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Manifold Shape: from Differential Geometry to Mathematical Morphology [*]

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Abstract. Much progress has been made in extending Euclidean mathematical morphology to more complex structures such as complete lattices or spaces with a non-commutative symmetry group. Such generalizations are important for practical situations such as translation and rotation invariant pattern recognition or shape description of patterns on spherical surfaces. Also in computer vision much use is made of spherical mappings to describe the world as seen by a human or machine observer. Stimulated by these developments the question is studied here of the shape description of patterns on arbitrary (smooth) surfaces based on mathematical morphology. The primary interest in this paper is to outline the mathematical structure of this description. Some concepts of differential geometry, in particular those of parallel transport and covariant differentiation, are used to replace the more restricted concept of invariance used so far in mathematical morphology. The corresponding morphological operators which leave the geometry on the surface invariant are then constructed.

Keywords: mathematical morphology, differential geometry, parallel transport, dilation, erosion, closing, opening, shape concepts, group invariance.

1 Introduction

Much progress has been made in extending Euclidean mathematical morphology as developed by Matheron and Serra [7, 12] to more complex structures such as complete lattices [13, 3, 11] or spaces with a non-commutative symmetry group [9, 10]. Such generalizations are important for practical situations like translation and rotation invariant pattern recognition or shape description of patterns on spherical surfaces (satellite data of the earth, microscopic images of virus particles, etc.). Also in computer vision and image understanding there is increasing use of group theoretical methods [4]. Stimulated by these developments the question is studied here of the shape description of patterns on arbitrary (smooth) surfaces based on mathematical morphology. The primary interest in

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this paper is to outline the mathematical structure of this description. It is clear that human observers are able to recognize patterns on a curved surface (say, patterns on ceramics) as 'similar'. This notion is quantified by introducing some concepts of differential geometry, in particular those of parallel transport and covariant differentiation which can be used to replace the more restricted concept of invariance groups used so far in mathematical morphology. When using a geometric concept of shape in line with F. Klein's Erlanger Programm [6] such invariance concepts form an essential ingredient in shape descriptions. Next, the corresponding morphological operators which leave the geometry on the surface invariant will be constructed. In view of the fact that both differential geometry and mathematical morphology start from *local* operations it is perhaps not too surprising that a connection between the two can be established. The present paper presents a first step in this direction.

The organization of this paper is as follows. In Sect. 2 a number of prerequisites are stated from mathematical morphology, with particular emphasis on symmetry properties and their role in shape description. Then in Sect. 3 the required differential geometric concepts are briefly introduced. The study is mainly restricted to smooth (hyper)surfaces in n-dimensional Euclidean space, although most of the results carry over to more general Riemannian manifolds as well. Finally, in Sect. 4 these differential geometric concepts are used to construct morphological operators on smooth surfaces which leave the geometry of the surface invariant. The results presented here are of a preliminary nature. Both the mathematical treatment and the question of the usefulness of the approach outlined here require a more detailed study.

2 Invariance Concepts in Mathematical Morphology

In [10] a study was made of a *homogeneous space* $(\star, M)$; that is, a set M on which a transitive but not necessarily commutative group $\star$ of invertible transformations is defined. Here *transitive* means that for any pair of points in the set there is a transformation in the group which maps one point on the other. If this mapping is unique the transformation group is called *simply transitive* [15]. Each element $g \in \star$ is a mapping $M \to M : x \mapsto g(x)$, satisfying

$$\text{(i)} \quad gh(x) = g(h(x)), \quad \text{(ii)} \quad e(x) = x,$$

where e is the unit element of $\star$ (i.e. the identity mapping $x \mapsto x$, $x \in M$), and gh denotes the product of two group elements g and h. The *inverse* of an element $g \in \star$ will be denoted by g^{-1}. Usually we will also write gx instead of $g(x)$. The *stabilizer* of $x \in M$ is the subgroup $\star_x := \{g \in \star : gx = x\}$. The object space by which binary images on M are modelled is the Boolean lattice $\mathcal{P}(M)$ of all subsets of M, ordered by set inclusion. A brief sketch will be given of the construction of morphological operations on this homogeneous space with full invariance under the acting group $\star$.

