L2-Gain Analysis of Nonlinear Systems and Nonlinear State Feedback H∞ Control

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Published in:
IEEE Transactions on Automatic Control

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Abstract—**Previously obtained results on L_2-gain analysis of smooth nonlinear systems are unified and extended using an approach based on Hamilton–Jacobi equations and inequalities, and their relation to invariant manifolds of an associated Hamiltonian vector field. Based upon these results a nonlinear analog is obtained of the simplest part of the recently developed state-space approach to linear H_∞ control, namely the state feedback H_∞ optimal control problem. Furthermore, the relation with H_∞ control of the linearized system is dealt with.**

I. Introduction

A RECENT breakthrough in linear control theory has been the derivation of state-space solutions to standard H_∞ optimal control problems; see [1] and the references quoted therein. A particularly nice feature of the approach taken is that it basically relies on simple state-space tools familiar from LQ and LQG theory, such as "completion of the squares" arguments, Riccati equations, and connections between frequency domain inequalities and spectral factorizations on the one hand, and Riccati equations and Hamiltonian matrices on the other hand (see, e.g., [1], [4], [35], [12]).

Now in the classical paper by Willems on least-squares optimal control [35] the relations of this machinery with the fundamental notion of *dissipativity* were being stressed, while in [36] dissipativity was not only defined for linear systems but as a general system property, generalizing the notions of passivity in (nonlinear) electrical networks, and Lyapunov and input–output stability of nonlinear (feedback) systems. Subsequently, in [23], [16] (see also [24]) the notion of dissipativity was crucially used in the stability analysis of (differentiable) nonlinear state-space systems. (Recently, the importance of these techniques for stabilization of nonlinear systems was recognized in [9].) The aim of the present paper is to continue this line of research, and to use this framework in order to develop a nonlinear analog of the most straightforward part of the current state-space theory of *linear H_∞* control, namely the solution of the state feedback H_∞ control problem (see, e.g., [21], [28], [40]). A word concerning terminology is certainly in order here, since the H_∞ norm is defined as a norm on transfer matrices and so does not directly generalize to nonlinear systems. However, when translated to the time domain, the H_∞ norm is nothing else than the L_2-induced norm (from the input time-functions to the output time-functions for initial state zero). This latter norm is also eminently suited to nonlinear systems, and in fact is commonly called the L_2-gain of the nonlinear system (see, e.g., [10], [33]). Hence, instead of using the terminology "nonlinear H_∞ optimal control" as in [6], [7], [30] or Section III of the present paper, it would be more correct (but maybe less clear) to use a phrase like "nonlinear L_2-gain optimal control."

Of course, the solution of the nonlinear state feedback H_∞ control problem constitutes only a first step in the development of a full state-space theory of nonlinear H_∞ control, which has to be concerned with the much more involved problem of dynamic *measurement* feedback (where the measurements are corrupted by disturbances). Indeed, in the original motivation for linear H_∞ design (see, e.g., [39], also [12], [19], and the references quoted therein) the assumption that only corrupted measurements are available for feedback (instead of full knowledge of the state) was essential. On the other hand, in the state-space approach to linear H_∞ control, the solution of the state feedback H_∞ problem has proved to be instrumental, and we expect this to be true in the nonlinear case as well.

A relatively new tool as compared to [35], [36], [23], [16], which will be employed throughout this paper, is the strict relation between Hamilton–Jacobi equations (the nonlinear analog of Riccati equations) and *invariant manifolds* of Hamiltonian vector fields, although in the linear case this relation reduces to the well-known connection between solutions of Riccati equations and existence of invariant subspaces of Hamiltonian matrices. In the context of the nonlinear infinite horizon optimal control problem, this relation has already been recognized in [22], [5] in deducing the existence of a locally well-behaved stable invariant manifold for the Hamiltonian vector field corresponding to the optimal Hamiltonian from the existence of the stabilizing solution to the Riccati equation corresponding to the linearized problem. However, we feel that the full (geometrical) significance of this relation has not been really appreciated also in nonlinear optimal control (see also [32]). Since the required mathematics for formulating this relation (stable and unstable invariant manifolds, Lagrangian submanifolds, generating functions) are not very well known we have devoted an Appendix to this material.

Another major theme in this paper will be the relation of the L_2-gain of a nonlinear system with the L_2-gain (or, H_∞ norm) of its linearized system. In fact, we will show that if, roughly speaking, the H_∞ control problem for the linearized system is solvable then *locally* one obtains a solution to the

Manuscript received April 24, 1991; revised October 23, 1991. Paper recommended by Associate Editor, J. Hammer.

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IEEE Log Number 9108278.

0018-9286/92$03.00

nonlinear H_∞ control problem. Some first results of the approach taken in the present paper have been already reported in [30]; see also [31].

Notation: The notation used is fairly standard. We denote by $z^T z$ or $\|z\|^2$ the square norm of a vector $z \in \mathbb{R}^k$. The notation $L_2(0, T)$ will be also used for *vector-valued* functions, i.e., we say that $z:(0,T) \to \mathbb{R}^k$ is in $L_2(0,T)$ if $\int_0^T \|z(t)\|^2 \, dt < \infty$. For a differentiable function $V: \mathbb{R}^n \to \mathbb{R}$ we denote by $(\partial V/\partial x)(x)$ the row-vector of partial derivatives, and by $(\partial^T V/\partial x)(x)$ the corresponding column-vector.

Furthermore, the solution at time t_2 of the system $\dot{x} = f(x) + g(x)u$ with initial condition $x(t_1) = x_1$ and input $u(\cdot):(t_1, t_2) \mathbb{R}^m$ will be denoted by $x(t_2) = \varphi(t_2, t_1, x_1, u)$. By $\mathbb{C}^-$ and $\mathbb{C}^+$ we denote the open left-half, respectively, open right-half, of the complex plane. Furthermore, $\sigma(A)$ denotes the set of eigenvalues of a square matrix A. $\mathbb{R}^+$ is the set of nonnegative reals.

Some additional notation concerning cotangent bundles, differential one-forms, and Hamiltonian vector fields will be given in the Appendix.

II. THE L_2-GAIN OF NONLINEAR SYSTEMS

In this paper we consider smooth, i.e., C^∞, nonlinear systems of the form (see, e.g., [18], [27])

$$\dot{x} = f(x) + \sum_{j=1}^{m} g_j(x) u_j, \qquad u = (u_1, \cdots, u_m) \in \mathbb{R}^m,$$
$$y_j = h_j(x), \quad j = 1, \cdots, p, \qquad y = (y_1, \cdots, y_p) \in \mathbb{R}^p \tag{1}$$

where $x = (x_1, \cdots, x_n)$ are local coordinates for a smooth state-space manifold denoted by M. Throughout, we assume the existence of an equilibrium $x_0 \in M$, i.e., $f(x_0) = 0$, and without loss of generality we assume that $h_j(x_0) = 0$, $j = 1, \cdots, p$. For simplicity of notation throughout we will abbreviate (1) as

$$\dot{x} = f(x) + g(x)u, \quad u \in \mathbb{R}^m, \quad y \in \mathbb{R}^p, \quad x \in M,$$
$$y = h(x), \quad f(x_0) = 0, \quad h(x_0) = 0 \tag{2}$$

where $g(x)$ is an $n \times m$ matrix with jth column given as $g_j(x)$. Following, e.g., [10], [17], [33] we will give the following definition of finite L_2-gain.

Definition 1: Let $\gamma \geq 0$. System (2) is said to have L_2-gain less than or equal to γ if

$$\int_0^T \|y(t)\|^2 \, dt \leq \gamma^2 \int_0^T \|u(t)\|^2 \, dt \tag{3}$$

for all $T \geq 0$ and all $u \in L_2(0, T)$, with $y(t) = h(\varphi(t, 0, x_0, u))$ denoting the output of (2) resulting from u for initial state $x(0) = x_0$. (Note that in particular this requires that $y(t)$ exists for all $t \in (0, T)$.) The system has L_2-gain $< \gamma$ if there exists some $0 \leq \tilde{\gamma} < \gamma$ such that (3) holds for $\tilde{\gamma}$.

Remark: Definition 1 is a special case of the general definition of dissipativity as given in [36], [35], [16], where instead of the *supply rate* $w(y, u) = \gamma^2 \|u\|^2 - \|y\|^2$ as used in Definition 1 arbitrary supply rates $w(y, u)$ are being considered.

Notice that condition (3) can be equivalently expressed as

$$\inf_{\substack{u \in L_2(0,T),\, T \geq 0 \\ x(0) = x_0}} \int_0^T (\gamma^2 \|u\|^2 - \|y\|^2) \, dt \geq 0. \tag{4}$$

Furthermore, it is easily seen (see, e.g., [35, remark 2 after theorem 2] that (4) is equivalent to

$$\lim_{T \to \infty} \inf_{\substack{u \in L_2(0,T) \\ x(0) = x_0}} \int_0^T (\gamma^2 \|u\|^2 - \|y\|^2) \, dt \geq 0. \tag{5}$$

We start with the following theorem, which is basically a restatement of fundamental results obtained in [36], [23], [16].

Theorem 2: Consider the system (2) and let $\gamma > 0$. We have the following list of implications $(A) \to (B) \leftrightarrow (C) \to (D)$.

(A): There exists a smooth solution $V: M \to \mathbb{R}^+$ (i.e., $V(x) \geq 0$ for all $x \in M$) of the Hamilton–Jacobi equation

$$\frac{\partial V}{\partial x}(x) f(x) + \frac{1}{2} \frac{1}{\gamma^2} \frac{\partial V}{\partial x}(x) g(x) g^T(x) \cdot \frac{\partial^T V}{\partial x}(x) + \frac{1}{2} h^T(x) h(x) = 0, \qquad V(x_0) = 0. \tag{6}$$

(B): There exists a smooth solution $V \geq 0$ of the Hamilton–Jacobi inequality

$$\frac{\partial V}{\partial x}(x) f(x) + \frac{1}{2} \frac{1}{\gamma^2} \frac{\partial V}{\partial x}(x) g(x) g^T(x) \cdot \frac{\partial^T V}{\partial x}(x) + \frac{1}{2} h^T(x) h(x) \leq 0, \qquad V(x_0) = 0. \tag{7}$$

(C): There exists a smooth solution $V \geq 0$ of the dissipation inequality

$$\frac{\partial V}{\partial x}(x) f(x) + \frac{\partial V}{\partial x}(x) g(x) u \leq \frac{1}{2} \gamma^2 \|u\|^2 - \frac{1}{2} \|y\|^2, \qquad V(x_0) = 0 \tag{8}$$

for all $u \in \mathbb{R}^m$, with $y = h(x)$.

(D): The system has L_2-gain less than or equal to γ.

In fact, any solution of (6) is a solution of (7), any solution of (7) is a solution of (8), and any solution of (8) is a solution of (7).

