Unilaterally constrained dynamical systems

Dam, Albert Anton ten

Appendix A

Descriptions of the contact and release sets

In this appendix first some relations are presented that exist between subsets of the boundary set in case of unilaterally constrained dynamical systems. Next, algorithms are derived that will, in principle, produce all the contact and release sets discussed in chapters 6 and 7 in a finite number of steps.

A.1 Linear systems

Lemma A.1.1 Let $\mathbb{X}_n(A, B, C)$ satisfy the assumptions. Then:
(i) $\mathcal{X}_{con} \cap \mathcal{X}_{rel} = \mathcal{V}^* \cup \mathcal{V}_g$.
(ii) $\mathcal{X}_{con,v} \cap \mathcal{X}_{rel} = \emptyset$, and $\mathcal{X}_{rel,v} \cap \mathcal{X}_{con} = \emptyset$.
(iii) $\mathcal{X}_{con} \cup \mathcal{X}_{rel} = \ker(C) \backslash \mathcal{V}_f$.
(iv) $\mathcal{V}_c \cap \mathcal{V}_g = \emptyset$.

Proof: (i, $\subseteq$): Suppose that $x \in \mathcal{X}_{con} \cap \mathcal{X}_{rel}$. If $r_C(x) = r_0$ then $x \in \mathcal{V}_c$. From lemma 6.4.6 it follows that $x \in \mathcal{V}^*$. If on the other hand, $r_C(x) \neq r_0$ then $r_C(x) < r_0$ c.f. corollary 6.4.2 (i). From $x \in \mathcal{X}_{rel}$, lemma 6.4.4 (ii), and corollary 6.4.2 (iii) it follows that $h_{r_C(x)}(x, \underline{u}) > 0$, $\forall \underline{u} \in \mathbb{U}^{\mathbb{N}}$. From $x \in \mathcal{X}_{con}$ it follows from lemma 6.4.4 (i) that $h_{r_C(x)}(x, \underline{u})$ is even. From definition 6.4.5 now follows that $x \in \mathcal{V}_g$. (i, $\supseteq$): From definition 6.4.5 and lemma 6.4.4 it follows that $\mathcal{V}_g \subseteq \mathcal{X}_{con} \cap \mathcal{X}_{rel}$. Suppose that $x \in \mathcal{V}_c$. Then $r_C(x) = r_0$ c.f. definition 6.4.5. From corollary 6.4.2 (ii) and definition 6.4.1 it follows that $h_i(x, \underline{u}) = 0$ $\forall i < r_0$, and $h_{r_0}(x, \underline{u}) = CA^{r_0}x + CA^{r_0-1}B\underline{u}_0$, with $CA^{r_0-1}B \neq 0$. By appropriate choice of $\underline{u}_0$ one can make $h_{r_0}(x, \underline{u})$ positive or negative, taking into account the parity of r_0. From lemma 6.4.4 now follows that $x \in \mathcal{X}_{con} \cap \mathcal{X}_{rel}$.
(ii): We only prove the first equality. The second equality can be proven analogously. Suppose $x \in \mathcal{X}_{con,v}$. Then $r_C(x) = 1$ c.f. definition 6.4.7. It follows from lemma 6.4.4 and

$CB = 0$ that $h_1(x, \underline{u}) < 0$, $\forall \underline{u} \in \mathbb{U}^{\mathbb{N}}$. Lemma 6.4.4 (ii) now gives that $x \notin \mathcal{X}_{rel}$.
(iii): Let $x \in \ker(C)$. First assume that $x \in \mathcal{V}_f$. Definition 6.4.5 (ii) gives that $\forall \underline{u} \in \mathbb{U}^{\mathbb{N}}$, $r_C(x)$ is even and $h_{r_C(x)}(x, \underline{u}) < 0$. From lemma 6.4.4 now follows that $x \notin \mathcal{X}_{con} \cup \mathcal{X}_{rel}$. Next assume that $x \in \ker(C) \backslash \mathcal{V}_f$. If $r_C(x)$ is odd and $h_{r_C(x)}(x, \underline{u}) < 0$ then it follows from lemma 6.4.4 that $x \in \mathcal{X}_{con}$. If $h_{r_C(x)}(x, \underline{u}) > 0$ then it follows from lemma 6.4.4 that $x \in \mathcal{X}_{rel}$. This leads to $x \in \mathcal{X}_{con} \cup \mathcal{X}_{rel}$.
(iv): Follows from the definitions. ◁