First recall the construction of dilations on $\mathcal{P}(M)$ without any invariance property, as given by Serra [13, Ch.2, Proposition 2.1] (a dilation/erosion is a mapping commuting with unions/intersections):

Proposition 1. *A mapping $\delta : \mathcal{P}(M) \to \mathcal{P}(M)$ is a dilation if and only if there exists a function $\gamma : M \to \mathcal{P}(M)$, called a 'structuring function', such that*

$$\delta(X) = \bigcup_{x \in X} \gamma(x). \tag{1}$$

This statement can be interpreted as follows. Attach to each point x of M a subset $\gamma(x)$ of M, that is, think of M as being completely 'covered' by a collection of subsets of itself. Then the dilation $\delta(X)$ is the union of all the subsets which are attached to points of X. On the complete lattice $\mathcal{L} = \mathcal{P}(M)$, there exists for every dilation $\delta : \mathcal{L} \to \mathcal{L}$ a unique erosion $\varepsilon : \mathcal{L} \to \mathcal{L}$, called the *adjoint* of δ, such that (ε, δ) is an adjunction. Here a pair (ε, δ) of mappings on $\mathcal{L}$ is called an *adjunction* if for every $X, Y \in \mathcal{L}$ the following equivalence holds:

$$\delta(X) \leq Y \iff X \leq \varepsilon(Y);$$

see [3, 11]. It is easy to see that the erosion ε associated (by adjunction) with δ is given by

$$\varepsilon(X) = \{y \in M : \gamma(y) \subseteq X\}. \tag{2}$$

Next morphological operators which possess invariance properties under a transformation group are considered. Assume that M is a homogeneous space under a group $\star$ which acts transitively on M. In that case it is appropriate to take a *fixed* set A (the 'structuring element') and attach to any $x \in M$ *all* the sets $gA := \{ga : a \in A\}$, where g runs over the complete collection of group elements which move a fixed point ω (called the 'origin') to x. The set gA is sometimes referred to as the *(group) translate of A (by g)*.

Example. The translation group on the plane
Consider the plane $M = \mathbb{R}^2$, acted upon by the commutative group $\mathbf{T}$ of translations. This is the classical case [12, 7]. Here one uses translates $\tau_x(A) = \{x + a : a \in A\}$ of a single set A, where τ_x is the unique (Euclidean) translation which maps the origin to the point x.

Example. The translation-rotation group on the plane
Consider the plane $M = \mathbb{R}^2$, acted upon by the group $\star$ generated by translations and rotations of the plane (Euclidean motion group, group of rigid motions), a noncommutative group. Let the origin ω be the point $(0, 0)$. The stabilizer Σ is equal to the group $\mathbf{R}$ of rotations around the origin. The collection of all group elements which map ω to x is the set $\{\tau_x s : s \in \Sigma\}$, where τ_x is the unique translation which maps the origin to the point x (see the previous example). The basic objects in defining morphological operations with respect to this group are formed by all translated and rotated copies of a single set A. An application which occurs in the problem of motion planning for robots has been considered in great detail in [9].

Example. The rotation group on the sphere ([8])
Consider the unit 2-sphere $M = S^2$, acted upon by the three-dimensional rotation group $\star = SO(3)$, also a noncommutative group. Let $\mathbf{v} = (x, v)$, $v \in S^1$,

be a unit tangent vector at $x \in S^2$. Choose the north pole $\mathcal{N}$ as the origin of the sphere, and define a base-vector **b** to be an arbitrary unit tangent vector at $\mathcal{N}$. Then the tangent vector **v** represents a unique rotation, i.e. the one which maps **b** to **v**. The stabilizer Σ is the set of rotations around the north pole $\mathcal{N}$. For a fixed set $A \subset S^2$ the set $\{g_x sA : x \in S^2, s \in \Sigma\}$, where for each $x \in S^2$, g_x is any particular group element (a representative) which maps ω to x, forms the basic collection from which morphological operations are constructed.

2.1 Morphological operators

In the classical case, one uses Euclidean translations to define dilations and erosions.

$$Minkowski\ addition: \quad X \oplus A = \{x + a : x \in X, a \in A\} = \bigcup_{a \in A} X_a, \quad (3)$$

$$Minkowski\ subtraction: \quad X \ominus A = \bigcap_{a \in A} X_{-a}, \quad (4)$$

where

$$X_a = \tau_a(X) = \{x + a : x \in X\}.$$

Next consider the case of a homogeneous space $(\star, M)$. First the following definition is needed.

Definition 2. Let $(\star, M)$ be a homogeneous space with Σ the stabilizer of the origin ω in M. A subset X of M is called Σ-*invariant* if $X = \overline{X}$, where $\overline{X} = \bigcup_{s \in \Sigma} sX$. If X is not Σ-invariant, $\overline{X}$ is called the Σ-*invariant extension* of X.

Let $A \subset M$. Then the mapping

$$\delta(X) := \bigcup_{x \in X} \bigcup_{\{g \in \Gamma : g\omega = x\}} gA \quad (5)$$

is a dilation δ which is $\star$-invariant; that is, $\delta(gX) = g\delta(X)$ for all $g \in \star$, $X \in \mathcal{P}(M)$. Moreover, all $\star$-invariant dilations are of this form [10]. Using Definition 2, (5) can be rewritten as

$$\delta(X) = \bigcup_{x \in X} \bigcup_{s \in \Sigma} g_x sA = \bigcup_{x \in X} g_x \overline{A}, \quad (6)$$

