Mixed Spectra and Rotational Symmetry

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1. Introduction

There are some physical experiments which produce time series whose power spectra seem to be superpositions of continuous spectra and delta functions: mixed spectra. On the other hand, there are no mathematical examples known of persistent attractors in dynamical systems with continuous time, having such spectra. (For the interpretation of power spectra in terms of dynamical systems see Section 2.) In this paper we show that persistent attractors with a mixed spectrum do exist in the context of dynamical systems with rotational or $SO(2)$-symmetry. The introduction of symmetry in the problem is motivated by the fact that one of the very accurate experiments, leading to mixed spectra, is the Couette-Taylor flow as reported in [BS, 1987].

Our $SO(2)$-equivariant attractor is closely related to the quasi-periodic attractors whose unfoldings and bifurcations were studied, e.g., in [BHTB, 1990]. The main difference, apart from having a mixed spectrum, is that our present example has sensitive dependence on initial conditions (in a topological sense), however, without having positive Lyapunov exponents or positive entropy. Because of this relation with quasi-periodic attractors and with the skew products in ergodic theory, we call them SQP (skew-quasi-periodic) attractors.

We also investigate the bifurcations leading to SQP attractors, in particular a variation of the Hopf bifurcation between quasi-periodic attractors of dimension two and three, the so-called skew Hopf bifurcation. In the presence of symmetry this bifurcation is fairly well understood. Interesting problems seem to arise when considering such transitions from a quasi-periodic attractor with two frequencies to an SQP attractor after a non-$SO(2)$-symmetric perturbation has been added. This seems to give a new route to chaos, in which a quasi-periodic attractor becomes chaotic. Here, however, we mainly have numerical results. Now we give a description of the content of the different sections.

In Section 2 we discuss the interpretation of the power spectrum in terms of the concepts of ergodic theory and recall the relevant results on the ergodic theory of differential dynamical systems. In this section there are no new results, but we include it because this interpretation of the power spectrum is, as far as we know, missing in the general texts dealing with the applications of the theory of dynamical systems to physical experiments.

In Section 3 we construct the SQP attractors and discuss their persistence. To indicate the relation with the usual quasi-periodic attractors, we recall that, for dynamical systems with discrete time, a quasi-periodic attractor with two frequencies admits coordinates x, y in $\mathbb{R}$ mod 1 such that the time evolution has the form $(x, y) \mapsto (x + \alpha, y + \beta)$, with α and β irrational and satisfying certain Diophantine conditions. An SQP attractor admits such coordinates for which the time evolution has the form $(x, y) \mapsto (x + \alpha, y + k \cdot x)$ for some $k \in \mathbb{Z} - \{0\}$. This map commutes with the standard $SO(2)$-action on the last coordinate. Corresponding attractors with continuous time are obtained by suspension. It is important to note that the map describing the time evolution of an SQP attractor is *not* homotopic to the identity. In the present case this implies that after suspension we get an attractor on which the $SO(2)$-action is *non-trivial*, i.e., the attractor cannot be decomposed as a product of a 2-manifold and $SO(2)$ in a way which is compatible with the $SO(2)$-action.

In Section 4 we discuss the bifurcational aspects of the SQP attractors related to the fact that they are non-trivial $SO(2)$-bundles. In order to make the paper self-contained, we start with a brief review of the global aspects of principal fibre bundles, and in particular principal $SO(2)$-bundles. (Here we have to assume that the reader is familiar with cohomology theory.) This theory of $SO(2)$-bundles then is used to show that certain bifurcations are impossible: In particular, a bifurcation from a quasi-periodic attractor with two frequencies, on which $SO(2)$ acts non-trivially, to an SQP attractor is ruled out; from this we conclude that the transition from quasi-periodic (modulated rotating wave, see [GSS, 1988]) to chaotic dynamics in the Couette-Taylor experiment cannot be explained in terms of SQP attractors. On the other hand, there exists a possibility of a bifurcation from a quasi-periodic attractor, on which $SO(2)$ acts *trivially*, to an SQP attractor.

The latter bifurcation, the skew Hopf bifurcation, has strong analogies with the generalized Hopf bifurcations for quasi-periodic attractors from two to three frequencies, studied in [BHTB, 1990]. In our Section 5 we construct the formal normal forms for skew Hopf bifurcations and in Section 6 analytical properties of this bifurcation, in particular, of the invariant tori, are investigated. We shall see that in the parameter-direction the complexity of 'Chenciner bubbles' arises.

In the final section, we comment on various aspects of the dynamics, mainly based on numerical simulations. First we discuss what happens when the SQP attractor, and the corresponding torus, are destroyed by resonance. Then we consider the skew Hopf bifurcation with a non-symmetric perturbation added: We describe the first obstruction to regaining the symmetry by a coordinate change (normal-form approach) and then show some of the attractors which are possible when the symmetry is violated.

This work was motivated by experimental results which we discuss now in some more detail. First, in the Couette-Taylor experiment (on the motion of fluid between two rotating cylinders) clearly an $SO(2)$-symmetry is present:

rotations around the common axis of the two cylinders. In fact there is more symmetry: a $\mathbb{Z}_2$-symmetry corresponding to reflection about the horizontal midplane. However, anything which occurs persistently in a situation with low symmetry also can occur persistently in a situation with higher symmetry. This holds in particular if the extra symmetry is discrete and if the dynamics is restricted to the part of the state space on which the action is free (i.e., if we consider attractors which are disjoint from their symmetric images). Besides this we have to point out that mixed spectra were also observed in situations without symmetry (compare the Rayleigh-Bénard instability, see [GB, 1980]).

We recall that this problem of mixed spectra was also considered in [FCFPS, 1980] for numerical simulations of a system without symmetry, where it was shown that certain attractors, especially versions of the Rössler attractor, have power spectra with sharp peaks, which, numerically, cannot be distinguished from delta functions. There are good reasons, though no proofs, to expect these attractors to have a continuous spectrum. This might also be the case for the Couette-Taylor and the Rayleigh-Bénard experiments: namely, that the delta functions, suggested by the experimental results, are just sharp (local) maxima in a continuous distribution.

In some sense our results support this latter idea: Although we succeed in making a persistent attractor with mixed spectrum for $SO(2)$-equivariant systems, our bifurcation results indicate that the results of BRANDSTATER & SWINNEY [BS, 1987] concerning the Couette-Taylor flow cannot be modelled by an SQP attractor.

2. Spectral theory

The purpose of this section is to interpret power spectra, as they are calculated routinely from experimental time series, in terms of the notions of ergodic theory, as applied to the dynamical system modelling the experiment giving rise to the time series. There are no new results in this section, but we include it since we do not know of any reference discussing this material, which is a combination of ergodic theory (SBR measures), functional analysis (spectral theorems), and Fourier theory, and which is fundamental for applying the mathematical theory of dynamical systems to the interpretation of (physical) experiments.

2.1. General setting

We begin by defining the notions of dynamical system, observable, attractor, and the physical, or SBR measure.

A *dynamical system* is a smooth flow $\Phi_t : M \to M$, where M is a differentiable manifold, usually finite-dimensional, and where Φ_t is a one-parameter group of diffeomorphisms, generated by a smooth vector field X on M. We think of the points $x \in M$ as possible states of a physical system of which Φ_t describes the time evolution. We assume that M is compact, or, if not, that each forward orbit $\mathscr{O}^+(x) = \{\Phi_t(x) \mid t \geqq 0\}$ has a compact closure. In some cases we shall also have to consider dynamical systems with discrete time $(t \in \mathbb{Z})$; they are generated by a diffeomorphism $\varphi = \Phi_1$.

Often one does not observe the complete state of a system, but only part of it. We incorporate this into our setting by including a function $Y: M \to \mathbb{R}$, the *observable*, assigning to each state $x \in M$ the value $Y(x)$, which we observe, or measure, when the system is in that state. So an evolution $x(t) = \Phi_t(x)$ leads to the time series $y(t) = Y(\Phi_t(x))$. Of course, one can measure more 'coordinates' of each state. This could be modelled by a function Y with values in $\mathbb{R}^q$. For simplicity we here only consider the case $q = 1$.

Roughly speaking, one says that a dynamical system, given by Φ_t, has an attractor if there is a set of positive measure so that any two points in that set asymptotically have the same future behaviour. We formalize this as follows. A compact set $K \subset M$, which is the support of a Borel probability measure μ, is an *attractor* if for some open neighbourhood B of K, and for some subset $B' \subset B$ such that $B - B'$ has Lebesgue measure zero, we have for each $x \in B'$ and each continuous $g: M \to \mathbb{R}$, that

$$\lim_{T \to \infty} T^{-1} \int_0^T g(\Phi_t(x))\, dt = \int_K g\, d\mu.$$

The set B is called the *basin* of the attractor. A few remarks are in order. It would be formally more correct to speak of an attractor (K, μ). However, in the cases we consider it will be clear what the measure μ is. Our definition differs from the usual ones which do not refer to a measure, but simple (i.e., stationary and periodic) attractors clearly have a probability measure with the above property. The corresponding measures, also called *SBR* (Sinaï-Bowen-Ruelle) or *physical* measures, were constructed by SINAÏ [S, 1968] for Anosov diffeomorphisms, by RUELLE [R, 1976] for Axiom A attractors of diffeomorphisms, and by BOWEN & RUELLE [BR, 1975] for Axiom A attractors of flows. For more information on these measures see also [R, 1980] and [R, 1989].

It may seem inconsistent that, on the one hand we take into account that not $x \in M$ but only $Y(x) \in \mathbb{R}$ can be observed, while, on the other hand we are using *arbitrary* continuous functions $g: M \to \mathbb{R}$ in the definition of an attractor. This can be justified, however, by observing that if we follow the evolution of a state $x \in M$, we obtain $Y(\Phi_t(x)) = (Y \circ \phi_t)(x)$ by looking at the measurement after time t. So not only Y, but also $Y \circ \Phi_t$ is 'observable'. Finally, under generic conditions on Φ_t (or X) and Y, any continuous function $g: M \to \mathbb{R}$ can be written as a continuous function of $Y \circ \Phi_{t_1}, \ldots, Y \circ \Phi_{t_q}$ for some finite sequence $t_1, \ldots, t_q$; see [T, 1981].

In the following subsections we assume M, Φ_t, X, K, μ, B, and B' to be as above. We then also consider the induced one-parameter group of unitary transformations ϕ_t in $\mathscr{L}^2_\mu(K)$, the space of square-integrable functions on K, defined by $\phi_t(g) = g \circ \Phi_t$. This one-parameter group has an infinitesimal generator $i\mathscr{X}$, where

$$\mathscr{X} = \lim_{t \to 0} \frac{-i}{t}(\phi_t - \mathrm{Id})$$

is a densely defined self-adjoint operator: For any C^1-function $g: M \to \mathbb{R}$, one has $\mathscr{X}(g|K) = -i \cdot X(g)|K$, where $X(g)$ is the function on M obtained by taking the derivative of g in the direction of X.