Note that the result in lemma A.1.1 (i) shows that example 6.3.2 is indeed non general. That example however does show that some of the subsets can be empty. For the sets $\mathcal{X}_g$, $\mathcal{X}_f$, $\ker(C)$ expressions in terms of constraint matrix C are given by (6.9), (6.10) and (6.11) respectively. Moreover, from $\mathcal{V}_c = \mathcal{V}^*$ it follows that ISA (6.7) yields the set $\mathcal{V}_c$ in a finite number of steps. Therefore, in the remainder we will concentrate on the subsets $\mathcal{V}_g$, $\mathcal{V}_f$, $\mathcal{X}_{con,v}$, $\mathcal{X}_{con,h}$, $\mathcal{X}_{rel,v}$ and $\mathcal{X}_{rel,h}$. Recall from theorem 6.4.9 that these sets are two by two disjunct. For the sets $\mathcal{X}_{con,v}$ and $\mathcal{X}_{rel,v}$ there holds $r_C(x) = 1$, by definition. From definition 6.4.1 now follows that $h_1(x, \underline{u}) = CAx$. This yields by lemma 6.4.4:

Proposition A.1.2 Let $\mathbb{X}_n(A, B, C)$ satisfy the assumptions. Then: $\mathcal{X}_{con,v} = \{x \in \mathcal{X} \mid Cx = 0 \wedge CAx < 0\}$, and $\mathcal{X}_{rel,v} = \{x \in \mathcal{X} \mid Cx = 0 \wedge CAx > 0\}$.

Proof: Omitted. ◁

For the subsets $\mathcal{V}_g$, $\mathcal{V}_f$, $\mathcal{X}_{con,h}$ and $\mathcal{X}_{rel,h}$ the derivation of alternative representations is based on the observation that either $r_C(x)$ must be odd or $r_C(x)$ must be even.

Proposition A.1.3 Let $\mathbb{X}_n(A, B, C)$ satisfy the assumptions. Then:
(i) $\mathcal{V}_g = \cup_{1 \le i < \frac{1}{2} r_0} \mathcal{V}_g^i$;
$\mathcal{V}_g^i = \{x \in \ker(C) \mid CA^{2i} x > 0\} \cap \{\cap_{0 \le j < 2i} A^{-j} \ker(C)\}$, $1 \le i < \frac{1}{2} r_0$.
(ii) $\mathcal{V}_f = \cup_{1 \le i < \frac{1}{2} r_0} \mathcal{V}_f^i$;
$V_f^i = \{x \in \ker(C) \mid CA^{2i} x < 0\} \cap \{\cap_{0 \le j < 2i} A^{-j} \ker(C)\}$, $1 \le i < \frac{1}{2} r_0$.
(iii) $\mathcal{X}_{con,h} = \cup_{1 \le i < \frac{1}{2}(r_0 - 1)} \mathcal{X}_{con,h}^i$;
$\mathcal{X}_{con,h}^i = \{x \in \ker(C) \mid CA^{2i+1} x < 0\} \cap \{\cap_{0 \le j \le 2i} A^{-j} \ker(C)\}$, $1 \le i < \frac{1}{2}(r_0 - 1)$.
(iv) $\mathcal{X}_{rel,h} = \cup_{1 \le i < \frac{1}{2}(r_0 - 1)} \mathcal{X}_{rel,h}^i$;
$\mathcal{X}_{rel,h}^i = \{x \in \ker(C) \mid CA^{2i+1} x > 0\} \cap \{\cap_{0 \le j \le 2i} A^{-j} \ker(C)\}$, $1 \le i < \frac{1}{2}(r_0 - 1)$.

