -\tilde{B}_{11}^t & 0_{r \times s} \\ 0_{s \times r} & 0_{s \times l} & -\tilde{B}_{12}^t & I_s \end{bmatrix}. \quad (30)$$

From Kirchhoff's laws:  $\tilde{B}v = 0$  and  $\tilde{Q}i = 0$ , and from the expressions of the fundamental loop and cutset matrices, one may deduce  $s$  constraint equations on the currents:  $i_{\mathcal{C}}^t = (i_{\mathcal{C}_1}^t, i_{\mathcal{C}_2}^t) = (\dot{q}_{\mathcal{C}_1}^t, \dot{q}_{\mathcal{C}_2}^t)$  at the ports of the capacitors and  $l$  constraint equations on the voltages:  $v_{\mathcal{L}}^t = (v_{\mathcal{L}_1}^t, v_{\mathcal{L}_2}^t) = (\dot{\phi}_{\mathcal{L}_1}^t, \dot{\phi}_{\mathcal{L}_2}^t)$  at the ports of the inductors:

$$-\tilde{B}_{12}^t \dot{q}_{\mathcal{C}_1} + \dot{q}_{\mathcal{C}_2} = 0 \quad (31)$$

$$B_{21} \dot{\phi}_{\mathcal{L}_1} + \dot{\phi}_{\mathcal{L}_2} = 0. \quad (32)$$

As the matrix  $\tilde{B}_{12}$  is constant, the previous equalities may be integrated to:

$$-\tilde{B}_{12}^t q_{\mathcal{C}_1} + q_{\mathcal{C}_2} = r_{\mathcal{C}} \quad (33)$$

$$B_{21} \phi_{\mathcal{L}_1} + \phi_{\mathcal{L}_2} = r_{\mathcal{L}} \quad (34)$$

where  $r_{\mathcal{C}}$  and  $r_{\mathcal{L}}$  are two constant vectors of dimension  $s$  and respectively  $l$ . These invariant functions constitute an independent set of invariants (or Casimir functions) of the Poisson bracket. Furthermore it is a generator set as the tree  $\mathcal{L}_1$  and the cotree  $\mathcal{C}_1$  are maximal.

*Proposition 5:* The Casimir functions (33)–(34) of the Poisson bracket corresponding to the structure matrix (24), are given by the integral of the relations (31), respectively (32), obtained by Kirchhoff's laws applied on the coloops defined by  $\mathcal{C}_2$  in  $\mathcal{C}$ , respectively by the loops defined by  $\mathcal{L}_2$  in  $\mathcal{L}$ .

*Example:* Consider the electrical circuit of example 2 proposed in [1] and represented in Fig. 1. The energy variables associated with the capacitors:  $C_i$ , respectively the inductors:  $L_i$ , are the charges:  $q_i$ , respectively the fluxes:  $\phi_i$  and grouped into a single vector of energy-variables  $x = \begin{pmatrix} q \\ \phi \end{pmatrix} \in \mathbb{R}^7$ . The total energy in the circuit is:

$$H(x) = \frac{b}{4} q_1^4 + \frac{1}{2C_2} q_2^2 + \frac{d}{6} q_3^6 + \frac{1}{2L_1} \phi_1^2 + k \sin \phi_2 + \frac{g_3}{4} \phi_3^4 + \frac{g_4}{4} \phi_4^4. \quad (35)$$

One may check that the capacitor's branches form a maximal tree  $\mathcal{C}$ , and the inductors' branches form the complementary cotree  $\mathcal{L}$ . Thus the capacitor's tree gives rise to the following matrix  $B_{\text{LC}}$ :

$$B_{\text{LC}} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix}. \quad (36)$$

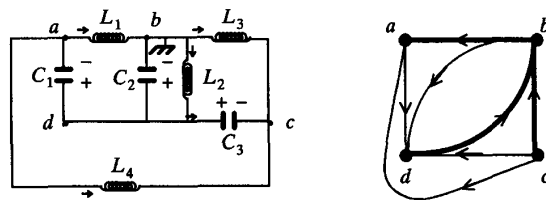


Fig. 1. Example of LC-circuit and its network graph with associated reference orientation.

