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A general KAM-theory for reversible systems is given. The cases of both maximal and lower-dimensional tori are covered. In some cases parameters are needed for persistence, therefore an unfolding theory is developed.

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1. INTRODUCTION

Reversibility of a dynamical system involves an involution of the phase space, which takes evolutions to evolutions, reversing the time-parametrization. It is known (see Arnold, 1984; Sevryuk, 1991a) that there exists a great similarity between Hamiltonian (symplectic) systems and reversible ones. This was illustrated by Moser (1973), who shows that the "classical" KAM-theorem also holds in the reversible setting. Here the tori are maximal in a sense to be explained now.

Indeed let $M = \mathbb{T}^n \times \mathbb{R}^m \times \mathbb{R}^{2p}$ be the phase space. The coordinates on $\mathbb{T}^n$ are denoted by $x = (x_1,..., x_n) \bmod 2\pi$, on $\mathbb{R}^m$ by $y = (y_1,..., y_m)$, and on $\mathbb{R}^{2p}$ by $z = (z_1,..., z_{2p})$. Given any linear involution $R: \mathbb{R}^{2p} \to \mathbb{R}^{2p}$ (i.e., with $R^2 = Id$), we define the involution $G: M \to M$ by $G(x, y, z) = (-x, y, Rz)$. We assume that G is of type $(n + p, m + p)$, meaning that its submanifold of fixed points has dimension $m + p$ or, equivalently, that R has the eigenvalue 1 with multiplicity p. A vector field X on M is called (G-) reversible if

$$G_*(X) = -X \tag{1}$$

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This implies that G maps trajectories of X to trajectories of X, reversing the time parametrization. Any (G-) equivariant transformation Φ of M (i.e., with $\Phi \circ G = G \circ \Phi$) preserves this reversibility property. If we write

$$X(x, y, z) = \sum f_j(x, y, z)\, \partial/\partial x_j + \sum g_k(x, y, z)\, \partial/\partial y_k + \sum h_l(x, y, z)\, \partial/\partial z_l$$

or, in shorthand notation, $X = f\partial/\partial x + g\partial/\partial y + h\partial/\partial z$, then Eq. (1) translates into

$$\begin{aligned} f(-x, y, Rz) &\equiv f(x, y, z) \\ g(-x, y, Rz) &\equiv -g(x, y, z) \\ h(-x, y, Rz) &\equiv -Rh(x, y, z) \end{aligned} \tag{2}$$

In this non-Hamiltonian context integrability means equivariance with respect to the natural $\mathbb{T}^n$-action on M (compare Broer *et al.*, 1990; Huitema, 1988). Therefore the reversible vector field X is integrable, whenever it has the x-independent form

$$X(x, y, z) = f(y, z)\, \partial/\partial x + g(y, z)\, \partial/\partial y + h(y, z)\, \partial/\partial z \tag{3}$$

where then $g(y, 0) \equiv 0$ by (2). This means that for any $y_0 \in \mathbb{R}^m$ with $h(y_0, 0) = 0$, the torus $\mathbb{T}^n \times \{y_0\} \times \{0\}$ is X-invariant. Observe that this torus also is (G-) invariant, while the dynamics inside the torus is conditionally periodic (or parallel) with frequency vector $f(y_0, 0)$.

Historically, the general question concerns the persistence of such tori under reversible perturbation, the perturbation not necessarily being integrable. In the case $p = 0$, the tori are called *maximal*.

The subcase of this, where $n = m$, was treated by Moser (1973): This is the "classical" KAM-theorem as mentioned before. Notice that now for an integrable vector field (3), the phase space M is completely foliated by invariant n-tori. Indeed, the integrable vector field has the form $X(x, y) = f(y)\, \partial/\partial x$, the involution being given by $G(x, y) = (-x, y)$. It is shown that many of these tori persist. As is well-known, these persisting tori have diophantine frequency vectors. We recall that $\omega = f(y_0)$ is *diophantine*, if for some $\tau > n - 1$ and $\gamma > 0$

$$|\langle \omega, k \rangle| \geqslant \gamma\, |k|^{-\tau} \tag{4}$$

for all $k \in \mathbb{Z}^n \setminus \{0\}$. Here we abbreviated $\langle \omega, k \rangle = \sum \omega_j k_j$ and $|k| = \sum |k_j|$. Observe that an integrable torus with $f(y_0) = \omega$ satisfying (4) has quasi-periodic dynamics. The persistence of these tori is established by constructing (equivariant) conjugacies with the integrable case. Hence, the perturbed dynamics again is quasi-periodic.

For τ and γ fixed, the set of all $\omega \in \mathbb{R}^n$ satisfying (4) is a "Cantor set," a closed and nowhere dense set, of positive measure. Pöschel (1982) further extends this result, showing that in the *nondegenerate* case, where the frequency map $y_0 \mapsto f(y_0)$ has maximal rank, these conjugacies are Whitney-smooth in y_0. Therefore the union of the persistent quasi-periodic tori keeps positive measure.

This result was further generalized by Parasyuk (1982) for the case $p=0$, $n>m$, by Arnold and Sevryuk (1986) and Sevryuk (1986) for the case $p=0$, $n \leqslant m$, and by Huitema (1988, Section 9b) for $p=0$ and general n, m. Huitema's theory requires introducing parameters, as will be explained below.

In the case $p>0$ the invariant n-tori are called *lower dimensional.* It has been shown that the Hamiltonian KAM-theorems on lower dimensional tori have analogues in the reversible setting. For an overview see Sevryuk (1990, 1991a,b, 1993). For more details, see below.

Example (Moser, 1966). Consider a weakly forced oscillator

$$\ddot{z}_1 + a^2 z_1 = \varepsilon f(t, z_1, \dot{z}_1)$$

which is reversible in the sense that $f(-t, z_1, -\dot{z}_1) \equiv f(t, z_1, \dot{z}_1)$. Also, the forcing is quasi-periodic, meaning that for rationally independent (or even diophantine) frequencies $\omega_1, \omega_2, \ldots, \omega_n$, we have $f(t, u, v) = F(t\omega_1, t\omega_2, \ldots, t\omega_n, u, v)$, for a function $F\colon \mathbb{T}^n \times \mathbb{R}^2 \to \mathbb{R}$. Presently the frequencies ω_j are fixed and the problem is to determine, for small $|\varepsilon|$, response solutions with these same frequencies.

The above equation of motion can be written as a vector field

$$\omega \partial/\partial x + a z_2 \partial/\partial z_1 + (-a z_1 + (\varepsilon/a)\, F(x, z_1, z_2))\, \partial/\partial z_2$$

with $\mathbb{T}^n \times \mathbb{R}^2$ as the phase space. This brings us into the present setup, with $m=0$ and $p=1$, where the involution R is given by $R(z_1, z_2) = (z_1, -z_2)$. The n-torus $\mathbb{T}^n \times \{0\}$ is invariant for the unperturbed system, where the dynamics is quasi-periodic with frequency vector ω. The above response problem now translates to the question of persistence of this torus for small values of the perturbation parameter ε, the dynamics in which remains conjugate to the unperturbed one. It turns out that the (normal) frequency a here also is needed as a parameter, suitably included in diophantine conditions. For details see below.

