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# Stable Kernel Representations as Nonlinear Left Coprime Factorizations

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## Abstract

A representation of nonlinear systems based on the idea of representing the input-output pairs of the system as elements of the kernel of a stable operator has been recently introduced [8, 9]. This has been denoted the kernel representation of the system. In this paper it is demonstrated that the kernel representation is a generalization of the left coprime factorization of a general nonlinear system in the sense that it is a dual operator to the right coprime factorization of a nonlinear system. The results obtainable in the linear case linking left and right coprime factorizations are shown to be reproduced within the kernel representation framework.

## 1. Introduction

In this paper the links between nonlinear right coprime factorizations and stable kernel representations (*skrs*) for nonlinear systems are explored. In the linear factorization theory it may be seen that right factorizations are a natural dual of left factorizations. This has led to a rich structure, yielding the Youla parameterizations, which may be derived in terms of the left or right coprime factorization framework, see e.g. [14, 12]. In the nonlinear case, such results have not been obtainable to date. This has led to a number of problems in attempts to generalize the results of the linear factorization theory to the nonlinear case.

Right factorizations for nonlinear systems have been extensively studied, see e.g. Verma [13] or Sontag [10], and the references contained therein. It was demonstrated that right factorizations for nonlinear systems could be derived if the system was stabilizable by a smooth state feedback controller. However, the structure of the Youla parameterization could not, in general, be obtained within this framework. Such results were only available for special cases in which either the plant or controller was linear.

Meanwhile left coprime factorizations were investigated, mainly from an input-output point of view. This work was initiated by Hammer [3, 4], and continued by Tay [11], Chen and de Figueiredo [1, 2] and Paice and Moore [5, 7].

Links between the right and left factorization frameworks were investigated by Paice and Moore [8, 6], and although right factorizations could be derived when the plant and controller had stable left coprime factorizations, the dual result could not be obtained.

Recently, the authors introduced the idea of representing

a nonlinear system in terms of being the kernel of some stable operator, see [8] or [9]. In this work it was demonstrated that by using *skrs* it is possible to derive the Youla parameterization, and furthermore to derive the class of all stabilizing Plant-Controller pairs, as was derived in [12] for the linear case. This was achieved without having to resort to the restrictive assumptions required in past attempts at this problem. Furthermore, a state-space derivation of a *skr* was presented for a general nonlinear system. This gave the first derivation of the Youla parameterization for a nonlinear system with a natural state-space interpretation. In this paper, the links between *skrs* and right coprime factorizations (*rcfs*) are further explored, and it is demonstrated that the dual structure between linear right and left factorizations is reproduced in the nonlinear case by the relationships between *skrs* and *rcfs*.

By applying the results of this paper to the state space expressions of the *rcfs* for a given nonlinear feedback system, it becomes possible to derive the Youla parameterization for that system. Further, given *skrs* for the plant and controller, it becomes possible to derive *rcfs* for that system. This is significant, as in the nonlinear case, the Youla parameterization is best described in terms of *skrs*, whereas stability of the system with respect to exogenous (additive) inputs is better characterized by the *rcf* description.

The paper is organized as follows. In Section 2, right coprime factorizations and stable kernel representations for nonlinear systems are reviewed. This leads to two characterizations of the well-posedness and stability of nonlinear feedback systems. Section 3 presents the main results of this paper, developing the links between right coprime factorizations and stable kernel representations, and unifying these apparently different ways of viewing a feedback system. In section 4 conclusions are drawn and remarks for further work are presented.

## 2. Preliminaries

In this section the definitions and concepts appropriate to the study of nonlinear feedback systems via *skrs* and *rcfs* are reviewed from [9] and [6], respectively. It should be noted from the outset that this seems to lead to two definitions of well-posedness and stability for a nonlinear feedback system, one including exogenous inputs, and one excluding such inputs. It will be shown in the Section 3 that these notions are consistent. In the meantime, it will

be clear from the context which definition is being used.

In the sequel the term system will be taken to denote a general (dynamical) system, and the terms feedback system or closed-loop system will be used to indicate an interconnection of such systems.

### Signal Spaces and Stability

In this paper we consider general input-output or dynamical systems between signal spaces. These mappings are dependent on spaces of initial conditions. Signal spaces are taken to be vector spaces of functions from a given time domain to a given Euclidean space, whereas the initial condition spaces are more commonly Euclidean vector spaces. Note that we do not make any distinction between discrete and continuous time systems.