2.2. The spectral theorem

Here we describe the spectral decomposition of a one-parameter group of unitary transformations $\phi_t: H \to H$ in a complex Hilbert space H with infinitesimal generator $i\mathscr{X}$. This is mainly based on on [CFS, 1982] and [M, 1963], but there are many more references, e.g., see [L, 1962]. First we have to define a measure whose values are projections in a Hilbert space H.

A *projection-valued measure* P on $\mathbb{R}$ assigns to each Borel subset $E \subset \mathbb{R}$ a projection $P(E)$ in a Hilbert space H, i.e., $P(E)$ is a self-adjoint idempotent map, such that
(i) $P(\emptyset) = 0$;
(ii) $P(\mathbb{R}) = \mathrm{Id}$;
(iii) $P(E_1) \circ P(E_2) = P(E_1 \cap E_2)$;
(iv) if $E_1, E_2, \dots$ is a countable collection of mutually disjoint Borel sets, then $P(\bigcup E_i) = \sum P(E_i)$.

For such a projection-valuded measure P and for $f \in H$, the Borel measure P_f is defined by $P_f(E) = \langle P(E) f, f \rangle$, where $\langle\ ,\ \rangle$ denotes the inner product on H. This measure is positive, and if $\|f\| = 1$, it is a probability measure. From these measures we obtain a quadratic map Q_P on H by setting

$$Q_P(f, f) = \int_{-\infty}^{+\infty} x \cdot dP_f(x) .$$

Note that Q_P may be unbounded (and only densely defined), but at least $\mathrm{Im}(P(E))$, for bounded E, is in the domain of Q_P. From Q_P we obtain a unique self-adjoint operator $\mathscr{X}_P$ by putting

$$\langle \mathscr{X}_P(f), F \rangle = Q_P(f, f)$$

for all f in the domain of Q_P.

According to the *spectral theorem*, for ϕ_t and $\mathscr{X}$ as in (2.1), there is a unique projection-valued measure P (with projections in $\mathscr{L}^2_\mu(K)$) such that $\mathscr{X} = \mathscr{X}_P$. It then follows from the above definitions and the fact that $\phi_t = e^{i\mathscr{X}t}$ (properly defined), that

$$\langle \phi_t(f), f \rangle = \int_{-\infty}^{+\infty} e^{i\mathscr{X}t} \cdot dP_f(x)$$

for all $f \in \mathscr{L}^2_\mu(K)$. We call P the *spectrum* of $\mathscr{X}$ or of the group $\{\phi_t\}$.

To clarify the notions introduced above, we discuss their analogues, in terms of eigenvalues and eigenspaces, for the case of a *finite-dimensional* Hilbert space H. In this case there are no problems with densely defined but unbounded linear maps. If $\mathscr{X}$ is self-adjoint, then it has a finite number of real eigenvalues $\lambda_1, \dots, \lambda_k$ with mutually orthorgonal eigenspaces $V_1, \dots, V_k$.

Then the corresponding projection-valued measure P is given by

$$P(E) = \sum_{i:\lambda_i \in E} P_{V_i},$$

where P_{V_i} is the orthogonal projection on V_i. For a vector $f = f_1 + \ldots + f_k$, $f_i \in V_i$, we have $P_f(\{\lambda_i\}) = \| f_i \|^2$, so P completely describes the decomposition of H into its eigenspaces.

The main difference in the infinite-dimensional case is that the measures need no longer be concentrated in points. In fact, for any positive Borel measure μ on $\mathbb{R}$ with $\mu(\mathbb{R}) < \infty$, there is a Hilbert space H, a self-adjoint operator $\mathscr{X}$ on H, and a vector $f \in H$, such that, in the above notation, $P_f = \mu$. Namely, take $H = \mathscr{L}^2_\mu(\mathbb{R})$, $(\mathscr{X}(g))(x) = x \cdot g(x)$, and $f(x) = 1$ for all $x \in \mathbb{R}$.

We say that the spectrum P of $\mathscr{X}$ as above is *continuous* if $P(E) = 0$ whenever E has Lebesgue measure zero; we say that P has a *pure point* spectrum if there is a countable set of points $\{\lambda_i\}_{i \in \mathbb{Z}}$ such that $P(\{\lambda_i\}_{i \in \mathbb{Z}}) = \mathrm{Id}$. We say that P has a *mixed* spectrum if there are both points $x \in \mathbb{R}$ such that $P(\{x\}) \neq 0$ and Borel subsets $B \subset \mathbb{R}$ on which P is continuous and non-zero. Of course, there are more possibilities: P may be concentrated on a set of Lebesgue measure zero without having atoms, i.e., without having points x with $P(\{x\}) \neq 0$.

For completeness, and for later reference, here we also state the spectral theorem for *unitary operators*: For each unitary operator $U: H \to H$ there is a unique projection-valued measure P_U on the complex unit circle $\mathbb{S}^1 \subset \mathbb{C}$ such that for each $f \in H$, one has $\langle U(f), f \rangle = \int_{\mathbb{S}^1} s \cdot dP_f(s)$. U is then isometrically equivalent with the operator $\mathscr{U}$ on $\mathscr{L}^2_{P_f}(\mathbb{S}^1)$ which maps $g(z)$ to $zg(z)$. Combining the two spectral theorems with $\phi_t = e^{i\mathscr{X}t}$, we see that the spectra $P_{\mathscr{X}}$ and P_{ϕ_t} of $\mathscr{X}$ and ϕ_t are related by $P_{\phi_t} = (\lambda_t)_* P_{\mathscr{X}}$, where $\lambda_t: \mathbb{R} \to \mathbb{S}^1$ is given by $\lambda_t(x) = e^{ixt}$. This implies that $\mathscr{X}$ has a continuous, a pure point, or a mixed spectrum if and only if the same holds for ϕ_t, for all $t \neq 0$.

2.3. The power spectrum

Let M, Φ_t, Y, K, μ, B etc. be as defined in Subsection 2.1. For each $s \in \mathbb{R}$ and $x \in B'$, we define

$$\sigma(s) = \lim_{T \to \infty} T^{-1} \int_0^T Y(\Phi_t(x)) \cdot Y(\Phi_{t+s}(x)) \cdot dt$$

$$= \int_K Y \cdot (Y \circ \phi_s) \cdot d\mu.$$

We call σ the autocorrelation function. Since $Y(x) \cdot Y(\Phi_s(x))$ as a function of (x, s) is smooth, so is σ. In terms of $\phi_s: \mathscr{L}^2_\mu(K) \to \mathscr{L}^2_\mu(K)$ as investigated in Subsection 2.2, we have

$$\sigma(s) = \langle \phi_s(Y), Y \rangle = \int_{-\infty}^{+\infty} e^{ixs} \cdot dP_Y(x).$$

So σ has a Fourier transform in the sense of a (positive) measure, which equals the spectral measure associated with the function Y as a vector in $\mathscr{L}^2_\mu(K)$.

From the theory of time series [P, 1981] we know that this Fourier transform P_Y of σ is the power spectrum (or the non-normalized power spectral density function, in this reference) of the time series $Y(\Phi_t(x))$ for any $x \in B'$. This can be interpreted as the energy density as a function of the frequency, and it is calculated (up to normalization) as the squared norm of the Fourier transform of the time series. For complete details see the last reference. Below we shall explain, without entering the convergence problems, why this relation holds.

First we recall that the Fourier transform $\hat{f}$ of f is given by

$$\hat{f}(\omega) = (2\pi)^{-1} \int_{-\infty}^{+\infty} e^{-i\omega t} f(t)\, dt,$$

which implies that

$$f(t) = \int_{-\infty}^{+\infty} e^{i\omega t} \hat{f}(\omega)\, d\omega.$$

We use the following notation. If $y(t)$ is a time series, for example, $y(t) = Y(\Phi_t(x))$, defined for all $t \in \mathbb{R}$, or for all $t \geqq 0$, then $y_T(t)$ is the time series defined by

$$y_T(t) = \begin{cases} y(t) & \text{if } t \in [0, T], \\ 0 & \text{otherwise}. \end{cases}$$

For such a time series $y(t)$ we define

$$\sigma_T(s) = T^{-1} \int_{-\infty}^{+\infty} y_T(t) \cdot y_T(t+s)\, dt,$$

and h_T by

$$h_T(\omega) = \frac{2\pi}{T} |\hat{y}_T(\omega)|^2.$$

Clearly, σ_T converges to the autocorrelation function σ as $T \to \infty$. The function $h_T(\omega)$, or rather its limit for $T \to \infty$, by definition is the non-normalized power spectral density function, although this limit in general only exists as a measure. We have

$$\begin{aligned} h_T(\omega) &= \frac{2\pi}{T} |\hat{y}_T(\omega)|^2 \\ &= (2\pi T)^{-1} \int \int_{-\infty}^{+\infty} e^{-i\omega(t'-t)} \cdot y_T(t') \cdot y_T(t)\, dt\, dt' \\ &= (2\pi)^{-1} \int_{-\infty}^{+\infty} e^{-i\omega s} \cdot T^{-1} \left(\int_{-\infty}^{+\infty} y_T(t) \cdot y_T(t+s)\, dt \right) ds = \hat{\sigma}_T(\omega). \end{aligned}$$

Passing to the limit, we get $h(\omega) = \hat{\sigma}(\omega)$. So for $x \in B'$, and $y(t) = Y(\Phi_t(x))$, the spectral density functions h_T converge, in the sense of measures, to the spectral measure P_Y as defined in Subsection 2.2.

2.4. Spectra of attractors of dynamical systems

It is not hard to see that periodic and quasi-periodic attractors have pure point spectra. This will be discussed in the next section. For non-periodic Axiom A attractors, e.g., see [B, 1977], the situation was investigated by SINAÏ [S, 1968], RUELLE [R, 1976], and BOWEN & RUELLE [BR, 1975].

For 'connected' Axiom A attractors of diffeomorphisms (see [R, 1976]) the autocorrelation of a smooth function (observable) converges exponentially to a constant value. This implies that the corresponding spectral density also is given by a smooth function, so we have a continuous spectrum. For a 'disconnected' Axiom A attractor, i.e., an attractor $K = K_1 \cup \ldots \cup K_l$ which is the disjoint union of compact sets K_i such that $\varphi(K_i) = K_{i+1}$, $i = 1, \ldots, l-1$, and $\varphi(K_l) = K_1$, the spectrum is the superposition of a continuous part, corresponding to $\varphi^l \mid K_1$, and a part concentrated in points, corresponding to the period l.

