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Cycles, L-functions and triple products of elliptic curves

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Abstract. A variant of a conjecture of Beilinson and Bloch relates the rank of the Griffiths group of a smooth projective variety over a number field to the order of vanishing of an L-function at the center of the critical strip. Presently, there is little evidence to support the conjecture, especially when the L-function vanishes to order greater than 1. We study 1-cycles on E^3 for various elliptic curves $E/\mathbb{Q}$. In each of the 76 cases considered we find that the empirical order of vanishing of the L-function is at least as large as our best lower bound on the rank of the Griffiths group. In 11 cases this lower bound is two.

0. Introduction

Let K be a number field and let W (or W_K if we wish to emphasize the base field) be a smooth, projective, geometrically irreducible K-variety. The free abelian group generated by scheme theoretic points of codimension r is denoted $Z^r(W_K)$ and called the group of codimension r cycles on W_K. We write $Z^r(W_K)_{\mathrm{rat}}$ (respectively $Z^r(W_K)_{\mathrm{alg}}$, $Z^r(W_K)_{\mathrm{hom}}$) for the subgroups of cycles which are rationally (respectively algebraically, homologically) equivalent to 0, so that

$$Z^r(W_K)_{\mathrm{rat}} \subset Z^r(W_K)_{\mathrm{alg}} \subset Z^r(W_K)_{\mathrm{hom}} \subset Z^r(W_K).$$

The purpose of this article is to test numerically some highly speculative conjectures (called, perhaps more aptly, "recurring fantasies" by Bloch [Bl]) concerning the following groups of cycle classes:

$$CH^r(W_K)_{\mathrm{hom}} := Z^r(W_K)_{\mathrm{hom}}/Z^r(W_K)_{\mathrm{rat}} \quad \text{and} \quad \mathrm{Griff}^r(W_K) := Z^r(W_K)_{\mathrm{hom}}/Z^r(W_K)_{\mathrm{alg}}.$$

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We begin with

Recurring fantasy 0.1 (Beilinson [Be], Bloch [Bl]).

$$\operatorname{rank} CH^r(W_K)_{\text{hom}} = \operatorname{ord}_{s=r} L_K(H^{2r-1}(W), s).$$

An attractive feature of 0.1 is that it raises the distant but tempting possibility that there is a relatively simple and occasionally computable formula for the left hand side. The L-function on the right is defined by an Euler product that converges to a holomorphic function for $\operatorname{Re}(s) > r + 1/2$. However it is conjectured to have an analytic continuation to an entire function. In some instances this conjecture has been verified and the right hand side of 0.1 is computable.

When $r = 1$, $CH^r(W_K)_{\text{hom}} \simeq \operatorname{Pic}^0(W_K)$, and $\operatorname{Griff}^r(W_K) = 0$; in this case 0.1 has received much attention. The case that $K = \mathbb{Q}$ and W is an elliptic curve was considered 30 years ago by Birch and Swinnerton-Dyer [B-Sw]. If W is a modular elliptic curve and the right hand side of 0.1 is at most 1, then 0.1 has recently been shown to hold [Gr]. Even when the right hand side is > 1 there is numerical evidence in favor of the conjecture [BuGrZa], [Br-Mc].

When $r > 1$, there is much less evidence for 0.1. We will investigate the case when $r = 2$ and W is a threefold defined over $\mathbb{Q}$. In order to eliminate, as much as is possible, phenomena which are explainable in terms of cycles of codimension one, we replace W by a motivic factor, M, which is defined using correspondences which annihilate all level one Hodge substructures of $H^3(W(\mathbb{C}), \mathbb{Q})$. In this situation we have (cf. [Bl])

Recurring fantasy 0.2.

$$\operatorname{rank} \operatorname{Griff}^2(M) = \operatorname{ord}_{s=2} L(H^3(M), s).$$

As a test case for 0.2 we consider the following situation: $E/\mathbb{Q}$ is an elliptic curve, $W = E^3$, $P \in Z^3(W \times_{\mathbb{Q}} W)$ is a correspondence satisfying $P_* H^3(W_{\bar{\mathbb{Q}}}) \simeq \operatorname{Sym}^3 H^1(E_{\bar{\mathbb{Q}}})$ and $P^2 = 3P$. Then $M = (W_{\mathbb{Q}}, \frac{1}{3}P)$ is a Chow motive in the sense of [Mu]. The cycle class groups and cohomology groups of M are defined by applying the projector to the corresponding groups of W. For example,

$$\operatorname{Griff}^2(M) \otimes \mathbb{Q} := P_*(\operatorname{Griff}^2(W) \otimes \mathbb{Q}) \quad \text{and} \quad H^\cdot(M, \mathbb{Q}_l) := P_* H^\cdot(W, \mathbb{Q}_l).$$