The concept of stability for general nonlinear operators and feedback systems is now defined implicitly via the notion of stability on the various input and output spaces of these operators. A signal space  $\mathcal{Z}$  is divided into two disjoint subsets as follows

$$\mathcal{Z} = \mathcal{Z}^s \cup \mathcal{Z}^u, \quad \mathcal{Z}^s \cap \mathcal{Z}^u = \emptyset,$$

where  $\mathcal{Z}^s$  denotes the set of all stable signals, and  $\mathcal{Z}^u$  the set of all unstable signals. For cross-product spaces  $\mathcal{Z}_{GK} = \mathcal{Z}_G \times \mathcal{Z}_K$ ,  $\mathcal{Z}_{GK}^s$  is defined to be  $\mathcal{Z}_G^s \times \mathcal{Z}_K^s$ , and  $\mathcal{Z}_{GK}^u$  is the remainder of the space.

Note that  $\mathcal{Z}$  may be partitioned in many ways. It is not assumed that  $\mathcal{Z}^s$  is a vector space, or that it is closed, although it is assumed that  $0 \in \mathcal{Z}^s$ . Commonly these sets are formed by defining a norm on the space  $\mathcal{Z}$ , and then defining a signal to be stable iff it has finite norm.

**Definition 2.1** An input-output map  $\Sigma : \mathcal{U} \rightarrow \mathcal{Y}$  is said to be stable if the image of  $\mathcal{U}^s$  under  $\Sigma$  is a subset of  $\mathcal{Y}^s$ .  $\square$

### Kernel Representations

In this section the notion of representing a general system,  $\Sigma$ , as being represented by the kernel of a family of operators, parameterized by the initial conditions of the system, is introduced. This is extended to give kernel representations of feedback systems. Definitions of well-posedness and stability of feedback systems are presented for use within this framework.

Consider the system  $\Sigma$ , with input and output signal spaces  $\mathcal{U}$  and  $\mathcal{Y}$  respectively, and initial condition space  $\mathcal{X}_\Sigma$ . It is assumed that every such system under consideration may be described by a family of maps

$$R_\Sigma^z : \mathcal{Y} \times \mathcal{U} \rightarrow \mathcal{Z}, \quad \forall x \in \mathcal{X}_\Sigma, \quad (1)$$

known as the *kernel representation* of  $\Sigma$ , such that all possible input-output pairs  $u, y$  for the system  $\Sigma$  with initial conditions  $x \in \mathcal{X}_\Sigma$  satisfy

$$R_\Sigma^z(y, u) = 0. \quad (2)$$

In general it is not possible to describe a kernel representation by a single map  $R_\Sigma : \mathcal{Y} \times \mathcal{U} \rightarrow \mathcal{Z}$ , however for brevity, we shall refer to the kernel representation  $R_\Sigma$ . The key to the development of the following results is to examine the solutions to

$$R_\Sigma^z(y, u) = z \quad (3)$$

where  $z$  is not necessarily equal to zero.

For arbitrary  $z$ , the input-output map induced by the solution pairs to (3) for a given initial condition  $x \in \mathcal{X}$  will be denoted by  $\Sigma_z(x) : \mathcal{U} \mapsto \mathcal{Y}$ . The input-output map  $\Sigma_0(x) : \mathcal{U} \mapsto \mathcal{Y}$  will be simply denoted by  $\Sigma(x)$ ; the input-output map of  $\Sigma$  for initial condition  $x$ .

The existence of  $\Sigma_z(x)$  for all  $z$  is necessary in the context of this paper, as we wish to consider the right factorizations of such systems. It is thus necessary to introduce the concept of well-definedness of the kernel representation  $R_\Sigma : \mathcal{Y} \times \mathcal{U} \rightarrow \mathcal{Z}$ .

**Definition 2.2** A kernel representation (1) is said to be well-defined if for each  $z \in \mathcal{Z}$ , and initial conditions  $x \in \mathcal{X}_\Sigma$ , the map  $\Sigma_z(x) : \mathcal{U} \rightarrow \mathcal{Y}$  exists, so that for all  $u \in \mathcal{U}$ ,  $y = \Sigma_z(x)u$  iff  $R_\Sigma^z(u, y) = z$ .  $\square$

Note that  $R_\Sigma$  can be well-defined for  $x \in \mathcal{X}_\Sigma$  only if the map

$$R_\Sigma^z(\cdot, u) : \mathcal{Y} \rightarrow \mathcal{Z} \quad (4)$$

is one to one, and onto, i.e. invertible. We denote this inverse

$$[R_\Sigma^z]^{-1}(\cdot, u) : \mathcal{Z} \rightarrow \mathcal{Y}. \quad (5)$$

This is summarized in the following result:

**Proposition 2.3** A given kernel representation (1) of  $\Sigma$  is well-defined iff for all  $x \in \mathcal{X}$  and all  $u \in \mathcal{U}$ , the map  $[R_\Sigma^z]^{-1}(\cdot, u)$  of (5) exists.  $\square$

**Remark 2.4** It may be shown that the well-definedness of a kernel representation may be derived by considering the behavior of the system when connected in feedback with the zero operator. See [9] for details.