In the case of Axiom A attractors of flows, the situation is more complicated; see [BR, 1975]. First we have to introduce the notion of C-denseness. We say that the flow Φ_t is C-dense on the Axiom A attractor K if for any $x \in K$, $W^u(x) \cap K$ is dense in K. Here $W^u(x)$ denotes the unstable manifold of x. For a C-dense Axiom A attractor, the autocorrelation of an 'observable' converges to a constant (but it is not known how fast). This implies that the spectrum has no atoms, but does not yet imply that the spectrum is continuous. Still we expect generic Axiom A attractors of flows only to have continuous spectra. The condition of C-denseness above is related to the fact that an Axiom A attractor K of a flow can have a mixed spectrum if it is a *pure suspension*, i.e., if there is a codimension-one manifold N, transverse to the flow and intersecting K, and a constant t_0 such that for all $x \in K \cap N$, $\Phi_{t_0}(x) \in K \cap N$. In this case the point spectrum corresponds to the period t_0.

For the more complicated attractors, e.g., the examples of HÉNON [H, 1976], LORENZ [L, 1963], and RÖSSLER [R, 1979], there are, as far as we known, no rigorous results. So there are no firm arguments for the sharp peaks, as reported in [FCFPS, 1980], to correspond to point spectra or not; however, it seems most likely that they do not.

3. A persistent $SO(2)$-invariant attractor with mixed spectrum

In this section we consider a sequence of examples, of increasing complexity, leading to the announced skew quasi-periodic attractor.

3.1. A 1-quasi-periodic attractor of a diffeomorphism – persistence

We consider a C^∞-diffeomorphism $\varphi : P \to P$ of a manifold, defining a dynamical system with discrete time. We say that φ has a *1-quasi-periodic attractor* if there is a smooth closed curve $S \subset P$ such that

(i) $\varphi(S) = S$;

(ii) $\varphi | S$ is smoothly conjugate to a rotation over an angle $2\pi\alpha$, with α irrational and, moreover, satisfying the Diophantine condition below;

(iii) for normal vectors $v \neq 0$, $v \in N_x(S) = T_x(P)/T_x(S)$, the sequence $\| d\varphi^n(v) \|$ decreases exponentially in n, uniformly in v.

We introduce the *Diophantine condition* or *DC* as follows: α satisfies the DC if for some $\gamma > 0$, $\sigma \geqq 2$ and if for all $k_1 \in \mathbb{Z} - \{0\}$, $k_2 \in \mathbb{Z}$,

$$\left| \alpha - \frac{k_2}{k_1} \right| \geqq \gamma |k_1|^\sigma .$$

Note that one also defines the notion of quasi-periodic attractor without including the DC. We included it in order to obtain persistence.

From the fact that $\varphi | S$, as above, is smoothly conjugate to a Diophantine irrational rotation, it follows that there is a unique invariant Borel probability measure on S, e.g., see [CFS, 1982], i.e., $\varphi | S$ is uniquely ergodic. This probability measure is denoted by m. It follows from the unique ergodicity and the theory of normal hyperbolicity [HPS, 1977] that for a full neighbourhood B of S we even have that

$$\lim_{N \to \infty} N^{-1} \sum_{i=0}^{N-1} g(\varphi^i(x)) = \int_S g \, dm$$

holds, for all $x \in B$ and all continuous $g : P \to \mathbb{R}$. So S, with the above measure m, is an attractor in the sense of the previous section.

Although a 1-quasi-periodic attractor is not *persistent* under small perturbations, it still is persistent in the following sense. For any generic k-parameter family $\varphi_{\mu_1, \ldots, \mu_k} : P \to P$ with $\varphi_{0, \ldots, 0} = \varphi$, we have, for any $\mu = (\mu_1, \ldots, \mu_k)$ near zero, a normally attracting smooth invariant closed curve S_μ. Furthermore, the set of μ-values for which $\varphi_\mu | S_\mu$ is smoothly conjugate to a rotation over $2\pi\alpha(\mu)$, with $\alpha(\mu)$ irrational and satisfying the DC, has $\mu = 0$ as a point of density in the sense of Lebesgue. The present generic condition can be made explicit in terms of the derivative of φ_μ with respect to μ at $\mu = 0$: Informally it means that the rotation number $\alpha(\mu)$ should have a non-zero derivative for $\mu = 0$. These persistence properties are discussed in [BHTB, 1990]. We point out that an (irrational) rotation on the circle has a pure point spectrum. Indeed, if $\varphi : \mathbb{S}^1 \to \mathbb{S}^1$ is given by $\varphi(x) = x + \alpha \bmod 1$, the Lebesgue measure m is invariant (if the rotation is irrational, it is the only invariant Borel probability measure) and there is an orthogonal basis of eigenvectors of $\mathscr{L}^2_m(\mathbb{S}^1)$:

$$\{ e^{2\pi i n x} \}_{n=-\infty}^{+\infty}$$

with eigenvalues $e^{2\pi i n \alpha}$.

3.2. *A 2-quasi-periodic attractor of a flow, a normal form*

The situation we consider here is the suspension of the example in the previous subsection. In other words, we consider a smooth vector field Z on a manifold Q, together with a codimension-one submanifold $P \subset Q$ with the following property. A *return* or *Poincaré map* $\varphi : P \to P$ is defined, i.e., a map φ such that the positive orbit of Z, starting in $x \in P$, has its first intersection with P in $\varphi(x)$. This return map φ need not be defined on all of P, but wherever it is defined, it is invertible. We say that Z has a 2-quasi-periodic attractor if a return map φ has a 1-quasi-periodic attractor. We assume this to be the case, and denote the attracting closed curve for φ by S and the corresponding attracting torus for Z by T. Such a 2-quasi-periodic attractor for a vector field is persistent in the same sense as the 1-quasi-periodic attractor discussed before. Compared with the previous subsection there is one complication: $Z|T$ is not completely described by $\varphi|S$. For a complete description we also need the *return time* $\tau : S \to \mathbb{R}_+$, which is formally defined as follows. If $x(t)$ is a Z-integral curve with $x(0) \in S$, then $\tau(x(0))$ is the smallest positive real number such that $x(\tau(x(0))) \in S$. We do *not* assume the return time to be *constant* (as we did for the 'pure suspension' mentioned in Subsection 2.4).

We shall now show that, by a proper choice of a (different) section $S' \subset T$, we can ensure the return map, with respect to S', is constant. We identify S with $\mathbb{R}/\mathbb{Z}$ such that $\varphi|S$ is given by $x \mapsto x + \alpha$, α satisfying the DC. For a function $g : S \to \mathbb{R}$ (or $g : \mathbb{R} \to \mathbb{R}$ with period one), a new section S_g in T is defined by

$$S_g = \{\Phi_{g(x)}(x) \,|\, x \in S\},$$

where Φ_t denotes the flow of Z. The return time for S_g is easily seen to be

$$\tau_g(x) = \tau(x) + g(x + \alpha) - g(x),$$

where also S_g is identified with $\mathbb{R}/\mathbb{Z}$ in such a way that $x \in S$ and $\Phi_{g(x)}(x)$ correspond to the same element of $\mathbb{R}/\mathbb{Z}$. This last equation is called the *homological equation*. We want to find, for a given function τ, a function g such that the corresponding function τ_g is constant.

Solution of the homological equation. The solution of this type of equation is discussed in many places, e.g., see [A, 1983]. Since we shall encounter this equation several times in this article, we include a discussion of the existence and uniqueness of its solutions.

We write in Fourier series:

$$\tau(x) = \sum a_n e^{2\pi i n x}, \qquad g(x) = \sum b_n e^{2\pi i n x}.$$

By the Paley-Wiener theorem, τ is in C^∞ if and only if for each k, $\lim_{n\to\pm\infty} a_n n^k = 0$. We assume τ to be in C^∞. In order to get τ_g constant, the Fourier coefficients b_n of g have to satisfy

$$a_n - b_n + b_n \cdot e^{2\pi i n\alpha} = 0$$

for all $n \neq 0$. So we have to take $b_n = a_n / (1 - e^{2\pi i n\alpha})$. Since we assumed that α satisfies the DC, this implies that also $\lim_{n\to\pm\infty} b_n \cdot n^k = 0$ for all k. So g, defined by the Fourier coefficients b_n, is in C^∞ and, up to an additive constant, the unique solution making τ_g constant.

Since we can make the return time constant, we can choose suitable coordinates $(x, y) \in \mathbb{R}^2/\mathbb{Z}^2$ on T such that $Z|T = a\dfrac{\partial}{\partial x} + b\dfrac{\partial}{\partial y}$, with $a/b = \alpha$. Now it follows that $Z|T$ is uniquely ergodic, meaning that it has only one invariant Borel probability measure m, which is the Lebesgue measure with respect to x, y. Therefore here we also have an attractor for which the sets B and B', as in the definition of Subsection 2.1, can be taken equal. If $i\mathscr{X}$ denotes the infinitesimal generator of the one-parameter group of unitary transformations in $\mathscr{L}^2_m(T)$ induced by $\Phi_t|T$, then for $\mathscr{X}$ we have an orthonormal basis of eigenfunctions $e^{2\pi i n x} \cdot e^{2\pi i m y}$ with eigenvalues $2\pi(na + mb)$: Here we also have a pure point spectrum.

3.3. *An SQP attractor for $SO(2)$-equivariant diffeomorphisms*

We return to diffeomorphisms, but now we assume that they are equivariant with respect to, i.e., commute with, a given $SO(2)$-action. So let N be a manifold with an $SO(2)$-action. We only consider the region where the action is free, so we may just as well assume the $SO(2)$-action to be free on all of N. We can then identify points on the same $SO(2)$-orbit, thus obtaining a projection $\pi: N \to P$. Clearly π is a circle bundle, even a principal $SO(2)$-bundle; e.g., see [H, 1966]. Let $\Psi: N \to N$ be a diffeomorphism commuting with the $SO(2)$-action, i.e., such that for all $g \in SO(2)$, $g \circ \Psi = \Psi \circ g$, where we identify the elements of $SO(2)$ with the corresponding transformations in N. For such a Ψ there is a 'projection' $\varphi: P \to P$ such that $\pi \circ \Psi = \varphi \circ \pi$. We now assume that φ has a 1-quasi-periodic attractor S as in Subsection 3.1. We denote the corresponding attracting set in N by $L = \pi^{-1}(S)$. This situation is as persistent as the 1-quasi-periodic attractor in Subsection 3.1.

For suitable coordinates $(x, z) \in \mathbb{R}^2/\mathbb{Z}^2$ on L we have

$$\pi(x, z) = x,$$

$$\Psi(x, z) = (x + \alpha, z + f(x)),$$

where α is an irrational number satisfying the DC, and where $f: \mathbb{R} \to \mathbb{R}$ is a smooth function such that $f(x+1) - f(x) \in \mathbb{Z}$, and hence is constant. We call this integer the *twisting constant* of the attracting set L; in the calculations below it is denoted by k. For most of what follows, we assume the twisting constant to be *non-zero*. Also this last assumption clearly is persistent under small pertubations of Ψ.