Associated to the third cohomology of M is a conductor, N. Standard conjectures imply that

$$\Lambda(H^3(M), s) := N^{s/2} (2\pi)^{-2s} \Gamma(s) \Gamma(s-1) L(H^3(M), s)$$

is an entire function which satisfies a functional equation

$$\Lambda(H^3(M), s) = w \Lambda(H^3(M), 4 - s), \quad w \in \{\pm 1\}. \tag{0.3}$$

This conjectured analytic continuation and functional equation have been proved by Gross and Kudla [Gr-Ku] when the level of the elliptic curve is square-free (so that by Wiles-

Taylor, it is modular). We shall not, however, restrict ourselves to this situation. Instead, for each test curve E we give numerical evidence, although no proof, that (0.3) holds. Then we ask the computer to evaluate the derivatives $L^{(j)}(H^3(M), 2)$ for $j \geqq 0$ until it finds one which appears to be non-zero. The numerical evidence is very convincing and gives us the probable or, as we will sometimes say, the "empirical" order of vanishing of $L(H^3(M), s)$ at $s = 2$.

In choosing test curves, we have concentrated on plane cubics of the form

$$E_a: \quad y^2 z = (a^2 - 4)x^3 + (2a^2 - 4a)x^2 z + (a^2 - 4)xz^2,$$

with $a \in \mathbb{Q}$ and $a \notin \{-1, \pm 2, 0, -3, -3/2\}$. Such curves are modular (Proposition 7.4). Furthermore we have a systematic method for constructing one, and in some cases two, classes in the Griffiths group of the motive M. Let **Rnk** denote the rank of the subgroup of $\mathrm{Griff}^2(M)$ which these classes generate, and let **Ord** denote the empirical order of vanishing of $L(H^3(M), s)$ at $s = 2$. A difficulty in computing **Ord** which we encountered frequently was that for many choices of the parameter $a \in \mathbb{Q}$ the conductor N was so large ($N > 10^{12}$) that the L-series calculations could not be done to the desired precision (10^{-6}) in the time available. No similar difficulties were encountered in the computation of **Rnk**.

The following table is a rough summary of our calculations. It combines results for 76 curves of the form E_a, and 16 curves of low conductor (chosen to test our programs, and to test hypotheses on the conductor N) for which we had no information on the Griffiths group. In the following table the entry in row **Rnk** and column **Ord** is the number of curves with lower bound **Rnk** on the rank of $\mathrm{Griff}^2(M)$ and empirical order of vanishing **Ord**.

Table 0.4. Summary of results.

Rnk\Ord	0	1	2	3	4
0	9	4	3	0	0
1	0	33	27	5	0
2	0	0	6	3	2

The conjecture 0.2 predicts that the entries below the main diagonal should be 0, which is certainly consistent with our calculations. If the two sides of 0.2 had nothing to do with each other, this would be a surprising result.

We now give a brief outline of the contents of the individual sections. The first six are devoted to computing the lower bound on the rank of $\mathrm{Griff}^2(M)$.

Section 1 recalls the definition and basic properties of the cycle class map for a smooth variety W over a field k of characteristic $\neq l$:

$$\text{(0.5)} \qquad \mathrm{cl}_0 : CH^r(W_k)_{\mathrm{hom}} \to H^1(G_k, H^{2r-1}(W_{\bar{k}}, \mathbb{Z}_l(r))).$$

This map is our main tool for showing that a given element of $CH^r(W_k)_{\mathrm{hom}}$ is non-zero.

The Griffiths group, $\mathrm{Griff}^r(W_k)$ is discussed in § 2. In particular it is shown how to use (0.5) and correspondences to detect non-trivial elements in the Griffiths group.