**Remark 2.5** Note that every system has a kernel representation, although it will not be unique, for example any input-output map  $\Sigma(x) : \mathcal{U} \mapsto \mathcal{Y}$  has kernel representation  $R_\Sigma^z(y, u) = F(y - \Sigma(x)u)$ , for any invertible  $F : \mathcal{Y} \mapsto \mathcal{Y}$  such that  $F(0) = 0$ .

**Definition 2.6** A kernel representation  $R_\Sigma : \mathcal{Y} \times \mathcal{U} \rightarrow \mathcal{Z}$  of  $\Sigma$  is called a *stable kernel representation* (skr) of  $\Sigma$  iff for all initial conditions  $x \in \mathcal{X}_\Sigma$ ,  $R_\Sigma^z(\cdot, \cdot)$  is a stable operator. That is, if  $y \in \mathcal{Y}^s$ ,  $u \in \mathcal{U}^s$ , then  $z = R_\Sigma^z(y, u) \in \mathcal{Z}^s$ .  $\square$

Unless otherwise stated, all kernel representations used in the sequel will be *skrs*.

The notion of interconnecting two systems, the plant and the controller, to form closed-loop or feedback systems is now introduced and developed for use within this framework. Note that it is common to allow for the introduction of external signals between the plant and controller so as to account for reference signals, or noise signals corrupting the control or measured signal. When considering *skrs* only the case where these external signals are zero will be considered, this is referred to as the noise-free case.

Consider a plant,  $G : \mathcal{U} \rightarrow \mathcal{Y}$ , and controller  $K : \mathcal{Y} \rightarrow \mathcal{U}$ , with kernel representations  $R_G : \mathcal{Y} \times \mathcal{U} \rightarrow \mathcal{Z}_G$ ,  $R_K : \mathcal{U} \times \mathcal{Y} \rightarrow \mathcal{Z}_K$  which are interconnected to form the system  $\{G, K\}$  as in Figure 1. The closed loop then has a kernel representation

$$R_{\{G, K\}} : \mathcal{Y} \times \mathcal{U} \rightarrow \mathcal{Z}_G \times \mathcal{Z}_K \quad (6)$$

$$R_{\{G, K\}}(y, u) = \begin{pmatrix} R_G(y, u) \\ R_K(u, y) \end{pmatrix}$$

as in Figure 2.

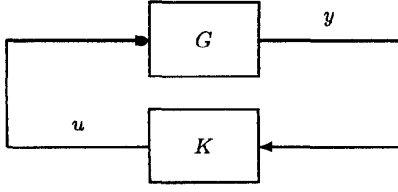


Figure 1: The system  $\{G, K\}$

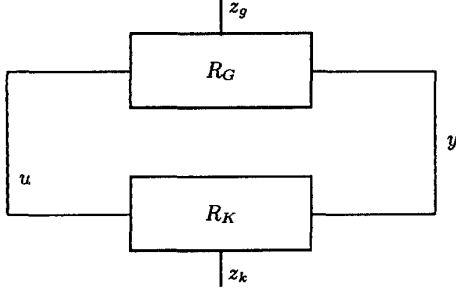


Figure 2: The Kernel Representation of  $\{G, K\}$ .

The existence of a solution pair  $(u, y)$  for a given  $(z_G, z_K)$  is not guaranteed. Thus, in order to work with feedback systems within this framework, we will need to assume that solutions exist. This property is known as well-posedness. **Definition 2.7** The system  $\{G, K\}$  is well-posed iff for all initial conditions,  $(x_G, x_K) \in \mathcal{X}_{GK}$ , and for all  $(z_G, z_K) \in \mathcal{Z}_{GK}$ , the solution  $(u, y)$  to (6) is unique. That is, for all  $(x_G, x_K) \in \mathcal{X}_{GK}$ ,

$$\left[ R_{\{G,K\}}^{z_G, z_K} \right]^{-1} : \mathcal{Z}_{GK} \rightarrow \mathcal{U} \times \mathcal{Y} \quad \text{exists.} \quad (7)$$

□

**Remark 2.8** The above definition of well-posedness of a feedback system, when specialized to linear systems, is very similar to the notion of *regular feedback interconnection*, as proposed in [15]. Note that the requirement of existence of unique solutions  $w = (y, u)^T$  for every  $x \in \mathcal{X}$  excludes the possibility of *singular feedback* ([15]).