The dynamical equations associated with this circuit are a Hamiltonian system according to (23) defined with respect to the Poisson bracket given by the structure matrix  $J$ :

$$J = \begin{bmatrix} 0_{3 \times 3} & B_{\text{LC}}^t \\ -B_{\text{LC}} & 0_{4 \times 4} \end{bmatrix}. \quad (37)$$

Since the matrix  $B_{\text{LC}}$  has rank 3, the structure matrix  $J$  has rank 6, and thus the Poisson bracket is degenerate. This means that its kernel has dimension 1 and there is one independent Casimir function which may be obtained from the topological constraints as above. The maximal subset  $\mathcal{C}_1$  of  $\mathcal{C}$  which forms a cotree is:  $\mathcal{C}_1 = \mathcal{C}$  (hence  $\mathcal{C} = \emptyset$ ). The maximal subset  $\mathcal{L}_1$  of  $\mathcal{L}$  which form a tree may be chosen to be the branches containing the inductors  $L_i, i \in \{1, 2, 3\}$  and corresponding to the bold edges in the network graph in Fig. 1. The complementary cotree  $\mathcal{L}_2$  of  $\mathcal{L}_1$  in  $\mathcal{L}$  is then composed of the edge  $(c, a)$ . Considering the cycle defined by this edge in  $\mathcal{L}$  and applying Kirchhoff's loop rule, one obtains:  $v_1 + v_3 - v_4 = 0$ . By integration the following invariant or Casimir function is obtained as:  $r_{\mathcal{L}} = \phi_1 + \phi_3 - \phi_4$ .

Bernstein and Lieberman [1] formulate the dynamics of an LC-circuit as a standard Hamiltonian system which actually corresponds to the reduced Hamiltonian system. They propose state-variables which are linear combinations of the capacitors' charges and the inductors' flux linkages and correspond to the canonical variables (12); they call the redundant coordinates  $r$  "nonactive" and the remaining coordinates  $(q, p)$  "active." A very interesting point is that these linear combinations are deduced from the partition of the ports and the associated fundamental loop and cutset matrices (29)–(30) of the network graph recalled above. In next section we develop these graphical aspects by proposing a graphical realization of both the Poisson bracket associated with the LC-circuit and a change of coordinates to canonical coordinates.

#### IV. NETWORK REALIZATION OF POISSON BRACKETS

We have seen in the previous section that the structure matrix of the Poisson bracket associated with an LC-circuit is determined by the network graph and the tree  $\mathcal{C}$  of edges connected to the capacitor's ports (or the complementary cotree  $\mathcal{L}$  connected to the inductors' ports). It defines anti-reciprocal relations on the port variables of the capacitors and inductors.

In the sequel we propose to use the antireciprocal network element called "gyrator" [5], to give a network realization of the Poisson bracket defined in (24).

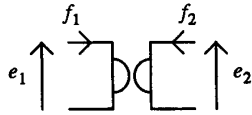


Fig. 2. Conventional diagram of a gyrator.

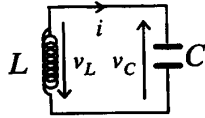


Fig. 3. An inductor and capacitor in series.

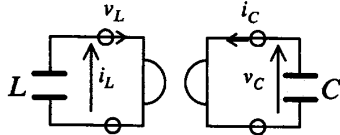


Fig. 4. Realization of a circuit composed of an inductor and a capacitor in series using a gyrator.

For the synthesis of electrical circuits Tellegen proposed the gyrator element [5], a two-port element which is anti-reciprocal and power-continuous. He deduced its constitutive relation which we recall hereafter with the modification that it is dimensionless and then is called "symplectic gyrator" [6].

**Definition 10:** A symplectic gyrator is an anti-reciprocal and power continuous element of which the conventional diagram is represented in Fig. 2 and of which the port variables verify the following constitutive relations:

$$\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \quad (38)$$

where  $(e_1, e_2)$  are the across variables at its ports and  $(f_1, f_2)$  are the through variables at its ports according to Fig. 2.