Proof: We will prove only (i). The other statements are proven analogously. From definition 6.4.5 it follows that: $\{x \in \mathcal{V}_g\} \Leftrightarrow \{r_C(x) = 2i \text{ and } h_{r_C(x)}(x, \underline{u}) > 0, 1 \le i < \frac{1}{2} r_0\}$. The result now follows from corollary (6.4.2) (ii) and (iii), and the definition of the map $h_i(x, \underline{u})$. ◁

In the remainder of this section we show that the calculation of the subspaces can be based on the Invariant Subspace Algorithm (ISA) (6.7). The integer r_0 is well defined as a consequence of the controllability of the pair (A, B). Recall from definition 6.7.2 that $r_1 := \min\{i \in \mathbb{N} \mid \cap_{0 \le j < i} A^{-j} \ker(C) \subseteq A^{-i} \ker(C)\}$, and $r_{min} := \min(r_0, r_1)$.

Lemma A.1.4 Let $\mathbb{X}_n(A, B, C)$ satisfy the assumptions. Let $\mathcal{V}^i$ denote the subspaces obtained in step i of ISA (6.7). Then:
(i) $\text{im}(B) \not\subseteq A^{-(r_0-1)} \ker(C)$;
(ii) $\text{im}(B) \subseteq A^{-(i-1)} \ker(C)$, $\forall i : 1 \leq i < r_0$;
(iii) $\mathcal{V}^i = \cap_{0 \leq j < i} A^{-j} \ker(C)$, $\forall i : 1 \leq i \leq r_0$;
(iv) $r_{min} = r_0$, and $\mathcal{V}^{r_0} = \mathcal{V}^*$.

Proof: (i) and (ii) follow immediately from the assumptions and the definitions. (iii): From ISA (6.7) it follows that for $i = 1$ the equality (iii) holds since $\mathcal{V}^1 = \ker(C)$. Now assume that $i = n < r_0$. Then, from (6.7): $\mathcal{V}^{n+1} = \ker(C) \cap (A^{-1}(\cap_{0 \leq j < n} A^{-j} \ker(C)) + \text{im}(B))$. Using (ii) now gives $\mathcal{V}^{n+1} = \ker(C) \cap (A^{-1} \cap_{0 \leq j < n} A^{-j} \ker(C)) = \cap_{0 \leq j < n+1} A^{-j} \ker(C)$. (iv): From (i), (ii), and definition 6.7.2 follows that $r_0 \leq r_1$. The second statement now follows from (i), (ii) and ISA. $\lhd$

Combining ISA (6.7) and lemma A.1.4 now gives:

Corollary A.1.5 Let $\mathbb{X}_n(A, B, C)$ satisfy the assumptions. Then the subspaces $\mathcal{V}^i$, calculated with ISA, have the following characteristics: $\mathcal{V}^0 = \mathcal{X}$; $\mathcal{V}^1 = \ker(C) = \mathcal{V}^0 \backslash (\mathcal{X}_g \cup \mathcal{X}_f)$; $\mathcal{V}^2 = \mathcal{V}^1 \backslash (\mathcal{X}_{con,v} \cup \mathcal{X}_{rel,v})$; $\mathcal{V}^3 = \mathcal{V}^2 \backslash (\mathcal{V}_g^1 \cup \mathcal{V}_f^1)$; $\mathcal{V}^4 = \mathcal{V}^3 \backslash (\mathcal{X}_{con,h}^1 \cup \mathcal{X}_{rel,h}^1)$; ... ; $\mathcal{V}^{r_0} = \mathcal{V}_c$.

Finally, combining corollary A.1.5 with propositions A.1.2 and A.1.3 and lemma A.1.4 yields the following algorithm, which produces all the required subsets in a finite number of steps. Since $\mathcal{V}^* = \mathcal{V}^{r_0}$ the algorithm can be terminated in at most r_0 steps.