where, again for each x, g_x is any particular group element which maps ω to x. The adjoint erosion of (5) is formed by associating with a subset X the collection of points $y \in M$ such that $gA \subseteq X$ for *all* $g \in \star$ which move the origin to y. For a representation of this erosion as an intersection of translated sets, see [10]. This shows that any $\star$-invariant dilation on $\mathcal{P}(M)$ can be reduced to a dilation δ_A^Γ involving a Σ-*invariant* structuring element A; the same is true for erosions. *Openings* by a subset A of M can be defined by

$$\gamma_A^\Gamma(X) = \bigcup_{g \in \Gamma} \{gA : gA \subseteq X\}, \quad (7)$$

which is the union of all translates gA of A which are included in X. Such $\star$-openings are generally not reducible to openings by a Σ-invariant structuring element, see [10].

2.2 The role of symmetry groups in shape description

Usually, 'shape' is defined as referring to those properties of geometrical figures which are invariant under the Euclidean similarity group [5]. Intuitively speaking, one first has to bring figures to a standard location, orientation and scale before being able to 'compare' them. The following definition generalizes this.

Definition 3. Let M be a set, $\star$ a group acting on M. Two subsets X, Y of M are said to have *the same shape with respect to* $\star$, or *the same $\star$-shape*, if they are $\star$-equivalent, meaning that there is a $g \in \star$ such that $Y = gX$. If no such $g \in \star$ exists, X and Y are said to have *different $\star$-shape*.

In essence this definition goes back to F. Klein's 'Erlanger Programm' (1872), which considers geometry to be the study of transformation groups and the properties invariant under these groups [6]. In Euclidean morphology, all translates of a set X by the Euclidean translation group $\mathbf{T}$ have the same $\mathbf{T}$-shape. After adding rotations to obtain the Euclidean motion group $\mathbf{M}$, rotated versions of X or its translates have the same $\mathbf{M}$-shape as X.

This notion of shape is still too restricted in the case of sets on arbitrary surfaces M, for in general no group $\star$ exists which acts transitively on M. Therefore a more general definition of shape equivalence will be sought using a number of concepts from differential geometry. To motivate this whole enterprise, a simple but important example of a morphological operation on an arbitrary surface will first be given.

2.3 Motivating example

Let M be a smooth surface in $\mathbb{R}^3$ supplied with the induced metric. That is, lengths are measured 'along the surface'. For any $x \in M$, let $D_r(x)$ be the disk of radius r centred at x. Then a dilation $\delta : \mathcal{P}(M) \to \mathcal{P}(M)$ can be defined as follows:

$$\delta(X) = \bigcup_{x \in X} D_r(x). \tag{8}$$

That is, $\delta(X)$ is the union of all points of M with distance smaller than r to some point of X. Comparing with the cases studied above one could say that a disk of radius r is used here as the 'structuring element'. In the same way one can define erosions or openings by a disk of radius r. The problem described in the rest of the paper essentially boils down to the question of how this can be generalized when the structuring element is not a disk.

3 Elementary Concepts from Differential Geometry

A brief outline is given here of some background material on Riemannian manifolds. Since it is the aim to develop in the next section some new concepts in a way which is intuitively clear, the discussion here is mainly restricted to the case of smooth surfaces in Euclidean 3-space. Thorpe [16] is followed as far as terminology and notation is concerned. For results in a more abstract setting see Boothby [1] or Helgason [2].

A *surface of dimension* n, or *n-surface*, in $\mathbb{R}^{n+1}$ is a non-empty subset M of $\mathbb{R}^{n+1}$ of the form $M = f^{-1}(c)$ where $f : U \to \mathbb{R}$, U open in $\mathbb{R}^{n+1}$, is a smooth function with the property that $\nabla f(p) \neq \mathbf{0}$ for all $p \in M$, and $c \in \mathbb{R}$, where ∇ is the gradient operator. That is, M is a *level set* of f at *height* c. The gradient property implies that all points of M are *regular*. A *vector* at a point $p \in \mathbb{R}^{n+1}$ is a pair $\mathbf{v} = (p, v)$ where $v \in \mathbb{R}^{n+1}$. The set of all vectors tangent to M at p equals $[\nabla f(p)]^-$ and is called the *tangent space* of M at p, denoted by M_p in the following. A parameterized curve, or 'curve' for short, in M is a smooth function $\alpha : I \to M$ where I is some open interval in $\mathbb{R}$. The space M_p consists of velocity vectors at p of all curves passing through p and is an n-dimensional vector subspace of the space of all vectors at p. The disjoint union of all tangent spaces,

$$T(M) = \bigcup_{p \in M} M_p,$$

is called the *tangent bundle* of M.