Consider the following simple LC-circuit depicted in Fig. 3, composed of an inductor and a capacitor in series. The topological relations are:

$$i_C = i_L \quad \text{and} \quad v_L = -v_C,$$

and, if  $H$  denotes the sum of the electric and magnetic energy in the circuit, the dynamic equations are:

$$\begin{pmatrix} \dot{q} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{bmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial \phi} \end{bmatrix}. \quad (39)$$

Using the symplectic gyrator the same circuit may be also realized by two capacitor elements, by dualizing the constitutive relation of the inductor according to Fig. 4.

Now the topological relations on the port variables of the inductor and the capacitor, represented by the structure matrix (24), are realized by a symplectic gyrator. Therefore the graphical representation of the current and voltage variables at the port of the inductor is also dualized: the current variable  $i_L$  is represented as an "across-variable" of the capacitor and the voltage variable  $v_L$  is represented as a "through-variable".

The LC-circuit may be realized by a network of capacitors and gyrators in the very general case [5] by using the

Fig. 5. A bond with effort variable  $e$  and flow variable  $f$ . (a) Single bond. (b) Multibond.

dualization of the inductors or more generally of some part of the circuit. But the effect of introducing the gyrators by dualization is to dualize not only the constitutive relations of the energy storage elements but also the representation of the interconnection, i.e., some part of the network graph has to be dualized. This may lead to representation problems because the algebraic dual of a nonplanar graphs may not be itself graphical [15].

Therefore in the sequel we shall use the bond graph formalism [7], [8] which is able to represent the constraint relations expressed in a network graph in the two dual ways [18]–[20].

In the bond graph formalism the interconnection constraints induced by an oriented network graph are represented by an oriented bond graph called "simple junction structure". It is a graphical representation of the two dual matroids [15] defined by the cycles and cocycles of the oriented network graph [18], [19], [21].

**Definition 11:** A simple junction structure is an oriented graph consisting of:

- nodes called *junctions*; they are either 1-junctions or 0-junctions (see Definition 12:) and are denoted by 0 and 1 respectively
- edges called *bonds* and denoted as in Fig. 5a, where the half arrows indicates the orientation of the bond; bonds connected only at one side to a junction are called *external bonds* and bonds connected at both sides are called *internal bonds*.

Two power variables are associated with each bond; one is called "effort variable" and placed on the opposite side of the half arrow and the other is called "flow variable" and placed on the side of the half arrow (see Fig. 5(a)). The power variables of the external bonds are the port variables of the simple junction structure. If the port variables of a simple junction structure may be identified with the across-variables of a network graph and the flow variables may be identified with the through variables, the simple junction structure is said "graphic"; in the dual case the simple junction structure is said to be "co-graphic". A simple junction structure may be neither graphic nor co-graphic.

**Definition 12:** The 1-junctions and 0-junctions represent the two dual elementary constraints on the set of power-variables at their ports.

A 1-junction imposes the following constraint relations on the power variables  $(e_1, \dots, e_n)$  and  $(f_1, \dots, f_n)$  at its ports:

$$\text{identity equations:} \quad f_1 = \dots = f_n \quad (40)$$

$$\text{and balance equation} \quad \sum_{i=1}^n \varepsilon_i e_i = 0 \quad (41)$$

where  $\varepsilon_i = 1$  if the bond is directed outward the junction, else  $\varepsilon_i = -1$ .

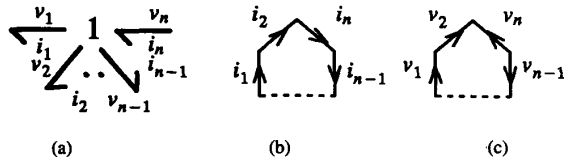


Fig. 6. 1-junction and associated current and voltage network graphs.

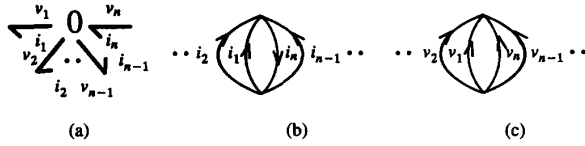


Fig. 7. 0-junction and associated current and voltage network graphs.

A 0-junction imposes the dual constraint relations on the power variables (i.e., interchanging the effort variables and the flow variables in (40) and in (41)).

The 1-junction may be considered as a graphic simple junction structure representing the topological constraints defined by a network graph consisting in a single loop (see Fig. 6), and the 0-junction, respectively, as representing the constraints associated with a single coloop (see Fig. 7). It may be noted that the reference directions of the current and voltage variable are associated if and only if the external bonds (i.e., bonds connected at one side only to a junction) are directed outside the junctions.