Although we are primarily interested in varieties defined over $\mathbb{Q}$, the actual computations with (0.5) will involve reducing modulo a prime of good reduction. When $H^{2r-1}(W_{\bar{\mathbb{F}}_p}, \mathbb{Z}_l(r))$ is torsion free there is a diagram,

$$(0.6)\quad \begin{array}{ccccc} CH^r(W_{\mathbb{Q}})_{\mathrm{hom}} & \to & CH^r(W_{\mathbb{Q}_p})_{\mathrm{hom}} & \xrightarrow{\mathrm{sp}} & CH^r(W_{\mathbb{F}_p})_{\mathrm{hom}} \\ \mathrm{cl}_{0\mathbb{Q}}\downarrow & & \mathrm{cl}_{0\mathbb{Q}_p}\downarrow & & \mathrm{cl}_{0\mathbb{F}_p}\downarrow \\ H^1(G_{\mathbb{Q}}, H^{2r-1}(W_{\bar{\mathbb{Q}}}, \mathbb{Z}_l(r))) & \to & H^1(G_{\mathbb{Q}_p}, H^{2r-1}(W_{\bar{\mathbb{Q}}_p}, \mathbb{Z}_l(r))) & \xrightarrow{\vartheta^{-1}} & H^1(G_{\mathbb{F}_p}, H^{2r-1}(W_{\bar{\mathbb{F}}_p}, \mathbb{Z}_l(r))) \,, \end{array}$$

in which sp is the specialization map and ϑ is the restriction map which results from identifying $H^{2r-1}(W_{\bar{\mathbb{F}}_p}, \mathbb{Z}_l(r))$ with $H^{2r-1}(W_{\bar{\mathbb{Q}}_p}, \mathbb{Z}_l(r))$ via base change isomorphisms [Mi], VI. 4.2. Section 3 is devoted to proving that the second square commutes. The first square commutes by 1.9 (5).

After these generalities the varieties, E_a and $W_a = E_a^3$, and the motive $M_a = (W_a, \frac{1}{3}P)$ are introduced in § 4. There is a genus 3 curve C_a and a map $\varrho : C_a \to E_a^3$. We define $\Xi_a = \varrho(C_a) - [-1]_* \varrho(C_a)$ and study the cycle class $P_* \Xi_a \in CH^2(M_a)_{\mathrm{hom}}$ by means of (0.5). Now $\mathrm{cl}_0(P_* \Xi_a)$ is not easy to compute directly, essentially because $P_* H^3(W_{a\bar{\mathbb{Q}}}, \mathbb{Q}_l(2))$ is an irreducible Galois representation with the property that for any curve $T/\bar{\mathbb{Q}}$ and any correspondence $\Gamma \in Z^2(W_a \times_{\mathbb{Q}} T)$ we have $\Gamma_* P_* H^3(W_{a\bar{\mathbb{Q}}}, \mathbb{Q}_l(2)) = 0$. Upon reduction mod p, E_a acquires complex multiplication and the Galois representation $P_* H^3(W_{a\bar{\mathbb{F}}_p}, \mathbb{Q}_l(2))$ is reducible. In fact there is a correspondence $\Gamma \in Z^2(W_{a\mathbb{F}_p} \times E_{a\mathbb{F}_p})$ and a commutative diagram,

$$(0.7)\quad \begin{array}{ccc} P_* CH^2(W_{a\mathbb{F}_p})_{\mathrm{hom}} & \xrightarrow{\Gamma_*} & CH^1(E_{a\mathbb{F}_p})_{\mathrm{hom}} \otimes \mathbb{Z}_l \\ \mathrm{cl}_0\downarrow & & \mathrm{cl}_0 \otimes 1 \downarrow \simeq \\ H^1(G_{\mathbb{F}_p}, P_* H^3(W_{\bar{\mathbb{F}}_p}, \mathbb{Z}_l(r))) & \xrightarrow{\Gamma_*} & H^1(G_{\mathbb{F}_p}, H^1(E_{a\bar{\mathbb{F}}_p}, \mathbb{Z}_l(1))) \,, \end{array}$$

in which the lower arrow is non-zero for appropriate choice of l and p. Since

$$H^1(G_{\mathbb{Q}}, P_* H^3(W_{\bar{\mathbb{Q}}}, \mathbb{Z}_l(r)))$$

is frequently torsion free (4.8), the problem of showing that $\mathrm{cl}_0(P_* \Xi_a)$ has infinite order has been reduced by (0.6) and (0.7) to the conceptually simpler problem of showing that the element $\Gamma_* P_* \Xi_a \in CH^1(E_{a\mathbb{F}_p})_{\mathrm{hom}} \otimes \mathbb{Z}_l$ is non-zero.