Conversely, suppose (D) holds, and assume the system is *reachable* from x_0 [i.e., for any $\bar{x} \in M$ there exists a $\bar{t} \geq 0$ and input u such that $\bar{x} = \varphi(\bar{t}, 0, x_0, u)$]. Then the functions

$$V_a(x) = -\inf_{\substack{u\in L_2(0,T),\, T\geq 0\\ x(0)=x}} \frac{1}{2}\int_0^T (\gamma^2\|u\|^2 - \|y\|^2)\, dt \quad (9)$$

$$V_r(x) = \inf_{\substack{u\in L_2(t_{-1},0),\\ t_{-1}\leq 0,\\ x=\varphi(0,t_{-1},x_0,u)}} \frac{1}{2}\int_{t_{-1}}^0 (\gamma^2\|u\|^2 - \|y\|^2)\, dt \quad (10)$$

are well defined for all $x \in M$ (i.e., are finite). Moreover, V_a and V_r satisfy

$$0 \leq V_a \leq V_r, \qquad V_a(x_0) = V_r(x_0) = 0. \quad (11)$$

Furthermore, any solution V of (6), (7), or (8) satisfies

$$0 \leq V_a \leq V \leq V_r. \quad (12)$$

The functions V_a and V_r satisfy the integral version of (8), i.e., the *integral dissipation inequality*

$$V(x(t_1)) - V(x(t_0)) \leq \frac{1}{2}\int_{t_0}^{t_1} (\gamma^2\|u(t)\|^2 - \|y(t)\|^2)\, dt, \qquad V(x_0) = 0 \quad (13)$$

for all $t_1 \geq t_0$ and all $u \in L_2(t_0, t_1)$, where $x(t_1) = \varphi(t_1, t_0, x(t_0), u)$.

If we assume that V_a and/or V_r are *smooth*, then V_a and/or V_r satisfy the Hamilton–Jacobi equation (6), and thus (A) holds.

Proof: $(A) \to (B)$ is trivial. For $(B) \leftrightarrow (C)$ let V satisfy (7). Then by "completing the squares" (see [30])

$$\frac{\partial V}{\partial x}f + \frac{\partial V}{\partial x}gu = -\frac{1}{2}\gamma^2\left\| u - \frac{1}{\gamma^2}g^T\frac{\partial^T V}{\partial x}\right\|^2 + \frac{\partial V}{\partial x}f + \frac{1}{2}\frac{1}{\gamma^2}\frac{\partial V}{\partial x}gg^T\frac{\partial^T V}{\partial x} + \frac{1}{2}\gamma^2\|u\|^2$$

and thus upon substituting (7)

$$\frac{\partial V}{\partial x}f + \frac{\partial V}{\partial x}gu \leq \frac{1}{2}\gamma^2\|u\|^2 - \frac{1}{2}\|y\|^2 - \frac{1}{2}\gamma^2\left\| u - \frac{1}{\gamma^2}g^T\frac{\partial^T V}{\partial x}\right\|^2$$

from which (8) follows. Conversely, let V satisfy (8). Then again by completing the squares

$$\frac{\partial V}{\partial x}f + \frac{1}{2}\|y\|^2 \leq \frac{1}{2}\gamma^2\left\| u - \frac{1}{\gamma^2}g^T\frac{\partial^T V}{\partial x}\right\|^2 - \frac{1}{2}\frac{1}{\gamma^2}\frac{\partial V}{\partial x}gg^T\frac{\partial^T V}{\partial x}$$

for all u, and thus also for $u = (1/\gamma^2)g^T(\partial^T V/\partial x)$, yielding (7).

$(C) \to (D)$ follows by integration of (8) (see [36])

$$V(x(T)) - V(x(0)) \leq \frac{1}{2}\gamma^2\int_0^T \|u(t)\|^2\, dt - \frac{1}{2}\int_0^T \|y(t)\|^2\, dt \quad (14)$$

where $x(T) = \varphi(t, 0, x(0), u)$ for any $u \in L_2(0, T)$. Taking $x(0) = x_0$ and using $V(x_0) = 0$ and $V \geq 0$ immediately yields (3). The well-definedness of V_a and V_r, the inequalities (11), (12), as well as the fact that V_a and V_r satisfy (13) have been shown in [36]. Finally, if V_a and/or V_r are smooth, then, (see, e.g., [23]) it follows from the theory of dynamic programming that V_a and/or V_r satisfy (6).

Remark 1: The functions V_a and V_r are called the *available storage*, respectively, *required supply* in [36].

Remark 2: If (C) holds, then (cf. [36]) V_a is well defined even *without* the reachability assumption.

Remark 3: For *linear* systems $\dot{x} = Ax + Bu$, $y = Cx$ the Hamilton–Jacobi equation (6) comes down to an algebraic Riccati equation, the Hamilton–Jacobi inequality (7) reduces to a quadratic matrix inequality [35], while the dissipation inequality (8) yields the linear matrix inequality of the bounded real or Kalman–Yacubovich–Popov lemma [1], [35], [37].

Remark 4: The dissipation inequality (8) may also be *factorized* as in [16], [23].

We collect the following facts from [16], [23], [35].

Proposition 3: Suppose x_0 is a globally asymptotically stable equilibrium of the drift vector field f in (2). Then any solution V of (6), (7), or (8) automatically satisfies $V \geq 0$. Furthermore, the available storage has the following modified expression:

$$V_a(x) = -\inf_{\substack{u\in L_2(0,T),\, T\geq 0\\ x(0)=x,\\ \lim_{T\to\infty} x(T)=x_0}} \frac{1}{2}\int_0^T (\gamma^2\|u\|^2 - \|y\|^2)\, dt. \quad (15)$$

Proof: The first part immediately follows from the inequality $(\partial V/\partial x)(x)f(x) \leq 0$ (see [23], [16]). Expression (15) follows from Remark 2 after [35, theorem 2]. □

Remark: Note that by Milnor's theorem global asymptotic stability of f implies that M is diffeomorphic to $\mathbb{R}^n$.

The "converse" of Proposition 3, finite gain implying (global) asymptotic stability, was being pursued in [23], [16], [17], [34], leading to the following definition and theorem.

Definition 4 [23], [16]: The system (2) is called *zero-state observable* if for any trajectory such that $u(t) \equiv 0$, $y(t) \equiv 0$ implies $x(t) \equiv x_0$, i.e., for all $x \in M$

$$h(\varphi(t, 0, x, 0)) = 0, \quad t \geq 0 \Rightarrow \varphi(t, 0, x, 0) = x_0, \quad t \geq 0. \quad (16)$$

(N.B. There is some confusion of terminology in the literature; we follow [9].)

Theorem 5 [16], [23]: Assume (2) is zero-state observable. Suppose there exists a smooth solution $V \geq 0$ to either

(6), (7), or (8). Then $V(x) > 0$, $x \neq x_0$, and the free system $\dot{x} = f(x)$ is locally asymptotically stable. Furthermore, assume that V is proper (i.e., for each $c > 0$ the set $\{x \in M \mid 0 \leq V(x) \leq c\}$ is compact), then $\dot{x} = f(x)$ is globally asymptotically stable.

Proof: (Sketch; see [16, lemma 1 and theorem 2] for details.) Let $V \geq 0$ satisfy (8). By zero-state observability we have $V_a(x) > 0$ and thus $V(x) > 0$ for $x \neq x_0$. Then for $u(t) \equiv 0$

$$\frac{\partial V}{\partial x} f \leq -\frac{1}{2} h^T h$$

and (global) asymptotic stability follows by LaSalle's invariance principle. □

Remark: Properness of V can be assured by requiring a stronger form of observability (see [16], [34]). For applications of Theorem 5 to the stability of interconnected systems we refer to, e.g., [25], [29], and the references quoted therein.

Theorem 2 is somewhat unsatisfactory in the sense that if we want to conclude (A) from (D), then we have to check smoothness of the functions V_a and V_r, which are usually not readily given (also verifying reachability of the system (2) is generally a difficult task). We will now show how the geometric approach as proposed in [30] yields at least the *local* existence of a smooth solution $V \geq 0$ of (6) if the L_2-gain is less than γ. First we have to relate the L_2-gain of the system (2) to the L_2-gain of the system *linearized* at x_0, i.e.,

$$\begin{aligned} \dot{\bar{x}} &= F\bar{x} + G\bar{u}, \qquad \bar{u} \in \mathbb{R}^m, \quad \bar{x} \in \mathbb{R}^n, \\ \bar{y} &= H\bar{x}, \qquad \bar{y} \in \mathbb{R}^p \end{aligned} \tag{17}$$

where

$$F = \frac{\partial f}{\partial x}(x_0), \quad G = g(x_0), \quad H = \frac{\partial h}{\partial x}(x_0).$$

Proposition 6: Suppose the system (2) has L_2-gain $\leq (<)\gamma$, then the linearized system (17) has L_2-gain $\leq (<)\gamma$.

Proof: Consider an input $\bar{u}(\cdot)$ to (17), and define $u(t, \epsilon) = \epsilon \bar{u}(t)$, ϵ small, as a one-parameter family of inputs to (2). Denote the resulting family of outputs of (2) for $x(0) = x_0$ by $y(t, \epsilon)$, and define $\bar{y}(t) = (\partial/\partial\epsilon) y(t, \epsilon)|_{\epsilon=0}$. Suppose (2) has L_2-gain $\leq \gamma$, then

$$\int_0^T \| y(t,\epsilon)\|^2 \, dt \leq \gamma^2 \int_0^T \| u(t,\epsilon)\|^2 \, dt, \qquad \forall \epsilon \text{ small}. \tag{18}$$

Differentiating (18) twice to ϵ, and setting $\epsilon = 0$, yields

$$\int_0^T \| \bar{y}(t)\|^2 \, dt \leq \gamma^2 \int_0^T \| \bar{u}(t)\|^2 \, dt \tag{19}$$

and thus (17) has L_2-gain $\leq \gamma$. If (2) has L_2-gain $< \gamma$, then (18), and thus (19), hold for some $\tilde{\gamma} < \gamma$, and we conclude that also (17) has L_2-gain $< \gamma$. □

For the linearized system (17) the following result is well known (see [35], [1], [11], [21], [14], and for the uncontrollable case [28]).

Theorem 7: Consider the linearized system (17), and assume F is asymptotically stable. The L_2-gain of (17) is less than or equal to γ if and only if there exists a solution $P \geq 0$ of the algebraic Riccati equation

$$F^T P + PF + \frac{1}{\gamma^2} PGG^T P + H^T H = 0. \tag{20}$$

Furthermore, the L_2-gain of (17) is less than γ if and only if there exists a solution $P \geq 0$ of (20) satisfying additionally

$$\sigma\left(F + \frac{1}{\gamma^2} GG^T P\right) \subset \mathbb{C}^-. \tag{21}$$

Remark: Notice that (20) is the Hamilton–Jacobi equation (6) for the linear system (17) and a quadratic function $V(\bar{x}) = \frac{1}{2}\bar{x}^T P \bar{x}$.