Algorithm A.1.6 *Computation of the subsets of the state-space.* Let $\mathbb{X}_n(A, B, C)$ satisfy the assumptions. Let $\mathcal{V}^i$ denote the subspaces obtained in step i of ISA (6.7). Let $\mathcal{V}^0 = \mathcal{X}$. Then:
(i) $\mathcal{V}^i = \{x \in \mathcal{V}^{i-1} \mid CA^{i-1}x = 0\}$, $1 \leq i \leq r_0$;
(ii) $\mathcal{X}_g = \{x \in \mathcal{V}^0 \mid Cx > 0\}$;
(iii) $\mathcal{X}_f = \{x \in \mathcal{V}^0 \mid Cx < 0\}$;
(iv) $\mathcal{X}_{con,v} = \{x \in \mathcal{V}^1 \mid CAx < 0\}$;
(v) $\mathcal{X}_{rel,v} = \{x \in \mathcal{V}^1 \mid CAx > 0\}$;
(vi) $\mathcal{V}_g = \cup_{1 \leq i < \frac{1}{2} r_0} \mathcal{V}_g^i$ with $\mathcal{V}_g^i = \{x \in \mathcal{V}^{2i} \mid CA^{2i}x > 0\}$, $1 \leq i < \frac{1}{2} r_0$;
(vii) $\mathcal{V}_f = \cup_{1 \leq i < \frac{1}{2} r_0} \mathcal{V}_f^i$ with $\mathcal{V}_f^i = \{x \in \mathcal{V}^{2i} \mid CA^{2i}x < 0\}$, $1 \leq i < \frac{1}{2} r_0$;
$(viii)$ $\mathcal{X}_{con,h} = \cup_{1 \leq i < \frac{1}{2}(r_0-1)} \mathcal{X}_{con,h}^i$ with
$\mathcal{X}_{con,h}^i = \{x \in \mathcal{V}^{2i+1} \mid CA^{2i+1}x < 0\}$, $1 \leq i < \frac{1}{2}(r_0 - 1)$;
(ix) $\mathcal{X}_{rel,h} = \cup_{1 \leq i < \frac{1}{2}(r_0-1)} \mathcal{X}_{rel,h}^i$ with
$\mathcal{X}_{rel,h}^i = \{x \in \mathcal{V}^{2i+1} \mid CA^{2i+1}x > 0\}$, $1 \leq i < \frac{1}{2}(r_0 - 1)$;
(x) $\mathcal{V}_c = \mathcal{V}^{r_0} (= \mathcal{V}^*)$.

Proof: Omitted. $\lhd$

A.2 Nonlinear systems

In this section we give some results for the nonlinear case. The proofs mimic the proofs given in the previous section for the linear case. We assume throughout that there is a single inequality constraint, i.e. $h : \mathcal{M} \mapsto \mathbb{R}$.

Corollary A.2.1 Let $\mathcal{M}_n(f, g, h)$ satisfy the assumptions. Then $\mathcal{N}_c = \mathcal{N}^*$. ◁

Lemma A.2.2 Let $\mathcal{M}_n(f, g, h)$ satisfy the assumptions. Then:
(*i*) $\mathcal{M}_{con} \cap \mathcal{M}_{rel} = \mathcal{N}^* \cup \mathcal{N}_g$.
(*ii*) $\mathcal{M}_{con,v} \cap \mathcal{M}_{rel} = \emptyset$, and $\mathcal{M}_{rel,v} \cap \mathcal{M}_{con} = \emptyset$.
(*iii*) $\mathcal{M}_{con} \cup \mathcal{M}_{rel} = \mathcal{M}_b \backslash \mathcal{N}_f$.
(*iv*) $\mathcal{N}_c \cap \mathcal{N}_g = \emptyset$.