The present paper is concerned with the case of arbitrary n, m, and p, but assumes the presence of sufficiently many external parameters. This *unfolding theory* is embedded in the general Lie algebra setup of Broer *et al.* (1990) and Huitema (1988), as will be explained now. For simplicity everything will be phrased in terms of vector fields, but a similar theory holds for reversible diffeomorphisms.

Remark. The results of Quispel and Sevryuk (1993), however, show that the diffeomorphism case has aspects that are more complicated.

2. SETUP AND RESULTS

A first general (i.e., not necessarily Hamiltonian) KAM-theory was given by Moser (1966, 1967). This theory is formulated at once for various contexts, e.g., for Hamiltonian or volume-preserving systems, but also for the general (dissipative) case. The idea is to express such a preservation of structure in terms of the Lie algebra of all vector fields and its subalgebras. Another idea is the introduction of modifying terms, *viz.*, parameters, needed for the persistence of the tori.

These ideas were taken up by Broer *et al.* (1990) and Huitema (1988), who have developed an unfolding theory of quasi-periodic tori in a general Lie algebra context. The main point is that the parameters allow for variation of all internal and normal frequencies of the integrable tori. This involves the unfolding theory of Arnold (1971), applied to the appropriate subalgebra of the general linear algebra.

The persistent, near-integrable n-tori then smoothly foliate over closed, nowhere dense "Cantor" subsets of the parameter space, that have positive measure. See above; here the techniques of Pöschel (1982) are used. Moreover, the results are formulated in terms of *structural stability*, where the (Whitney-) smooth conjugacies restrict to a suitable union of quasi-periodic (diophantine) invariant tori. For the occasion we speak of *quasi-periodic stability*. A discussion of the relation with the modifying terms of Moser (1966, 1967) is given by Broer *et al.* (1990) and Huitema (1988, Section 7).

However, since the reversible vector fields do not form a Lie algebra, these formulations do not apply directly. Nevertheless, as we shall see below, the proof of the main result of Broer *et al.* (1990) and Huitema (1988) to this setting also applies here. For an indication of this, see also Moser (1966, 1967).

To be precise, as before $M = \mathbb{T}^n \times \mathbb{R}^m \times \mathbb{R}^{2p}$ is our phase space, with coordinates (x, y, z). We also introduce a finite-dimensional parameter space P, with coordinate μ. Instead of individual vector fields, from now on we consider families of those, parametrized by $\mu \in P$. So we are dealing with a family X, given in shorthand notation by

$$X(x, y, z, \mu) = f(x, y, z, \mu)\, \partial/\partial x + g(x, y, z, \mu)\, \partial/\partial y + h(x, y, z, \mu)\, \partial/\partial z \quad (5)$$

where G-reversibility means that for each value of μ, the relations (2) hold. As in (3), integrability amounts to x-independence of X. For simplicity we assume real analyticity in all arguments, noting that a straightforward adaptation exists for the case of C^k, with $k \leqslant \infty$ sufficiently large. Compare Pöschel (1982) or the appendix of Broer *et al.* (1990) and Huitema (1988).

A first, naïeve starting point for the KAM perturbation analysis is an integrable family X, with an invariant n-torus of the form

$$V_{y_0,\mu_0} = \mathbb{T}^n \times \{y_0\} \times \{0\} \subset \mathbb{T}^n \times \mathbb{R}^m \times \mathbb{R}^{2p}$$

so with $h(y_0, 0, \mu_0) = 0$. Observe that V_{y_0,μ_0} is G-invariant. We shall investigate the persistence of such tori under small reversible perturbation.

In the present real analytic setting we use a topology that is natural for the perturbation analysis; compare Broer *et al.* (1990) and Huitema (1988). In fact, we consider M as the real part of $\tilde{M} = (\mathbb{C}^n/(2\pi\mathbb{Z})^n) \times \mathbb{C}^m \times \mathbb{C}^{2p}$. Any of our vector fields then has a complex analytic extension to a neighborhood of M in $\tilde{M}$, and likewise for the parameter space P. In the complex analytic setting we consider the usual *compact-open* topology, for our real analytic families just taking the restriction. We refer to this as the *real analytic topology*.

2.1. Nonisolatedness and Localization

In our reversible case, the invariant n-tori V_{y_0} in integrable systems are not isolated. In fact, as in the Hamiltonian case, these tori occur in continua, parametrized by y_0. This is illustrated by the following example.

Example (Sevryuk (1991a,b). Let Y be a vector field and G an involution on $\mathbb{R}^{n+m+2p}$, with $n \leqslant m$. Let us assume that Y has the origin as an equilibrium, which is fixed by G. Moreover, we assume the following.

1. The linear part $D_0 Y$ has the eigenvalue 0 with multiplicity $m-n$; all other eigenvalues are different from 0 and simple.
2. $D_0 Y$ has at least n pairs of purely imaginary eigenvalues.
3. The type of G is $(n+p, m+p)$.

Then, under generic conditions, involving finitely many nonresonance conditions on the eigenvalues of $D_0 Y$, the following normal forms can be obtained for Y and G. The vector field Y, up to a small pertubation, has the G-reversible, integrable form

$$X(x, y, z) = \omega(y)\,\partial/\partial x + \Omega(y)\,z\partial/\partial z$$

for $(x, y, z) \in \mathbb{T}^n \times \mathbb{R}^m \times \mathbb{R}^{2p}$, where the involution G has the familiar form $G\colon (x, y, z) \mapsto (-x, y, Rz)$ with R of type (p, p). We conclude that the invariant n-tori of the vector field X form a continuum parametrized by y, which fills up the manifold $\{z = 0\}$.

Remarks.

(i) For $m > n$ the above condition 1 on the eigenvalues of $D_0 Y$ is still open (cf. Sevryuk, 1992), also compare the lemma below. In the Hamiltonian setup, however, multiple eigenvalues always mean positive codimension and bifurcation. For a Hamiltonian analogue with a double-zero eigenvalue, e.g., see Broer *et al.* (1993).

(ii) In earlier work of Scheurle (1987), a similar case is studied, where $m = n$, and where a parameter μ is included of at least dimension $n - 1$. In fact, here $D_0 Y$ must have at least n pairs of simple, purely imaginary eigenvalues, while the other eigenvalues should be away from the imaginary axis (but do not have to be simple). In this case, again under generic assumptions, a similar reversible, integrable form is obtained, but now both ω and Ω depend only on μ (rather than on y).

Next let us return to our general situation. The unperturbed problem concerns an integrable, reversible family X, compare (3) and (5), with an invariant n-torus V_{y_0,μ_0} contained in $\{z = 0\}$.

Lemma 1. *Given the integrable family $X(x, y, z, \mu) = f(y, z, \mu)\,\partial/\partial x + g(y, z, \mu)\,\partial/\partial y + h(y, z, \mu)\,\partial/\partial z$ of reversible vector fields, with the invariant n-torus $V_{y_0,\mu_0} = \mathbb{T}^n \times \{y_0\} \times \{0\}$. Assume that $\det h_z(y_0, 0, \mu_0) \neq 0$. Then V_{y_0,μ_0} is embedded in a smooth continuum of invariant n-tori, parametrized by (y, μ). Moreover, there exists an equivariant (local) change of coordinates, after which this continuum coincides with the submanifold $\{z = 0\}$.*

Proof. For the duration of this proof we suppress the parameter μ. By assumption there exists a direct sum splitting $\mathbb{R}^{2p} = \mathbb{R}^p \oplus \mathbb{R}^p$, $z = (z_1, z_2)$, in which the involution R takes the form $R\colon (z_1, z_2) \mapsto (z_1, -z_2)$. In accordance with this splitting we also decompose $h = (h_1, h_2)$.