We consider now the properties of the operators  $\Sigma_z$  which are defined by *skrs*. As these operators are implicitly defined it cannot be expected that (3) will yield a stable operator for all possible  $x \in \mathcal{X}$  and  $z \in \mathcal{Z}$ , even when the original operator  $\Sigma$  is stable. Thus we make the following definition.

**Definition 2.9** A system  $\Sigma$  with stable kernel representation  $R_\Sigma(\cdot, \cdot)$ , as in (1), is stable over the set  $\mathcal{B} \subset \mathcal{Z}^s \times \mathcal{X}_\Sigma$  if for all  $(z, x) \in \mathcal{B}$  the input-output map  $\Sigma_z(x)$  is stable iff  $(z, x) \in \mathcal{B}$ . □

The system  $\Sigma$  with *skr* (1) is called *generally stable*, or simply *stable*, if it is stable over  $\mathcal{Z}^s \times \mathcal{X}$ .

The definition of stability is now extended to include closed loop systems.

**Definition 2.10** The closed loop system  $\{G, K\}$  with *skr*  $R_{\{G,K\}}$  as in (6) is stable over  $\mathcal{B}_{GK} \subset \mathcal{Z}_{GK}^s \times \mathcal{X}_{GK}$  if it is

well-posed, and for all pairs  $(z_{GK}, x_{GK}) \in \mathcal{Z}_{GK}^s \times \mathcal{X}_{GK}$  the solution  $(y, u)$  to (6) is stable, iff  $(z_{GK}, x_{GK}) \in \mathcal{B}_{GK}$ .

A section of  $\mathcal{B}_{GK}$  corresponding to the initial condition  $x$  is defined as  $\mathcal{B}_{GK}^x = \{z : (z, x) \in \mathcal{B}_{GK}\}$ . □

The system  $\{G, K\}$  is said to be *generally stable*, or simply *stable*, if it is stable over  $\mathcal{Z}_{GK}^s \times \mathcal{X}_{GK}$ .

**Lemma 2.11** The system  $\{G, K\}$  is well-posed and stable over  $\mathcal{B} \subset \mathcal{Z}_{GK}^s \times \mathcal{X}_{GK}$  iff for all  $x_{GK} \in \mathcal{X}_{GK}$  the map

$$\left[ R_{\{G,K\}}^{x_{GK}} \right]^{-1} : \mathcal{Z} \rightarrow \mathcal{Y} \times \mathcal{U} \quad \text{exists,} \quad (8)$$

$$\text{and } \left[ R_{\{G,K\}}^{x_{GK}} \right]^{-1} \mathcal{B}^{x_{GK}} \subset \mathcal{Y}^s \times \mathcal{U}^s \quad (9)$$

□

The proof arises out of the definitions and is left to the reader.

**Remark 2.12** Note that for linear systems, stable kernel representations reduce to stable left factorizations for the system in the following sense. Suppose that  $G = \tilde{M}^{-1}\tilde{N}$  is a stable left factorization for  $G$ , then  $R(y, u) = \tilde{M}y - \tilde{N}u$  is a well-defined and stable kernel representation for  $G$ .

**Remark 2.13** In the linear case Lemma 2.11 reduces to the well-known result that the feedback system  $\{G, K\}$  is well-posed and stable iff

$$\begin{bmatrix} \tilde{V} & -\tilde{U} \\ -\tilde{N} & \tilde{M} \end{bmatrix}^{-1} \quad \text{exists and is stable,}$$

where  $G = \tilde{M}^{-1}$  and  $K = \tilde{V}^{-1}\tilde{U}$ .

In analogy with linear systems, we introduce the following definition of coprimeness for *skrs*.

**Definition 2.14** A stable kernel representation  $R_\Sigma(\cdot, \cdot)$ , as in (1), is said to be *coprime* if it has a right inverse. That is, there exists a stable operator  $T : \mathcal{Z} \rightarrow \mathcal{Y} \times \mathcal{U}$  such that

$$R \circ T = Z : \mathcal{Z} \mapsto \mathcal{Z}, \quad \text{unimodular} \quad (10)$$

where  $\circ$  denotes composition of operators. □

**Remark 2.15** Note that coprimeness of the *skrs* is necessary for the feedback system to be well-posed, thus in the sequel, all *skrs* are assumed to be coprime.

**Remark 2.16** Note that (10) is a nonlinear form of the Bezout Identity, and that the operator  $Z$  may be taken to be the identity without loss of generality.

## Right Coprime Factorizations

In this section the concept of right coprime factorizations (*rcfs*) for nonlinear systems are defined, and a characterization of well-posedness and stability of feed-back systems with exogenous inputs is given in terms of these factorizations.