A *geodesic* in M is a parameterized curve $\alpha : I \to M$ whose acceleration is everywhere orthogonal to M, that is, $\ddot{\alpha}(t) \in M^-_{\alpha(t)}$ for all $t \in I$. Geodesics have constant speed,

$$\frac{d}{dt} \|\dot{\alpha}(t)\|^2 = \frac{d}{dt} \langle \dot{\alpha}, \dot{\alpha} \rangle = 2 \langle \dot{\alpha}, \ddot{\alpha} \rangle = 0,$$

since $\dot{\alpha} \in M_{\alpha(t)}$ and $\ddot{\alpha}(t) \in M^-_{\alpha(t)}$ (angular brackets denote inner products). Given any $p \in M$ and any $\mathbf{v} \in M_p$ there exists a geodesic passing through p with velocity $\mathbf{v}$ at p. When the domain I of α is chosen as large as possible, the resulting geodesic is called a *maximal geodesic*. For each $p \in M, \mathbf{v} \in M_p$, there is a *unique* maximal geodesic α with $\alpha(0) = p$, $\dot{\alpha}(0) = \mathbf{v}$. For example, on the sphere S^2 the geodesics are great circles (or single points); on the cylinder they are straight lines, circles, or spirals (or points). An n-surface M is said to be *geodesically complete* if every maximal geodesic in M has domain $\mathbb{R}$. For example, the n-sphere is geodesically complete, the n-sphere with north pole deleted is not.

3.1 Parallel transport

In Euclidean space one knows how to transport vectors from one point to another by using the operation of translation. On an n-surface a comparable operation can be defined, which is called *parallel transport* or *parallel translation*. The concepts are developed in a few steps.

A vector field $\mathbf{X}$ on $U \subset \mathbb{R}^{n+1}$ is a function $p \to \mathbf{X}(p) = (p, X(p))$ for some function $X : U \to \mathbb{R}^{n+1}$. A vector field $\mathbf{X}$ is smooth if the components of the function $X : U \to \mathbb{R}^{n+1}$ are all smooth, that is, have continuous partial derivatives of all orders. A vector field along the curve $\alpha : I \to M$ is a function $t \mapsto \mathbf{X}(t)$ where $\mathbf{X}(t) \in \mathbb{R}^{n+1}$; if $\mathbf{X}(t) \in M_{\alpha(t)}$ for all $t \in I$, the vector field is called *tangent to M along α*. If $\mathbf{X}$ is a tangent vector field along a curve $\alpha : I \to M$ then the derivative $\dot{\mathbf{X}}$ is generally not tangent to M. To obtain a vector field tangent to M one has to project $\dot{\mathbf{X}}$ orthogonally onto $M_{\alpha(t)}$. This operation is called *covariant differentiation* and the resulting vector field

$$\mathbf{X}'(t) = \dot{\mathbf{X}} - \langle \dot{\mathbf{X}}(t), \mathbf{N}(\alpha(t)) \rangle \mathbf{N}(\alpha(t)),$$

where

$$\mathbf{N}(p) = \frac{\nabla f(p)}{\|\nabla f(p)\|}, \quad p \in M,$$

is a unit normal vector field, is called the covariant derivative of $\mathbf{X}$. The covariant derivative measures the rate of change of $\mathbf{X}$ as seen from the surface M. A curve $\alpha : I \to M$ is a geodesic if and only if the covariant acceleration $(\dot{\alpha})'$ is zero along α. A smooth vector field $\mathbf{X}$ tangent to M along α is called *constant* or *(Levi-Civita) parallel* if $\mathbf{X}' = 0$. If $\mathbf{X}$ and $\mathbf{Y}$ are parallel vector fields along α, then $\langle \mathbf{X}, \mathbf{Y} \rangle' = 0$, so $\langle \mathbf{X}, \mathbf{Y} \rangle$ is constant along α. In particular, $\mathbf{X}$ and $\mathbf{Y}$ have constant length; therefore, also the angle between $\mathbf{X}$ and $\mathbf{Y}$ is constant. For an example see Fig. 1. The velocity vector field along a parameterized curve in M
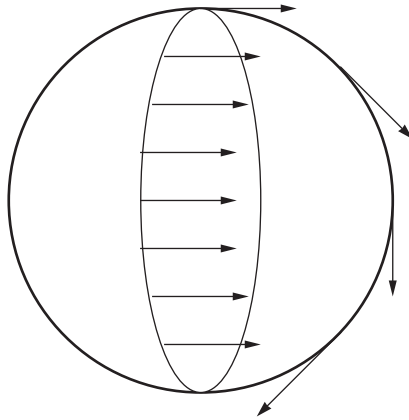

Fig. 1. Parallel vector fields along geodesics in the 2-sphere.

is parallel if and only if α is a geodesic. The following theorem is fundamental [16]:

Theorem 4. *Let M be an n-surface in $\mathbb{R}^{n+1}$, $\alpha : I \to M$ a parameterized curve in M, let $t_0 \in I$ and $\mathbf{v} \in M_{\alpha(t_0)}$. Then there exists a unique vector field $\mathbf{V}$, tangent to M along α, which is parallel and has $\mathbf{V}(t_0) = \mathbf{v}$.*

A simple characterization is possible for a 2-surface M: a vector field tangent to M along a geodesic α is parallel if and only if both $\|\mathbf{X}\|$ and the angle between $\mathbf{X}$ and $\dot{\alpha}$ are constant along α.