Now the basic observation made in [30] is the following; see the Appendix for further details. Consider the nonlinear system (2), and let $\gamma > 0$. Define the following *Hamiltonian function* on T^*M, with natural coordinates $(x, p) = (x_1, \cdots, x_n, p_1, \cdots, p_n)$ (see Appendix I)

$$H_\gamma(x, p) = p^T f(x) + \frac{1}{2}\frac{1}{\gamma^2} p^T g(x) g^T(x) p + \frac{1}{2} h^T(x) h(x). \tag{22}$$

The corresponding Hamiltonian vector field X_{H_γ} on T^*M (see Appendix III), in local coordinates (x, p) given as

$$\begin{aligned} \dot{x}_i &= \frac{\partial H_\gamma}{\partial p_i}(x, p) \\ \dot{p}_i &= -\frac{\partial H_\gamma}{\partial x_i}(x, p) \end{aligned} \qquad i = 1, \cdots, n, \tag{23}$$

has equilibrium $(x_0, 0)$, and the linearization of X_{H_γ} in $(x_0, 0)$ is given by the linear Hamiltonian differential equation

$$\begin{bmatrix} \dot{\bar{x}} \\ \dot{\bar{p}} \end{bmatrix} = \begin{bmatrix} F & \frac{1}{\gamma^2} GG^T \\ -H^T H & -F^T \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{p} \end{bmatrix} \tag{24}$$

with F, G, and H as in (17). Now it is well known, and can be readily checked (see, e.g., [12]), that $P = P^T$ is a solution of (20) if and only if

$$\begin{bmatrix} F & \frac{1}{\gamma^2} GG^T \\ -H^T H & -F^T \end{bmatrix} \begin{bmatrix} I \\ P \end{bmatrix} = \begin{bmatrix} I \\ P \end{bmatrix} \left(F + \frac{1}{\gamma^2} GG^T P\right)$$

and thus $P = P^T$ is a solution of (20), (21) if and only if

$$\operatorname{span} \begin{bmatrix} I \\ P \end{bmatrix} = \text{stable eigenspace of} \begin{bmatrix} F & \frac{1}{\gamma^2} GG^T \\ -H^T H & -F^T \end{bmatrix} \tag{25}$$

and in particular the Hamiltonian matrix in (25) does not have imaginary eigenvalues (see, e.g., [12]). Since (24) is the linearization of (23), we conclude from Appendix III (Proposition A.7) that the existence of a solution $P = P^T$ of (20) *implies* that the *stable invariant manifold* N^- of X_{H_γ} through $(x_0, 0)$ is n-dimensional and is *tangent* at $(x_0, 0)$ to span $\begin{bmatrix} I \\ P \end{bmatrix}$. Furthermore, by Appendix III it follows that locally around x_0 the manifold N^- is given as

$$N^- = \left\{ \left(x, p = \frac{\partial^T V}{\partial x}(x) \right) \middle| x \text{ around } x_o \right\} \tag{26}$$

where V is a (local) solution of the Hamilton–Jacobi equation (6), with $(\partial^2 V/\partial x^2)(0) = P$ (cf. Proposition A.8). We obtain the following theorem, which extends [30, lemma 5].

Theorem 8: Consider the linearized system (17), and let $\gamma > 0$. Assume that F is asymptotically stable. Suppose the L_2-gain of (17) is less than γ, then there exists locally about x_0 a smooth solution $V > 0$ of the Hamilton–Jacobi equation (6). Furthermore, there exists a solution $V \geq 0$ such that the vector field $f + (1/\gamma^2) g g^T (\partial^T V/\partial x)$ is locally exponentially stable.

Proof: By Theorem 7 there exists a solution $P \geq 0$ of (20), (21). By Proposition A.7 this yields the local existence of a smooth solution V to (6) such that $f + (1/\gamma^2) g g^T (\partial^T V/\partial x)$ is locally exponentially stable. Since F is asymptotically stable the vector field $\dot{x} = f(x)$ is locally asymptotically stable, and thus by Proposition 3 we have locally $V \geq 0$. □

We call the vector field X_{H_γ} *hyperbolic* in $(x_0, 0)$ if its linearization at $(x_0, 0)$ does not have purely imaginary eigenvalues. From Proposition 6 and Theorem 8 we immediately deduce the following.

Corollary 9: Consider (2) and its linearization (17). Let $\gamma > 0$. Suppose (2) has L_2-gain $< \gamma$ and F is asymptotically stable. Then there exists a neighborhood W of x_0 and a smooth function $V: W \to \mathbb{R}^+$, satisfying (6) on W. Also $f + (1/\gamma^2) g g^T (\partial^T V/\partial x)$ is locally exponentially stable on W, and X_{H_γ} is hyperbolic in $(x_0, 0)$.

Remark 1: If the linear system (17) is detectable, then the asymptotic stability of F is *implied* by the fact that (17) has L_2-gain $< \gamma$ (compare to Theorem 5).

Remark 2: If (2) has L_2-gain $\leq \gamma$ (instead of $< \gamma$) then the vector field X_{H_γ} need not be hyperbolic, i.e., the Hamiltonian matrix in (24) may have purely imaginary eigenvalues. (In fact (see [35]) this will be the case if and only if the L_2-gain of (17) *equals* γ.) If this happens, then there exists a nontrivial *center manifold* of X_{H_γ} which may, or may not, yield a solution of the Hamilton–Jacobi equation.

We have the following converses to Corollary 9.

Theorem 10: Consider the system (2) and its linearization (17). Assume F is asymptotically stable. Suppose that X_{H_γ} is hyperbolic in $(x_0, 0)$. Then there exists a neighborhood W of x_0 and a smooth function $V^- \geq 0$ on W satisfying (6), and such that $f + (1/\gamma^2) g g^T (\partial^T V^-/\partial x)$ is locally exponentially stable on W. Furthermore,

$$\int_0^T \| y(t) \|^2 \, dt \leq \gamma^2 \int_0^T \| u(t) \|^2 \, dt \tag{27}$$

for all $T \geq 0$ and all $u \in L_2(0, T)$ such that the state-space trajectories $x(t) = \varphi(t, 0, x_0, u)$, $t \in (0, T)$ do not leave W. (In other words, the system *restricted* to W has L_2-gain $\leq \gamma$.) Furthermore, if additionally (F, G) is controllable, then there exists a neighborhood $\tilde{W}$ of x_0 and a smooth solution $V^+ \geq 0$ of (6) on $\tilde{W}$ such that $-(f + (1/\gamma^2) g g^T (\partial^T V^+/\partial x))$ is locally exponentially stable on $\tilde{W}$, and (27) holds with W replaced by $\tilde{W}$.

Proof: By hyperbolicity it follows that

$$\mathrm{Ham}_\gamma = \begin{bmatrix} F & \dfrac{1}{\gamma^2} G G^T \\ -H^T H & -F^T \end{bmatrix} \tag{28}$$

does not have imaginary eigenvalues. It is well known (see, e.g., [11, lemma 2]) that, since F is asymptotically stable, this implies the existence of a $P = P^T \geq 0$ to (20), (21), and thus the L_2-gain of (17) is less than γ. Then the local existence of a smooth $V^- \geq 0$ satisfying (6) such that $f + \dfrac{1}{\gamma^2} g g^T (\partial^T V^-/\partial x)$ is locally exponentially stable follows from Theorem 8 (see also Appendix III). The inequality (27) follows by the same arguments as used in the implication $(C) \to (D)$ in the proof of Theorem 2, cf. (14). Finally, if (F, G) is controllable, then it is easily seen from, e.g., [11, lemma 2], that, since Ham_γ does not have imaginary eigenvalues, there exists a solution $P = P^T$ to (20) satisfying $\sigma(F + \dfrac{1}{\gamma^2} G G^T P) \subset C^+$. By the same arguments as used in the proof of Theorem 8, but now with regard to the *unstable* invariant manifold, the final statements of Theorem 10 follow. □

Corollary 11: Suppose X_{H_γ} is hyperbolic and F is asymptotically stable. Let W and $\tilde{W}$ be the neighborhoods of x_0 as constructed in Theorem 10. Then the stable invariant manifold N^- of X_{H_γ} is such that $\pi: N^- \cap T^*W \to W$ is a diffeomorphism (with $\pi: T^*W \to W$ denoting the canonical projection $(x, p) \mapsto x$; see Appendix I). If, additionally, (F, G) is controllable, then the unstable invariant manifold N^+ of X_{H_γ} is such that $\pi: N^+ \cap T^*\tilde{W} \to \tilde{W}$ is a diffeomorphism.

Proof: $N^- \cap T^*W = \{(x, (\partial V^-/\partial x)(x)) \mid x \in W\}$ with V^- as obtained in Theorem 10. Analogously for N^+. □

We have the following global version of Theorem 10.

Proposition 12: Consider the system (2). Let $\gamma > 0$. Assume f is globally asymptotically stable. Suppose X_{H_γ} is hyperbolic and its stable invariant manifold N^- is such that

$$\pi: N^- \subset T^*M \to M \tag{29}$$

is a diffeomorphism. Then there exists a global smooth solution $V^- \geq 0$ to (6) (and thus the system has gain $\leq \gamma$ by Theorem 2). The same conclusion follows (with V^- replaced by V^+) if (29) holds for the unstable invariant manifold N^+.

Proof: By Proposition A.5 N^- is a Lagrangian submanifold of T^*M. Since $\pi: N^- \to M$ is a diffeomorphism it follows that N^- is the graph of a closed one-form σ^- on M. By global asymptotic stability of f we have $M \simeq \mathbb{R}^n$, and thus by Poincare's lemma there exists a smooth function $V^- : M \to \mathbb{R}$ such that $\sigma^- = dV^-$. It follows by Proposition A.6 that V^- satisfies the Hamilton–Jacobi equation (6). Finally, by Proposition 3 $V^- \geq 0$. The same holds for N^+, leading to a global solution $V^+ \geq 0$ of (6). □

Actually, the functions V^-, respectively, V^+, corresponding to the stable, respectively, unstable, invariant manifold of X_{H_γ} have a very clear interpretation, as shown by the following theorem.

Theorem 13: Take the same assumptions as in Proposition 12. Then the function V_a [cf. (9)] is well defined and actually $V_a = V^-$. Furthermore, assume additionally that the function V_r [cf. (10)] is well-defined, then $V_r = V^+$. Hence, the stable and unstable invariant manifolds of X_{H_γ} are

$$N^- = \left\{ \left(x, \frac{\partial^T V_a}{\partial x}(x) \right) \middle| x \in M \right\}$$

$$N^+ = \left\{ \left(x, \frac{\partial^T V_r}{\partial x}(x) \right) \middle| x \in M \right\}. \tag{30}$$

In particular, V_a and V_r are smooth.

Remark 1: In the linear case, the equalities $V_a = V^-$, $V_r = V^+$, were established in [35].

Remark 2: Recall that V_r is well defined if additionally (2) is reachable from x_0. Note that Theorem 13 makes the slight asymmetry in the definition of V_a and V_r clear.