The reversibility conditions (2) now imply that

$$g(y, z_1, -z_2) \equiv -g(y, z_1, z_2)$$
$$h_1(y, z_1, -z_2) \equiv -h_1(y, z_1, z_2)$$
$$h_2(y, z_1, -z_2) \equiv h_2(y, z_1, z_2)$$

It follows that both $g(y, z_1, 0) \equiv 0$ and $h_1(y, z_1, 0) \equiv 0$. Hence $\partial h_1/\partial z_1(y_0, 0, 0) = 0$, and the assumption of the lemma implies that $\det \partial h_2/\partial z_1(y_0, 0, 0) \neq 0$. By the Implicit Function Theorem we then locally solve the equation $h_2(y, z_1, 0) = 0$ for $z_1 = Z_1(y)$, with $Z_1(y_0) = 0$. This indeed means that y parametrizes a local m-parameter continuum of invariant n-tori. Finally, it is easy to check that

$$(x, y, z_1, z_2) \mapsto (x, y, z_1 - Z_1(y), z_2)$$

is a coordinate change as desired. □

Remark. In the corresponding general Hamiltonian case one has $m = n$ with symplectic form $\sum_{i=1}^{n} dx_i \wedge dy_i + \sum_{j=1}^{p} dz_j \wedge dz_{p+j}$. In that case the normalization of Lemma 1 holds automatically, e.g., compare Broer *et al.* (1990) and Huitema (1988, Sections 3 and 6).

In the sequel our unperturbed system will be the integrable, reversible family X in the generic setting of Lemma 1, so with the corresponding properties $g(y, 0, \mu_0) = 0$ and $h(y, 0, \mu_0) = 0$, for y near y_0. We then say that the family X is *normalized* at the torus V_{y_0,μ_0}. Also, we shall use the notations $V_\mu = \bigcup_y V_{y,\mu}$ and $V = \bigcup_\mu V_\mu$. So for each fixed value of μ, the family X has an m-parameter family $V_\mu = \bigcup_y V_{y,\mu}$ of invariant n-tori, parametrized by y. For technical reasons, however, it is more convenient to have only one torus per parameter value.

The latter situation can be achieved by the following localization, e.g., compare Broer *et al.* (1990) and Huitema (1988, Section 5b). First we introduce P_{loc} as an open subset of $\mathbb{R}^m$ and $V_{\mathrm{loc}} = \{((x, y, z), v, \mu) \in M \times P_{\mathrm{loc}} \times P \mid y = v \text{ and } z = 0\}$. Also, we define the variable $y_{\mathrm{loc}} = y - v$ giving a local analysis near V_{loc} in the variables x, y_{loc}, z, v, and μ. Notice that the involution G now gets the form $(x, y_{\mathrm{loc}}, z) \mapsto (-x, y_{\mathrm{loc}}, Rz)$. Thus we obtain, from our integrable and reversible family X, a family X_{loc}, defined by $X_{\mathrm{loc}}(x, y_{\mathrm{loc}}, z, v, \mu) = X(x, y_{\mathrm{loc}} + v, z, \mu)$, which is again integrable and reversible.

From now on we work in this localized situation, for simplicity writing y again, instead of y_{loc}. Note that the phase space is still the same manifold $M = \mathbb{T}^n \times \mathbb{R}^m \times \mathbb{R}^{2p}$. Here we are dealing with an integrable family

$X_{\text{loc}} = X_{\text{loc}}(x, y, z, \nu, \mu)$ with invariant tori that are now given by the equations $y = 0$, $z = 0$. Persistence results in the localized setting are easily translated to the original one.

2.2. Normal Linearization

It is technically convenient to transfer this perturbation problem to the *normal bundle* $N(V_{\nu,\mu})$ of $V_{\nu,\mu}$ in M. Compare Broer *et al.* (1990) and Huitema (1988, Sections 2a, 6b).

To this end we consider the following situation, for the moment dropping all subscripts loc. Let $X = X(x, y, z, \nu, \mu)$ be a general, i.e., not necessarily integrable, reversible family (5), with $V_{\nu,\mu}$ as an invariant torus. This means that

$$g(x, 0, 0, \nu, \mu) = 0 = h(x, 0, 0, \nu, \mu)$$

for all $x \in \mathbb{T}^n$. On a neighborhood of $V_{\nu,\mu}$ in M and for $\varepsilon > 0$, we now define a scaling operator $D_\varepsilon \colon M \to N(V_{\nu,\mu})$, given by

$$D_\varepsilon \colon (x, y, z) \mapsto (x, \varepsilon^{-1} y, \varepsilon^{-1} z)$$

Here we identify $N(V_{\nu,\mu})$ with a neighborhood of $V_{\nu,\mu}$. One easily defines an involution $N(G)$ on $N(V_{\nu,\mu})$, such that $N(G) \circ D_\varepsilon = D_\varepsilon \circ G$. This means that the scaling D_ε does not take us out of the reversible setting. Suppressing all parameters, we get

$$(D_\varepsilon)_* X(x, y, z) = f(x, \varepsilon y, \varepsilon z)\, \partial/\partial x + \varepsilon^{-1} g(x, \varepsilon y, \varepsilon z)\, \partial/\partial y$$
$$+ \varepsilon^{-1} h(x, \varepsilon y, \varepsilon z)\, \partial/\partial z$$

Expanding

$$g(x, y, z) = g_y(x, 0, 0)\, y + g_z(x, 0, 0)\, z + O(|y|^2 + |z|^2)$$
$$h(x, y, z) = h_y(x, 0, 0)\, y + h_z(x, 0, 0)\, z + O(|y|^2 + |z|^2)$$

it follows that $\lim_{\varepsilon \to 0} (D_\varepsilon)_* X$ exists as a normal linear, reversible vector field $N(X)$ on $N(V_{\nu,\mu})$, given by

$$N(X)(x, y, z) = f(x, 0, 0)\, \partial/\partial x + (g_y(x, 0, 0)\, y + g_z(x, 0, 0)\, z)\, \partial/\partial y$$
$$+ (h_y(x, 0, 0)\, y + h_z(x, 0, 0)\, z)\, \partial/\partial z$$

In the integrable case, this normal linear form again is integrable, i.e., x-independent, in which case we speak of a *Floquet form*. Let us study these forms further. In the fiber direction this involves the (infinitesimally) reversible Floquet matrix

$$\begin{pmatrix} g_y(0,0) & g_z(0,0) \\ h_y(0,0) & h_z(0,0) \end{pmatrix} \tag{6}$$

From the reversibility condition (2) and the normalization of Lemma 1, we here obtain that both $g_y(0,0)=0$ and $h_y(0,0)=0$.

The perturbation problem then is transferred to the normal bundle $N(V)=\bigcup_{\nu,\mu} N(V_{\nu,\mu})$ as follows. Any perturbation of the integrable family X, for small $\varepsilon>0$, by $(D_\varepsilon)_*$ is transformed into a perturbation of the corresponding Floquet form $N(X)$. Persistence concerning the union $V=\bigcup_{\nu,\mu} V_{\nu,\mu}$ of tori then translates to that of the zero section of $N(V)$. Finally, an extra scaling $y=\bar{y}$, $z=\varepsilon\bar{z}$, allows us to replace (6) by

$$\begin{pmatrix} 0 & 0 \\ 0 & h_z(0,0) \end{pmatrix} \tag{7}$$

where, again by (2), the matrix $h_z(0,0)\in gl(2p,\mathbb{R})$ is (infinitesimally) reversible. From now on we denote the set of all such reversible matrices by $gl_{-R}(2p,\mathbb{R})$.