Parallelism can be used to transport tangent vectors from one point of an n-surface to another.

Definition 5. *Let $p, q \in M$ and let $\alpha : [a, b] \to M$ be a parameterized curve from $\alpha(a) = p$ to $\alpha(b) = q$. For $\mathbf{v} \in M_p$ let $\mathbf{V}$ be the unique parallel vector field along α with $\mathbf{V}(a) = \mathbf{v}$. The map $P_\alpha : M_p \to M_q$ determined by*

$$P_\alpha(\mathbf{v}) = \mathbf{V}(b)$$

is called parallel transport *from p to q, and $P_\alpha(\mathbf{v})$ the* parallel translate *of $\mathbf{v}$ along α to q.*

Parallel transport from p to q is path dependent: if α and β are two curves from p to q then, in general, $P_\alpha(\mathbf{v}) \neq P_\beta(\mathbf{v})$ (an exception occurs for surfaces of zero curvature, such as the Euclidean plane). More precisely, $P_\beta(\mathbf{v})$ differs from $P_\alpha(\mathbf{v})$ by a rotation around the normal to M at q. When a vector in M_p is transported along a closed curve beginning and ending in p, it will carry out a rotation in M_p. The set of such rotations of M_p generated by parallel translation along closed curves is called the *holonomy group* at p. Holonomy groups at different points of M are isomorphic.

The following result will be needed later [16]:

Theorem 6. *Let M be an n-surface in $\mathbb{R}^{n+1}$, $p, q \in M$ and α a piecewise smooth curve from p to q. Then parallel translation $P_\alpha : M_p \to M_q$ along α is a vector space isomorphism which preserves inner products:*

1. P_α is a linear map;
2. P_α is 1-1 and onto;
3. $\langle P_\alpha(\mathbf{v}), P_\alpha(\mathbf{w}) \rangle = \langle \mathbf{v}, \mathbf{w} \rangle$ for all $\mathbf{v}, \mathbf{w} \in M_p$.

To study questions concerning lengths of curves in M, it is convenient to parameterize curves by arc length, that is, choose a reparameterization such that α has unit speed. A well-known result concerning geodesics then asserts that if α is a shortest unit-speed curve from p to q in M, then α is a geodesic (the reverse is in general not true, consider e.g. geodesics on a sphere).

3.2 The exponential map

Above we have seen how to transport vectors from one point of an n-surface to another. The next question is how to map points of M to vectors in the tangent bundle $T(M)$. This then will enable us below to transport subsets of M from one point to another.

Definition 7. For $\mathbf{v}$ in the tangent bundle $T(M)$, let $\alpha_{\mathbf{v}}$ denote the unique maximal geodesic in M with $\dot{\alpha}_{\mathbf{v}}(0) = \mathbf{v}$. Let $U = \{\mathbf{v} \in T(M) : 1 \in \text{domain } \alpha_{\mathbf{v}}\}$. The map $\mathsf{Exp} : U \to M$ defined by $\mathsf{Exp}(\mathbf{v}) = \alpha_{\mathbf{v}}(1)$ is called the *exponential map* of M.

For $p \in M$, we will also write Exp_p to denote the mapping $M_p \to M : \mathbf{v} \mapsto \mathsf{Exp}(\mathbf{v})$. Since geodesics have constant speed, $\mathsf{Exp}_p(\mathbf{v})$ is the point on the unique geodesic determined by $\mathbf{v}$ whose distance from p along the geodesic is precisely $\|\mathbf{v}\|$; cf. Fig. 2. The following theorem summarizes the most important properties
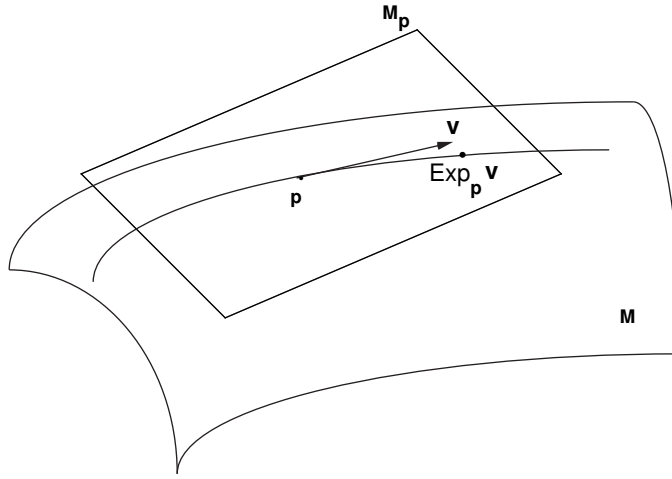

Fig. 2. The exponential map maps a tangent vector $\mathbf{v} \in M_p$ to the point lying at a distance $\|\mathbf{v}\|$ from p on the unique geodesic through p with initial velocity $\mathbf{v}$.

of the exponential map [16].