Proof: The theorem follows from Pontryagin's minimum principle by showing that

$$V^-(x) = -\min_{\substack{u \in L_2 \\ x(0)=x \\ x(\infty)=x_0}} \frac{1}{2}\int_0^\infty \left(\gamma^2 \|u(t)\|^2 - \|y(t)\|^2\right) dt$$

$$V^+(x) = \min_{\substack{u \in L_2 \\ x(-\infty)=x_0 \\ x(0)=x}} \frac{1}{2}\int_{-\infty}^0 \left(\gamma^2 \|u(t)\|^2 - \|y(t)\|^2\right) dt \tag{31}$$

However, a completely elementary proof can be given as follows. We have by (6)

$$\frac{\partial V^-}{\partial x}\left(f + \frac{1}{\gamma^2} g g^T \frac{\partial^T V^-}{\partial x}\right) = -\frac{1}{2}h^T h + \frac{1}{2}\frac{1}{\gamma^2}\frac{\partial V^-}{\partial x} g g^T \frac{\partial^T V^-}{\partial x}.$$

It follows that for any $t_1 \geq t_0$

$$V^-(x(t_1)) - V^-(x(t_0)) = \int_{t_0}^{t_1}\left(-\frac{1}{2}h^T h + \frac{1}{2}\frac{1}{\gamma^2}\frac{\partial V^-}{\partial x} g g^T \frac{\partial^T V^-}{\partial x}\right) dt \tag{32}$$

where $x(t_1) = \varphi(t_1, t_0, x(t_0), u(t) = (1/\gamma^2) g^T(x(t)) (\partial^T V/\partial x)(x(t))))$, i.e., $x(t_1)$ is the solution at $t = t_1$ of

$$\dot{x} = f(x) + \frac{1}{\gamma^2} g(x) g^T(x) \frac{\partial^T V^-}{\partial x}(x) \tag{33}$$

with initial condition $x(t_0)$, and where the right-hand side of (32) is evaluated along the solution of (33) for $t \in (t_0, t_1)$. Since V^- satisfies (6) it follows by Theorem 2 and Remark 2 after Theorem 2 that V_a is well defined and satisfies the integral dissipation inequality (14) for every u. Thus, by taking the feedback control $u(t) = (1/\gamma^2) g^T(x(t))(\partial^T V/\partial x)(x(t))$, $x(t)$ being the solution of (33), we obtain (replacing 0 and T in (14) by t_0, respectively, t_1)

$$V_a(x(t_1)) - V_a(x(t_0)) \leq \int_{t_0}^{t_1}\left(\frac{1}{2}\gamma^2 \frac{1}{\gamma^4}\frac{\partial V^-}{\partial x} g g^T \frac{\partial^T V^-}{\partial x} - \frac{1}{2}h^T h\right) dt. \tag{34}$$

Subtracting (32) from (34) we obtain

$$(V_a - V^-)(x(t_1)) - (V_a - V^-)(x(t_0)) \leq 0. \tag{35}$$

Letting $t_1 \to \infty$, and using global asymptotic stability of $f + (1/\gamma^2) g g^T (\partial^T V^-/\partial x)$ (since V^- corresponds to the stable invariant manifold) we obtain

$$V_a(\bar{x}) \geq V^-(\bar{x}) \tag{36}$$

for every point $\bar{x} = x(t_0) \in M$. Now by global asymptotic stability of f, we have (Proposition 3) $V^- \geq 0$, and thus by Theorem 2 we obtain $V_a \leq V^-$. Combining this with (36) leads to the desired equality $V_a = V^-$. The equality $V_r = V^+$ follows similarly by noting that for any $t_1 \geq t_0$,

$$(V_r - V^+)(x(t_1)) - (V_r - V^+)(x(t_0)) \leq 0$$

where $x(t_1)$ is now the solution at $t = t_1$ of

$$\dot{x} = f(x) + \frac{1}{\gamma^2} g(x) g^T(x) \frac{\partial^T V^+}{\partial x}(x) \tag{37}$$

with initial condition $x(t_0)$, and letting $t_0 \to -\infty$. □

Finally, we have the following proposition (compare to the linear statement in [35]).

Proposition 14: Take the same assumptions as in Theorem 13. Then $V_a = V^-$ and $V_r = V^+$ satisfy

$$V_r(x) > V_a(x), \qquad \text{for all } x \neq x_0. \tag{38}$$

Proof: Since V^- and V^+ satisfy the Hamilton–Jacobi equation (6) it follows that $P^- := (\partial^2 V^-/\partial x^2)(x_0)$, $P^+ := (\partial^2 V^+/\partial x^2)(x_0)$ satisfy the algebraic Riccati equation (20); see Proposition A.8. Furthermore, $\sigma(F + (1/\gamma^2) G G^T P^-) \subset \mathbb{C}^-$ and $\sigma(F + (1/\gamma^2) G G^T P^+) \subset \mathbb{C}^+$. It follows from $V_r \geq V_a$ that $P^+ \geq P^-$. However, by hyperbolicity P^- (P^+) corresponds to the stable (unstable) eigenspace of (24). Since the intersection of the stable and unstable eigenspace is just the origin we have $P^+ > P^-$. Hence, near x_0 we have $V^+(x) > V^-(x)$, $x \neq x_0$. Furthermore, by using (6)

$$\frac{\partial(V^+ - V^-)}{\partial x}\left[f + \frac{1}{\gamma^2} g g^T \frac{\partial^T V^-}{\partial x}\right] = -\frac{1}{2}\frac{1}{\gamma^2}\left\|\frac{\partial(V^+ - V^-)}{\partial x} g\right\|^2 \tag{39}$$

and thus $V^+ - V^-$ is nonincreasing along solutions of the globally asymptotic stable vector field $f + (1/\gamma^2)gg^T(\partial^T V^-/\partial x)$. It follows that $V^+(x) > V^-(x)$, for *all* $x \neq x_0$. □

Remark 1: By using (6) we also have

$$\frac{\partial(V^+ - V^-)}{\partial x}\left[f + \frac{1}{\gamma^2}gg^T\frac{\partial^T V^+}{\partial x}\right] = \frac{1}{2}\frac{1}{\gamma^2}\left\|\frac{\partial(V^+ - V^-)}{\partial x}g\right\|^2. \quad (40)$$

Comparing (40) to (39) we obtain the following nonlinear analog of [35, lemma 8]:

$$\frac{\partial(V^+ - V^-)}{\partial x}\left[f + \frac{1}{\gamma^2}gg^T\frac{\partial^T V^+}{\partial x}\right] + \frac{\partial(V^+ - V^-)}{\partial x}\left[f + \frac{1}{\gamma^2}gg^T\frac{\partial^T V^-}{\partial x}\right] = 0. \quad (41)$$

Remark 2: In analogy with the linear case we can expect that violation of the strict inequality (38) corresponds to nonhyperbolicity of X_{H_γ}; see also Remark 2 after Corollary 9.

III. Nonlinear State Feedback H_∞ Optimal Control

Let us now consider a smooth nonlinear system (2), which additionally is affected by (unknown) disturbances d in the following way:

$$\dot{x} = f(x) + g(x)u + k(x)d, \quad u \in \mathbb{R}^m, \quad d \in \mathbb{R}^q, \quad y \in \mathbb{R}^p, \quad x \in M,$$
$$y = h(x), \quad f(x_0) = 0, \quad h(x_0) = 0 \quad (42)$$

with $k(x)$ denoting an $n \times q$ matrix with entries depending smoothly on x. Now for any smooth state feedback

$$u = l(x), \quad l(x_0) = 0 \quad (43)$$

we can consider the closed-loop system (42), (43) and consider its L_2-gain from the disturbances d to the block vector of outputs $y = h(x)$ and inputs $u = l(x)$, i.e., the L_2-gain of

$$d \overset{\text{closed-loop system}}{\rightarrow} \begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} h(x) \\ l(x) \end{bmatrix}. \quad (44)$$

In analogy with the linear state-space H_∞ theory (see, e.g., [21], [28], [40], [11]) we will define the (standard) nonlinear *state feedback* H_∞ optimal control problem as follows.

Definition 15—Nonlinear State Feedback H_∞ Optimal Control Problem: Find, if existing, the smallest value $\gamma^* \geq 0$ such that for any $\gamma > \gamma^*$ there exists a state feedback (43) such that the L_2-gain from d to $\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right]$ is less than or equal to γ.

This definition is somewhat different from the definition used in linear H_∞ control where it is also required that the closed-loop system is *asymptotically stable*. Certainly also in the nonlinear case we would like the closed-loop system considered in Definition 15 to be asymptotically stable in some sense; however, as in Theorem 5, often asymptotic stability will be *implied* by the finite gain property of the closed-loop system (see Corollary 17), and we will find it easier to consider first the H_∞ control problem as formulated in Definition 15 without any *a priori* conditions on closed-loop stability.

We start with the following theorem, which extends a result obtained in [30].

Theorem 16: Consider the nonlinear system with disturbances (42). Let $\gamma > 0$. Suppose there exists a smooth solution $V \geq 0$ to the Hamilton–Jacobi equation

$$\frac{\partial V}{\partial x}(x)f(x) + \frac{1}{2}\frac{\partial V}{\partial x}(x) \cdot \left[\frac{1}{\gamma^2}k(x)k^T(x) - g(x)g^T(x)\right]\frac{\partial^T V}{\partial x}(x) + \frac{1}{2}h^T(x)h(x) = 0, \quad V(x_0) = 0 \quad (45)$$

or to the Hamilton–Jacobi inequality

$$\frac{\partial V}{\partial x}(x)f(x) + \frac{1}{2}\frac{\partial V}{\partial x}(x) \cdot \left[\frac{1}{\gamma^2}k(x)k^T(x) - g(x)g^T(x)\right]\frac{\partial^T V}{\partial x}(x) + \frac{1}{2}h^T(x)h(x) \leq 0, \quad V(x_0) = 0 \quad (46)$$

then the closed-loop system for the feedback

$$u = -g^T(x)\frac{\partial^T V}{\partial x}(x) \quad (47)$$

has L_2-gain (from d to $\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right]$) less than or equal to γ.

Proof: By "completing the squares" and using (45) or (46) we obtain [30]

$$\frac{d}{dt}V = \frac{\partial V}{\partial x}f + \frac{\partial V}{\partial x}gu + \frac{\partial V}{\partial x}kd \leq \frac{1}{2}\left\|u + g^T\frac{\partial^T V}{\partial x}\right\|^2 - \frac{1}{2}\gamma^2\left\|d - \frac{1}{\gamma^2}k^T\frac{\partial^T V}{\partial x}\right\|^2 - \frac{1}{2}\|y\|^2 - \frac{1}{2}\|u\|^2 + \frac{1}{2}\gamma^2\|d\|^2 \quad (48)$$

where $y = h(x)$. Choosing u as in (47), and integrating from $t = 0$ to $t = T \geq 0$, starting from $x(0) = x_0$, we obtain for all $T \geq 0$

$$\frac{1}{2}\int_0^T (\|y(t)\|^2 + \|u(t)\|^2)\, dt \leq \frac{1}{2}\gamma^2 \int_0^T \|d(t)\|^2\, dt + V(x_0) - V(x(T))$$

and since $V(x_0) = 0$ and $V \geq 0$ the result follows. □

Remark: Note that for an *arbitrary* initial condition $x(0)$ we obtain the useful inequality

$$\int_0^T (\|y(t)\|^2 + \|u(t)\|^2)\, dt \leq \gamma^2 \int_0^T \|d(t)\|^2\, dt + 2V(x(0)). \quad (49)$$

In particular, by letting $T \to \infty$ in (49) we immediately obtain that for every $d \in L_2(0, \infty)$ the resulting input function $u(t) = -g^T(x(t))(\partial^T V/\partial x)(x(t))$ and output function $y(t) = h(x(t))$ are in $L_2(0, \infty)$ for every $x(0)$.