Proof: The proof is similar to the proof of A.1.1. ◁

Proposition A.2.3 Let $\mathcal{M}_n(f, g, h)$ satisfy the assumptions. Then:
(*i*) $\mathcal{M}_{con,v} = \{x \in \mathcal{M} \mid h(x) = 0 \wedge L_f h(x) < 0\}$,
(*ii*) $\mathcal{M}_{rel,v} = \{x \in \mathcal{M} \mid h(x) = 0 \wedge L_f h(x) > 0\}$,
(*iii*) $\mathcal{N}_g = \cup_{1 \le i < \frac{1}{2}r_0} \mathcal{N}_g^i$;
$\mathcal{N}_g^i = \{x \in \mathcal{M}_b \mid L_f^{2i} h(x) > 0\} \cap \{\cap_{0 \le j < 2i} L_f^j h(x) = 0\}$, $1 \le i < \frac{1}{2}r_0$.
(*iv*) $\mathcal{N}_f = \cup_{1 \le i < \frac{1}{2}r_0} \mathcal{N}_f^i$;
$\mathcal{N}_f^i = \{x \in \mathcal{M}_b \mid L_f^{2i} h(x) < 0\} \cap \{\cap_{0 \le j < 2i} L_f^j h(x) = 0\}$, $1 \le i < \frac{1}{2}r_0$.
(*v*) $\mathcal{M}_{con,h} = \cup_{1 \le i < \frac{1}{2}(r_0 - 1)} \mathcal{M}_{con,h}^i$;
$\mathcal{M}_{con,h}^i = \{x \in \mathcal{M}_b \mid L_f^{2i+1} h(x) < 0\} \cap \{\cap_{0 \le j \le 2i} L_f^j h(x) = 0\}$, $1 \le i < \frac{1}{2}(r_0 - 1)$.
(*vi*) $\mathcal{M}_{rel,h} = \cup_{1 \le i < \frac{1}{2}(r_0 - 1)} \mathcal{M}_{rel,h}^i$;
$\mathcal{M}_{rel,h}^i = \{x \in \mathcal{M}_b \mid L_f^{2i+1} h(x) > 0\} \cap \{\cap_{0 \le j \le 2i} L_f^j h(x) = 0\}$, $1 \le i < \frac{1}{2}(r_0 - 1)$.

Proof: The proof runs analogously to the proofs of propositions A.1.2 and A.1.3. ◁

The above result can be rewritten to obtain a result analoguus to algorithm A.1.6, but we omit the details.

Notation

Notation

Below the notation which is used in this thesis is given. A short explanation is included.

Equation refers to equalities and inequalities alike. All inequalities are componentwise, with the notable exception of the notation $\overset{e}{\geq}$ (see below).

General	
$\underline{n}$	$\{1, 2, \ldots, n\}$.
$\mathbb{N}$	The natural numbers $\{1, 2, 3, \ldots\}$.
$\mathbb{R}$	The reals.
$\mathbb{Z}$	The integers.
$\mathbb{C}$	The complex numbers.
$\mathbb{Z}_+$	The nonnegative integers $\{0, 1, 2, 3, \ldots\}$.
$\mathbb{R}_+$	The nonnegative reals $[0, \infty)$.
$\mathbb{R}^q$	The q-dimensional real vectors.
$\mathbb{R}^q_+$	The q-dimensional real vectors with nonnegative real coefficients.
$\mathbb{R}^{n \times m}$	The $(n \times m)$ matrices with real coefficients.
$\mathbb{R}^{n \times m}_+$	The $(n \times m)$ matrices with nonnegative real coefficients.
$\mathbb{T}$	The time set.
$\mathbb{W}^{\mathbb{T}}$	The set of all maps from $\mathbb{T}$ to $\mathbb{W}$.
$\mathbb{R}[s]$	The polynomials with real coefficients and positive powers in the indeterminate s.
$\mathbb{R}^{n \times m}[s]$	The $(n \times m)$ polynomial matrices in the indeterminate s.
$\mathbb{R}[s, s^{-1}]$	The polynomials with real coefficients and positive and negative powers in the indeterminate s.
$\mathbb{R}^{n \times m}[s, s^{-1}]$	The $(n \times m)$ polynomial matrices with real coefficients and positive and negative powers in the indeterminate s.
$\mathbb{R}^{n \times m}_+[s, s^{-1}]$	Idem as $\mathbb{R}^{n \times m}[s, s^{-1}]$ but with nonnegative coefficients only.
$\mathbb{R}(s)$	The rational functions in the indeterminate s.