Although the computation of $\Gamma_* P_* \Xi_a$ involves nothing more than points on an elliptic curve over the algebraic closure of a finite field, it is nonetheless computationally non-trivial. The methods used to evaluate $\Gamma_* P_* \Xi_a$ are discussed in § 5.

The sixth section focuses on an isogeny $\phi : E_{-3-a} \to E_a$. It is defined over $\mathbb{Q}$ for infinitely many values $a \in \mathbb{Q}$. Using $\phi^3 : E_{-3-a}^3 \to E_a^3$, we move $P_* \Xi_{-3-a}$ from W_{-3-a} to W_a. With the help of this second cycle we verify that $\mathrm{rank}(\mathrm{Griff}^2(M_{a\mathbb{Q}})) \geqq 2$ for infinitely many $a \in \mathbb{Q}$. This completes the discussion of the Chow groups.

The computations of the leading terms of the L-series at the central point are discussed in § 7. The basic idea is that the functional equation implies that the value (or leading term) of the L-series can be expressed in terms of sums involving special functions related to K-Bessel functions. We discuss the computation of the coefficients of the L-series, and the computation of the values of the special functions.

Section eight is devoted to showing that the conductor of M is well-defined and readily computable. In the case of the curves E_a we use this to verify a remarkably simple relationship between the conductor of M and the conductor of E_a that was first noticed empirically.

Finally, in the last section we give three different tables summarizing our various computational results.

The recurring fantasies 0.1 and 0.2 have also been investigated in the context of complex multiplication cycles on Kuga-Satake varieties (see [Bes], [Br], [Ne 1], [Ne 2], [Sch] and [Zh]). These cycles are defined in modular terms and seem to exhibit behaviour analogous to Heegner cycles on modular elliptic curves. In particular when the L-function vanishes to order greater than one, it seems plausible that averaging Heegner cycles over the Galois group always produces a cycle which is torsion in the Chow group. By contrast the non-modular cycles studied here give elements of infinite order in the Chow group even when the L-function vanishes to high order. In particular they provide the first evidence for 0.2 when the order of vanishing is greater than one.

This paper is a generalization of [Bl] which treats the case of the CM-elliptic curve E_0. Many essential ideas which play a role here appeared for the first time in [Bl]. We have also borrowed from [T] where the remaining cases for which $E_a/\mathbb{Q}$ has CM-type are investigated. The projector Q in § 4 was borrowed from Gross [Gr-Sch]. The specialization technique of [Bl] and (0.6) has a geometric precursor (cf. [Ce]).

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Notations.

$A[l^n] = \mathrm{Ker}: A \xrightarrow{l^n} A$ where A is an Abelian group.

$A[l^\infty] = \bigcup_n A[l^n]$.

$\bar{k}$ = separable closure of the field k.

$G_k = \mathrm{Gal}(\bar{k}/k)$.

1. Preliminaries on the cycle class map

Let W be a smooth variety of dimension d over a field k. For each prime l distinct from the characteristic there is a cycle class map [Bl], §1,

$$\text{(1.1)} \qquad \mathrm{cl}: Z^r(W) \to \varprojlim H^{2r}(W_k, \mathbb{Z}/l^n(r)) =: H^{2r}(W_k, \mathbb{Z}_l(r)),$$

which annihilates $Z^r(W)_{\mathrm{rat}}$. Write $\bar{k}$ for a separable closure of k. Composing (1.1) with the map on cohomology induced by extension of scalars gives rise to the familiar cycle class map [Mi], VI.9,

$$\text{(1.2)} \qquad \mathrm{cl}_{\bar{k}}: Z^r(W) \to H^{2r}(W_{\bar{k}}, \mathbb{Z}_l(r)).$$

Using (1.2) for varying l we may define

$$\text{(1.3)} \qquad Z^r(W)_{\mathrm{hom}} = \mathrm{Ker}[Z^r(W) \to \prod_{l \neq \mathrm{char}(k)} H^{2r}(W_{\bar{k}}, \mathbb{Z}_l(r))].$$

In this paper we shall make use of a cycle class map

$$\text{(1.4)} \qquad \mathrm{cl}_0^n: Z^r(W)_{\mathrm{hom}} \to H^1(G_k, H^{2r-1}(W_{\bar{k}}, \mathbb{Z}/l^n(r))),$$

and the inverse limit

$$\text{(1.5)} \qquad \mathrm{cl}_0: Z^r(W)_{\mathrm{hom}} \to \varprojlim H^1(G_k, H^{2r-1}(W_{\bar{k}}, \mathbb{Z}/l^n(r))) \simeq H^1(G_k, H^{2r-1}(W_{\bar{k}}, \mathbb{Z}_l(r))).$$