The system  $\Sigma : \mathcal{U} \rightarrow \mathcal{Y}$  has a *stable right factorization* if there exist stable operators  $D : \mathcal{Z}_r \rightarrow \mathcal{U}$ , invertible, and  $N : \mathcal{Z}_r \rightarrow \mathcal{Y}$  such that  $\Sigma = ND^{-1}$ .

The following definition was first presented in [5], and was developed from the point of view of preventing the nonlinear equivalent of unstable pole-zero cancellations. Motivation for taking this approach to coprimeness may be found in Hammer [4].

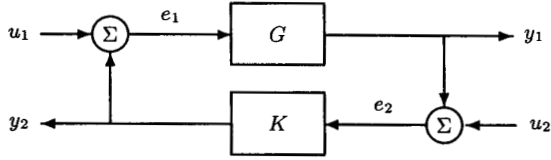


Figure 3: The feedback system  $\{G, K\}$ .

**Definition 2.17** Let  $M, N$  be a right factorization for  $G: \mathcal{U} \rightarrow \mathcal{Y}$

$$G = NM^{-1}, N: \mathcal{S}^r \rightarrow \mathcal{Y}, M: \mathcal{S}^r \rightarrow \mathcal{U} \quad (11)$$

where  $M$  and  $N$  are BIBO stable. Then  $M, N$  is a right coprime factorization of  $G$  (rcf) iff for all unbounded inputs  $s \in \mathcal{S}_u^r$ ,  $M s$  or  $N s$  is unbounded.  $\square$

We first review the connection between right coprime factorizations and the Bezout identity.

**Lemma 2.18**[7] Given a stable right factorization of  $G$ , as in (11), suppose that there exists a bounded-input bounded-output (BIBO) stable mapping  $L: \mathcal{U} \times \mathcal{Y} \rightarrow \mathcal{S}^r$  such that

$$L \begin{bmatrix} M \\ N \end{bmatrix} = I \quad (12)$$

Then  $G = NM^{-1}$  is a right coprime factorization for  $G$   $\square$

In the context of rcfs we consider that the closed loop  $\{G, K\}$  has external inputs, as in Figure 3. This leads to the following definitions of well-posedness and stability.

**Definition 2.19** The system  $\{G, K\}$  is well-posed if the closed-loop system input-output operator from  $u_1, u_2$  to  $e_1, e_2$ , namely

$$\begin{bmatrix} I & -K \\ -G & I \end{bmatrix}^{-1} \text{ exists.} \quad (13)$$

$\square$

**Definition 2.20** The system  $\{G, K\}$ , assumed well-posed, is said to be internally stable iff for all bounded inputs  $u_1, u_2$  the outputs  $y_1, y_2$  and  $e_1, e_2$  are bounded. This is equivalent to

$$\begin{bmatrix} I & -K \\ -G & I \end{bmatrix}^{-1} \text{ is BIBO stable.} \quad (14)$$

$\square$

These definitions may be given equivalent forms by considering the rcfs of the plant and controller, as follows.

**Theorem 2.21** [6]

Given  $\{G, K\}$ , and  $G = NM^{-1}$  and  $K = UV^{-1}$  rcfs, then  $\{G, K\}$  is well-posed iff

$$\begin{bmatrix} M & -U \\ -N & V \end{bmatrix}^{-1} \text{ exists} \quad (15)$$

and is internally stable iff

$$\begin{bmatrix} M & -U \\ -N & V \end{bmatrix}^{-1} \text{ is BIBO stable} \quad (16)$$

$\square$

Hence the stability and well-posedness of the system depends on the existence and stability of the operator  $\begin{bmatrix} M & -U \\ -N & V \end{bmatrix}^{-1}$ . In fact the relationship is somewhat stronger, coprimeness also results from the stability of this operator.

**Lemma 2.22** Suppose we have  $G = NM^{-1}$  and  $K = UV^{-1}$ , such that the operators  $M, N, U, V$  are BIBO stable. Then these are rcfs for  $G$  and  $K$  if they satisfy (16).  $\square$

**Remark 2.23** These results are exactly the same as those obtained in the linear theory.

**Remark 2.24** When *skrs* are considered as nonlinear left coprime factorizations it is immediate that the above results are dual results to Definition 2.7 and Lemma 2.11. Previously such results were not obtainable for left coprime factorizations.

### 3. Relationships to Coprime Factorizations

In [8, 9] it may be seen that the main results obtained in nonlinear factorization theory using left factorizations are duplicable using stable kernel representations. We now further explore the relationship between *skrs* and coprime factorizations, and demonstrate that the *skr* of a general operator is a generalization of its left coprime factorization. It is shown that any operator with a left coprime factorization has a stable kernel representation, and that the results derived linking linear left and right coprime factorizations may also be obtained using *skrs*.