Next we want to find coordinates

$$\tilde{x} = x + c \bmod 1, \quad \tilde{z} = z + h(x) \bmod 1,$$

on L such that the function $\tilde{f}$, corresponding to the function f in the above representation of Ψ, gets simplified. In the $\tilde{x}$, $\tilde{y}$ coordinates we have

$$\Psi(\tilde{x}, \tilde{z}) = \left(\tilde{x} + \alpha, \tilde{z} + f(\tilde{x} - c) + h(\tilde{x} - c + \alpha) - h(\tilde{x} - c)\right),$$

so that

$$\tilde{f}(\tilde{x}) = f(\tilde{x} - c) + h(\tilde{x} - c + \alpha) - h(\tilde{x} - c).$$

As in the previous subsection, using Fourier expansions for $h(\tilde{x})$ and $(\tilde{f}(\tilde{x}) - k\tilde{x})$, where k is the twisting constant, and the Payley-Wiener theorem, we find c and h such that for $k = 0$, $\tilde{f}(\tilde{x})$ is constant, and for $k \neq 0$, $\tilde{f}(\tilde{x}) = k \cdot \tilde{x}$.

For $k \neq 0$, the above normal form shows that $\Psi|L$ is a skew product; see [CFS, 1982]. This implies that $\Psi|L$ is uniquely ergodic (implying that the attracting set L has an SBR measure and hence is an attractor) and that $\Psi|L$ has a mixed spectrum. This last fact can be understood in the following way: Because L, including its dynamics, projects on a 1-quasi-periodic attractor, there are atoms in the spectrum; the continuous part is due to the fact that, given the present value of the $\tilde{z}$-coordinate, the next value of this coordinate is completely unpredictable (or that successive values of the $\tilde{z}$-coordinate are completely uncorrelated). For a more formal proof we construct a basis for $\mathscr{L}^2_m(L)$. Denote the coordinates for which the normal form holds by x and z (instead of $\tilde{x}$ and $\tilde{z}$), so that we have $\Psi(x, z) = (x + \alpha, z + kx)$. Then the invariant measure m is the Lebesgue measure with respect to x and z. We take the basis

$$b_{l,n} = e^{2\pi i l x} \cdot e^{2\pi i n z}$$

with $l, n \in \mathbb{Z}$. Then Ψ^* transforms $b_{l,n}$ to $e^{2\pi i \alpha l} \cdot b_{(l+kn),n}$. This implies, for $n = 0$, that $e^{2\pi i \alpha l}$, $l \in \mathbb{Z}$, are eigenvalues. On the other hand, for $n \neq 0$, we find orthonormal systems

$$\ldots, e^{2\pi i \alpha k n} \cdot b_{-kn,n}, b_{0,n}, b_{kn,n}, \ldots$$

such that Ψ^* maps each element to the next. The action of Ψ^*, restricted to the linear span of such an orthonormal system, is called a Lebesgue component and has a continuous spectrum.

This last point follows from the spectral theorem and the fact that $H = \mathscr{L}^2_m(\mathbb{S}^1)$, m the Lebesgue measure, has an orthonormal base $\{e^{2\pi i n s}\}$ such that the transformation $\mathscr{Y}$, which maps the function $g(s)$ to $e^{2\pi i s} g(s)$ (or $g(z)$ to $zg(z)$ if we consider $\mathbb{S}^1$ as the unit circle in $\mathbb{C}$), also maps each base vector to its successor.

We call L a *skew-quasi-periodic* (SQP) attractor.

Summary. We constructed an $SO(2)$-invariant attractor of a diffeomorphism Ψ with a mixed spectrum that is as persistent as a 1-quasi-periodic attractor for diffeomorphisms without symmetry. On such an attractor there are C^∞-coordinates x and z with values in $\mathbb{R}$ mod 1 such that $\Psi(x, z) = (x + \alpha, z + kx)$ for some integer $k \neq 0$. Such an attractor is called an SQP attractor.

3.4. An SQP attractor for an SO(2)-invariant flow

As we suspended the 1-quasi-periodic attractor of a diffeomorphism to obtain a 2-quasi-periodic attractor of a flow, we now consider a suspension of the SQP attractor constructed in the previous subsection. The relation between all these constructions is given in Figure 1.

So let M be a manifold with a free $SO(2)$-action, an $SO(2)$-invariant vector field X, i.e., for $g \in SO(2)$, $g_*(X) = X$, and a codimension-one $SO(2)$-invariant submanifold N. Let $\Psi : N \to N$ be a return map for X. The orbit manifolds of M and N are denoted by Q and P, respectively; the induced map in P is denoted by φ and the induced vector field in Q by Z.

$$\begin{tikzcd}[row sep=large]
\text{diffeomorphisms (return maps)} & & \text{vector fields} & \\
L \subset N^{\Psi} \arrow[d, "\pi"] & \subset & M^X \supset K \arrow[d, "\pi"] & \text{free } SO(2)\text{-action} \\
 & & & \text{projections to orbit manifolds} \\
S \subset P_{\varphi} & \subset & Q_Z \supset T & \text{no group action}
\end{tikzcd}$$

Figure 1. Relation between different attractors: left diffeomorphisms, right vector fields, below with no group action, above with free $SO(2)$-action.

Now we impose the following conditions. The map φ has a 1-quasi-periodic attractor S with rotation number α satisfying the DC, and $\Psi | \pi^{-1}(S)$ has a non-zero twisting constant k. We have seen that when these conditions are satisfied, they are persistent in the sense of Subsection 3.1. We denote the corresponding attractors of Ψ, Z and X by L, T and K, respectively. It follows from Subsection 3.2 that for suitable coordinates $(x, y) \in \mathbb{R}^2/\mathbb{Z}^2$ on T, we have $Z | T = a_1 \partial/\partial x + a_2 \partial/\partial y$ with $a_1/a_2 = \alpha$. By rechoosing the sections P and N, if necessary, we obtain the situation where $S = T \cap P = \{y = 0\}$. Then for suitable coordinates $(x, y) \in \mathbb{R}^2/\mathbb{Z}^2$ on L, where we identify x and $x \circ \pi$, we have

$$\Psi(x, z) = (x + \alpha, z + kx) \bmod \mathbb{Z}^2,$$

with $k \neq 0$. Then it is clear from the theory of skew products (e.g., see [CFS, 1982]) that $X | K$ is uniquely ergodic, that K is an attractor in the sense of Subsection 2.1, and that it has a mixed spectrum, see Subsection 3.3. Also K is called *skew-quasi-periodic* or SQP.

A final remark is the following. The projection $\pi | K : K \to T$ is an $SO(2)$-bundle. Since the twisting constant is non-zero, this bundle is *non-trivial*. So not every manifold with an $SO(2)$-action admits an SQP attractor. We return to this in the next section.

4. Global considerations

We here review some general results from the theory of principal fibre bundles, which will only be formulated for $SO(2)$-bundles, and discuss the consequences for our SQP attractors in terms of possible existence and bifurcations. As a general reference for (principal) fibre bundles we recommend [H, 1966], which contains all the proofs of the facts we recall here. Except when explicitly stated otherwise, all the topological spaces considered are manifolds, either finite-dimensional or modelled on Hilbert or Banach spaces.

4.1. SO(2)-bundles

We say that a space E with a *free* $SO(2)$-action, which is *locally trivial*, is a *(principal) SO(2)-bundle*. The action being free means that for $g \in SO(2)$ and $e \in E$, $g(e) = e$ implies that g is the identity element in $SO(2)$. The action is locally trivial if each orbit, or fibre $F_0 = SO(2) \cdot e_0$, has a neighbourhood W which is homeomorphic to $U \times SO(2)$, for some space U, and if there is a homeomorphism $h: U \times SO(2) \to W$ such that $g_1(h(u, g_2)) = h(u, g_1 g_2)$ for all $u \in U$, $g_1, g_2 \in SO(2)$. We say that the $SO(2)$-bundle is *trivial* if the neighbourhood W above can be taken equal to E. For an $SO(2)$-bundle as above, the space B of $SO(2)$-orbits is called its *base space*, and its *projection* is $\pi: E \to B$, mapping points to their orbit; E is the *total space*. Sometimes the bundle is denoted by its projection.

For a continuous map $f: X \to B$, with B the base space of an $SO(2)$-bundle as above, there is an induced $SO(2)$-bundle with total space

$$f^*(E) = \{(x, e) \in X \times E \mid f(x) = \pi(e)\},$$

and with the obvious $SO(2)$-action. If the maps $f_1, f_2: X \to B$ are *homotopic*, i.e., if they can be continuously deformed into each other, then $f_1^*(E)$ and $f_2^*(E)$ are *equivalent* in the sense that there is a homeomorphism $h: f_1^*(E) \to f_2^*(E)$ such that for all $e \in f_1^*(E)$, $g \in SO(2)$,
(i) $\pi_1(e) = \pi_2(h(e))$,
(ii) $h(g(e)) = g(h(e))$,
where π_i is the projection of $f_i^*(E)$ to X.

We say that an $SO(2)$-bundle E, with projection $\pi: E \to B$, is *universal* if for each space X there is a one-to-one correspondence between homotopy classes of continuous maps $f: X \to B$ and equivalence classes, in the above sense, of $SO(2)$-bundles with base space X. There is a general theorem asserting the existence of such universal bundles – these universal bundles are characterized by the property that their total space is *contractible*, in the sense that all their homotopy groups are zero. An explicit construction of a universal $SO(2)$-bundle with total space, projection, and base $E_{SO(2)}$, $\pi_{SO(2)}$, and $B_{SO(2)}$, respectively, is the following. Let H be a complex separable Hilbert space. We take $E_{SO(2)}$ to be the unit sphere in H, which happens to be contractible. The $SO(2)$-action is defined by the scalar multiplication with complex numbers of norm one. The base space $B_{SO(2)}$ then is the infinite-dimensional complex projective space. Its fundamental group is trivial, implying that any $SO(2)$-bundle over the circle $\mathbb{S}^1$ is trivial, and $H^2(B_{SO(2)}; \mathbb{Z}) \cong \mathbb{Z}$. For any orientable closed surface M and any continuous $f: M \to B_{SO(2)}$, the bundle $f^*(E_{SO(2)})$, and hence the homotopy class of f, is completely determined by $f^*(c) \in H^2(M; \mathbb{Z}) \cong \mathbb{Z}$, where c is a fixed generator of $H^2(B_{SO(2)}; \mathbb{Z})$. The element $f^*(c)$ is called the *Chern class* of the bundle induced by f.

Combining the above facts, we see that for any $SO(2)$-bundle $\pi : E \to B$, there is a continuous $f: B \to B_{SO(2)}$ such that $f^*(E_{SO(2)})$ is equivalent to E. If now f is a homeomorphism onto its image, then the bundle E is even equivalent to the *subbundle* $\pi_{SO(2)}^{-1}(f(B))$. If B is a closed 2-manifold, then every continuous $f: B \to B_{SO(2)}$ can be approximated by an embedding, because $B_{SO(2)}$ is infinite-dimensional. So any $SO(2)$-bundle over a closed 2-manifold is a subbundle of the universal bundle, in fact of any universal bundle whose base space is a manifold.