2.3. Nondegeneracy and Diophantine Conditions

To summarize the above preliminaries, our perturbation problem technically lives on $N(V_{\text{loc}})$. Here, as the unperturbed system, a reversible family of Floquet forms

$$N(X_{\text{loc}})(x,y,z,\nu,\mu)=f_{\text{loc}}(0,0,\nu,\mu)\,\partial/\partial x+h_{\text{loc},z}(0,0,\nu,\mu)\,z\partial/\partial z$$

is considered, where the interest is with the persistence of the zero section $y=0$, $z=0$. (As before we keep writing y instead of y_{loc}, omitting all bars on the coordinates.)

We now need to introduce a generalization of the nondegeneracy concept met before, concerning the maximal rank of the frequency map. To this end we define maps ω: $P_{\text{loc}}\times P\to\mathbb{R}^n$ and Ω: $P_{\text{loc}}\times P\to gl(2p,\mathbb{R})$ by

$$\omega(\nu,\mu)=f_{\text{loc}}(0,0,\nu,\mu) \qquad \text{and} \qquad \Omega(\nu,\mu)=h_{\text{loc},z}(0,0,\nu,\mu) \tag{8}$$

The generalization involves the eigenvalues of the matrices $\Omega(\nu,\mu)$; compare Broer *et al.* (1990), Huitema (1988), and Moser (1967). As we saw before, $\Omega(\nu,\mu)$ is reversible, i.e., $\Omega(\nu,\mu)\in gl_{-R}(2p,\mathbb{R})$. This is easily seen to imply that if λ is an eigenvalue of Ω, then so is $-\lambda$.

For $(\nu_0, \mu_0) \in P_{\text{loc}} \times P$, we assume that $\Omega(\nu_0, \mu_0)$ has only simple eigenvalues. Notice that by continuity, this property is persistent for small variation of the parameters. Moreover, by the evenness of the dimension $\det \Omega(\nu, \mu) \neq 0$, for all (ν, μ) near (ν_0, μ_0). Let us consider the spectrum of such matrices $\Omega(\nu, \mu)$. This consists of the eigenvalues $\pm\delta_1, \pm\delta_2, \ldots, \pm\delta_{N_1}$; $\pm i\varepsilon_1, \pm i\varepsilon_2, \ldots, \pm i\varepsilon_{N_2}$; $\pm\alpha_1 \pm i\beta_1, \pm\alpha_2 \pm i\beta_2, \ldots, \pm\alpha_{N_3} \pm i\beta_{N_3}$, depending on (ν, μ), where $p = N_1 + N_2 + 2N_3$ and where all δ_j, ε_j, α_j, and β_j are taken positive. (The simplicity also implies mutual distinctness.)

We then consider the map spec: $P_{\text{loc}} \times P \to \mathbb{R}^p$, defined by

$$\text{spec}: (\nu, \mu) \mapsto (\delta_{n_1}, \varepsilon_{n_2}, \alpha_{n_3}, \beta_{n_3})(\nu, \mu) \tag{9}$$

where $1 \leqslant n_j \leqslant N_j$ for $j = 1$, 2, and 3.

The invariant torus V_{ν_0, μ_0} is said to be *nondegenerate* if the product map $\omega \times \text{spec}: P_{\text{loc}} \times P \to \mathbb{R}^n \times \mathbb{R}^p$, at (ν_0, μ_0) has a surjective derivative. In the next section we shall meet a more conceptual approach to this property, using versality of the unfolding $\Omega(\nu, \mu)$; compare Arnold (1971), Broer *et al.* (1990), Huitema (1988), and Sevryuk (1992).

Let Γ be a connected neighborhood of (ν_0, μ_0) in $P_{\text{loc}} \times P$, such that for each $(\nu, \mu) \in \Gamma$ the eigenvalues of $\Omega(\nu, \mu)$ are simple. Then for $(\nu, \mu) \in \Gamma$, let $\omega_1^N(\nu, \mu), \ldots, \omega_r^N(\nu, \mu)$ be the positive imaginary parts of the eigenvalues of $\Omega(\nu, \mu)$, where we refrain from counting double. So, in the above terms, the ω_j^N consist of the ε_j and β_j, and $r = N_2 + N_3$. These numbers are called the *normal frequencies* of the torus $V_{\nu, \mu}$. Thus we obtain a frequency map

$$\mathscr{F}: \Gamma \to \mathbb{R}^n \times \mathbb{R}^r, (\nu, \mu) \mapsto (\omega(\nu, \mu), \omega^N(\mu, \nu)) \tag{10}$$

with $\omega^N = (\omega_1^N, \ldots, \omega_r^N)$. In the nondegenerate case, by taking Γ sufficiently small, we can ensure $\mathscr{F}$ to be a submersion.

We recall that the dynamics in $V_{\nu,\mu}$ is quasi-periodic if the (internal) frequencies $\omega_1(\nu, \mu), \ldots, \omega_n(\nu, \mu)$ are diophantine (in fact rational independence would suffice for this). Presently, however, diophantine conditions of type (4) are not sufficient, but we need to include the normal frequencies ω^N into these. To be precise, we fix $\tau > n - 1$ and consider $\gamma > 0$ as a parameter. Then we require that

$$|\langle \omega, k \rangle + \langle \omega^N, l \rangle| \geqslant \gamma\, |k|^{-\tau} \tag{11}$$

for all $k \in \mathbb{Z}^n \backslash \{0\}$ and all $l \in \mathbb{Z}^r$ with $|l| \leqslant 2$. Here we use the same notation as in (4). By $(\mathbb{R}^n \times \mathbb{R}^r)_\gamma$ we denote the set of all $(\omega, \omega^N) \in \mathbb{R}^n \times \mathbb{R}^r$ satisfying (11). Also, we consider the pullback

$$\Gamma_\gamma = \mathscr{F}^{-1}((\mathbf{R}^n \times \mathbf{R}^r)_\gamma) \tag{12}$$

to $P_{\text{loc}} \times P$. As before, Γ_γ is a "Cantor set," i.e., a closed, nowhere dense subset of Γ of positive measure. Indeed, it is easy to see that for bounded sets in $\Gamma \backslash \Gamma_\gamma$, the measure is of order γ as $\gamma \downarrow 0$.

2.4. The Stability Result

We are now ready to formulate the main result of this paper.