The cohomology group on the right is computed with continuous cochains where $H^{2r-1}(W_{\bar{k}}, \mathbb{Z}_l(r))$ has the inverse limit topology [Ta]. This map was considered by Bloch in the paper which forms the inspiration for the present work [Bl]. Bloch constructed (1.4) from (1.1) using the Hochschild-Serre spectral sequence (see also [Ra]). We use an alternative construction for (1.4) by means of extensions [J], §9. To describe this begin with $Z \in Z^r(W)_{\mathrm{hom}}$. Write $|Z|$ for the support of Z and define

$$\text{(1.6)} \quad H^{2r}_{|Z|}(W_{\bar{k}}, \mathbb{Z}/l^n(r))_0 = \mathrm{Ker}[H^{2r}_{|Z|}(W_{\bar{k}}, \mathbb{Z}/l^n(r)) \to H^{2r}(W_{\bar{k}}, \mathbb{Z}/l^n(r))].$$

By purity [Mi], VI.9.1,

$$H^{2r-1}_{|Z|}(W_{\bar{k}}, \mathbb{Z}/l^n(r)) = 0.$$

There results a short exact sequence of G_k-modules,

$$\text{(1.7)} \qquad 0 \to H^{2r-1}(W_{\bar{k}}, \mathbb{Z}/l^n(r)) \to H^{2r-1}((W - |Z|)_{\bar{k}}, \mathbb{Z}/l^n(r)) \to H^{2r}_{|Z|}(W_{\bar{k}}, \mathbb{Z}/l^n(r))_0 \to 0.$$

An element of

$$\mathrm{Ext}^1_{\mathbb{Z}/l^n[G_k]}(\mathbb{Z}/l^n, H^{2r-1}(W_{\bar{k}}, \mathbb{Z}/l^n(r)))$$

is obtained from (1.7) by pull back with respect to the map $\mathbb{Z}/l^n \to H^{2r}_{|Z|}(W_{\bar{k}}, \mathbb{Z}/l^n(r))_0$ which takes 1 to the fundamental class $[Z]$ of Z [Mi], VI. 6. Via the canonical identification,

$$(1.8)\quad \mathrm{Ext}^1_{\mathbb{Z}/l^n[G_k]}(\mathbb{Z}/l^n, H^{2r-1}(W_{\bar{k}}, \mathbb{Z}/l^n(r))) \simeq H^1(G_k, H^{2r-1}(W_{\bar{k}}, \mathbb{Z}/l^n(r))),$$

this element is taken to $\delta([Z])$, where δ is the first coboundary in the long exact G_k-cohomology sequence associated to (1.7). The Hochschild-Serre approach and the extension approach give rise to the same map, cl_0^n, at least up to a factor ± 1, which we shall ignore [J], 9.4.

Proposition 1.9. (1) cl_0^n *is functorial with respect to smooth pullback in the category of smooth k-varieties.*

(2) cl_0^n *is functorial with respect to proper direct image in the same category.*

(3) *Given* $Z' \in Z^s(W)$ *define* $Z^r_{Z'}(W)_{\mathrm{hom}}$ *to be the subgroup of nullhomologous cycles all of whose components meet* Z' *properly. The following diagram commutes*:

$$\begin{array}{ccc} Z^r_{Z'}(W)_{\mathrm{hom}} & \xrightarrow{\;.Z'\;} & Z^{r+s}(W)_{\mathrm{hom}} \\ \mathrm{cl}_0^n \downarrow & & \mathrm{cl}_0^n \downarrow \\ H^1(G_k, H^{2r-1}(W_{\bar{k}}, \mathbb{Z}/l^n(r))) & \xrightarrow{H^1(\,,\cup\, \mathrm{cl}_{\bar{k}}(Z'))} & H^1(G_k, H^{2r+2s-1}(W_{\bar{k}}, \mathbb{Z}/l^n(r+s))) . \end{array}$$

(4) cl_0^n *annihilates cycles rationally equivalent to* 0.