With regard to (global) *asymptotic stability* of the closed-loop system, we obtain the following from Theorem 5.

Corollary 17: Suppose there exists a solution $V \geq 0$ to (45) or (46). Assume the system $\dot{x} = f(x)$ with outputs $y = h(x)$, $u = -g^T(x)(\partial^T V/\partial x)(x)$ is zero-state observable (see Definition 4). Then $V(x) > 0$ for $x \neq x_0$ and the closed-loop system (42), (47) (with $d(t) \equiv 0$) is locally asymptotically stable. Assume additionally that V is proper, then the closed-loop system is globally asymptotically stable.

Proof: Consider the closed-loop system

$$\dot{x} = \left[f - gg^T \frac{\partial^T V}{\partial x}\right](x) + k(x)d$$

$$\begin{cases} y = h(x) \\ u = -g^T(x)\dfrac{\partial^T V}{\partial x}(x) \end{cases} \quad (50)$$

with inputs d, and outputs (y, u). Clearly, the system is zero-state observable. Application of Theorem 5 to this system now yields the result. [Note that by (45) or (46) $(\partial V/\partial x)[f - gg^T(\partial^T V/\partial x)] \leq -(1/2)(\partial V/\partial x)gg^T \cdot(\partial^T V/\partial x) - (1/2)h^T h$.] □

Let us now proceed to a (partial) converse of Theorem 16.

Theorem 18: Consider the system (42), and let $\gamma > 0$. Suppose there exists a smooth feedback $u = l(x)$, $l(x_0) = 0$, such that the L_2-gain [from d to (y, u)] of the closed-loop system

$$\dot{x} = f(x) + g(x)l(x) + k(x)d$$

$$\begin{cases} y = h(x) \\ u = l(x) \end{cases} \quad (51)$$

[with inputs d and outputs (y, u)] is less than or equal to γ. Assume additionally that (51) is reachable from x_0. Then by Theorem 2, V_a and V_r for (51) are well defined. Assume V_a or V_r are smooth. Then $V := V_a$ or $V := V_r$ satisfies the Hamilton–Jacobi inequality (46). Furthermore, the closed-loop system (50) for the feedback $u = -g^T(x)(\partial^T V/\partial x)(x)$ also has L_2-gain $\leq \gamma$.

Proof: By Theorem 2 $V \geq 0$ satisfies the Hamilton–Jacobi equation

$$\frac{\partial V}{\partial x}(f + gl) + \frac{1}{2}\frac{1}{\gamma^2}\frac{\partial V}{\partial x}kk^T\frac{\partial^T V}{\partial x} + \frac{1}{2}h^T h + \frac{1}{2}l^T l = 0 \quad (52)$$

and thus

$$\begin{aligned} \frac{\partial V}{\partial x}\left[f - gg^T\frac{\partial^T V}{\partial x}\right] &= \frac{\partial V}{\partial x}[f + gl] + \frac{\partial V}{\partial x}g\left[-g^T\frac{\partial^T V}{\partial x} - l\right] \\ &= -\frac{1}{2}\frac{1}{\gamma^2}\frac{\partial V}{\partial x}kk^T\frac{\partial^T V}{\partial x} - \frac{1}{2}h^T h \\ &\quad -\frac{1}{2}\left\|l + g^T\frac{\partial^T V}{\partial x}\right\|^2 - \frac{1}{2}\frac{\partial V}{\partial x}gg^T\frac{\partial^T V}{\partial x}. \end{aligned}$$

Therefore,

$$\frac{\partial V}{\partial x}\left[f - gg^T\frac{\partial^T V}{\partial x}\right] + \frac{1}{2}\frac{1}{\gamma^2}\frac{\partial V}{\partial x}kk^T\frac{\partial^T V}{\partial x} + \frac{1}{2}h^T h + \frac{1}{2}\frac{\partial V}{\partial x}gg^T\frac{\partial^T V}{\partial x} \leq 0 \quad (53)$$

and the Hamilton–Jacobi inequality (46) follows. The last statement immediately follows from Theorem 16. □

Remark 19: The existence of a smooth solution $V \geq 0$ to the Hamilton–Jacobi equation (45) may be pursued by the following iterative procedure. Consider the system (51) with l replaced by $l_1(x) := -g^T(x)(\partial^T V/\partial x)(x)$, i.e., (50). By (53) this system has gain $\leq \gamma$ (Theorem 2), and thus by Theorem 2, if we *assume* that the available storage is *smooth*, there exists a smooth solution $V_1 \geq 0$ to

$$\frac{\partial V_1}{\partial x}[f + gl_1] + \frac{1}{2}\frac{1}{\gamma^2}\frac{\partial V_1}{\partial x}kk^T\frac{\partial^T V_1}{\partial x} + \frac{1}{2}h^T h + \frac{1}{2}l_1^T l_1 = 0. \quad (54)$$

Since V satisfies the corresponding inequality (i.e., $= 0$ replaced by ≤ 0) Theorem 2 yields $V_1 \leq V$. Now, using (54) we obtain

$$\begin{aligned} \frac{\partial V_1}{\partial x}\left[f - gg^T\frac{\partial^T V_1}{\partial x}\right] &= \frac{\partial V_1}{\partial x}[f + gl_1] + \frac{\partial V_1}{\partial x}g\left[-g^T\frac{\partial^T V_1}{\partial x} - l_1\right] \\ &= -\frac{1}{2}\frac{1}{\gamma^2}\frac{\partial V_1}{\partial x}kk^T\frac{\partial^T V_1}{\partial x} - \frac{1}{2}h^T h \\ &\quad -\frac{1}{2}\left\|l_1 + g^T\frac{\partial^T V_1}{\partial x}\right\|^2 - \frac{1}{2}\frac{\partial V_1}{\partial x}gg^T\frac{\partial^T V_1}{\partial x} \end{aligned}$$

and thus, compare (53),

$$\frac{\partial V_1}{\partial x}\left[f - gg^T\frac{\partial^T V_1}{\partial x}\right] + \frac{1}{2}\frac{1}{\gamma^2}\frac{\partial V_1}{\partial x}kk^T\frac{\partial^T V_1}{\partial x} + \frac{1}{2}h^Th + \frac{1}{2}\frac{\partial V_1}{\partial x}gg^T\frac{\partial^T V_1}{\partial x} \le 0. \quad (55)$$

Defining $l_2(x) := -g^T(x)(\partial^T V_1/\partial x)(x)$, we conclude that the system (51) with l replaced by l_2 again has gain $\le \gamma$. Thus, by Theorem 2, if we *assume* that the available storage for this system is *smooth*, there exists a smooth solution $V_2 \ge 0$, with $V_2 \le V_1$, to

$$\frac{\partial V_2}{\partial x}[f + gl_2] + \frac{1}{2}\frac{1}{\gamma^2}\frac{\partial V_2}{\partial x}kk^T\frac{\partial^T V_2}{\partial x} + \frac{1}{2}h^Th + \frac{1}{2}l_2^Tl_2 = 0. \quad (56)$$

The iterative procedure is now clear, and if the smoothness assumption at every step of this procedure is satisfied, then we obtain a sequence of smooth functions

$$V \ge V_1 \ge V_2 \ge \cdots \ge V_i \ge \cdots \ge 0 \quad (57)$$

satisfying

$$\frac{\partial V_i}{\partial x}\left[f - gg^T\frac{\partial^T V_{i-1}}{\partial x}\right] + \frac{1}{2}\frac{1}{\gamma^2}\frac{\partial V_i}{\partial x}kk^T\frac{\partial^T V_i}{\partial x} + \frac{1}{2}h^Th + \frac{1}{2}\frac{\partial V_{i-1}}{\partial x}gg^T\frac{\partial V_{i-1}^T}{\partial x} = 0. \quad (58)$$

By (57) we have pointwise convergence to a function $V^* \ge 0$, i.e., $V^*(x) = \lim_{i\to\infty} V_i(x)$, and if we assume V^* to be smooth, then, taking the limit $i \to \infty$ in (58), we see that V^* satisfies the Hamilton–Jacobi equation (45). Furthermore, it follows from Theorem 2 and Theorem 13 that if all the involved Hamiltonians $H^i = p^T[f + gl_i] + (1/2)(1/\gamma^2)p^Tkk^Tp + (1/2)h^Th + (1/2)l_i^Tl_i$ are hyperbolic, then all the vector fields $f + gl_i + (1/\gamma^2)kk^T(\partial^T V_i/\partial x)$ are globally asymptotically stable. Assuming additionally that the Hamiltonian $H^* = p^T[f + gl_*] + (1/2)(1/\gamma^2)p^Tkk^Tp + (1/2)h^Th + (1/2)l_*^Tl_*$, with $l_* = -g^T(\partial^T V^*/\partial x)$, is hyperbolic, it thus follows that $f - gg^T(\partial^T V^*/\partial x) + (1/\gamma^2)kk^T(\partial^T V^*/\partial x)$ is globally asymptotically stable.

Of course, the assumptions made in the previous theorem (especially reachability from x_0 by the disturbances) are not particularly satisfying. Partial remedy is provided by the following analogs of Corollary 9 and Proposition 12.

Proposition 20: Suppose the L_2-gain of (51) is less than γ, and assume $F + GL$ with $L = (\partial l/\partial x)(x_0)$ is asymptotically stable, then there exists a neighborhood W of x_0 and a smooth function $V \ge 0$ on W satisfying (46). Alternatively, assume $f + gl$ is globally asymptotically stable. Define the Hamiltonian

$$H_\gamma(x, p) = p^T[f(x) + g(x)l(x)] + \frac{1}{2}\frac{1}{\gamma^2}p^Tk(x)k^T(x)p + \frac{1}{2}h^T(x)h(x) + \frac{1}{2}l^T(x)l(x) \quad (59)$$

and suppose X_{H_γ} is hyperbolic, and its stable invariant manifold is diffeomorphic to M under the canonical projection $\pi : T^*M \to M$. Then there exists a global solution $V \ge 0$ to (46).