Main Theorem. *Let X be a real analytic integrable, reversible family of vector fields on M, parametrized over P. Also, let V and V_μ, for $\mu \in P$ be as before. For $(y_0, \mu_0) \in \mathbb{R}^m \times P$ assume that X has an invariant n-torus $V_{\mu_0} \cap \{y = y_0\}$, at which it is both normalized and nondegenerate. Moreover, assume that the matrix $\Omega(y_0, \mu_0)$ has only simple eigenvalues.*

Then for $\gamma > 0$ sufficiently small, there exists a neighborhood Γ of (y_0, μ_0) in $\mathbb{R}^m \times P$ and a neighborhood $\mathscr{V}$ of $N(X_{\text{loc}})$ in the real analytic topology on the reversible families of vector fields on $N(V_{\text{loc}})$, such that the following holds. For all $\tilde{X} \in \mathscr{V}$ there exists a map $\Phi: N(V_{\text{loc}}) \to N(V_{\text{loc}})$ such that

1. *The restriction of Φ to $(\nu, \mu) \in \Gamma$ is an equivariant C^∞-diffeomorphism onto its image, C^∞-near the identity map. Moreover, Φ preserves the projection to $P_{\text{loc}} \times P$ and to the zero section of V_{loc}. Also, Φ is real analytic in x and affine in the variables y and z.*
2. *A further restriction of Φ to $(\nu, \mu) \in \Gamma_\gamma$ takes the zero section of $N(V_{\text{loc}})$ to an $\tilde{X}$-invariant manifold $\tilde{V}_{\text{loc}}$, inducing a conjugacy $N(\Phi): N(V_{\text{loc}}) \to N(\tilde{V}_{\text{loc}})$ between $N(X)$ and $N(\tilde{X})$.*

A proof of the Main Theorem will be given in the next section, by reducing it to that of Broer *et al.* (1990) and Huitema (1988, Theorem 8.1). From this theorem we can deduce a stability result on $M \times P_{\text{loc}} \times P$, using the scaling operator D_ε, for $\varepsilon > 0$ and small. From here we can project back to $M \times P$, so obtaining an answer to the original perturbation problem. This will now be summarized.

Corollary. *Let X and Γ be as in the Main Theorem. Then for all real analytic reversible families $\tilde{X}$, sufficiently near X in the real analytic topology, the following holds. There exists an $\tilde{X}$-invariant "Cantor set" $\tilde{V} \subset M \times P$, which is a C^∞-near-identity-image of $V \cap \{(y, \mu) \in \Gamma_\gamma\}$ and hence a union of n-tori. Inside the tori this diffeomorphism induces a real analytic conjugacy between X and $\tilde{X}$. Moreover, the diffeomorphism is equivariant and preserves the (reversible) normal linear behavior.*

2.5. Concluding Remarks

We recall the remarks, made in Section 1, regarding structural stability, in particular quasi-periodic stability: The perturbed family $\tilde{X}$ has an invariant "Cantor set," being a union of quasi-periodic tori, on which it is conjugate to the restriction $X | V \cap \{(y, \mu) \in \Gamma_\gamma\}$. In particular, the property of having such an invariant set of positive measure in the product $M \times P$ is open in the real analytic topology. This despite the infinite codimension of integrable families.

We note, however, that, although the conjugacy between X and $\tilde{X}$ is equivariant, it does not necessarily preserve the projection to the parameter space P.

Remarks.

(i) A direct adaptation of these results exists in the forced oscillator case quoted in Section 1. Here we have that $\omega^N = a$, while the n-vector ω remains fixed. In fact, this problem has the parameter(s) $\mu = (\varepsilon, a)$ and we consider the frequency map $\mathscr{F}: (\varepsilon, a) \mapsto a$, defined on some bounded, say, rectangular set Γ. Condition (11) defines a Cantor set on the a-axis and Γ_γ then is the product of an ε-interval with this Cantor set. We consider the Floquet form

$$\omega \partial/\partial x + \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} z \partial/\partial z$$

with invariant n-tori V_μ, given by $z = 0$. The perturbation $(\varepsilon/a) F$ only slightly distorts this family of n-tori, as well as the Cantor set. In fact, the conjugacy Φ acts as the identity in the x-direction and is analytic in ε.

(ii) In the case $p = 0$ the phase space reduces to $M = \mathbb{T}^n \times \mathbb{R}^m$. As we saw in Section 1, this means that the involution G simply is $(x, y) \mapsto (-x, y)$, while the integrable family X has the form $X(x, y, \mu) = f(y, \mu)\, \partial/\partial x$. Since there are no normal frequencies, nondegeneracy of X involves only f, which then must have maximal rank. In the case $p = 0$, $m \geqslant n$, the projection to P can be preserved by the conjugacies, provided that for fixed μ the map $y \mapsto f(y, \mu)$ already has maximal rank. In that case, for each fixed μ, there exists an invariant set as described above (i.e., a union of quasi-periodic tori of positive measure), which is contained in M.

The present approach needs a lot of parameters: Nondegeneracy can hold true only in the presence of at least $n + p$ parameters. However, there exist variations of the above results where the conditions are relaxed somewhat. For a discussion see Broer *et al.* (1990) and Huitema (1988, Section 7).

One simplification "gets rid" of all hyperbolic eigenvalues reducing to an equivariant center manifold. In that case the corresponding parameters can be dropped. Then the regularity of the conjugacy generically decreases to finite differentiability. Compare Broer *et al.* (1990), Huitema (1988), and Pöschel (1988). Also, the normal linear behavior will only be partially preserved while conjugating.

A further reduction passes from conjugacies to equivalences, only considering ratios of frequencies. This "gets rid" of one further parameter. The case where m, n, and p are arbitrary with $N_2 = 0$, and where n parameters are present, has been considered by Bibikov (1991).

As said in Section 2.1, the results of Sevryuk (1991a, b) concern individual systems instead of families. In the cases where the nondegeneracy already can be fulfilled with help of the parameter $\nu \in P_{\mathrm{loc}}$ alone, we get results on individual systems particularizing as before. In that case our results are related to Sevryuk (1991a, b), where it is necessary that $m \geqslant n$. Our approach, however, needs the stronger assumption that $m \geqslant n + p$, which, according to the earlier remark about reduction to a center manifold, can be relaxed to $m \geqslant n + N_2 - 1$.

The results of Parasyuk (1982) for the case $p = 0$, $m \leqslant n$, also concern individual systems. The corresponding approach is based on a special technique of diophantine approximations on submanifolds of $\mathbb{R}^n$ and will not be discussed here.

At this point a recent development in classical KAM-theory should be mentioned, e.g., see Xiu *et al.* (1994). Here small divisor methods are presented for nearly integrable Hamiltonian systems with degeneracy, the weaker nondegeneracy conditions translating into a smaller number of required parameters. These methods also may be useful in the present setting, probably leading to an unfolding theory with fewer parameters. In future research we will come back to this.

3. PROOF

3.1. Introduction

Our proof of the Main Theorem very closely follows that of Broer *et al.* (1990) and Huitema (1988, Theorem 8.1), but many ideas are given by Moser (1966, 1967). Let us describe what is going on. First, in order to avoid heavy notation, we assume that the family $X = X(x, y, z, \mu)$ already is localized in the sense of Section 2.1. This means that we are considering one invariant n-torus $V_\mu = \mathbb{T}^n \times \{y = 0\} \times \{z = 0\}$, for each $\mu \in P$.