We shall also use the condensed notation of multibond graphs associated with variables in real valued vector spaces  $\mathbb{R}^n$ . An array of  $n$  bonds taken together is then represented by a multibond (see Fig. 5(b)) and an array of  $n$  junctions is represented by a 0 or 1 with underscore and indexed by  $n$ .

Complex interconnection constraints are obtained by interconnecting 0- and 1-junctions by bonds [18], [19], [21], i.e., constructing a simple junction structure. There exists several systematic procedures for constructing an oriented simple junction structure representing the topological constraints defined by a network graph based either on the node method [22] or on the mesh method, or on more general sets of cycles or cocycles families in the network graph [19], [18].

However, a simple junction structure may also represent constraint relations defined by matrices, representing matroids over the field  $\{-1, 0, 1\}$  [15]. Such relations are more general than the topological relations associated with an oriented network graph [23], [15]. Therefore in the sequel we shall use the construction of simple junction structures from matrices which allows to represent in a uniform way the constraints associated with matroids being graphical or not.

**Proposition 6:** A simple junction structure representing at its ports (denoted by 1 and 2), the fundamental loop matrix:  $B = (I_{n_1 \times n_1} \ B_{12})$  and the fundamental cutset matrix:  $Q = (-B_{12}^t I_{n_2 \times n_2})$  is realizable for any matrix  $B_{12}$  with coefficients in  $\{-1, 0, 1\}$  in the following way:

- Write an array of  $n_1$  1-junctions and an array of  $n_2$  0-junctions.
- Connect to each junction an external bond directed outwards the junction.


 Fig. 8. Simple junction structure representing the fundamental loop matrix  $B = (I_{n_1 \times n_1} \ B_{12})$ .

$$E_C(q) :: \mathbb{C} \begin{array}{c} v_C = \left( \frac{\partial E_C}{\partial q_i} \right)_{i=1..n} \\ \overline{\underline{\quad n \quad}} \\ i_C = \dot{q} \end{array} \quad E_L(\phi) :: \mathbb{I} \begin{array}{c} v_L = \dot{\phi} \\ \overline{\underline{\quad n \quad}} \\ i_L = \left( \frac{\partial E_L}{\partial \phi_i} \right)_{i=1..n_L} \end{array}$$

Fig. 9. Bond graph representation of inductors and capacitors.

$$\begin{array}{c} e_1 = f_2 \\ \overline{\underline{\quad n \quad}} \\ f_1 \end{array} \begin{array}{c} J_n^s \\ \overline{\underline{\quad n \quad}} \\ GY \end{array} \begin{array}{c} e_2 = -f_1 \\ \overline{\underline{\quad n \quad}} \\ f_2 \end{array}$$

 Fig. 10. Bond graph representation of an array of  $n$  symplectic gyrators.

—For every nonnull coefficient  $b_{ij} \in B_{12}$  create a bond relating the  $i$ -th 1-junction to the  $j$ -th 0-junction directed toward, respectively outwards, the 0-junction if  $b_{ij} = 1$ , respectively  $b_{ij} = -1$ .

Such a simple junction structure will be denoted as shown in Fig. 8.

With the graphic identification of the power variables, the energy storage elements capacitor, respectively inductor, are represented as 1-port storage elements denoted by  $C$  with energy function:  $E_C(q) \in C^\infty(\mathbb{R}^n)$ , respectively by  $I$  with energy function:  $E_L(\phi) \in C^\infty(\mathbb{R}^n)$  according to Fig. 9.

Finally an array of  $n$  symplectic gyrators with constitutive relation generalized from (38):

$$\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = J_n^s \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \quad (42)$$

with

$$J_n^s = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \quad (43)$$

is represented according to the Fig. 10.

Using the bond graph notation, the realization of an LC-circuit by a network of capacitors and gyrators may be generalized from the example in Figs. 3 and 4 to the general LC-circuits defined in Section III.

The bond graph in Fig. 11 represents two arrays of energy-storage elements denoted by  $C$ , storing the total electrical and magnetic energy of the circuit respectively. They are interconnected by a generalized junction structure composed of three simple junction structures and an array of  $r$  symplectic gyrators interconnected by 0- and 1-junctions.