(5) cl_0^n *is functorial with respect to extension of the base field.*

(6) *Let* W' *be smooth and* W *smooth and projective over* k. *Then a correspondence* $\Gamma \in Z^{d+s-r}(W \times W')$ *gives rise to a commutative diagram*:

$$\begin{array}{ccc} CH^r(W)_{\mathrm{hom}} & \xrightarrow{\;\Gamma_*\;} & CH^s(W')_{\mathrm{hom}} \\ \mathrm{cl}_0 \downarrow & & \mathrm{cl}_0 \downarrow \\ H^1(G_k, H^{2r-1}(W_{\bar{k}}, \mathbb{Z}_l(r))) & \xrightarrow{H^1(\,,\mathrm{cl}_{\bar{k}}(\Gamma)_*)} & H^1(G_k, H^{2s-1}(W'_{\bar{k}}, \mathbb{Z}_l(s))) . \end{array}$$

(7) *Let* W/k *be smooth and projective. When* $r = 1$ *the cocycle* $\delta(Z)$ *may be represented by the crossed homomorphism* $G_k \to \mathrm{Pic}(W_{\bar{k}})[l^n]$, $\sigma \to D - \sigma D$, *where* $l^n D \sim_{\mathrm{rat}} Z$.

Proof. (1) This follows easily from the extension definition and the functoriality of the cycle class map to local cohomology with respect to pullback by smooth morphisms [Mi], VI. 6c, 9.2.

(2) By Poincaré duality (1.7) may be rewritten in terms of homology [J], 9.0.1. The assertion follows from the functoriality of the fundamental homology class with respect to proper direct image.

(3) See [J], 10.6.

(4) A cycle on W which is rationally equivalent to zero may be written as $\mathrm{pr}_{W*}(\Gamma \cdot \mathrm{pr}^*_{\mathbb{P}^1}(z))$, where $\Gamma \in Z(\mathbb{P}^1 \times W)$, $z \in Z^1(\mathbb{P}^1)$, and Γ meets $\mathrm{pr}^*_{\mathbb{P}^1}(z)$ properly. Since $H^1(\mathbb{P}^1_{\bar{k}}, \mathbb{Z}/l^n(1)) \simeq 0$, (4) follows from (1), (2), and (3).

(5) Both (1.7) and the cycle class map to local cohomology are functorial with respect to extension of the base field.

(6) $\Gamma_*(z) := \mathrm{pr}_{W'*}(\Gamma \cdot \mathrm{pr}^*_W(z))$. When $\mathrm{pr}^*_W(z)$ meets Γ properly, the assertion follows from (1), (2), (3). In general one may replace $\mathrm{pr}^*(z)$ by z' in the same rational equivalence class such that $\Gamma \cdot z'$ is well defined. Furthermore the rational equivalence class of the intersection is independent of the choice of z' [Rob]. The assertion follows from (1), (2), (3), and (4).

(7) Define

$$(1.10) \qquad \mathscr{K}_W := \mathrm{Ker}[\bar{k}(W)^*/\bar{k}(W)^{*l^n} \xrightarrow{\mathrm{div}} \mathrm{Div}(W)_{\bar{k}}/l^n \mathrm{Div}(W)_{\bar{k}}] .$$

For a proper closed subset, $Z \subset W$, $H^1_Z(W_{\bar{k}}, \mathbb{Z}/l^n(1)) \simeq 0$ (purity) and the sequence

$$0 \to H^1(W_{\bar{k}}, \mathbb{Z}/l^n(1)) \to H^1((W-Z)_{\bar{k}}, \mathbb{Z}/l^n(1)) \to H^2_Z(W_{\bar{k}}, \mathbb{Z}/l^n(1))$$

is exact. Taking the limit over all such Z and using $H^1(\bar{k}(W), \mathbb{Z}/l^n(1)) \simeq \bar{k}(W)^*/(\bar{k}(W)^*)^{l^n}$ gives rise to an identification

$$\mathscr{K}_W \simeq H^1(W_{\bar{k}}, \mathbb{Z}/l^n(1)) .$$

Let $(\mathrm{Div}_{|Z|}(W_{\bar{k}})/l^n)_0$ denote the group of divisors on $W_{\bar{k}}$, with $\mathbb{Z}/l^n$-coefficients, whose class in the cohomology group $H^2(W_{\bar{k}}, \mathbb{Z}/l^n(1))$ vanishes and whose support is contained in $|Z|$. When $r = 1$ (1.7) may be rewritten as

$$(1.11) \qquad 1 \to \mathscr{K}_W \to \mathscr{K}_{W-|Z|} \to (\mathrm{Div}_{|Z|}(W_{\bar{k}})/l^n)_0 \to 0 .$$