Finally, for later use we observe that as long as we are interested only in $SO(2)$-bundles having as their base a (closed) surface we may use, instead of a universal bundle, any $SO(2)$-bundle whose total space is 2-connected, meaning that its first and second homotopy groups are zero.

4.2. Special SO(2)-actions

For physical systems in the three-dimensional Euclidean space having rotational symmetry, say around the z-axis, and whose time evolution can be described by partial differential equations, the state space is usually as described below. In particular, this description is valid for the Couette-Taylor flow.

Let D be a compact domain (i.e., D is the closure of its interior) in $\mathbb{R}^3$, invariant under rotations around the z-axis. We think of D as the spatial domain of our system; each possible state is given by a function, or a vector field, or another such 'field' defined on D, possibly satisfying some rotationally invariant boundary conditions. These fields together form the state space $\mathscr{H}$. We assume that $\mathscr{H}$ has the structure of an infinite-dimensional linear or affine space. The $SO(2)$-action (rotation around the z-axis) induces an affine $SO(2)$-action in $\mathscr{H}$. The topology of $\mathscr{H}$, as a topological vector space, will be chosen so that the evolution equation makes sense. In all the usual cases, however, corresponding to, e.g., L^p or Sobolev norms, we have the following property (P):

(P) Let

$$\mathscr{H}_0 = \{X \in \mathscr{H} \mid X \text{ is not invariant under any } g \in SO(2),\ g \neq \mathrm{Id}\},$$

$$\mathscr{H}_1 = \{X \in \mathscr{H} \mid X \text{ is invariant under all } g \in SO(2)\},$$

and for $i > 2$,

$$\mathscr{H}_i = \{X \in \mathscr{H} \mid X \text{ is invariant under the subgroup of } SO(2) \text{ of order } i\}.$$

Then each $\mathscr{H}_i$, for $i \geqq 1$, is a closed linear, or affine, subspace of infinite dimension and infinite codimension. Moreover $\bigcup_{i=1}^{\infty} \mathscr{H}_i$ is closed.

We assert that $\mathscr{H}_0$ contains every $SO(2)$-bundle (over a surface) as a subbundle. Indeed from (P) and the fact that $\mathscr{H}_0$ is the complement of $\bigcup_{i=1}^{\infty} \mathscr{H}_i$ it follows that $\mathscr{H}_0$ is contractible: Any k-sphere in $\mathscr{H}_0$ can be contracted in $\mathscr{H}$, and by transversality this contraction can be made to avoid each $\mathscr{H}_i$, for $i \geqq 1$. On the other hand, $\mathscr{H}_0$ is just the part on which $SO(2)$ acts freely, so, apart from the local triviality, $\mathscr{H}_0$ is a universal bundle and hence contains any $SO(2)$-bundle (over a closed surface) as a subbundle. Because the $SO(2)$-action induced in the above way is not differentiable, local triviality is harder to prove, and therefore we proceed differently. For the finite-dimensional case see [W, 1969].

We assume, which is realistic in the present context, that there are infinite-dimensional linear subspaces $F_n \subset \mathscr{H}$, $n = 0, 1, 2, \ldots$ such that

- $F_0 = \mathscr{H}_1$;
- F_n is $SO(2)$-invariant and the $SO(2)$-action on F_n is equivalent (in the sense of *real* vector spaces) with complex multiplication by z^n in a complex Banach space (with $SO(2)$ identified with the complex numbers with norm one);
- each element in $\mathscr{H}$ can be approximated by elements which are finite sums of elements $f_0 \in F_0$, $f_1 \in F_1$, etc.

In the case where $\mathscr{H}$ is a space of functions on D, F_n contains those functions which, in cylindrical coordinates, have the form $g_1(z, r) \cos n\varphi + g_2(z, r) \sin n\varphi$. It is clear that the $SO(2)$-action is smooth on each F_n, and also on each $\bar{F}_n = \bigoplus_{m=0}^{n} F_m$. Furthermore, apart from $F_0 = \mathscr{H}_1$, we have for $i \geqq 2$, and $n \geqq 1$, that $\bar{F}_n \cap \mathscr{H}_i$ has infinite codimension in $\bar{F}_n$, so $\bar{F}_n \cap \mathscr{H}_0$ is contractible for $n \geqq 1$, and hence a universal $SO(2)$-bundle. Hence $\mathscr{H}_0$ contains any $SO(2)$-bundle as a subbundle.

4.3. Obstruction to bifurcations

The purpose of this subsection is to show, under some mild conditions, that, in an $SO(2)$-equivariant dynamical system, no direct bifurcation exists from a 2-quasi-periodic attractor with non-trivial $SO(2)$-symmetry to an SQP attractor.

First, we have to say something about the manifolds (and $SO(2)$-actions) to be allowed as state spaces. In the previous subsection we argued that we have to allow Banach spaces with continuous but non-differentiable $SO(2)$-actions. Technically this is inconvenient. However, near compact attractors (and their bifurcations) one often can construct finite-dimensional invariant manifolds, attracting all evolutions, so-called *centre manifolds*, see [HPS, 1977]. Also, for certain partial differential equations, there is even a global analogue in the form of *inertial manifolds*; see [MS, 1987]. In all these cases one can replace the infinite-dimensional state space by a finite-dimensional one, in which the $SO(2)$-action and the evolution are smooth. In the following we assume such a reduction to have been carried out and hence assume our state space to be a finite-dimensional manifold and both our flow and $SO(2)$-action to be smooth.

A bifurcation of a 2-quasi-periodic attractor to an SQP attractor in a manifold M with $SO(2)$-action would lead to a one-parameter family K_μ of compact sets, varying continuously with respect to the Hausdorff metric, such that for $\mu \leqq 0$, K_μ is a torus, invariant under the $SO(2)$-action, and for $\mu > 0$, K_μ is a non-trivial $SO(2)$-bundle over a torus. Below we shall prove that the existence of such a family K_μ is impossible if we require that the induced $SO(2)$-action on K_μ, for $\mu \leqq 0$, is non-trivial.

First, in the case where the action on K_0 is free, and hence free in a neighbourhood of K_0, the argument is simple. We restrict our attention to the part $M_0 \subset M$ on which the action is free. Let $\pi : M_0 \to B_0$ denote the projection on the orbit space. The projection $\pi(K_0)$ of K_0 is a closed curve and has a neighbourhood U such that $\pi^{-1}(U)$ is a trivial $SO(2)$-bundle. But then $\pi^{-1}(U)$ cannot contain a non-trivial subbundle like K_μ, $\mu > 0$.

In the case where the action on K_0 is not free, we consider the vector field Y on M which generates the $SO(2)$-action, i.e., if $SO(2)$ is identified with the complex numbers of norm one, $Y(x)$ is the tangent vector of the curve $s \mapsto e^{is}(x)$. Since we assume that the $SO(2)$-action on K_0 is neither trivial nor free, there is an integer l such that for each $x \in K_0$, $e^{is}(x) = x$ for $s = 2\pi/l$ and $e^{is}(x) \neq x$ for $x \in (0, 2\pi/l)$. Now we take a smooth closed curve $S \subset K_0$, everywhere transversal to Y and intersecting each $SO(2)$-orbit only once. Then we extend S to a codimension-one (open) manifold Σ in M, transversal to K_0, and to Y; we define U to be the union of all $SO(2)$-orbits through Σ. Clearly, U is a neighbourhood of K_0. Finally we define a projection

$$\Pi : \Sigma \times SO(2) \to U$$

by $\Pi(x, e^{is}) = e^{is}(x)$, again identifying $SO(2)$ with the complex numbers with norm one. It is not hard to see that the number $l(x)$ of points in $\Pi^{-1}(x)$ is the largest integer such that $e^{i(2\pi/l(x))}(x) = x$. So $\tilde{K}_\mu = \Pi^{-1}(K_\mu)$ for $\mu > 0$ is homeomorphic to K_μ. Also $\tilde{K}_\mu = \Pi^{-1}(K_\mu)$ for $\mu \leqq 0$ is a torus, but now with a free $SO(2)$-action. Still $\tilde{K}_\mu$ depends continuously on μ. We saw before that this is impossible.

4.4. Skew Hopf bifurcation

In this subsection we show that it is possible to have a bifurcation of a 2-quasi-periodic attractor with *trivial* $SO(2)$-action to an SQP attractor. This is called a skew Hopf bifurcation. In the usual Hopf bifurcation the dimension of the attractor goes up by one, and one frequency is added; here the dimension also goes up by one, but the attractor becomes a *skew* product. For persistence and normal forms of this bifurcation, see the next section.

As in the previous subsection, M is a manifold with an $SO(2)$-action, while $M_1 \subset M$ denotes the set of points in M on which $SO(2)$ acts trivially. We assume that M_1 contains a 2-torus T. Also we assume that we can choose in the normal bundle of T a trivial subbundle N of dimension 4 on which $SO(2)$ acts freely. Note that it is possible to find in $\bar{F}_n$, $n \geqq 2$, as defined in Subsection 4.2, a manifold M with induced $SO(2)$-action in which all the above objects can be realized. Since the $SO(2)$-action is free in N, we can introduce a complex structure in its fibres so that the $SO(2)$-action corresponds to complex multiplication. Since N is trivial, we can split it into two trivial 1-dimensional complex bundles N_1 and N_2 which we identify with $T \times \mathbb{C}$. Now we want to construct a *non-trivial* complex 1-dimensional subbundle W in N. For this we need a map $w: T \to P^1(\mathbb{C})$, $P^1(\mathbb{C})$ being the complex projective line, whose elements we denote by pairs $[z_1 : z_2]$, not both equal to zero, while $[z_1 : z_2]$ and $[\lambda z_1 : \lambda z_2]$ define the same element. The bundle corresponding to w is $W = \{(x, u_1, u_2) \,|\, z_1(x) \cdot u_1 + z_2(x) \cdot u_2 = 0\}$, where $w(x) = [z_1(x) : z_2(x)]$, and where $x \in T$, $u_1, u_2 \in \mathbb{C}$, with (x, u_1), (x, u_2) representing elements in N_1, N_2 respectively. Then W is non-trivial if and only if the degree of w is non-zero, i.e., if and only if w is not homotopic to zero. In fact, $P^1(\mathbb{C})$ can be interpreted as the base of a 'universal' $SO(2)$-bundle for $SO(2)$-bundles over surfaces (see the remark at the end of Subsection 4.1), the total space of this 'universal' bundle being the unit sphere in $\mathbb{C}^2$, and hence 2-connected. Since W is a *complex* bundle, it is, as a real bundle with 2-dimensional fibres, invariant under the action of $SO(2)$.