The three simple junction structures are defined according to the partition  $\{C_1, C_2, L_1, L_2\}$  defined in Section III and may



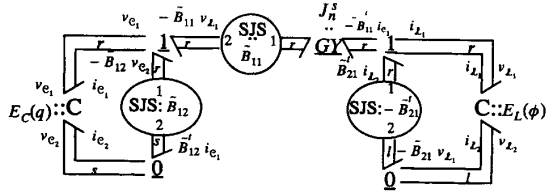


Fig. 11. Bond graph realization of the Poisson bracket associated with an LC-circuit according to the tree  $\mathcal{L}_1 \cup \mathcal{L}_2$ .

be realized according to Proposition 6 from the associated fundamental loop matrix  $\tilde{B}$  and the fundamental cutset matrix  $\tilde{Q}$  defined in (29)–(30). The constraint equations defined by these matrices, i.e., Kirchhoff's laws, may be read by considering the balance equations on all the junctions of the bond graph in Fig. 11.

In order to make explicit the antireciprocal relations between the port variables of the capacitors and the inductors, the inductors are dualized into storage elements denoted by  $C$  and accordingly the through- and across-variables in the part of the network graph corresponding to  $\mathcal{L}_1 \cup \mathcal{L}_2$ , i.e., describing the interconnection constraints among the inductors. These constraints are expressed by the loop matrix:  $(\tilde{B}_{21}^t I_l)$  in the network graph or by the loop matrix  $(I_r - \tilde{B}_{21}^t)$  after its partial dualization. This leads to the cographic simple junction structure defined by  $-\tilde{B}_{21}^t$  which is realizable according to Proposition 6.

The array of  $r$  symplectic gyrators relates the graphic and cographic simple junction structures and completes the generalized junction structure relating the two arrays of energy-storage elements denoted by  $C$ . This generalized junction structure is a bond graph realization of the antireciprocal relations between the port variables of the inductors and capacitors, i.e., it is a realization of the Poisson bracket matrix (24).

It may be noted that an equivalent realization of the Poisson bracket matrix by a network graph exists if and only if the algebraic dual of the part of the network graph defined by  $\mathcal{L}_1 \cup \mathcal{L}_2$  is graphic, i.e., if the network graph defined by  $\mathcal{L}_1 \cup \mathcal{L}_2$  is planar.

Consider now the port variables of the symplectic gyrators. Their flow variables are:

$$\begin{bmatrix} v_{\mathcal{L}_1} \\ \tilde{B}_{11}^t i_{c_1} \end{bmatrix}. \quad (44)$$

These variables are the time-derivatives of the coordinates of the reduced Hamiltonian system (called active variables) in the second change of coordinate (denoted by II) defined in [1], which leaves invariant the  $r$  fluxes of the inductors on the branches  $\mathcal{L}_1$ . Thus the variables at the ports of the symplectic gyrators correspond to the reduced Hamiltonian system with the array of  $r$  symplectic gyrators representing indeed the associated symplectic structure of rank  $2r$ .

It may also be noted that there exists an analogous bond graph realization corresponding to the first change of coordinates (denoted by I) defined in [1], which may be obtained by

dualizing the simple junction structure defined by the matrix  $\tilde{B}_{11}$  in the bond graph in Fig. 11.

Thus in the network representations, i.e., in the bond graph or the network graph if it exists, only the active variables are explicitly represented by their time-derivative as the flow variables at the ports of the symplectic gyrators. The redundant or nonactive variables are given by integrating the balance (31)–(32) read at the 0-junctions of the bond graph in Fig. 11, but are not explicitly represented as power variables.

In the sequel we propose a bond graph representation including a representation of the redundant variables (or more exactly of their time-derivatives). As the network formalism automatically generates the adjoint or conjugated variables  $s_1, \dots, s_l$  (see (17)), the proposed bond graph actually represents the embedding Hamiltonian system (18).