In the sequel, all statements will be assumed to hold for arbitrary initial conditions, so the superscripts denoting the initial conditions have been suppressed. However, it should be noted that attention must be paid to initial conditions, as the validity of the factorization of an operator is initial condition dependent.

The system  $\Sigma: \mathcal{U} \rightarrow \mathcal{Y}$  has a *stable left factorization* if there exist stable operators

$$\tilde{D}: \mathcal{Y} \rightarrow \mathcal{Z}_l \text{ invertible, } \tilde{N}: \mathcal{U} \rightarrow \mathcal{Z}_l \quad (17)$$

such that  $\Sigma = \tilde{D}^{-1}\tilde{N}$ .

The following result establishes that the *skr* of a system is a generalization of the left factorization of a system.

**Proposition 3.1** A system  $\Sigma: \mathcal{U} \rightarrow \mathcal{Y}$  will have a *stable left factorization* (17) iff there exists a *stable kernel representation*  $R_\Sigma$  for  $\Sigma$  (1) which is well-defined and separable in the sense that

$$R_\Sigma(y, u) = R_y(y) - R_u(u). \quad (18)$$

The *stable left factorization* will be given by

$$\tilde{D} = R_y \quad \tilde{N} = R_u \quad (19)$$

$\square$

**Proof.** Suppose that  $\Sigma = \tilde{D}^{-1}\tilde{N}$  is a *stable left factorization*, then it is straightforward to see that

$$R_\Sigma: \mathcal{Y} \times \mathcal{U} \rightarrow \mathcal{Z}, \quad R(y, u) = \tilde{D}y - \tilde{N}u \quad (20)$$

is a stable kernel representation for  $\Sigma$ . Further, as  $\tilde{D}$  is invertible, the operator  $[R_\Sigma]^{-1}(\cdot, u)$  exists, and by Proposition 2.3  $\Sigma$  is well-defined for this *skr*.

Conversely, suppose that  $R_\Sigma : \mathcal{Y} \times \mathcal{U}$  is a stable kernel representation for  $\Sigma$  which is well-defined and separable in the sense of (18). The operators  $\tilde{D}$  and  $\tilde{N}$  of (19) will be stable, and to prove that this is a stable left factorization of  $\Sigma$  it only remains to show that  $\tilde{D}$  is invertible. By Proposition 2.3, the operator  $[R_\Sigma]^{-1}(\cdot, u)$  exists, that is, once  $u$  is fixed, there exists a one-to-one and onto mapping between  $z$  and  $y$ . It is straightforward to see that this implies that  $\tilde{D}$  is invertible, and the proof is complete. ■

Note that in the linear case, all *skrs* are separable, and thus equivalent to left factorizations.

Left coprime factorizations of a given system are now defined in terms of a Bezout identity. Right factorizations have been defined from a set-theoretic point of view, but as seen in Theorem 2.21 and Lemma 2.22, this is equivalent to the Bezout based definition. For left coprime factorizations, the connection between set-based and Bezout definitions is not well established, thus we take a Bezout based approach.

**Definition 3.2** Consider a system  $\Sigma : \mathcal{U} \rightarrow \mathcal{Y}$  which has a stable left factorization, (17). Then  $\Sigma = \tilde{D}^{-1}\tilde{N}$  is a left coprime factorization iff there exists a stable operator  $T : \mathcal{Z}_l \rightarrow \mathcal{Y} \times \mathcal{U}$  such that

$$\left[ \begin{array}{c} \tilde{D} \\ \tilde{N} \end{array} \right] T = Z : \mathcal{Z}_l \rightarrow \mathcal{Z}_l, \text{ unimodular} \quad (21)$$

□

**Remark 3.3** The operator  $Z$  may be taken to be the identity without loss of generality.

**Remark 3.4** Note that the operator  $T$  of (21) may be written as  $\left[ \begin{array}{c} T_y \\ T_u \end{array} \right]$ , where  $T_y : \mathcal{Z}_l \rightarrow \mathcal{Y}$ ,  $T_u : \mathcal{Z}_l \rightarrow \mathcal{U}$ . Then if  $T_y$  is invertible, (21) implies that  $T_u T_y^{-1}$  gives a right coprime factorization of some other operator.

Thus the existence of a left coprime factorization implies the existence of a right factorization for some other operator. This is exploited in [6] to show that if the plant and controller of a stable and well-posed system have left factorizations, then they will also have right factorizations. This result may also be proven for a well-posed and stable system  $\{G, K\}$  with *skr*  $R_{\{G, K\}}$ .