As said before, we consider a perturbation $\tilde{X}$ of the integrable normal linear vector field

$$N(X)(x, y, z, \mu) = \omega(\mu)\, \partial/\partial x + \Omega(\mu)\, z\partial/\partial z$$

Using a proper identification of the normal bundle $N(V)$ with a neighborhood of V in $M \times P$ and, for simplicity, writing X instead of $N(X)$, the perturbation $\tilde{X}$ gets the form

$$\tilde{X} = X + f\partial/\partial x + g\partial/\partial y + h\partial/\partial z \tag{13}$$

with f, g, and h small in the real analytic topology. We now have to produce a transformation Φ: $(\xi, \eta, \zeta, \kappa) \mapsto (x, y, z, \mu)$, satisfying the nonlinear conjugacy equation

$$\begin{aligned}\Phi^*(\tilde{X})(\xi, \eta, \zeta, \kappa) = X(\xi, \eta, \zeta, \kappa) &+ O(|\eta|, |\zeta|)\, \partial/\partial\xi \\ &+ O(|\eta|, |\zeta|)\, \partial/\partial\eta + O(|\eta|, |\zeta|^2)\, \partial/\partial\zeta\end{aligned} \tag{14}$$

Here $\Phi^* = (\Phi^{-1})_*$. Notice that Φ serves to reduce the perturbation terms in (13) to the small O-terms in the right-hand side of (14). In the solution of Eq. (14) we shall make use of the diophantine conditions (11) that define the "Cantor set" $\Gamma_\gamma \subset P$; see Section 2.3. The map Φ is constructed as an infinite product

$$\Phi = \Psi_0 \circ \Psi_1 \circ \cdots \tag{15}$$

corresponding to a Newtonian iteration process. Here the following holds. The Ψ_j are defined as analytic maps on neighborhoods of $V \cap \{\mu \in \Gamma_\gamma\}$, that shrink in an appropriate way with j. These maps and neighborhoods have to be described carefully, in order to ensure the convergence of the product to a Whitney-differentiable map. Moreover, each Ψ_j is determined from a linearized version of Eq. (14), where the perturbation terms [again see (13)] are reduced "rapidly" with increasing index j.

In the Lie algebra setting of Broer *et al.* (1990) and Huitema (1988, Section 8) (also see Moser, 1966, 1967), all Ψ_j, and also Φ, are taken from the Lie group, corresponding to the Lie algebra at hand. In fact, in each iteration step Ψ_j is generated from this Lie algebra.

Presently this procedure has to be slightly changed, since the reversible vector fields with $G_*(Y) = -Y$ [see (1)] do not form a Lie algebra, although they do form a linear subspace. The equivariant vector fields, defined by $G_*(Y) = Y$, however, do form a Lie algebra. This algebra generates the group of equivariant maps, i.e., maps Φ commuting with G. As said in Section 1, if Y is a reversible vector field and Φ an equivariant transformation, then also $\Phi_*(Y)$ is reversible.

The main difference with Broer *et al.* (1990) and Huitema (1988, Section 8) is that we will carry out the iteration within the general algebra of vector fields, without explicitly keeping track of G. It then will turn out that, by the reversibility of the vector fields under iteration, the transformations Ψ_j automatically are equivariant. First, this implies that such a Newtonian iteration process is possible in the world of reversible systems. Second, and moreover, it implies that the convergence proof in Broer *et al.* (1990) and Huitema (1988, Section 8b) also applies for this case, which provides us with a full proof of the Main Theorem.

3.2. Preliminaries

First we need some properties related to the classes of vector fields at hand. We list them in a lemma, which is easily proven, mainly using the homogeneity of the involution.

To this end, for any vector field $Y = F\partial/\partial x + G\partial/\partial y + H\partial/\partial z$, we consider the linearization

$$\begin{aligned} Y_{\text{lin}}(x, y, z) = F(x, 0, 0)\, \partial/\partial x &\\ + (G(x, 0, 0) + G_y(x, 0, 0)\, y + G_z(x, 0, 0)\, z)\, \partial/\partial y &\\ + (H(x, 0, 0) + H_y(x, 0, 0)\, y + H_z(x, 0, 0)\, z)\, \partial/\partial z & \quad (16) \end{aligned}$$

denoting the space of all such, real analytic linearizations by $\mathscr{L}$. Also, we consider its Fourier truncations Y_d, $d \geqslant 0$, defined by

$$Y_d(x, y, z) = \sum_{|k| \leqslant d} Y_k(y, z)\, e^{i\langle x, k\rangle} \quad (17)$$

In particular, we shall need the truncations $Y_{\text{lin},d}$ of Y_{lin}, the space of which will be denoted by $\mathscr{L}_d$.

We recall the notation $gl_{-R}(2p, \mathbb{R})$ for the reversible matrices. Also, we introduce $gl_R(2p, \mathbb{R})$ for the equivariant matrices, i.e., the matrices that commute with R. Note that $gl_R(2p, \mathbb{R})$ is a Lie algebra. By $GL_R(2p, \mathbb{R}) = GL(2p, \mathbb{R}) \cap gl_R(2p, \mathbb{R})$ we denote the corresponding Lie group.

Lemma 2.

1. *If Y is a reversible, viz., an equivariant vector field, then so are both Y_{lin} and all Fourier truncations Y_d, respectively.*
2. *For any $\omega \in \mathbb{R}^n$, the normal linear system $\omega\partial/\partial x + \Omega z\partial/\partial z$ is a reversible vector field if and only if $\Omega \in gl_{-R}(2p, \mathbb{R})$.*
3. *The group $GL_R(2p, \mathbb{R})$ is algebraic.*

The first item of the lemma permits us to introduce $\mathscr{L}_{-G}$ and $\mathscr{L}_G$ for the reversible and the equivariant vector fields in $\mathscr{L}$, as well as $\mathscr{L}_{-G,d}$ and $\mathscr{L}_{G,d}$ for the corresponding Fourier truncations. The last statement implies that the orbits of the adjoint action of $GL_R(2p, \mathbb{R})$, are smooth submanifolds.

We introduce some further ingredients. To begin with we need the following direct sum spittings:

$$\begin{gathered} gl_R(2p, \mathbb{R}) = gl_{-R}(2p, \mathbb{R}) \oplus gl_R(2p, \mathbb{R}) \\ \mathscr{L} = \mathscr{L}_{-G} \oplus \mathscr{L}_G \end{gathered} \tag{18}$$

and similar for $\mathscr{L}_d$. We now turn to the normal linear unfolding $X(x, y, z, \mu) = \omega(\mu)\, \partial/\partial x + \Omega(\mu)\, z\partial/\partial z$, recalling that we unfold around the value μ_0. Since this form is reversible, so are the matrices $\Omega(\mu)$ by the previous lemma. From now on the parameter μ often is indicated as a superscript.

In the proof we have to consider the adjoint action ad X^μ, defined by the Lie bracket $Y \mapsto [X^\mu, Y]$. Observe that by the normal linearity of X^μ, the adjoint action ad X^μ, by (co-)restriction, induces well-defined linear maps $\mathscr{L} \to \mathscr{L}$ and $\mathscr{L}_d \to \mathscr{L}_d$. The natural counterpart of this action at the level of matrices is that of ad Ω^μ, acting on $gl_{-R}(2p, \mathbb{R})$.