Let us call  $s_C$ , respectively  $s_L$ , the vector of variables conjugated to the redundant variables  $r_C$ , respectively  $r_L$  according to (17). Their time-derivative is given by (18):

$$\dot{s}_i = \frac{\partial H_e}{\partial r_i}(q, p, r, s) = \frac{\partial H}{\partial r_i}(q, p, r).$$

Using (25), (33), and (34), a straightforward calculus shows that:

$$\frac{\partial H}{\partial r_C}(x) = (-\tilde{B}_{12}^t I_s \ 0_r \ 0_l) \begin{bmatrix} v_{c_1} \\ v_{c_2} \\ i_{\mathcal{L}_1} \\ i_{\mathcal{L}_2} \end{bmatrix} = -\tilde{B}_{12}^t v_{c_1} + v_{c_2} \quad (45)$$

$$\frac{\partial H}{\partial r_L}(x) = (0_r \ 0_s \ \tilde{B}_{21} I_l) \begin{bmatrix} v_{c_1} \\ v_{c_2} \\ i_{\mathcal{L}_1} \\ i_{\mathcal{L}_2} \end{bmatrix} = \tilde{B}_{21} i_{\mathcal{L}_1} + i_{\mathcal{L}_2}. \quad (46)$$

The identification of  $(\dot{r}_L, \frac{\partial H}{\partial r_C})$  as a vector of additional voltages and of  $(\dot{r}_C, \frac{\partial H}{\partial r_L})$  as a vector of additional currents, leads to consider the extension of the fundamental loop matrix (29) and the cutset matrix (30) to the following fundamental loop matrix:

$$\hat{B} = \begin{bmatrix} \tilde{B}_{11} & 0_{r \times l} & I_r & \tilde{B}_{12} & 0_l & 0_l \\ \tilde{B}_{21} & I_l & 0_{l \times r} & 0_{l \times s} & I_l & 0_s \\ 0_r & 0_l & \tilde{B}_{12}^t & -I_s & 0_s & I_s \end{bmatrix} \quad (47)$$

and the following fundamental cutset matrix:

$$\hat{Q} = \begin{bmatrix} I_r & -\tilde{B}_{21}^t & -\tilde{B}_{11}^t & 0_{r \times s} & 0_{r \times s} & 0_{r \times l} \\ 0_{s \times r} & 0_{s \times l} & -\tilde{B}_{12}^t & I_s & I_s & 0_{s \times l} \\ -\tilde{B}_{21} & -I_l & 0_{l \times r} & 0_{l \times s} & 0_{l \times s} & I_l \end{bmatrix}. \quad (48)$$

These extended loop and cutset matrices may be realized by augmenting the bond graph of Fig. 11 with an additional junction structure represented in bold lines in the bond graph in Fig. 12.

The bond graph realization of the embedding system is now completed by two arrays of symplectic gyrators representing the definition of the symmetry variable with respect to the Poisson bracket of the embedding space given in (17). Two

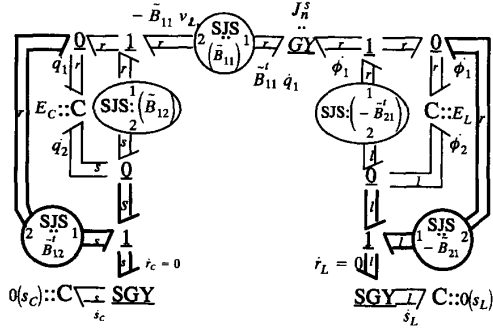


Fig. 12. Bond graph realization of the change of variable to the canonical coordinates according to the change of coordinates II.

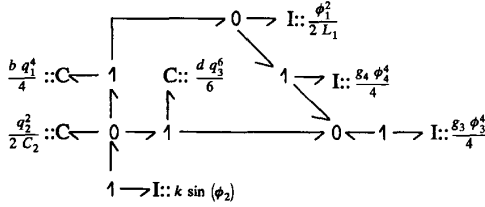


Fig. 13. Generalized bond graph model of the LC-circuit of Fig. 1 obtained by dualization at the ports of the fields of inductors.

energy-storage  $C$ -elements with energy-variables  $s_C$  and  $s_L$  and energy functions being identically zero represent the symmetry of the embedding system with respect to these variables.

Analogously to the generalized bond graph representation of the LC-circuit in Fig. 11, one reads at the ports of the symplectic gyrators the embedding Hamiltonian system in the canonical coordinates.