Proof: Apply Corollary 9 and Proposition 12 to the system (51), yielding the local (Corollary 9) or the global (Proposition 12) existence of a smooth solution to (52). □

Finally, we will give the analogs of Theorems 7 and 8 to this situation. Denote the linearization of (42) at x_0 as

$$\dot{\bar{x}} = F\bar{x} + G\bar{u} + K\bar{d}, \quad u \in \mathbb{R}^m, \quad \bar{d} \in \mathbb{R}^q, \quad \bar{x} \in \mathbb{R}^n,$$
$$\bar{y} = H\bar{x}, \quad \bar{y} \in \mathbb{R}^p \quad (60)$$

where F, G, H are defined in (17), while $K = k(x_0)$. First we recall from linear H_∞ theory (cf. [28], [38], [21], [11]) the following basic theorem (compare to Theorem 7 and the previous Theorems 16, 18, and Remark 19).

Theorem 21: Assume (H, F) is detectable. Let $\gamma > 0$. Then there exists a linear feedback

$$\bar{u} = L\bar{x} \quad (61)$$

such that the closed-loop system (60), (61) [with inputs $\bar{d}$ and outputs $(\bar{y}, \bar{u})$] is asymptotically stable and has L_2-gain less than or equal to γ if and only if there exists a solution $P \ge 0$ to the algebraic Riccati equation

$$F^TP + PF + P\left(\frac{1}{\gamma^2}KK^T - GG^T\right)P + H^TH = 0. \quad (62)$$

Furthermore, the L_2-gain is less than γ, if and only if there exists a solution $P \ge 0$ to (62), additionally satisfying

$$\sigma\left(F - GG^TP + \frac{1}{\gamma^2}KK^TP\right) \subset \mathbb{C}^-. \quad (63)$$

Moreover, if $P \ge 0$ is a solution to (62), then if we choose

$$L = -G^TP \quad (64)$$

the closed-loop system (60), (61) is asymptotically stable and has L_2-gain $\le \gamma$ [respectively $< \gamma$ if there exists a solution $\tilde{P}$ to (62), (63)].

We state the following analog of Proposition 6 with regard to state feedback H_∞ control of (42) and its linearization (60).

Proposition 22: Let $\gamma > 0$. Suppose there exists a smooth feedback $u = l(x)$, $l(x_0) = 0$, for (42) such that the L_2-gain of the nonlinear closed-loop system (51) is $\le (<)\gamma$. Then the linear feedback $\bar{u} = L\bar{x}$, with $L := (\partial l/\partial x)(x_0)$, for (60) results in a linear closed-loop system having gain $\le (<)\gamma$.

Proof: Notice that the linearization of the nonlinear closed-loop system (42) with $u = l(x)$ is given as the closed-loop system (60) with $\bar{u} = L\bar{x}$. Then apply Proposition 6. □

The analog of Theorem 8 was basically already derived in [30].

Theorem 23: Consider the linearized system (60), and assume (H, F) is detectable. Let $\gamma > 0$. Suppose there exists a feedback $\bar{u} = L\bar{x}$ such that the L_2-gain of the closed-loop system [from $\bar{d}$ to $(\bar{y}, \bar{u})$] is less than γ and the closed-loop system is asymptotically stable. Then there exists a neighborhood W of x_0 and a smooth function $V \geq 0$ defined on W such that V is a solution of the Hamilton–Jacobi equation (45). Furthermore, the feedback $u = -g'(x)(\partial^T V/\partial x)(x)$ for (42) has the property that the closed-loop system has locally L_2-gain $\leq \gamma$, in the sense that (with $x(0) = x_0$)

$$\int_0^T (\|y(t)\|^2 + \|u(t)\|^2)\, dt \leq \gamma^2 \int_0^T \|d(t)\|^2\, dt \quad (65)$$

for all $T \geq 0$ and all $d \in L_2(0, T)$ such that the state-space trajectories $x(t)$ starting from $x(0) = x_0$ do not leave W (i.e., the state feedback H_∞ control problem for γ is solved on W).

Proof: By Theorem 21 there exists a solution $P \geq 0$ to (62), (63). It follows (Appendix III) that $(x_0, 0)$ is a hyperbolic equilibrium of X_{H_γ}, where

$$H_\gamma(x, p) := p^T f(x) - \frac{1}{2} p^T \cdot \left(\frac{1}{\gamma^2} k(x) k^T(x) - g(x) g^T(x) \right) p + \frac{1}{2} h^T(x) h(x) \quad (66)$$

and that its stable invariant manifold N^- is tangent to span $\begin{bmatrix} I \\ P \end{bmatrix}$ in $(x_0, 0)$. This implies that locally about x_0 there exists a smooth solution V to (45) satisfying $(\partial^2 V/\partial x^2)(x_0) = P$. Furthermore, since $F - GG^T P$ is asymptotically stable (see Theorem 21) the vector field $f - gg^T(\partial^T V/\partial x)$ is locally asymptotically stable. Rewriting (45) as

$$\frac{\partial V}{\partial x}\left[f - gg^T \frac{\partial^T V}{\partial x} \right] + \frac{1}{2} \frac{1}{\gamma^2} \frac{\partial V}{\partial x} kk^T \frac{\partial^T V}{\partial x} + \frac{1}{2} \frac{\partial V}{\partial x} gg^T \frac{\partial^T V}{\partial x} + \frac{1}{2} h^T h = 0 \quad (67)$$

this implies by Proposition 3 that locally about x_0 $V \geq 0$, and by Theorem 2 that the closed-loop system has gain $\leq \gamma$ for all disturbance functions $d(\cdot)$ such that $x(t)$ remains in the domain where V is defined and is nonnegative (see also the proof of Theorem 10). □

Remark: Notice that the linearization of the feedback $u = -g^T(x)(\partial^T V/\partial x)(x)$ at x_0 is given as $\bar{u} = -G^T P\bar{x}$, which is (cf. 64) a solution of the state feedback H_∞ problem for the linearized system.

The main problem with Theorem 23 is that it does not give us any *a priori* information about the size of W. In particular, if γ approaches the optimal value γ^*, then W may even converge to just the point x_0, as shown by the following.

Example [31]: Consider the nonlinear system

$$\dot{x} = u + xd, \qquad y = x. \quad (68)$$

Clearly, its linearization in $x_0 = 0$ is not affected by disturbances, and thus $\gamma^* = 0$. Furthermore, the Riccati equation (60) for every γ is $P^2 = 1$, yielding the stabilizing solution $P = 1$. The Hamilton–Jacobi equation (45) takes the form

$$\left[\frac{\partial V}{\partial x}(x) \right]^2 \left[1 - \frac{x^2}{\gamma^2} \right] = x^2, \qquad V(0) = 0 \quad (69)$$

and clearly only has a solution for $|x| < \gamma$, namely $V(x) = \gamma^2 - \gamma^2(1 - x^2/\gamma^2)^{1/2}$, yielding the feedback

$$u = -\left(1 - x^2/\gamma^2\right)^{1/2} x. \quad (70)$$

Remark 24: With regard to Theorem 23 we also note that generally the feedback $u = -g^T(x)(\partial^T V/\partial x)(x)$ is not the only feedback $u = l(x)$ resulting in a closed-loop system (51) having locally gain $\leq \gamma$. In fact, also the *linear* feedback

$$u = -G^T P x \quad (71)$$

with $P \geq 0$ a solution of the Riccati equation (62) will result in a closed-loop system having locally gain $\leq \gamma$. (This observation is due to Prof. J. W. Grizzle and Prof. P. P. Khargonekar [15].) Indeed, the Hamilton–Jacobi equation (6) for the closed-loop system resulting from this linear feedback is given as

$$\frac{\partial \tilde{V}}{\partial x}\left[f - gG^T P x \right] + \frac{1}{2} \frac{1}{\gamma^2} \frac{\partial \tilde{V}}{\partial x} kk^T \frac{\partial^T \tilde{V}}{\partial x} + \frac{1}{2} x^T P G G^T P x + \frac{1}{2} h^T h = 0 \quad (72)$$

which by Theorems 8 and 21 has a local solution $\tilde{V} \geq 0$ of the form $\tilde{V}(x) = (1/2)x^T P x + h.o.t$ (see also Proposition A.8). However, we *conjecture* that the domain of definition of $\tilde{V}$ will be always contained (and in general be *strictly* contained) in the domain of definition of the solution $V^- \geq 0$ of (45) such that $f - gg^T(\partial^T V^-/\partial x) + (1/\gamma^2)kk^T(\partial^T V^-/\partial x)$ is asymptotically stable. A clue to a proof of this conjecture may be given by the fact that by rewriting (72) $\tilde{V}$ satisfies the Hamilton–Jacobi inequality

$$\frac{\partial \tilde{V}}{\partial x} f + \frac{1}{2} \frac{\partial \tilde{V}}{\partial x}\left[\frac{1}{\gamma^2} kk^T - gg^T \right] \cdot \frac{\partial^T \tilde{V}}{\partial x} + \frac{1}{2} h^T h = -\frac{1}{2}\left\| \frac{\partial(\tilde{V}(x) - \frac{1}{2}x^T P x)}{\partial x} g \right\|^2 \leq 0. \quad (73)$$

Now by Remark 19 it follows basically that $\tilde{V} \geq V^-$. Since the main obstruction for extending a *local* solution $V \geq 0$ of (45) into a *global* one seems to be the fact that $V(x)$ can become infinite for finite x (see the previous example), this suggests that the solution V^- (and thus the resulting feedback $u = -g^T(\partial^T V^- / \partial x)$) has the largest domain of definition. Note that for the preceding example the linear feedback (70) is given as $u = -x$, resulting in the Hamilton–Jacobi equation (72)

$$\frac{\partial \tilde{V}}{\partial x}(x)(-x) + \frac{1}{2}\frac{1}{\gamma^2}\left[\frac{\partial \tilde{V}}{\partial x}(x)\right]^2 x^2 + \frac{1}{2}x^2 + \frac{1}{2}x^2 = 0 \quad (74)$$

which only has a solution $\tilde{V} \geq 0$ for $|x| < \frac{1}{2}\sqrt{2}\gamma$, i.e., a domain of definition which is indeed smaller than the one obtained for (69).

We will now indicate how Theorem 23, showing that solvability of the state feedback H_∞ control problem for the linearized system (60) implies *local* solvability of the state feedback H_∞ control problem for the nonlinear system (42), can be extended to the case of H_∞ control by *measurement feedback*. To this aim let us consider (42) together with the additional measurement equations

$$y_m = c(x) + v, \qquad y_m \in \mathbb{R}^s, \quad v \in \mathbb{R}^s, c(x_0) = 0 \quad (75)$$

where y_m are the measured outputs and v are extra disturbances, leading to linearized measurement equations

$$\bar{y}_m = C\bar{x} + \bar{v}, \qquad \bar{y}_m \in \mathbb{R}^s, \quad \bar{v} \in \mathbb{R}^s, \quad C = \frac{\partial c}{\partial x}(x_0) \quad (76)$$

for the linearized system (60).