Lemma 3. *Under the hypotheses of the Main Theorem, for all $\mu \in \Gamma$,*

1. ad Ω^μ *is semisimple;*
2. ad Ω^μ *interchanges the properties "reversible" and "equivariant," i.e.,* ad $\Omega^\mu(gl_{-R}(2p, \mathbb{R})) \subset gl_R(2p, \mathbb{R})$ *and* ad $\Omega^\mu(gl_R(2p, \mathbb{R})) \subset gl_{-R}(2p, \mathbb{R})$;
3. *the adjoint action of $GL_R(2p, \mathbb{R})$ on $gl_R(2p, \mathbb{R})$ respects the first direct sum splitting of (18).*

Proof. The first statement follows from the fact that Ω^μ has only simple eigenvalues. The second and third items run like this: Suppose that $RA = -AR$, then $[\Omega, A]\, R = \Omega AR - A\Omega R = -\Omega RA + AR\Omega = R\Omega A - RA\Omega = R[\Omega, A]$, etc. □

Remark. The kernel ker ad Ω^{μ_0} also is called the centralizer, denoted by $C(\Omega^{\mu_0})$, and by the first statement of the lemma we have another direct sum splitting,

$$gl(2p, \mathbb{R}) = C(\Omega^{\mu_0}) \oplus \text{im ad } \Omega^{\mu_0} \tag{19}$$

We now explore the nondegeneracy of X^μ at the torus V_{μ_0} in terms of transversality of the matrix unfolding Ω^μ, where we consider the adjoint action of $GL_R(2p, \mathbb{R})$ on $gl_{-R}(2p, \mathbb{R})$. Compare Broer *et al.* (1990), Huitema (1988, Section 8a), and Sevryuk (1992). To this purpose we first define $C_{-R}(\Omega^{\mu_0}) := C(\Omega^{\mu_0}) \cap gl_{-R}(2p, \mathbb{R})$ as a linear space.

Lemma 4. *Under the hypothesis that Ω^{μ_0} has only simple eigenvalues,*

1. *The torus V_{μ_0} is nondegenerate if and only if $\mu \mapsto \omega(\mu)$ has a surjective derivative and if the unfolding $\Omega(\mu)$ is transversal to the orbit of Ω^{μ_0}.*
2. *$C_{-R}(\Omega^{\mu_0})$ is minitransversal to this orbit and has a linear isomorphic parametrization by $\mathbb{R}^p$, which yields a universal unfolding of Ω^{μ_0}.*

Proof. The first statement follows from the considerations in Section 2.3, in particular, of the map spec. It directly follows that the codimension of Ω^{μ_0} equals p. Also, by (19) we have

$$gl_{-R}(2p, \mathbb{R}) = C_{-R}(\Omega^{\mu_0}) \oplus \operatorname{ad} \Omega^\mu(gl_R(2p, \mathbb{R})) \quad \Box \tag{20}$$

In our context of finding conjugacies we now may simplify our unfolding as follows. We take $\mu = (\omega, \lambda)$, writing

$$X^{\omega,\lambda}(x, y, z) = \omega\partial/\partial x + \Omega(\lambda)\, z\partial/\partial z \tag{21}$$

where $\lambda \in \mathbb{R}^p \mapsto \Omega(\lambda) \in C_{-R}(\Omega^{\mu_0})$ is the linear centralizer unfolding of the above lemma. Here we unfold around $\lambda_0 \neq 0$, where $\Omega(\lambda_0) =: \Omega_0$. In this way, the parameter space becomes $P \subset \mathbb{R}^n \times \mathbb{R}^p$, as an open subset.

Remarks.

(i) For the linear centralizer unfolding one, moreover, has $C_{-R}(\Omega(\lambda)) = C_{-R}(\Omega_0)$, for all λ in a neighborhood of λ_0. This yields the direct sum splitting

$$gl_{-R}(2p, \mathbb{R}) = C_{-R}(\Omega_0) \oplus \operatorname{ad} \Omega(\lambda)(gl_R(2p, \mathbb{R})) \tag{22}$$

for all λ in a neighborhood of λ_0. This implies that the family $\Omega(\lambda)$ is a universal unfolding for each of its members.

(ii) With the above "definition" of nondegeneracy, it can be of interest to abandon the hypothesis that Ω_0 has only simple eigenvalues, and study problems where eigenvalues bifurcate. For some examples, e.g., see Braaksma *et al.* (1990).

By Lemma 2, the reversibility of the matrices $\Omega(\lambda)$ implies reversibility of the vector fields $X^{\omega,\lambda}$. Next we turn to the map ad X^μ.

Lemma 5. *Under the conditions of the Main Theorem, for* $\mu \in \Gamma_\gamma$,

1. ad X^μ: $\mathscr{L} \to \mathscr{L}$ *is semisimple;*
2. $N^\mu \in \ker \operatorname{ad} X^\mu \cap \mathscr{L}_{-G}$ *if and only if* N^μ *has the equivariant (or integrable) form* $N^\mu(x, y, z) = c_1(\mu)\, \partial/\partial x + C_2(\mu)\, z\partial/\partial z$ *with* $C_2(\mu) \in \ker \operatorname{ad} \Omega^\lambda \cap C_{-R}(\Omega_0)$;
3. ad X^μ *interchanges the properties "reversible" and "equivariant."*

Proof. Let us specify $Y \in \mathscr{L}$ by $Y^\mu(x, y, z) = u(x, \mu)\, \partial/\partial x + v(x, y, z, \mu)\, \partial/\partial y + w(x, y, z, \mu)\, \partial/\partial z$; see (16). Then

$$\operatorname{ad} X(Y) = u_x \omega \partial/\partial x + (v_x \omega + v_z \Omega z)\, \partial/\partial y + (w_x \omega + w_z \Omega z - \Omega w)\, \partial/\partial z \tag{23}$$

A straightforward computation in terms of Fourier coefficients (e.g., see Moser, 1967; Broer *et al.*, 1990; Huitema, 1988, Section 8b) now provides a basis of eigenvectors. Here we need the nonresonance conditions that follow from (11). In particular, we find the kernel by solving the equation ad $X(Y) = 0$. Indeed, for $\mu \in \Gamma_\gamma$ one gets the x-independent form

$$\begin{aligned} u(x, \mu) &= u_0(\mu) \\ v(x, y, z, \mu) &= v_0(\mu) + v_1(\mu)\, y + v_2(\mu)\, z \\ w(x, y, z, \mu) &= w_0(\mu) + w_1(\mu)\, y + w_2(\mu)\, z \end{aligned}$$

where

$$\Omega w_0 = 0, \qquad v_2 \Omega = 0, \qquad \Omega w_1 = 0, \qquad \Omega w_2 - w_2 \Omega = 0$$

We recall that by the simplicity of its eigenvalues it follows that det $\Omega \neq 0$ and hence that $w_0 = 0$, $v_2 = 0$, and $w_1 = 0$. By the reversibility of Y it further follows that $v_0 = 0$ and $v_1 = 0$. So we take $c_1 = u_0$ and $C_2 = w_2$.

The final statement of the lemma is an easy consequence of the reversibility of X. Indeed, one has $G_*[X, Y] = -[X, G_*(Y)]$, which directly implies the assertion. Also, compare Lemma 3. □

3.3. The Iteration Step

This section deals with the determination of the maps Ψ_j (see above); in particular, see the product (15). To this end write $\Phi_j = \Psi_0 \circ \Psi_1 \circ \cdots \Psi_{j-1}$, and $\tilde{X}_j = \Phi_j^*(\tilde{X})$, noting that $\Phi_0 = Id$ and therefore $\tilde{X}_0 = \tilde{X}$. Also, notice that $\Phi_{j+1} = \Phi_j \circ \Psi_j$. The jth iteration step now concerns the relation $\tilde{X}_{j+1} = \Psi_j^*(\tilde{X}_j)$.