The energy-variables of the generalized bond graph realization of the embedding system are the charges and fluxes of the LC-circuit together with the variables  $s_C$  and  $s_L$ . The invariants of the system are then expressed by the fact that the energy function does not depend on the symmetry variables. This means that the invariants could be removed by introducing some extra term in the energy function depending on these variables, i.e., adding an electrical and magnetic energy. However it is interesting to note that this augmented system may not be realizable by a LC-circuit as the augmented simple junction structure of the model of the embedding model may not have a network graph equivalent, as will now be illustrated on the example.

*Example (continued):* Using the node method [22], the electrical circuit in Fig. 1 has the bond graph representation given in Fig. 13. The inductors are represented by  $I$ -elements, the capacitors by  $C$ -elements and the simple junction structure represents the network graph of the circuit with the graphic identification of the power variables. Voltages are denoted by effort-variables and currents by flow-variables.

Dualizing the  $I$ -elements and partly dualizing the simple junction structure using *graphical* transformations [6], [8]

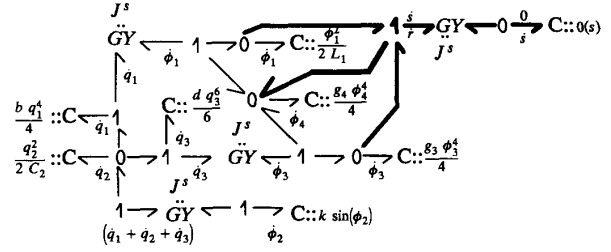


Fig. 14. Bond graph model of the LC-circuit of Fig. 1 according to the node method.

leads to the generalized bond graph model in Fig. 14 (excluding the bold part). This bond graph is the particular realization of the general model in Fig. 11 for the example. Three symplectic gyrators connect the field of inductors to the field of capacitors, revealing that the dimension of the Poisson bracket is 6. At the ports of the gyrators the flow variables are:  $\dot{q}_1$ ,  $(\dot{q}_1 + \dot{q}_2 + \dot{q}_3)$ ,  $\dot{q}_3$  and  $\phi_1$ ,  $\phi_2$ ,  $\phi_3$ , which correspond to the change of variables II to canonical coordinates in [1].

The embedding system is obtained by adding the bold part in the bond graph in Fig. 14 (corresponding to the general model in Fig. 12). The cutset matrix corresponding to the simple junction structure of the extended generalized bond graph is:

$$\hat{Q} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & -1 & -1 \end{bmatrix}. \quad (49)$$

Iri's algorithm [23] applied to the loop matrix with the graphic identification or the cographic identification shows that in both cases that this simple junction structure is not realizable as a network graph, and thus the embedding system is not realizable by an extended LC-circuit.

## V. CONCLUSION

The dynamic equations of (nonlinear) LC-circuits have been formulated as Hamiltonian systems generated by the total energy of the circuit with respect to a Poisson bracket uniquely defined by the network graph of the circuit. The state variables are the capacitors' charges and the inductors' flux linkages which define without ambiguity the energy of the circuit. The Poisson bracket is deduced from the fundamental loop matrix associated with the set of capacitors. Thereby the invariants of the systems due to Kirchhoff's laws are fully captured in the Poisson bracket, i.e., in the geometry of the state-space. It is an important feature of the proposed formulation that two independently defined mathematical objects, Hamiltonian function and Poisson bracket, correspond to the two independently defined network objects, the energy functions and the network graph.

The Poisson bracket was further investigated from a network perspective and realized by a network composed of gyrators and a partially dualized network graph. The bond graph notation was used in order to ensure a graphical representation of the Poisson bracket. From the network realization of the Poisson bracket, the reduced standard Hamiltonian system was

deduced as well as the realization of the embedding standard Hamiltonian system. It was illustrated on an example that the latter may not have a network graph realization but always has a bond graph realization.

The LC-circuits considered here do not contain any excess elements, but an analogous construction holds by using a parametrization of the state-space deduced from the network graph as it has been proposed in [9] in a more general context. Sources could also be included by considering so-called port-controlled Hamiltonian systems [10].

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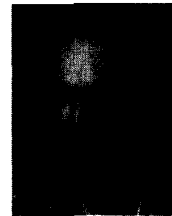
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