Matrices

I	The identity matrix.
A^T	The transpose of matrix A.
A_i	The ith row of matrix A.
${}_jA$	The jth column of matrix A.
${}^jA$	$row({}_1A, \dots, {}_{j-1}A, {}_{j+1}A, \dots, {}_sA)$.
(A_{ij})	The ijth element of matrix A.
$row(A, B)$	$[A : B]$.
$col(A, B)$	$[A^T : B^T]^T$.
$\ker(A)$	Kernel: $\{x \mid Ax = 0\}$.
$\mathrm{im}(A)$	Image: $\{y \mid \exists x \text{ such that } y = Ax\}$.
$A \geq 0$	$(A_{ij}) \geq 0$ for all i, j.
$A \overset{e}{\geq} 0$	$(A_{ij}) \geq 0$ for all $i \neq j$.

Other notation

σ^t	The (backward) t-shift.
$R(\sigma, \sigma^{-1})$	A polynomial operator in the shift.
$\mathfrak{P}$	A convex polyhedral set, $\mathfrak{P} \subseteq \mathbb{R}^n$.
$\mathfrak{P}^{\#}$	Polar set.
$\mathfrak{B}$	A behaviour, $\mathfrak{B} \subseteq \mathbb{W}^{\mathbb{T}}$.
$\mathbb{L}^q$	$(\mathbb{R}^q)^{\mathbb{T}}$ equipped with the topology of pointwise convergence.
L^q	The collection of all linear closed shift-invariant subspaces of $\mathbb{L}^q$.
$\mathfrak{L}_1^{loc}(\mathbb{R}, \mathbb{R}^q)$,$\mathfrak{L}_1^{loc}$	The space of locally integrable functions.
$\mathfrak{L}_{1,+}^{loc}$	$\mathfrak{L}_1^{loc}(\mathbb{R}_+, \mathbb{R}^q)$.
$\mathbb{U}(\mathbb{R}_+, \mathbb{R}^m)$	The collection of piecewise C^∞ signals.
$\Sigma_1 \wedge \Sigma_2$	The interconnection of systems Σ_1 and Σ_2.

Polyhedral sets

$cone\{v_1, \dots, v_m\}$	$\{\sum_i \lambda_i v_i \mid \lambda_i \geq 0, i \in \underline{m}\}$.
$\mathfrak{P}_E(A)$	$\{x \in \mathbb{R}^n \mid Ax = 0\}$.
$\mathfrak{P}_I(B, b)$	$\{x \in \mathbb{R}^n \mid Bx \geq b\}$.
$\mathfrak{P}_I(B)$	$\{x \in \mathbb{R}^n \mid Bx \geq 0\}$.
$\mathfrak{P}_{EI}(A; B, b)$	$\{x \in \mathbb{R}^n \mid Ax = 0 \wedge Bx \geq b\}$.
$\mathfrak{P}_{EI}(A^k; B, b)$	$\{x \in \mathbb{R}^n \mid A_i x = 0, Bx \geq b, i \neq k\}$.
$\mathfrak{P}_{EI}(A; B^l, b^l)$	$\{x \in \mathbb{R}^n \mid Ax = 0, B_j x \geq b_j, j \neq l\}$.
$\mathfrak{N}(M, N, L)$	$\{x \in \mathbb{R}^n \vert x = M\ell_1 + N\ell_2 + L\ell_3, \ell_2 \geq 0, \sum_i \ell_{3,i} = 1\}$.
$\mathfrak{N}_{EI}(M; N)$	$\{x \in \mathbb{R}^n \mid x = Ml_1 + Nl_2, l_2 \geq 0\}$.
$\mathfrak{N}_E(M)$	$\{x \in \mathbb{R}^n \mid x = Ml_1\}$.
$\mathfrak{N}_I(N)$	$\{x \in \mathbb{R}^n \mid x = Nl_2, l_2 \geq 0\}$.

Abbreviations

BFD	Backward Difference Formula
DAE	Differential Algebraic Equation
DASSL	Differential/Algebraic System Solver
FE	Forward-Euler
ISA	Invariant Subspace Algorithm
MPUM	Most Powerful Unfalsified Model
NLR	National Aerospace Laboratory NLR
ODE	Ordinary Differential Equation
PDE	Partial Differential Equation
SES	Static Equality System
SEIS	Static Equality/Inequality System
SIS	Static Inequality System
SISO	Single Input Single Output

Index