Let x_j, y_j, z_j, ω_j, and λ_j be the component functions of the inverse Φ_j^{-1}, then, in coordinates, we have to specify

$$\Psi_j\colon (x_{j+1}, y_{j+1}, z_{j+1}, \omega_{j+1}, \lambda_{j+1}) \mapsto (x_j, y_j, z_j, \omega_j, \lambda_j)$$

in terms of the vector field $\tilde{X}_j$. From now on $j \geqslant 0$ is fixed. In order to avoid clumsy notation, for the duration of this one iteration step, we drop the index j, writing $(x, y, z, \omega, \lambda)$ and $(\xi, \eta, \zeta, \sigma, \nu)$ instead of $(x_j, ..., \lambda_j)$ and $(x_{j+1}, ..., \lambda_{j+1})$, respectively.

As announced before, the map $\Psi = \Psi_j$ is infinitesimally generated from a vector field, to be denoted $\bar{\Psi}$. Also, a shift in the parameters will be included. To be precise, we further specify the unknown $\bar{\Psi} \in \mathscr{L}$ by

$$\bar{\Psi}(\xi, \eta, \zeta, \sigma, \nu) = \bar{U}(\xi, \sigma, \nu)\, \partial/\partial\xi + \bar{V}(\xi, \eta, \zeta, \sigma, \nu)\, \partial/\partial\eta + \bar{W}(\xi, \eta, \zeta, \sigma, \nu)\, \partial/\partial\zeta$$

with

$$\begin{aligned} \bar{V}(\xi, \eta, \zeta, \sigma, \nu) &= \bar{V}_0(\xi, \sigma, \nu) + \bar{V}_1(\xi, \sigma, \nu)\, \eta + \bar{V}_2(\xi, \sigma, \nu)\, \zeta \\ \bar{W}(\xi, \eta, \zeta, \sigma, \nu) &= \bar{W}_0(\xi, \sigma, \nu) + \bar{W}_1(\xi, \sigma, \nu)\, \eta + \bar{W}_2(\xi, \sigma, \nu)\, \zeta \end{aligned} \tag{24}$$

Next unknown parameter shifts are specified to have the form

$$\sigma \mapsto \omega(\sigma, \nu) = \sigma + \Lambda_1(\sigma, \nu), \qquad \nu \mapsto \lambda(\sigma, \nu) = \nu + \Lambda_2(\sigma, \nu) \tag{25}$$

Then if $\tilde{X} = \tilde{X}_j$ has the form (13), we first introduce the vector field

$$L^{\sigma,\nu}(\xi, \eta, \zeta) = (\tilde{X} - X)_{\mathrm{lin},d}\,(\xi, \eta, \zeta, \sigma, \nu) \tag{26}$$

In the Newton procedure $L \in \mathscr{L}_d$ replaces the perturbation terms of (13). The order of truncation d is determined in the proof of Broer *et al.* (1990) and Huitema (1988, Theorem 8.1), but the present considerations are independent of this choice. Second, regarding (25), we introduce a vector field $N \in \mathscr{L}_0 \subset \mathscr{L}_d$ by

$$N^{\sigma,\nu}(\xi, \eta, \zeta) = \Lambda_1(\sigma, \nu)\, \partial/\partial\xi + \Omega(\Lambda_2(\sigma, \nu))\, \zeta\partial/\partial\zeta \tag{27}$$

The linearized version of the conjugacy equation (14) now becomes

$$\mathrm{ad}\, X(\bar{\Psi}) = N + L \tag{28}$$

which has to be solved in $\bar{\Psi}$ and N. This is the so-called homological equation. The parameters (σ, ν) vary over a neighborhood of Γ_γ, where the diophantine conditions (11) hold only for $|k| \leqslant d$. The homological equation is solved by comparing coefficients of the (trigonometric) polynomials; see the proof of Lemma 5.

Observe that both L and N are reversible; cf. Lemma 2. The general idea is to determine $N \in \ker \operatorname{ad} X$ such that $N + L \in \operatorname{im} \operatorname{ad} X$. This means that N follows from integrability conditions. By Lemma 5 it follows that (27) is the typical form of a reversible element of $\ker \operatorname{ad} X$. A straightforward computation shows that

$$N^{\sigma,\nu}(\xi, \eta, \zeta) = -([f(\cdot, 0, 0, \sigma, \nu)]\, \partial/\partial\xi + [h_\zeta(\cdot, 0, 0, \sigma, \nu)]_{C_{-R}(\Omega_0)}\, \zeta\partial/\partial\zeta) \tag{29}$$

where $[\cdot]$ denotes the ξ-average over $\mathbb{T}^n$ and where the subscript $C_{-R}(\Omega_0)$ refers to the corresponding part in the direct sum splitting of $gl_{-R}(2p, \mathbb{R})$; see (22).

This leaves us with the remaining Fourier coefficients, i.e., with the solution of $\bar{\Psi} \in \mathscr{L}_d \subset \mathscr{L}$ from Eq. (28), which now involves only a straightforward computation; compare (23) (e.g., see Broer *et al.*, 1990; Huitema, 1988, Section 8b). The following lemma concludes these considerations, establishing the claim, as stated in Section 3.1, that the proof of Broer *et al.*, 1990; Huitema, 1988, Theorem 8.1) also applies here.

Lemma 6. Under the conditions of the Main Theorem, suppose that the diophantine conditions (11) are fulfilled for all $|k| \leqslant d$. Then Eq. (28), with N as in (29), has a unique solution $\bar{\Psi} \in \operatorname{im} \operatorname{ad} X \cap \mathscr{L}_{G,d}$.

Proof. By Broer *et al.* (1990) and Huitema (1988, Section 8b) it follows that (28) has a unique solution $\bar{\Psi} \in \operatorname{im} \operatorname{ad} X \cap \mathscr{L}_d$. Since the (co-)restriction of $\operatorname{ad} X$ to its image $\operatorname{im} \operatorname{ad} X \cap \mathscr{L}_d$ is invertible, the reversibility of the right-hand side, $N + L$, by Lemma 5.3 implies that $\bar{\Psi}$ is G-equivariant. □

3.4. The Maximal Case

In the maximal case, i.e., where $p = 0$, now almost no work is needed. See Huitema (1988, Section 9b).

From remark (ii), following the corollary (cf. Section 2.5), we recall that here $G(x, y) = (-x, y)$. The present analogue of the Floquet form (21) is the n-parameter family

$$X^{\omega}(x, y) = \omega\partial/\partial x$$

As said earlier, the proof of Broer *et al.* (1990) and Huitema (1988, Theorem 8.1) also applies here. Equation (28), in coordinates, gets even a simpler form, as does the direct sum splitting of $\mathscr{L}_{(d)}$. This implies that the typical form of a reversible element $N \in \ker \operatorname{ad} X$ is $N^\sigma(\xi, \eta) = \Lambda_1(\sigma, \nu)\, \partial/\partial\xi$; compare (27). Also, we give the present analogue of (29), which reads

$$N^\sigma(\xi, \eta) = -[f(\cdot, 0, \sigma)]\, \partial/\partial\xi$$

Another thing is that, instead of (11), we here need only the simpler diophantine conditions (4). The heart of the argument, however, remains the same, namely, that, iterating in the space $\mathscr{L}_d$, for $N + L$ reversible, the solution $\bar{\Psi}$ is automatically equivariant.

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