Theorem 8. *The exponential map* $\mathsf{Exp} : U \to M$ *of an* n-*surface in* R^{n+1} *has the following properties:*

1. *The domain* U *of* Exp *is an open set in* $T(M)$.
2. *If* $\mathbf{v} \in U$ *then* $t\mathbf{v} \in U$ *for* $0 \leq t \leq 1$.
3. Exp *is a smooth map.*
4. *For each* $p \in M$ *and* $\mathbf{v} \in M_p$, *the maximal geodesic* $\alpha_{\mathbf{v}}$ *with* $\dot{\alpha}_{\mathbf{v}}(0) = \mathbf{v}$ *is given by the formula* $\alpha_{\mathbf{v}}(t) = \mathsf{Exp}_p(t\mathbf{v})$.
5. *For* $\epsilon > 0$ *sufficiently small,* Exp_p *maps the* ϵ-*ball* $B_\epsilon = \{\mathbf{v} \in M_p : \|\mathbf{v}\| < \epsilon\}$ *diffeomorphically onto an open subset* U_ϵ *of* M *containing* p. *For* $q \in U_\epsilon$ *the curve* $\alpha_{\mathbf{v}}(t) = \mathsf{Exp}_p(t\mathbf{v})$ $(0 \leq t \leq 1)$ *with* $\mathsf{Exp}_p(\mathbf{v}) = q$ *is the* unique *geodesic joining* p *and* q; *it lies in* U_ϵ *and has length shorter than that of any other curve joining* p *and* q.

This theorem says that geodesics through $p \in M$ are images under Exp_p of the rays $\alpha(t) = t\mathbf{v}$ in M_p; see Fig. 3. In the case of the 2-sphere S^2, Exp_p maps the ball $\{\mathbf{v} \in S_p^2 : \|\mathbf{v}\| < \pi\}$ diffeomorphically onto $S^2 \setminus q$, where q is the antipodal point of p. In M_p, the geodesics through p are the orthogonal trajectories of
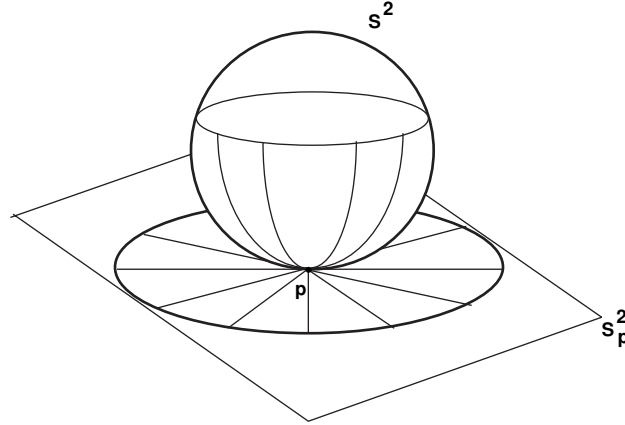

Fig. 3. The geodesics in S^2 through p are the images under the exponential map of the rays through $\mathbf{0}$ in S_p^2.

hypersurfaces

$$\left\{ \mathsf{Exp}_p(\mathbf{v}) : \mathbf{v} \in M_p, \|\mathbf{v}\| = constant \right\}.$$

If $p \in M$, the set $U_\delta(p)$ of points within distance δ of p is called a *spherical neighbourhood* of p or a *disk* of radius δ at p. A neighbourhood $U_\delta(p)$ such that there exists at most (at least) one geodesic segment contained in $U_\delta(p)$ joining any pair of points in $U_\delta(p)$ is called *simple* (*convex*). For sufficiently small δ, any neighbourhood $U_\delta(p)$ is simple and convex [2].

In general the domain and range of the exponential map is restricted. On so-called *geodesically complete* surfaces, such as compact surfaces, every maximal geodesic on M has domain $\mathbb{R}$, that is, can be infinitely extended. In that case the domain of the exponential map is all of $T(M)$. In certain cases the exponential map on a geodesically complete surface maps M_p diffeomorphically *onto* M for any $p \in M_p$, so that a 1-1 correspondence between points of M and points of M_p exists; that is, geodesics between any two points are unique. Examples of such spaces are simply connected geodesically complete surfaces of negative curvature; see Theorem 13.3 of Helgason [2].