**Theorem 3.5**

Consider a system  $\{G, K\}$ , with *skr* (6), which is well-posed and generally stable, and that  $G$  and  $K$  are well-defined. Then there exist right coprime factorizations for  $G$  and  $K$ ,  $G = NM^{-1}$   $N : \mathcal{Z}_K \mapsto \mathcal{Y}$   $M : \mathcal{Z}_K \mapsto \mathcal{U}$   $K = UV^{-1}$   $U : \mathcal{Z}_G \mapsto \mathcal{U}$   $V : \mathcal{Z}_G \mapsto \mathcal{Y}$  which satisfy the generalized Bezout identities

$$\begin{array}{l} R_G \left[ \begin{array}{c} V \\ U \end{array} \right] = I_{\mathcal{Z}_G} \\ R_K \left[ \begin{array}{c} V \\ U \end{array} \right] = 0 \end{array} \quad \begin{array}{l} R_G \left[ \begin{array}{c} N \\ M \end{array} \right] = 0 \\ R_K \left[ \begin{array}{c} N \\ M \end{array} \right] = I_{\mathcal{Z}_K} \end{array} \quad (22)$$

□

**Proof.** Consider a feedback system  $\{G, K\}$  with *skr*  $R_{\{G, K\}}$ , which is well-posed and generally stable. By

Definition 2.7, the operator  $[R_{\{G, K\}}]^{-1}$  exists and is stable, and thus the operators  $T_G : \mathcal{Z}_G \rightarrow \mathcal{Y} \times \mathcal{U}$  and  $T_K : \mathcal{Z}_K \rightarrow \mathcal{Y} \times \mathcal{U}$  defined by

$$T_G z_G = [R_{\{G, K\}}]^{-1} \left[ \begin{array}{c} z_G \\ 0 \end{array} \right], \quad T_K z_K = [R_{\{G, K\}}]^{-1} \left[ \begin{array}{c} 0 \\ z_K \end{array} \right], \quad (23)$$

are also be stable. The stable operators

$$\begin{array}{ll} M : \mathcal{Z}_K \rightarrow \mathcal{U} & N : \mathcal{Z}_K \rightarrow \mathcal{Y} \\ V : \mathcal{Z}_G \rightarrow \mathcal{Y} & U : \mathcal{Z}_G \rightarrow \mathcal{U} \end{array}$$

are now defined by

$$\left[ \begin{array}{c} N \\ M \end{array} \right] = T_K, \quad \left[ \begin{array}{c} V \\ U \end{array} \right] = T_G. \quad (24)$$

By the definitions of  $T_G$  and  $T_K$ , the following identities hold

$$\begin{array}{ll} R_G T_G = R_G \left[ \begin{array}{c} V \\ U \end{array} \right] = I_{\mathcal{Z}_G}, & R_G T_K = R_G \left[ \begin{array}{c} N \\ M \end{array} \right] = 0, \\ R_K T_G = R_K \left[ \begin{array}{c} V \\ U \end{array} \right] = 0 & R_K T_K = R_K \left[ \begin{array}{c} N \\ M \end{array} \right] = I_{\mathcal{Z}_K} \end{array} \quad (25)$$

To prove the proposition, it only remains to prove that  $M$  and  $V$  are invertible and thus give right factorizations for  $G$  and  $K$ .

We first prove that  $M$  is invertible. Suppose that  $M$  were not injective, then there exist  $z_1, z_2 \in \mathcal{Z}$ ,  $z_1 \neq z_2$  such that  $Mz_1 = Mz_2 = u$ . By well-definedness of  $G$ , there exists a unique  $y \in \mathcal{Y}$  such that  $R_G(y, u) = 0$ . Thus,

$$[R_{\{G, K\}}]^{-1} \left( \begin{array}{c} 0 \\ z_1 \end{array} \right) = [R_{\{G, K\}}]^{-1} \left( \begin{array}{c} 0 \\ z_2 \end{array} \right) = \left( \begin{array}{c} y \\ u \end{array} \right).$$

However, by well-posedness of the system, this implies that  $z_1 = z_2$ . Thus  $M$  is injective. Given any  $u \in \mathcal{U}$ , there exists a  $y \in \mathcal{Y}$  such that  $R_G(y, u) = 0$ , and thus a  $z_K = R_K(u, y)$ . Note that for this  $z_K$ ,  $u = Mz_K$ . Thus  $M$  is surjective, and is thus invertible.

Further for all  $u \in \mathcal{U}$ ,  $y = NM^{-1}u$  satisfies  $R_G(y, u) = 0$ , and thus  $y = Gu$ , and so  $G = NM^{-1}$  is a stable factorization. As  $M$  and  $N$  also satisfy the Bezout identity (22) this is a coprime factorization.