A bifurcating attractor now can be obtained as follows. Take a one-parameter family X_μ of $SO(2)$-invariant vector fields on M such that T is invariant and such that on T we have a quasi-periodic flow. Furthermore, we take X_μ such that the bundle W is invariant under the derivative of the flow defined by X_μ. Finally we arrange that the (normal) derivative of the flow is attracting to T (in all directions) for $\mu < 0$ and such that for $\mu = 0$ the attractions in the W directions, and only in these directions, become repelling. Furthermore, we assume the higher-order terms to be such that for $\mu = 0$, X_0 is still attracting towards T. It is clear that such a one-parameter family X_μ can be constructed. The corresponding attractors K_μ then are:

- for $\mu \leqq 0$: $K_\mu = T$;
- for $\mu > 0$: K_μ is an $SO(2)$-bundle over T, which is non-trivial since the bundle W is non-trivial.

For $\mu > 0$, the dynamics on K_μ is indeed the dynamics of an SQP attractor if the induced flow in the orbit space $K_\mu / SO(2)$ is 2-quasi-periodic, which is persistent in the sense dicussed in Section 3. This concludes the proof that a 2-quasi-periodic attractor can become skew-quasi-periodic.

We return now to the Couette-Taylor experiment and the possibility of explaining the transition of the 2-quasi-periodic dynamics (modulated rotating wave) to the chaotic dynamics with mixed spectrum. In the 2-quasi-periodic situation, the states are not in the fixed-point set of the circle action (this is clear from the photographs of the experiment which show patterns which are not rotationally symmetric). This means that the dynamics is described here by a 2-quasi-periodic attractor on which the symmetry does not act trivially. Thus there is no possibility of an SQP attractor bifurcating off in a continuous way (continuous in the Hausdorff sense for the attracting sets), and the experiment suggests strongly that we have here a soft bifurcation (i.e., that here the attracting set changes continuously). This means that, though the skew Hopf bifurcation describes a persistent way to go from 2-quasi-periodic to SQP, it is probably not the explanation for the Couette-Taylor phenomenon.

5. Skew Hopf bifurcation and normal forms

We consider the transition from a 2-quasi-periodic attractor to an SQP attractor, as in Subsection 4.4, but now from the point of view of normal forms. In order to simplify our presentation we only discuss normal forms for the corresponding return maps. So for such a return map we consider an attracting circle losing its stability; we assume, however, that it is still contained in a 3-dimensional normally attracting invariant manifold (this assumption is at least persistent; see also [BHTB, 1990]). In our further discussion we shall restrict our attention to this invariant manifold, to be denoted by N. We assume the bifurcation to take place in the presence of $SO(2)$-symmetry: The attracting manifold then can be chosen to be $SO(2)$-invariant. From the discussion in the previous section it follows that we have to assume that $SO(2)$ acts trivially on the (attracting) invariant circle S and that it acts freely on its normal bundle. This means that we have the following normal form for S and the $SO(2)$-action on N near S: There are coordinates $(x, z_1, z_2) \in \mathbb{R}/\mathbb{Z} \times \mathbb{R}^2 = \mathbb{S}^1 \times \mathbb{R}^2$ on N such that $S = \mathbb{S}^1 \times \{0\}$ and such that $SO(2)$ acts by rigid rotations in $\mathbb{R}^2$.

In this section we often identify $\mathbb{R}^2$ with $\mathbb{C}$ through $(z_1, z_2) \cong z_1 + iz_2 = re^{2\pi is}$ with $s \in \mathbb{R}/\mathbb{Z}$. The diffeomorphism (return map) Ψ, due to the fact that it commutes with the $SO(2)$-action, has the form

$$\Psi(x, r, s) = \left(f(x, r^2), rg_1(x, r^2), s + g_2(x, r^2)\right),$$

where both x and s are in $\mathbb{R}/\mathbb{Z}$, $f(x+1, r^2) = f(x, r^2) + 1$ and $g_2(x+1, r^2) = g_2(x, r^2) + k$, and, as in Subsection 3.3, $k \in \mathbb{Z}$ is the twisting constant which is supposed to be non-zero.

5.1. Normal form for Ψ

The main purpose of putting Ψ in normal form is to make the x-dependence of $\Psi(x, r, s)$ as simple as possible. Restricting to $r = 0$ we have $\Psi(x, 0, s) = \left(f(x, 0), 0, s + g_2(x, 0)\right)$ (for notation see above). The map $x \mapsto f(x, 0)$ is an $\mathbb{S}^1$ diffeomorphism. Let α be its rotation number. We assume α to satisfy the DC. Then by [H, 1979] and [Y, 1982] we can choose a new C^∞ x-coordinate so that $f(x, 0) = x + \alpha$ (mod 1). Now, exactly as in Subsection 3.3, by changes of coordinates consisting of

(i) adding a constant to x (mod 1);

(ii) applying an x-dependent rotation, i.e., replacing s by $s + S(x)$, with $S(x+1) = S(x)$;

we obtain $g_2(x, 0) = kx$. Next we show how to make $g_1(x, 0)$ independent of x. We replace r by $\bar{r} = rG(x)$, where G is a positive function with $G(x+1) = G(x)$. This corresponds to a smooth change of coordinates in $\mathbb{S}^1 \times \mathbb{R}^2 : y_1, y_2$ are replaced by $\bar{y}_i = y_i G(x)$. The expression for Ψ, with respect to $(x, \bar{r}, s)$ now is

$$\Psi(x, \bar{r}, s) = \left(x + \alpha + O(\bar{r}^2), G^{-1}(x)\ G(x+\alpha) \cdot g_1(x, 0) \cdot \bar{r} + O(\bar{r}^3), s + kx + O(\bar{r})\right).$$

So we want to find a positive function G such that $G^{-1}(x) \cdot G(x+\alpha) \cdot g_1(x, 0)$, or $-\ln G(x) + \ln G(x+\alpha) + \ln g_1(x, 0)$, is constant. But this is exactly the problem we solved in Subsection 3.2 using Fourier series. In the same way we obtain here a solution for G which is unique up to a multiplicative constant. So again writing r for $\bar{r}$ we have the following asymptotic expression for f, g_1, and g_2 (i.e., up to terms which are infinitely flat in r):

$$f(x, r^2) = x + \alpha + \sum_{i=1}^{\infty} \alpha_i(x)\, r^{2i},$$

$$g_1(x, r^2) = \beta_0 + \sum_{i=1}^{\infty} \beta_i(x)\, r^{2i},$$

$$g_2(x, r^2) = kx + \sum_{i=1}^{\infty} \gamma_i(x)\, r^{2i}.$$

With the next change of coordinates we replace α_1, β_1, and γ_1 by constants. Consider the following change of coordinates, corresponding to a C^∞ transformation for x, y_1, y_2:

$$\bar{x} = x + a_1(x)\, r^2, \quad \bar{r} = r + b_1(x)\, r^3, \quad \bar{s} = s + c_1(x)\, r^2.$$

With respect to these coordinates we have

$$\Psi(\bar{x}, \bar{r}, \bar{s}) = \left(\bar{f}(\bar{x}, \bar{r}^2), \bar{r}\bar{g}_1(\bar{x}, \bar{r}^2), \bar{s} + \bar{g}_2(\bar{x}, \bar{r}^2)\right)$$

with

$$\bar{f}(\bar{x}, \bar{r}^2) = \bar{x} + \alpha + \left(\alpha_1(\bar{x}) + a_1(\bar{x}+\alpha) - a_1(\bar{x})\right) \bar{r}^2 + O(\bar{r}^4),$$

$$\bar{g}_1(\bar{x}, \bar{r}^2) = \beta_0 + \left(\beta_1(\bar{x}) + b_1(\bar{x}+\alpha) - b_1(\bar{x})\right) \bar{r}^2 + O(\bar{r}^4),$$

$$\bar{g}_2(\bar{x}, \bar{r}^2) = k\bar{x} + \left(\gamma_1(\bar{x}) - k \cdot a_1(\bar{x}) + c_1(\bar{x}+\alpha) - c_1(\bar{x})\right) \bar{r}^2 + O(\bar{r}^4).$$

It is again a matter of solving the same problem as in Subsection 3.2 to make the coefficients of $\bar{r}^2$ independent of $\bar{x}$, using the fact that α satisfies the DC. Note that if $|\beta_0| \neq 1$, we even do not need α to satisfy the DC when simplifying g_1. However, the case $|\beta_0| = 1$ corresponds to the situation where the invariant circle is losing its stability, in which we are most interested. Going on inductively, and calling the new variables again x, r, s, we obtain, up to terms of order r^{2N+2}:

$$f(x, r^2) = x + \alpha + \sum_{i=1}^{N} \bar{\alpha}_i r^{2i},$$

$$g_1(x, r^2) = \beta_0 + \sum_{i=1}^{N} \bar{\beta}_i r^{2i},$$

$$g_2(x, r^2) = kx + \sum_{i=1}^{N} \bar{\gamma}_i r^{2i}.$$

5.2. Normal forms with parameters

Here we consider the case where the map Ψ, as described in the introduction of this section, depends on parameter(s) $\mu = (\mu_1, \ldots, \mu_q) \in \mathbb{R}^q$. Using the same arguments as in the previous subsection one can prove that the map can be put in the form

$$\Psi_\mu(x, r, s) = \left(f(\mu, x, r^2), rg_1(\mu, x, r^2), s + g_2(\mu, x, r^2)\right),$$

where f, g_1, and g_2 are, up to terms of order $O(|\mu|^N + |r|^N)$, independent of x. So up to these higher-order terms we have

$$\begin{aligned} f(\mu, x, r^2) &= x + \alpha(\mu) + \tilde{f}(\mu, r^2), \\ g_1(\mu, x, r^2) &= \beta_0(\mu) + \beta_1(\mu)\, r^2 + \tilde{g}_1(\mu, r^2)\, r^4, \\ g_2(\mu, x, r^2) &= kx + \tilde{g}_2(\mu, r^2)\, r^2. \end{aligned}$$

See also [BT, 1989], e.g.

Without loss of generality we may assume that $\beta_0 > 0$. As we observed before, the stability of the invariant circle $\{r = 0\}$ changes for $\beta_0(\mu) = 1$. We assume that $\beta_0(0) = 1$, i.e., we assume that the normal form is centred at a point where the stability changes. Furthermore, we assume that $d\beta_0/d\mu(0) \neq 0$. For $\mu = 0$, the stability of $\{r = 0\}$ is determined by $\beta_1(0)$: It is stable for $\beta_1(0) < 0$ and unstable for $\beta_1(0) > 0$. The generic assumption is that $\beta_1(0) \neq 0$; since we want to investigate a bifurcation to a *nearby attractor,* we (have to) assume that $\beta_1(0) < 0$. A final generic assumption is that $d\alpha/d\mu(0)$ and $d\beta_0/d\mu(0)$ are linearly independent.