Example. The one-sheeted hyperboloid defined by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \tag{9}$$

has curvature

$$K = -\frac{1}{a^2 b^2 c^2}\left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right)^{-2},$$

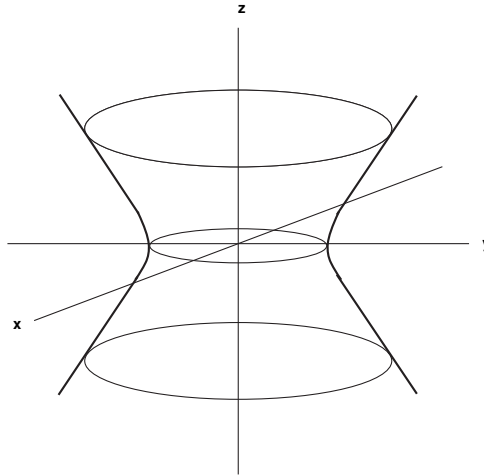which is everywhere negative [14]. For a sketch of this (ruled) surface, see Fig. 4.

Fig. 4. Sketch of the one-sheeted hyperboloid defined by (9).

4 Mathematical Morphology on Smooth Surfaces

In this section a sketch is given how a morphological description of binary images on smooth surfaces can be developed. The general construction of dilations on complete lattices by Serra (see Prop. 1) holds also, of course, for the special case of the lattice $\mathcal{P}(M)$, where M is a smooth surface in $\mathbb{R}^3$ and $\mathcal{P}(M)$ denotes the set of all subsets of M. The problem is to define morphological operators satisfying some form of invariance. In Sect. 2 we have seen how to handle the case when a transitive group action on M exists. It will be shown that this theory carries over to a large extent to the case when M is an arbitrary surface by replacing group translations by parallel translations, which are based upon the concept of covariant differentiation. The resulting morphological transformations may thus be referred to as 'covariant' operations.

The basic problem is how to 'transport' subsets of M from one location to another while preserving as many geometric properties as possible. Let X be a neighbourhood of the point $p \in M$. To transport this set X from the point p to another point $q \in M$ we perform the following steps. First map X to the tangent

space M_p by using the inverse of the exponential map: the image under this map is denoted by $\tilde{X}$. Then use parallel translation from p to q along a curve with initial point p and endpoint q. This maps $\tilde{X}$ to a neighbourhood, say $\tilde{Y}$, of q. Finally map $\tilde{Y}$ back to M by the exponential map, thus obtaining a subset Y of q; see Fig. 5.
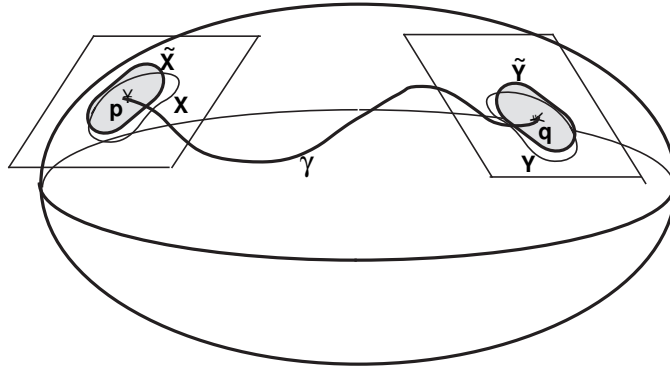

Fig. 5. Parallel transport of subsets of a surface.

To formalize this, let $\gamma = \gamma_{[p,q]}$ be a curve from p to q. Then an operator τ_γ can be defined by

$$Y = \tau_\gamma(X) = \mathsf{Exp}_q P_\gamma \mathsf{Exp}_p^{-1}(X), \tag{10}$$

where P_γ is the parallel transport of tangent vectors from M_p to M_q along γ (see Definition 5) and $P_\gamma(\tilde{X})$ is simply the union of all translated vectors $P_\gamma \mathbf{v}$ when $\mathbf{v}$ runs over $\tilde{X} = \mathsf{Exp}_p^{-1}(X)$. By transporting the initial set X at a fixed point ω along all possible curves to other points of M we cover M by an infinite collection of diffeomorphic copies of X, which in addition preserve several metrical properties (lengths, angles of tangent vectors). It may be verified that the operation (10) reduces to Euclidean translation when M is a plane, and to rotation in the case of the sphere (in the latter case one has to take for X a subset of the sphere not containing the antipodal point of ω in order for $\mathsf{Exp}_\omega^{-1}(X)$ to be well defined).

The following points should now be made. First, the exponential map is in general only invertible (in fact, a diffeomorphism) for a sufficiently small neighbourhood of the origin in M_p, although on some manifolds the inverse exists for arbitrary neighbourhoods of the origin in M_p, so that there is a 1-1 correspondence between the neighbourhoods of a point $p \in M$ and the neighbourhoods of the point $\mathbf{0} \in M_p$; see the example at the end of Sect. 3. Therefore we will take as the basic 'structuring element' a subset, not of M but of the tangent space at a given point ω of M. If $\tilde{A}$ is such a subset of M_ω, then an operator $\tilde{\tau}_\gamma$ (also

referred to as 'parallel translation along γ') can be defined by

$$\tilde{\tau}_\gamma(\tilde{A}) = \mathsf{Exp}_p P_\gamma(\tilde{A}), \tag{11}$$

where P_γ is the parallel transport of tangent vectors along a curve γ from ω to p.