A dual argument exploiting the well-definedness of  $K$  shows that  $V$  is also invertible, and that  $K = UV^{-1}$  is a right coprime factorization, and the proof is complete. ■

The dual result, showing that if the plant and controller of a well-posed and stable system have right factorizations, then there will exist left factorizations, holds in the linear case, where

$$\left[ \begin{array}{cc} V & N \\ U & M \end{array} \right]^{-1} = \left[ \begin{array}{cc} \tilde{M} & -\tilde{N} \\ -\tilde{U} & \tilde{V} \end{array} \right].$$

However, in the nonlinear case, the lack of a separability property, as in Theorem 3.5, means that this dual result is not available for nonlinear systems described by left coprime factorizations. However, when *skrs* are used instead of *lcfs*, dual results are immediately obtainable.

We now show that, dually to Theorem 3.5, coprime *skrs* may be derived for a plant and controller with right coprime factorizations when the closed loop is well-posed and stable.

### Theorem 3.6

Let  $\{G, K\}$  be a well-posed and stable feedback system, in the sense of Section 2., and suppose that  $G$  and  $K$  have right coprime factorizations,  $G = NM^{-1}$ ,  $K = UV^{-1}$ . Then, by Theorem 2.21,

$$\begin{bmatrix} M & -U \\ -N & V \end{bmatrix}^{-1} \quad \text{exists and is stable.}$$

Define the functions  $R_K : \mathcal{U} \times \mathcal{Y} \rightarrow \mathcal{Z}_K$  and  $R_G : \mathcal{U} \times \mathcal{Y} \rightarrow \mathcal{Z}_G$  by the following equations

$$\begin{bmatrix} R_K(-u - M(0), y + N(0)) \\ R_G(-y - V(0), u + U(0)) \end{bmatrix} = \begin{bmatrix} M & -U \\ -N & V \end{bmatrix}^{-1} \begin{pmatrix} u \\ y \end{pmatrix}. \quad (26)$$

Then  $R_G$  and  $R_K$  are well-defined, coprime stable kernel representations for  $G$  and  $K$ , respectively.  $\square$

**Proof.** Consider  $R_K(u, y) = 0$ . Then, defining  $z_2 = R_G(-y + N(0) - V(0), -u - M(0) + U(0))$ , (26) becomes

$$\begin{pmatrix} 0 \\ z_2 \end{pmatrix} = \begin{bmatrix} M & -U \\ -N & V \end{bmatrix}^{-1} \begin{pmatrix} -u + M(0) \\ y - N(0) \end{pmatrix}$$

Simple algebraic manipulations show that  $z_2 = V^{-1}y$ , and  $u = Uz_2 = Ky$ .

Thus  $R_K(u, y) = 0$  implies that  $u = Ky$ . Stability of  $\{G, K\}$  implies stability of this kernel representation of  $K$ . Simple algebraic manipulations of (26) prove that this  $skr$  is well-defined, and that further, the operator  $K_z$  induced by  $R_K(u, y) = z$  is given by

$$K_z y = K(y - N(0) + N(z)) + M(0) - M(z). \quad (27)$$

Dually, considering  $R_G(y, u) = 0$  and  $z_1 = R_K(-u - M(0) + U(0), -y + N(0) - V(0))$  shows that  $R_G$  is a well-defined stable kernel representation for  $G$ .

Coprime follows from the cases already considered. Consider that  $u = Uz_2$ ,  $y = Vz_2$ , then  $R_K(u, y) = 0$  and the following equations hold

$$\begin{aligned} z_2 &= R_G(-y + N(0) - V(0), -u - M(0) + U(0)) \\ &= R_G(-Vz_2 + N(0) - V(0), -Uz_2 - M(0) + U(0)). \end{aligned} \quad (28)$$

Note that (28) defines a bezout identity, that is it defines a stable right inverse for  $R_G$ , proving its coprimeness. A dual result proves coprimeness of  $R_K$ .  $\blacksquare$

### 4. Conclusion

In this paper we have developed the links between right coprime factorizations of nonlinear systems and stable kernel representations for nonlinear systems. Specifically, we have shown, Theorem 3.6, that if a system is stable and well-posed and has right coprime factorizations, then there exists a stable kernel description of the system. Dually, Theorem 3.5, if a system with a stable kernel representation is well posed and stable, then the kernel representations are well-defined and coprime, and there are right coprime factorizations for the plant and controller. Thus it has been demonstrated that the stable kernel representations are the dual to right coprime factorizations for nonlinear systems, and thus represent the appropriate generalization of left coprime factorizations for linear systems to the nonlinear arena.

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