Under the simplifying assumptions that the terms of order $O(|\mu|^N + |r|^N)$ are zero, we can give the following description of the dynamics. The parameter space is divided (near $\mu = 0$) into two parts, separated by the codimension-one manifold $H = \{\mu \in \mathbb{R}^q \mid \beta_0(\mu) = 1\}$. On one side of H where $\beta_0(\mu) < 1$, the invariant circle is attracting, on the other side it is repelling. But on the side where $\beta_0(\mu) > 1$, there is another attracting set, defined by $\{r = R(\mu)\}$, where $R(\mu)$ is the positive solution, near zero, of

$$\beta_0(\mu) + \beta_1(\mu)\, r^2 + \tilde{g}_1(\mu, r^2)\, r^4 = 1,$$

so

$$R(\mu) \approx \sqrt{(1 - \beta_0(\mu))/\beta_1(\mu)}\,.$$

This attracting set is a torus T_μ and is an SQP attractor whenever $\alpha(\mu) + \tilde{f}(\mu, R(\mu)^2)$ is irrational and satisfies the DC.

Of course this simple picture changes somewhat if the terms of order $O(|\mu|^N + r^N)$ are non-zero. However, a number of qualities are persistent:

- If $\{r = 0\}$ is hyperbolically attracting or repelling, this property remains under small perturbations.
- The torus T_μ is normally attracting and hence persistent under small perturbations (though the persistence becomes weaker as μ approaches H).
- The irrationality (including DC) of rotation numbers of $\Psi_\mu|\{r = 0\}$ and of the map, induced by Ψ_μ in the space of $SO(2)$-orbits of T_μ, is persistent in the sense of Section 3.

In fact in Section 6 we shall show that the above description of the dynamics remains valid for a set of μ-values (near zero) with large positive measure.

6. Invariant tori

In this section we investigate the existence of invariant 2-tori in our family

$$\Psi_\mu(x, r, s) = (f(\mu, x, r^2), rg_1(\mu, x, r^2), s + g_2(\mu, x, r^2))$$

of maps. The starting point of this study is the parameter-dependent normal form as obtained in Subsection 5.2. Indeed, we shall establish the persistence of the tori T_μ as found in the truncated normal form.

The parameters were named $\mu \in \mathbb{R}^q$ in a general way. From now on we simplify slightly, making use of the (generic) conditions given in Subsection 5.2 on the coefficients α and β_0 of the normal form. In fact, we take $q = 2$, writing $\mu = (\lambda, \alpha)$, where $\lambda := 1 + \beta_0$. The parameters λ and α here play a different rôle: λ is a local parameter as before, while α is allowed to vary globally over some compact set. To be more precise, in the parameter plane we fix a rectangle of the form $R := [-\lambda_0, \lambda_0] \times [\alpha_-, \alpha_+]$, from now on restricting our attention to the case where $\mu \in R$.

Let us consider the parameter-dependent normal form of Subsection 5.2 in some detail. The form given there holds on the nowhere dense, perfect subset R_{DC} of R, defined by the DC. As is shown in [BB, 1987], [BHTB, 1990], the normalizing transformation depends on the parameter α in a Whitney-C^∞ manner. This means that this transformation can be extended over the continuum R as a C^∞-diffeomorphism. In this way, the functions $\tilde{f}$, $\tilde{g}_1$ and $\tilde{g}_2$ of the normalized part in the α-direction extend in a C^∞-way. Moreover, the perturbation terms now obtain the form

$$(O(|\lambda|^N + r^N) + P_\mu(x, r, s)),$$
$$r(O(|\lambda|^N + r^N) + Q_{1,\mu}(x, r, s)),$$
$$(O(|\lambda|^N + r^N) + Q_{2,\mu}(x, r, s)),$$

where P, Q_1 and Q_2 are infinitely flat on R_{DC}. We note that all estimates in the above formulae are uniform for $\mu \in R$, as well as in the angles x and s.

6.1. Parameter domain of the tori

We recall the obvious fact that for all $\mu \in R$ the circle $r = 0$ is invariant. From now on we assume that the normal form coefficient $\beta_1(\mu)$ is negative on the whole of R, if necessary taking R somewhat smaller. In this case, as λ changes from negative to positive, our invariant circle $r = 0$ changes from attracting to repelling and an attracting 2-torus may appear near T_μ. Indeed, using hyperbolicity on the above normal form, we investigate the existence of invariant 2-tori of the family Ψ_μ, for $\mu \in R$, where λ ranges over some right-hand neighbourhood of the line $\lambda = 0$. Here we use a well-known contraction method on cone-fields; compare [RT, 1971] and [L, 1973]. For other methods also compare, e.g., [I, 1979] and [BHTB, 1990].

To this end we first fix some $\alpha_0 \in (\alpha_-, \alpha_+)$ satisfying the DC. That the functions P and Q_j, $j = 1, 2$, are flat on R_{DC} then implies that there exists an infinitely flat function, $p: (\mathbb{R}_+, 0) \to (\mathbb{R}_+, 0)$, such that

$$|P|, |Q_j| = p(|\alpha - \alpha_0|)\, O(|z|^0 + |\lambda|^0),$$

$j = 1, 2$, where the estimate is uniform in x, s and α_0. Next we consider an open disc of the form

$$\mathscr{D}_{\alpha_0, a} := \{(\lambda, \alpha) \in R \,|\, 0 < \lambda < a, p(|\alpha - \alpha_0|) < \lambda^3\},$$

determined by positive constant a.

From now on, we restrict our attention to the case where $\mu \in \mathscr{D}_{\alpha_0, a}$. Observe that this disc $\mathscr{D}_{\alpha_0, a}$ is contained in a right-hand neighbourhood of the line $\lambda = 0$, while its boundary touches this line with an infinite order of contact; see Figure 2.

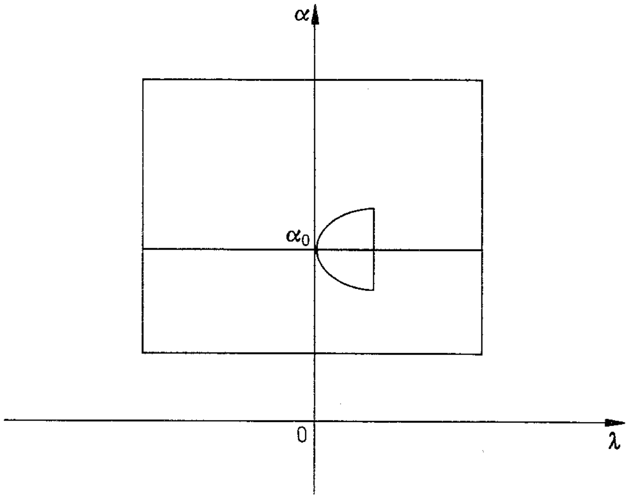

Figure 2. A disc $\mathscr{D}_{\alpha_0, a}$.

Note that in this way the size of the perturbation terms is controlled by the constant a. The main result of this section states that, for sufficiently small a, for parameter values $\mu \in \mathscr{D}_{\alpha_0, a}$, our family Ψ_μ indeed has a unique, attracting invariant 2-torus. Here the choice of the constant, among other things, depends on the degree of differentiability of these tori, which can be any finite number. By the uniformity in α_0, we obtain an uncountable union of such discs, which covers a large part of a right-hand neighbourhood of the line $\lambda = 0$.

In the complement of this union, near this line we meet so-called Chenciner bubbles, or resonance holes; e.g., compare [C, 1985a], [C, 1985b], [C, 1988], [BB, 1987], [IL, 1988] or [BHTB, 1990]. In the present case, the dynamics for parameter values inside these 'bubbles' still is not quite understood; for a few remarks on this subject we refer to Subsection 7.1, below.

6.2. Scaling

In order to find the tori we first perform a standard scaling, writing

$$r = (1 + y)\sqrt{\mu/(-\beta_1(0, \alpha))}\ .$$

From now on we work in the coordinates (x, s, y). After some computation, we find the following form for our family Ψ_μ:

$$\begin{aligned}\Psi_\mu(x, s, y) = \big(&x + \alpha + a(\lambda)\,\lambda(1 + y)^2 + F,\ s + kx \\ &+ b(\mu)\,\lambda(1 + y)^2 + G_2,\ (1 - 2\lambda)\,y - 3\lambda y^2 - \lambda y^3 + G_1\big)\,,\end{aligned}$$

where

$$\begin{aligned}F_\mu(x, s, y) &= O(\lambda^2) + p(|\alpha - \alpha_0|)\,O(\lambda^0) = G_{2,\mu}(x, s, y)\,,\\ G_{1,\mu}(x, s, y) &= O(\lambda^2) + p(|\alpha - \alpha_0|)\,O(\lambda^{-1/2})\end{aligned}$$

as $\lambda \downarrow 0$. The search for the invariant 2-tori then is near $y = 0$. To be more precise, we shall look in a neighbourhood given by $|y| \leqq b\lambda$, for an appropriate, positive constant b.

It is not difficult to show that, for b sufficiently small, the region $|y| \leqq b\lambda$ is positively invariant under Ψ_μ. Moreover, its derivative satisfies the estimate

$$D_{x,s,y}\Psi_\mu(x, s, y) = \begin{pmatrix} 1 & 0 & 0 \\ k & 1 & 0 \\ 0 & 0 & 1 - 2\lambda \end{pmatrix} + \begin{pmatrix} O(\lambda^2) & O(\lambda^2) & O(\lambda) \\ O(\lambda^2) & O(\lambda^2) & O(\lambda) \\ O(\lambda^2) & O(\lambda^2) & O(\lambda^2) \end{pmatrix}$$

as $\lambda \downarrow 0$. Again uniformity holds on the compact domain under consideration. We note that the terms $O(\lambda)$ in the third column of the perturbation matrix are due to the lower-order terms of the normal form.

6.3. Cone-fields

In order to prove the existence of the invariant 2-tori, we construct a cone-field in the tangent-bundle of the phase-space. First we adopt the convention that tangent vectors in the point (x, s, y) are written as (ξ, σ, η), employing the usual connotation. We now consider such a field given by

$$\xi^2 + \lambda^2\sigma^2 \geqq c^2\eta^2\,,$$

where c is a positive constant. It is not hard to verify that for an appropriate choice of constants, this cone-field is invariant under the derivative $D_{x,s,y}\Psi_\mu(x, s, y)$. The constant c should be large enough in order to compensate for the $O(\lambda)$-terms in the perturbation, as mentioned before. Also, the constant a, which bounds λ, should be small enough now to compensate for the effect of the 'nilpotent' term k.

Moreover, the derivative contracts this cone-field, making it narrower in the η-direction. Here we employ a weighted norm

$$\sqrt{\xi^2 + \lambda^2\sigma^2 + c^2\eta^2}\ .$$

By a standard argument, this yields the unique existence of an invariant 2-torus, as desired. See the references quoted above. To begin with, this method yields the regularity of the torus is C^1. However, using its uniqueness and considering its normal hyperbolicity, one can obtain any finite degree of differentiability. For this purpose, the parameter domain may have to be shrunk further.

7. Discussion of the dynamics

In this section we briefly discuss the dynamics of the family Ψ_μ, for parameter values μ in some neighbourhood of the line $\lambda = 0$. This discussion is mainly based on numerical simulation. In a forthcoming work, partly in the spirit of [C, 1985b] and [C, 1988], we plan a more thorough mathematical investigation of this.