Second, the image of the set X under parallel translation from p to q will in general depend on which path is taken. This, however, is a situation which we have already encountered when discussing mathematical morphology on spaces with a noncommutative group action; see Sect. 2. The solution found in that case works here as well: simply consider *all* possible paths from p to q.

Now it is possible to define dilations and erosions. Let $\tilde{A}$ (the 'structuring element') be a subset of the tangent space M_ω, ω an arbitrary but fixed point of M. Then define a mapping $\delta : \mathcal{P}(M) \to \mathcal{P}(M)$ by

$$\delta_{\tilde{A}}(X) = \bigcup_{x \in X} \bigcup_{\gamma} \mathsf{Exp}_x P_{\gamma_{[\omega, x]}}(\tilde{A}), \tag{12}$$

where the second union runs over all curves $\gamma = \gamma_{[\omega, x]}$ from ω to x. This can be rewritten as follows. Choose for every $x \in M$ a particular curve ('representative') from ω to x. Let $\tilde{\tau}_x$ denote parallel translation along this particular curve. Then, if Σ is the holonomy group at ω, which for the surfaces in $\mathbb{R}^3$ considered here is simply the group of rotations around the normal at ω, (12) can be written as

$$\delta_{\tilde{A}}(X) = \bigcup_{x \in X} \bigcup_{s \in \Sigma} \tilde{\tau}_x(s\tilde{A}). \tag{13}$$

It is obvious that this mapping is a dilation, either by direct proof or through the invocation of Prop. 1. Since parallel translation commutes with unions—both Exp_p and the vector space isomorphism P_γ do—we also can write (13) in the form

$$\delta_{\tilde{A}}(X) = \bigcup_{x \in X} \tilde{\tau}_x(\overline{A}), \tag{14}$$

where

$$\overline{A} := \bigcup_{s \in \Sigma} s\tilde{A} \tag{15}$$

may be called the *Σ-invariant extension* of $\tilde{A}$. For example, if $\tilde{A}$ is a line segment of length r starting at ω then $\overline{A}$ is a disk of radius r centred at ω. The similarity of these expressions with the results in Sect. 2 is clear. Erosions can be defined in a similar way. If $\overline{A}$ is a Σ-invariant structuring element then the mapping

$$\varepsilon_{\overline{A}}(X) = \{x \in M : \tilde{\tau}_x(\overline{A}) \subseteq X\}, \tag{16}$$

is an erosion which extracts all the points x of M such that the parallel translate of $\overline{A}$ from ω to x fits in X.

Openings can also be easily defined, where one does not have to restrict oneself to Σ-invariant structuring elements. For any neighbourhood $\tilde{A}$ in M_ω, let

$$\alpha_{\tilde{A}}(X) = \bigcup_{x \in M} \{\tilde{\tau}_x(\tilde{A}) : \tilde{\tau}_x(\tilde{A}) \subseteq X\} \qquad (17)$$

be the union of all parallel translates of $\tilde{A}$ along curves starting at ω which are included in X. It is obvious that this is an opening. Closings can be defined similarly.

Example. This example was already discussed in Sect. 2. Take for $\tilde{A}$ a disc of radius r centred at the origin in M_ω. Then the parallel translate $\tilde{\tau}_x(\tilde{A})$ is a spherical neighbourhood at $x \in M$. The dilation $\delta_{\tilde{A}}(X)$ by $\tilde{A}$ is the union of all points of distance smaller than r to some point of X, and the opening by $\tilde{A}$ extracts from X all spherical neighbourhoods of radius r which fit into X.

Example. Take for $\tilde{A}$ a straight line segment of length L through the origin in M_ω. Then the parallel translates of $\tilde{A}$ are geodesic segments and the opening by $\tilde{A}$ extracts from X all geodesic segments of length L which fit into X.

5 Discussion

In this paper the study of shape description of patterns on arbitrary (smooth) surfaces based on mathematical morphology has been initiated. The main aim has been to give an outline of the mathematical structure of this description based on concepts of differential geometry, in particular those of parallel transport and covariant differentiation which can be used to replace the more restricted concept of invariance groups used so far in mathematical morphology. Various morphological operators have been constructed on a surface M which are defined in terms of neighbourhoods of M which are obtained by parallel translation of a single set $A \subseteq M$ (the 'structuring element'). If M is Euclidean space or a sphere then these morphological operations reduce to the known ones which are invariant under the appropriate group (translations, rotations). What has not been discussed here is a precise formulation—in algebraic terms—of the invariance properties satisfied by the operators introduced here for arbitrary surfaces. This is an open problem which requires a more detailed study.

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