Proposition 25 (Based on [15]): Consider the nonlinear system (42), (75), and its linearization (60), (76). Assume the triples (F, G, H) and (F, K, C) are stabilizable and detectable. Let $\gamma > 0$. Suppose there exists a solution $P \geq 0$ to (62), (63), or equivalently (cf. Theorem 23), a neighborhood W of x_0 and a smooth function $V \geq 0$ on W such that V is a solution of (45) with $(\partial^2 V / \partial x^2)(x_0) = P$. Suppose additionally there exists $Q \geq 0$ satisfying

$$FQ + QF^T - Q\left(C^TC - \frac{1}{\gamma^2}H^TH\right)Q + KK^T = 0$$
$$\sigma\left(F - QC^TC + \frac{1}{\gamma^2}H^TH\right) \subset \mathbb{C}^- \quad (77)$$

and such that the maximal singular value of PQ is less than γ^2. Consider the following compensator:

$$\dot{z} = \left(F + \frac{1}{\gamma^2}KK^TP - GG^TP\right)z + \left(I - \frac{1}{\gamma^2}QP\right)^{-1} QC^T(y_m - Cz)$$
$$u = -g^T(z)\frac{\partial^T V}{\partial z}(z), \qquad z \in \mathbb{R}^n. \quad (78)$$

Then there exists a neighborhood $\tilde{W} \subset M \times \mathbb{R}^n$ of $(x_0, z = 0)$, such that the closed-loop system (42), (75), (78) is asymptotically stable on $\tilde{W}$ and

$$\int_0^T (\|y(t)\|^2 + \|u(t)\|^2)\,dt \leq \gamma^2 \int_0^T (\|d(t)\|^2 + \|v(t)\|^2)\,dt \quad (79)$$

for all $T \geq 0$ and all d, $v \in L_2(0, T)$ such that the trajectories $(x(t), z(t))$ of the closed-loop system starting from $x(0) = x_0$, $z(0) = 0$ do not leave $\tilde{W}$.

Proof: Consider the closed-loop system

$$\begin{cases} \dot{x} = f(x) - g(x)g^T(z)\dfrac{\partial^T V}{\partial z}(z) + k(x)d \\ \dot{z} = \left(F + \dfrac{1}{\gamma^2}KK^TP - GG^TP\right)z \\ \qquad + \left(I - \dfrac{1}{\gamma^2}QP\right)^{-1} QC^T(c(x) - Cz + v) \end{cases}$$
$$\begin{cases} y = h(x) \\ u = -g^T(z)\dfrac{\partial^T V}{\partial z}(z) \end{cases}. \quad (80)$$

The linearization of (80) at $(x_0, z = 0)$ is precisely the closed-loop system that would arise if we apply the compensator (78) with $u = -g^T(z)(\partial^T V / \partial z)(z)$ replaced by $\bar{u} = -G^TPz$ to the linearized system (60), (76). However, this is precisely the "central controller" of [11], and thus this linear closed-loop system is asymptotically stable and has L_2-gain [from $(\bar{d}, \bar{v})$ to $(\bar{y}, \bar{u})$] less than γ. The result now follows from Theorem 8.

Remark 26: The construction of the compensator (78) is not really satisfactory. Indeed we could have taken any compensator whose linearization *equals* the linearization of (78) in order that Proposition 25 continues to hold. The issue is therefore (see Remark 24) to choose a compensator which maximizes the size of $\tilde{W}$. This is certainly related to finding the proper nonlinear analog of the "dual" Riccati equation (77).

Finally, from Lukes [22] (see also [13]) we recall the following approach for obtaining an *approximate* solution of the Hamilton–Jacobi equation (45). (In [22] the Hamilton–Jacobi–Bellman equation from optimal control was considered; however, the approach remains the same.) Suppose there exists a solution $P \geq 0$ to (62), (63). Now write

$$V(x) = \tfrac{1}{2}x^TPx + V_h(x)$$
$$f(x) = Fx + f_h(x)$$
$$\frac{1}{2}\left(\left(\frac{1}{\gamma^2}k(x)k^T(x) - g(x)g^T(x)\right)\right) = \frac{1}{2}\left(\frac{1}{\gamma^2}KK^T - GG^T\right) + R_h(x)$$
$$\tfrac{1}{2}h^T(x)h(x) = \tfrac{1}{2}x^TH^THx + \theta_h(x) \quad (81)$$

where $V_h(x)$, $f_h(x)$, $R_h(x)$, and $\theta_h(x)$ contain higher order terms (beginning with degrees 3, 2, 1, and 3, respectively). Then the Hamilton–Jacobi equation (45) splits into two parts: the first part is the Riccati equation (62) while the second part is the higher order equation

$$-\frac{\partial V_h}{\partial x}(x)F_* x = \frac{\partial V}{\partial x}(x)f_h(x) + \frac{1}{2}\frac{\partial V_h}{\partial x}(x) \cdot \left[\frac{1}{\gamma^2}KK^T - GG^T\right]\frac{\partial^T V_h}{\partial x}(x) + \frac{1}{2}\frac{\partial V}{\partial x}(x)R_h(x)\frac{\partial^T V}{\partial x}(x) + \theta_h(x) \tag{82}$$

where $F_* := F - GG^TP + (1/\gamma^2)KK^TP$. The mth order terms $V^{(m)}(x)$ of $V(x)$ can now be computed inductively for $m = 3, 4, \cdots$, as follows. Denote the mth order terms on the right-hand side of (82) by $H_m(x)$. It follows that

$$-\frac{\partial V^{(m)}}{\partial x}(x)F_* x = H_m(x) \tag{83}$$

and thus by (63) we have that $V^{(m)}(x) = \int_0^\infty H_m(e^{F_* t}x)\,dt$, and so $V^{(m)}(x)$ is determined by $H_m(x)$. Now it is easily seen that $H_m(x)$ only depends on $V^{(m-1)}, V^{(m-2)}, \cdots, V^{(2)} = \frac{1}{2}x^TPx$, and therefore (83) determines $V^{(m)}(x)$ inductively starting from $V^{(2)}(x) = (1/2)x^TPx$. This approximation scheme is especially useful in the present context of (state feedback) H_∞ control since (see, e.g., [40]) the existence of a solution $P \geq 0$ to (62), (63) is equivalent to the existence of a solution $P > 0$ to (62) with equality replaced by strict inequality $<$. Hence, there exists $\epsilon > 0$ such that also the *modified* Riccati equation

$$F^TP + PF + P\left(\frac{1}{\gamma^2}KK^T - GG^T\right)P + H^TH + \epsilon^2 I = 0 \tag{84}$$

has a solution $P \geq 0$ satisfying (63). Using the above approximation scheme we may now obtain approximate solutions of the *modified* Hamilton–Jacobi equation

$$\frac{\partial V}{\partial x}(x)f(x) + \frac{1}{2}\frac{\partial V}{\partial x}(x) \cdot \left[\frac{1}{\gamma^2}k(x)k^T(x) - g(x)g^T(x)\right]\frac{\partial^T V}{\partial x}(x) + \frac{1}{2}h^T(x)h(x) + \frac{1}{2}\epsilon^2 x^Tx = 0 \tag{85}$$

and for x sufficiently small these will be solutions to the Hamilton–Jacobi inequality (46).

IV. Conclusions

We have unified the results obtained in [36], [26], [23] for finite L_2-gain systems. Furthermore, by using the properties of invariant manifolds of Hamiltonian vector fields, we have been able to extend these results in several directions and have provided a geometrical interpretation of them. This also allowed us to generalize some linear results of [35] to the nonlinear case. Further work needs to be done on the optimal case (L_2-gain $= \gamma$), where stable and unstable manifolds partly deteriorate into center manifolds (see the Remark after Corollary 9).

In Section III we have treated the standard (i.e., nonsingular) nonlinear state feedback H_∞ optimal control problem using the results of Section II. A major challenge will be the extension to the nonlinear H_∞ problem with dynamic measurement feedback and the nonlinear H_∞ filtering problem (see, e.g., [11], respectively, [20]). Among the next problems to be tackled are the *singular* nonlinear state feedback H_∞ problem (i.e., the L_2-gain from disturbances d to outputs y and a *part* of the inputs u is considered), and its relation with the nonlinear (almost) disturbance decoupling problem (see, e.g., [18], [27], [26]). On the theoretical side the connections with an operator and game-theoretic approach (see, e.g., [6]–[8]) need to be investigated.

Also much work has to be done regarding the *formulation* of the nonlinear H_∞ problem, since in most applications the system (42) will not be the real physical system but instead some redefined system (as in the mixed sensitivity problem; see, e.g., [12], [19]).

Finally, it is hoped that the nonlinear state feedback H_∞ optimal controllers as obtained in Section III have favorable robustness properties; however, this needs further investigation (see also [13] for a survey on the robustness of state feedbacks obtained from nonlinear optimal control).

Appendix

In Appendices I and II some mathematical background is summarized; for details we refer to, e.g., [3], [2]. The material in Appendix III is also partly taken from [30].

Appendix I

Preliminaries (Cotangent Bundle, One-Forms, Symplectic Forms)

Let M be an n-dimensional manifold, and let T_qM denote the *tangent space* to M at $q \in M$. Since T_qM is a linear space (in fact, $T_qM \simeq \mathbb{R}^n$) we can define its dual T_q^*M, called the *cotangent space* at $q \in M$.

The space $T^*M := \bigcup_{q\in M} T_q^*M$ is called the *cotangent bundle* over M, just like $TM := \bigcup_{q \in M} T_qM$ is called the tangent bundle.

Recall that a smooth vector field X on M is defined by a smooth mapping $X: M \to TM$ satisfying $\pi \circ X =$ identity, where $\pi: TM \to M$ is the natural projection taking an element $X_q \in T_qM \subset TM$ to its base point $q \in M$. Similarly, a *one-form* (or covector field) σ on M is defined by a smooth mapping $\sigma: M \to T^*M$ satisfying $\pi \circ \sigma =$ identity, where $\pi: T^*M \to M$ is a similarly defined projection. (All this simply means that $X(q) \in T_qM$ and $\sigma(q) \in T_q^*M$ for every $q \in M$, and that $X(q)$ and $\sigma(q)$ depend on q in a smooth manner.)

Let $x_1, \cdots, x_n$ be a set of local coordinates for M, then for every q in the coordinate neighborhood U we have a basis $\{(\partial/\partial x_1)|_q, \cdots, (\partial/\partial x_n)|_q\}$ for T_qM, where $(\partial/\partial x_i)|_q f = (\partial f/\partial x_i)(x_1(q), \cdots, x_n(q))$ for any smooth function f on M. Thus, a vector field X on M can be locally (i.e., on a coordinate neighborhood U) expressed as $X(q) = X_1(x(q))(\partial/\partial x_1)|_q + \cdots + X_n(x(q))(\partial/\partial x_n)|_q$, for certain smooth functions $X_1, \cdots, X_n$ on $\mathbb{R}^n$. We also denote $X = (X_1, \cdots, X_n)^T$. Similarly, if we denote the *dual basis* for T_q^*M by $\{dx_1|_q, \cdots, dx_n|_q\}$, then a one-form σ on M can be expressed on U as $\sigma(q) = \sigma_1(x(q))dx_1|_q + \cdots + \sigma_n(x(q))dx_n|_q$, for certain smooth functions $\sigma_1, \cdots, \sigma_n$. We call $\sigma = (\sigma_1, \cdots, \sigma_n)$ the local coordinate expression of σ. In local coordinates we define for any σ on M the *two-form* $d\sigma$ as $d\sigma(q) = \sum_{i,j=1}^n (\partial\sigma_i/\partial x_j)(x(q))dx_i \wedge dx_j|_q$. Here $dx_i \wedge dx_j|_q$ denotes the skew-symmetric bilinear form on T_qM defined by $dx_i \wedge dx_j|_q(X^1(q), X^2(q)) = X_i^1(x(q))X_j^2(x(q)) - X_i^2(x(q))X_j^1(x(q))$ for any two vectors $X^1(q), X^2(q) \in T_qM$. We say that σ is *closed* if $d\sigma = 0$. Clearly, this is equivalent to $(\partial\sigma_i/\partial x_j) = (\partial\sigma_j/\partial x_i)$ for $i, j = 1, \cdots, n$. Then locally (or globally if M is simply connected, e.g., if $M \simeq \mathbb{R}^n$) Poincare's lemma yields that $\sigma_i(x) = (\partial V/\partial x_i)(x)$, $i = 1, \cdots, n$, for some function V, and we write $\sigma = dV$ $(= (\partial V/\partial x_1)dx_1 + \cdots + (\partial V/\partial x_n)dx_n)$.