7.1. The present symmetric case

We start with our symmetric family Ψ_μ, for parameter values μ in a neighbourhood of the line $\lambda = 0$ in the rectangle R; see Figure 3.

As said before, the circle $r = 0$ always is invariant, attracting for $\lambda < 0$ and repelling for $\lambda > 0$. As we saw in the previous section, for $\lambda > 0$ sufficiently small and in the union of the discs $\mathscr{D}_{\alpha_0, a}$, over α_0 satisfying the DC, there exists a unique 2-torus attractor.

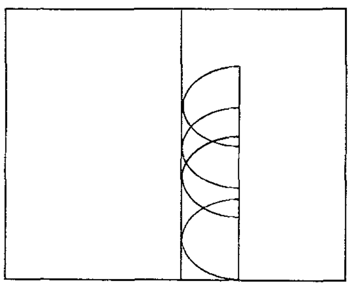

Figure 3. Union of discs, 'bubbles' in the complement.

The generic dynamics of the corresponding family of circle maps is well known, e.g., compare [A, 1983] or [BT, 1989]. There exists a C^∞-small perturbation of the set R_{DC}, containing parameter values μ for which this dynamics is quasi-periodic. This perturbed set still has large measure in the rectangle R; e.g., compare [BHTB, 1990].

In the corresponding 2-tori the dynamics is SQP; see above. For μ-values in the complement of this perturbed set, the circle dynamics is of 'phase-lock' type: These resonant circles contain periodic attractors and repellors, arranged in an alternate way. The corresponding dynamics in the 2-tori also is of 'phase-lock' type, quite comparable to what happens in the usual quasi-periodic Hopf-bifurcation; again, e.g., see [BHTB, 1990].

As numerical simulations suggest, and probably due to the resonances, the dynamics for μ-values inside the 'bubbles' is more involved. Here the 2-torus may decay to a surface with a non-regular projection on the circle $r = 0$, while for other parameter values it completely disappears. As we saw before, the SQP attractors do not have positive Lyapunov exponents. One question to ask is whether, say in the 'bubbles', evolutions can occur that do have a positive Lyapunov exponent.

7.2. Near symmetry

Here we consider the skew Hopf bifurcation after a non-symmetric perturbation has been added. A first question is whether non-symmetric terms can be transformed away with coordinate transformations like those we applied in order to obtain (formal) normal forms. It turns out that this is impossible. We demonstrate this by explicitly displaying the obstructions in the simplest case:

We consider transformations on $\mathbb{S}^1 \times \mathbb{R}^2$, that do not commute with the rotations in $\mathbb{R}^2$. We use complex coordinates $(x, z) \in \mathbb{S}^1 \times \mathbb{C} \cong \mathbb{S}^1 \times \mathbb{R}^2$. In the rotationally symmetric case, with an irrational and DC rotation in $\mathbb{S}^1 \times \{0\}$ we obtained the normal form

$$\Psi(x, z) = \left(x + \alpha + O(|z|^2), (\beta_0 e^{2\pi ikx} + O(|z|^2))\, z\right).$$

The lowest-order term in z, not commuting with the $SO(2)$-action, that can be used as a perturbation, has the form $(0, h(x))$, with $h: \mathbb{S}^1 = \mathbb{R}/\mathbb{Z} \to \mathbb{C}$. So, as the simplest perturbation away from the symmetric case, we consider

$$\Psi_\varepsilon(x, z) = \left(x + \alpha + O(|z|^2), \varepsilon h(x) + (\beta_0 e^{2\pi ikx} + O(|z|^2))\, z\right),$$

ε being a small parameter. In accordance with the above, we assume h to be in C^∞.

If the term $\varepsilon h(x)$ can be transformed away (modulo terms of the order $O(|\varepsilon|^2 + |z|)$), then this can be done by a coordinate change of the form $\tilde{z} = z + \varepsilon g(x)$. For such g we have, up to terms of the order ε^2:

$$\begin{aligned}\Psi_\varepsilon(x, \tilde{z}) = \Big(x + \alpha + O(|\tilde{z}|^2),\ &(\beta_0 e^{2\pi ikx} + O(|\tilde{z}|^2))\, \tilde{z} \\ &+ (h(x) + g(x + \alpha) - \beta_0 e^{2\pi ikx} g(x))\, \varepsilon\Big).\end{aligned}$$

So the problem is to find, for a given h, a function g satisfying the equation

$$h(x) + g(x + \alpha) - \beta_0 e^{2\pi ikx} g(x) = 0 \qquad (*)$$

Since Ψ is a diffeomorphism, we may assume that $\beta_0 > 0$. For $\beta_0 \neq 1$ the equation can be solved. However, this would correspond to the case of an invariant circle which is normally hyperbolic, while we are interested in the invariant circle at the moment that it loses its stability. So we assume that $\beta_0 = 1$.

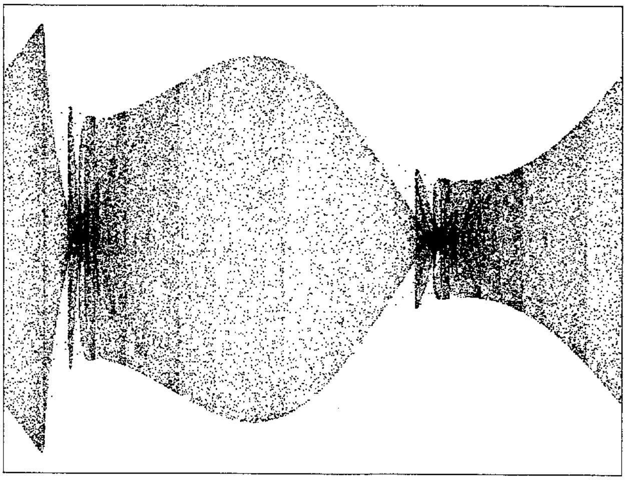

Figure 4. Numerical simulations in the symmetric case.

Next we write h and g as Fourier series:

$$h(x) = \sum h_n e^{2\pi inx}, \qquad g(x) = \sum g_m e^{2\pi imx}$$

and we have to solve

$$h_n + g_n e^{2\pi inx} - g_{n-k} = 0. \qquad (**)$$

Since we assumed h to be in C^∞, we have $\lim_{n\to\pm\infty} h_n n^k = 0$ for all k. On the other hand, in order to have the solution g of (*) continuous, the solutions $\{g_k\}$ of (**) should at least satisfy $\lim_{n\to\pm\infty} g_n = 0$. Clearly we can construct a unique solution $\{g_n^-\}$ of (**) such that $\lim_{n\to-\infty} g_n^- = 0$ and another unique solution $\{g_n^+\}$ for which $\lim_{n\to\infty} g_n^+ = 0$. These two solutions are equal if and only if $g_1^- = g_1^+, \ldots, g_k^- = g_k^+$. In that case, denoting the coinciding solutions by $\{\bar{g}_n\}$, we have $|\bar{g}_n| \leqq \Sigma_{m\leqq n} |h_m|$ and $|\bar{g}_n| \leqq \Sigma_{m\geqq n} |h_m|$. This implies that $\lim_{n\to\pm\infty} \bar{g}_n n^k = 0$ for all k, and hence that the corresponding function g is in C^∞. In the case where $\{g_n^-\}$ and $\{g_n^+\}$ are not equal, there is not even a continuous function g satisfying (*). So our normal form problem has an obstruction

$$(\lambda_1, \ldots, \lambda_k) \in \mathbb{C}^k, \qquad \lambda_i = g_i^- - g_i^+.$$

Observe that by allowing for rotations and scalar multiplications in the z-coordinate, we can transform the obstruction $(\lambda_1, \ldots, \lambda_k)$ to $(c\lambda_1, \ldots, c\lambda_k)$, for any $c \in \mathbb{C}\backslash\{0\}$. Moreover, it turns out that if one tries to remove other non-symmetric terms, the problem always reduces to equations of the form (*) and (**), leading to the same type of constructions.

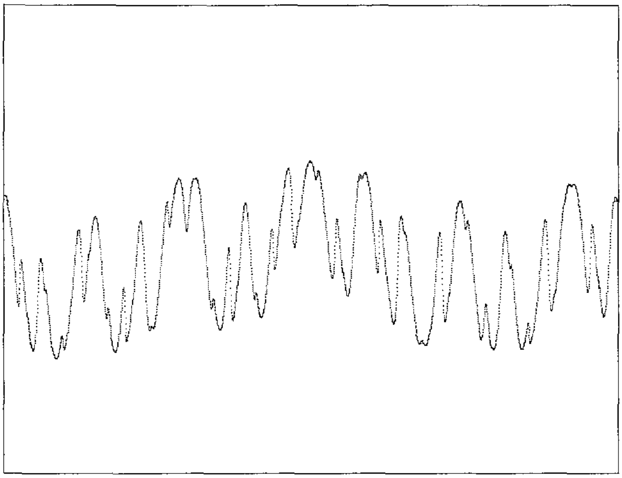

Figure 5. Invariant attracting curve losing its differentiability near a non-symmetric Hopf bifurcation. The horizontal coordinate is the x-coordinate on $\mathbb{S}^1$, the vertical coordinate is the real part of z.

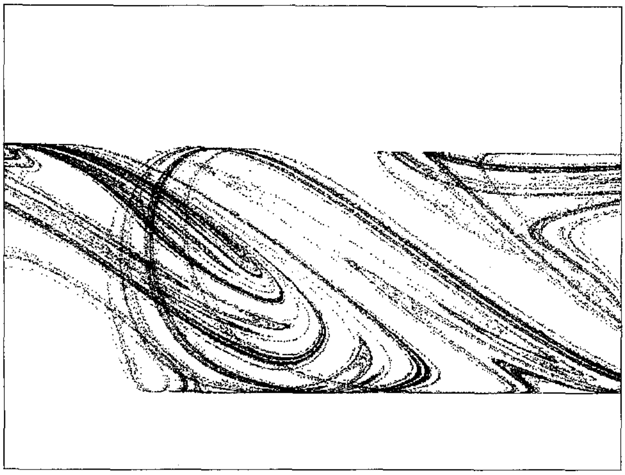

Figure 6. A Hénon-like attractor on an attracting invariant torus – coordinates as in the previous figure.

The above calculation also shows that the invariant circle $\{z = 0\}$, as a differentiable invariant closed curve, in general, is not persistent under non-symmetric perturbations. This is illustrated in the accompanying figures, obtained by numerically detecting the attracting set. Here a non-symmetric perturbation term was used, as in the above example.

The above discussion applies for the case of $\lambda < 0$, before the skew Hopf bifurcation takes place. We point out that, also after the skew Hopf bifurcation has taken place, so that $\lambda > 0$, non-symmetric perturbations seem to have important consequences. In Figure 6 below we show the attractor in an invariant torus, as obtained by a numerical simulation. This structure looks like the closure of some unstable separatrix – locally it looks like the Hénon attractor.

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