Given local coordinates $x_1, \cdots, x_n$ for M, we define *natural coordinates* for T^*M by attaching to any $\sigma(q) \in T_q^*M$ the coordinate values $(x_1(q), \cdots, x_n(q), \sigma_1(x(q)), \cdots, \sigma_n(x(q)))$ (where $\sigma = (\sigma_1, \cdots, \sigma_n)$ is the local coordinate expression of σ). We denote the natural coordinate functions on T^*M as $x_1, \cdots, x_n, p_1, \cdots, p_n$, i.e., $x_i(\sigma(q)) = x_i(q)$, $p_i(\sigma(q)) = \sigma_i(x(q))$, $i = 1, \cdots, n$.

In the above natural coordinates for T^*M we define two natural objects

a) the canonical one-form θ on T^*M given as

$$\theta(\sigma(q)) = \sum_{i=1}^n p_i(\sigma(q))dx_i|_q \tag{A.1}$$

b) the canonical *two-form* $\omega = d\theta$, i.e.,

$$\omega(\sigma(q)) = \sum_{i=1}^n dp_i \wedge dx_i|_{\sigma(q)}. \tag{A.2}$$

Appendix II

Lagrangian Submanifolds

Definition A.1: A submanifold $N \subset T^*M$ is called a *Lagrangian* submanifold if $\dim N = \dim M$ and ω restricted to N is zero [i.e., $\omega(q)(X_1(q), X_2(q)) = 0$ for every $q \in N$ and every $X_1(q), X_2(q) \in T_qN(\subset T_qT^*M)$].

Now let us consider submanifolds $N \subset T^*M$ which are *projectable* on M in the sense that $\pi: N \to M$ is a diffeomorphism (with $\pi: T^*M \to M$ being the natural projection); in particular $\dim N = \dim M$. Then it is clear that $N =$ graph σ for some one-form σ on M.

Proposition A.2: Consider a submanifold $N \subset T^*M$ which is projectable on M, i.e., $N =$ graph σ. Then N is a Lagrangian submanifold if and only if σ is a closed one-form.

Proof: $\omega = d\theta$ being zero on graph σ means precisely that $d\sigma = 0$, since $\theta\,|$graph $\sigma = \sigma$. □

Corollary A.3: Suppose $N \subset T^*M$ is projectable on M and Lagrangian. Furthermore, assume M is simply connected. Then $N =$ graph dV, for some smooth function $V: M \to \mathbb{R}$. (V is called the *generating function* of N).

Appendix III

Invariant Manifolds of Hyperbolic Hamiltonian Vector Fields

Consider T^*M, $\dim M = n$, with symplectic form ω. Let $H: T^*M \to \mathbb{R}$ be a smooth function, called a Hamiltonian function. Then the Hamiltonian vector field X_H on T^*M corresponding to H is defined by setting $\omega(X_H, Z) = -dH(Z)$ for every vector field Z on T^*M. In natural coordinates (x, p) for T^*M we obtain the familiar Hamiltonian equations

$$\begin{aligned} \dot{x}_i &= \frac{\partial H}{\partial p_i}(x, p) \\ \dot{p}_i &= -\frac{\partial H}{\partial x_i}(x, p) \end{aligned} \qquad i = 1, \cdots, n.$$

Suppose $(x_0, p_0) \in T^*M$ is an equilibrium for X_H, or equivalently $dH(x_0, p_0) = 0$, then the linearization of X_H at (x_0, p_0) is given by the Hamiltonian matrix

$$DX_H(x_0, p_0) = \begin{bmatrix} \dfrac{\partial^2 H}{\partial x \partial p} & \dfrac{\partial^2 H}{\partial p^2} \\ -\dfrac{\partial^2 H}{\partial x^2} & -\dfrac{\partial^2 H}{\partial p \partial x} \end{bmatrix}(x_0, p_0). \tag{A.3}$$

(Notice that $DX_H(x_0, p_0)$ defines a *linear* Hamiltonian vector field corresponding to the quadratic Hamiltonian given by the quadratic terms in the Taylor expansion of $H(x, p)$ around (x_0, p_0).)

The equilibrium (x_0, p_0) is called *hyperbolic* if $DX_H(x_0, p_0)$ does not have purely imaginary eigenvalues. By the fact that $DX_H(x_0, p_0)$ is Hamiltonian it follows that $DX_H(x_0, p_0)$ has n eigenvalues in $\mathbb{C}^-$ and n eigenvalues in $\mathbb{C}^+$ (in fact if λ is eigenvalue, then so is $-\lambda$).

Proposition A.4: Suppose (x_0, p_0) is a hyperbolic equilibrium for X_H. Then there exists a unique maximal (immersed) submanifold $N^- \subset T^*M$ through (x_0, p_0) satisfying the following:

1) N^- is invariant for X_H (i.e., $X_H(q) \in T_qN^-$ for every $q \in N^-$);

2) X_H restricted to N^- is globally asymptotically stable [with regard to (x_0, p_0)]. (In fact, N^- is the set of all points T^*M converging to (x_0, p_0).)

Furthermore, N^- satisfies the following tangency property.

3) N^- is tangent at (x_0, p_0) to the stable eigenspace of $DX_H(x_0, p_0)$ (cf. A.3).

N^- is called the *stable invariant manifold* of X_H. Analogously, there exists a unique maximal (immersed) submanifold $N^+ \subset T^*M$ through (x_0, p_0) satisfying the following:

1) N^+ is invariant for X_H;

2) $-X_H$ restricted to N^+ is globally asymptotically stable;

3) N^+ is tangent at (x_0, p_0) to the unstable eigenspace of $DX_H(x_0, p_0)$.

N^+ is called the *unstable invariant manifold* of X_H.

Proposition A.5 [30]: Suppose (x_0, p_0) is a hyperbolic equilibrium for X_H. Then the stable and unstable invariant manifolds N^- and N^+ are Lagrangian submanifolds.

Proof: See [30] for N^-. Note that N^+ is the *stable* invariant manifold for the Hamiltonian vector field $X_{-H} = -X_H$. □

Combining Proposition A.5 and Corollary A.3 we will show that the stable and unstable invariant manifolds of X_H correspond to solutions of a Hamilton–Jacobi equation.

Proposition A.6 [30]: Suppose (x_0, p_0) is a hyperbolic equilibrium for X_H. Suppose N^- and N^+ are projectable on M, with M simply connected. Then $N^- =$ graph dV^-, $N^+ =$ graph dV^+ for some smooth functions $V^-, V^+ : M \to \mathbb{R}$, satisfying the Hamilton–Jacobi equation

$$H\left(x, \frac{\partial V}{\partial x}(x)\right) = H(x_0, p_0) \qquad x \in M. \quad \text{(A.4)}$$

In particular, if $H(x, p)$ is of the form $H(x, p) = p^T f(x) + (1/2) p^T R(x) p + s(x)$, with $f(x_0) = 0$, $s(x_0) = 0$ ($R(x)$ being a symmetric $n \times n$ matrix), then V^- and V^+ are solutions of

$$\frac{\partial V}{\partial x}(x) f(x) + \frac{1}{2}\frac{\partial V}{\partial x}(x) R(x) \frac{\partial^T V}{\partial x}(x) + s(x) = 0,$$
$$V(x_0) = 0, \quad dV(x_0) = 0. \quad \text{(A.5)}$$

Remark: Conversely, if V is a solution of (A4), then the submanifold $N = \{(x, p = (\partial^T V/\partial x)(x)) \mid x \in M\}$ is an invariant submanifold for X_H, cf. [30].

Using the fact that N^- and N^+ are *tangent* at (x_0, p_0) to the stable, respectively, unstable, eigenspace of $DX_H(x_0, p_0)$ we immediately obtain the following *local* statement.

Proposition A.7: Suppose (x_0, p_0) is a hyperbolic equilibrium for X_H. Suppose the stable and unstable eigenspace of $DX_H(x_0, p_0)$ are of the form span $\begin{bmatrix} I \\ P^- \end{bmatrix}$, respectively, span $\begin{bmatrix} I \\ P^+ \end{bmatrix}$, for some matrices P^-, respectively, P^+. Then there exists a neighborhood W of x_0 and functions V^-, V^+ defined on W such that Proposition A.6 holds on W.

Proof: By tangency of N^- and N^+ to span $\begin{bmatrix} I \\ P^- \end{bmatrix}$, respectively, span $\begin{bmatrix} I \\ P^+ \end{bmatrix}$, it follows that there exists a neighborhood W of x_0 such that $N^- \cap T^*W$, respectively, $N^+ \cap T^*W$ are projectable on W.

Finally, we have the following linearization result.

Proposition A.8: Let V satisfy (A.4). Then $P = (\partial^2 V/\partial x^2)(x_0)$ satisfies the algebraic Riccati equation

$$A^T P + PA + PRP + Q = 0 \qquad \text{(A.6)}$$

with $A := (\partial^2 H/\partial x \partial p)(x_0, p_0)$, $R := (\partial^2 H/\partial p^2)(x_0, p_0)$, $Q := (\partial^2 H/\partial x^2)(x_0, p_0)$. In particular, $P^- := (\partial^2 V^-/\partial x^2)(x_0)$, $P^+ := (\partial^2 V^+/\partial x^2)(x_0)$ satisfy (A.6) and are such that span $\begin{bmatrix} I \\ P^- \end{bmatrix}$ and span $\begin{bmatrix} I \\ P^+ \end{bmatrix}$ are the stable, respectively, unstable, eigenspace of $DX_H(x_0, p_0)$.

Acknowledgment

This research has been performed while visiting the Center for Systems Science and Engineering, Arizona State University, Tempe. The hospitality of Dr. P. E. Crouch is gratefully acknowledged.

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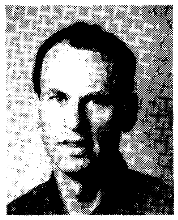

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