Now $\delta(Z)$ is represented by the crossed homomorphism, $G_k \to \mathscr{K}_W$, $\sigma \to \sigma f/f$, where $f \in \bar{k}(W)^*$ and $\mathrm{div}(f) = Z - l^n D$. The assertion follows from the isomorphism

$$\mathscr{K}_W \simeq \mathrm{Pic}(W_{\bar{k}})[l^n], \quad h \to \frac{1}{l^n}\mathrm{div}(h) .$$

Lemma 1.12. *Let k be a finite field and l a prime distinct from the characteristic of k. Then the map* $\mathrm{cl}_0 : CH^1(W_k)_{\mathrm{hom}} \otimes \mathbb{Z}_l \to H^1(G_k, H^1(W_{\bar{k}}, \mathbb{Z}_l(1)))$ *is an isomorphism.*

Proof. The second map in the sequence,

$$CH^1(W_k) \simeq H^1(W_k, \mathbb{G}_m) \to H^1(W_{\bar{k}}, \mathbb{G}_m)^{G_k},$$

is an isomorphism by Hilbert's Theorem 90 and the triviality of the Brauer group of k. Kummer theory gives an identification

$$(H^1(W_{\bar{k}}, \mathbb{G}_m)[l^\infty])^{G_k} \simeq H^1(W_{\bar{k}}, \mathbb{Q}_l/\mathbb{Z}_l(1))^{G_k} .$$

By [Co-Sa-So], p. 744, there is an exact sequence

$$(1.13) \qquad 0 \to H^1(W_{\bar{k}}, \mathbb{Z}_l(1))$$
$$\to H^1(W_{\bar{k}}, \mathbb{Q}_l(1)) \xrightarrow{\varpi} H^1(W_{\bar{k}}, \mathbb{Q}_l/\mathbb{Z}_l(1)) \to H^2(W_{\bar{k}}, \mathbb{Z}_l(1)).$$

The preceding isomorphisms identify the torsion group, $CH^1(W_k)_{\mathrm{hom}} \otimes \mathbb{Z}_l$, with $(\mathrm{im}(\varpi))^{G_k}$. The coboundary map associated with (1.13),

$$(1.14) \qquad (\mathrm{im}(\varpi))^{G_k} \to H^1(G_k, H^1(W_{\bar{k}}, \mathbb{Z}_l(1))),$$

is an isomorphism because 1 is not an eigenvalue of Frobenius operating on $H^i(W_{\bar{k}}, \mathbb{Q}_l(1))$ when $i \in \{0,1\}$. It follows from the construction of (1.13) that the map (1.14) agrees up to sign with $\varprojlim_n$ applied to 1.9 (7) and hence with cl_0.

2. Preliminaries on the Griffiths group

Let W/k be a smooth projective variety of dimension d. Given a smooth projective curve X/k and $\Gamma \in Z^r(X \times W)$ whose components are flat over X, there is a map $\Gamma_* : Z^1(X) \to Z^r(W)$ defined by $\Gamma_* z = \mathrm{pr}_{W*}(\Gamma \cdot \mathrm{pr}_X^*(z))$. Since the cycle class map (1.2) is compatible with correspondences [Fu], 19.2.7, we get a map

$$\Gamma_* : Z^1(X)_{\mathrm{hom}} \to Z^r(W)_{\mathrm{hom}}.$$

Define

$$(2.1) \qquad Z^r(W)_{\mathrm{alg}} = \sum \Gamma_*(Z^1(X)_{\mathrm{hom}}) \subset Z^r(W)_{\mathrm{hom}}$$

and

$$\mathrm{Griff}^r(W) = Z^r(W)_{\mathrm{hom}} / Z^r(W)_{\mathrm{alg}}.$$

The sum is taken over all pairs (X, Γ) as above. We call $\mathrm{Griff}^r(W)$ the Griffiths group of codimension r cycles on W.

Lemma 2.2. *Suppose $k \subset \mathbb{C}$. Suppose furthermore that a correspondence*

$$P \in Z^d(W \times W)$$

is given with the property that the $r+1$-level of the Hodge filtration, $F^{r+1} P_* H^{2r-1}(W(\mathbb{C}))$, does not vanish. If

(1) *the rational Hodge structure $P_* H^{2r-1}(W(\mathbb{C}))$ is irreducible, or*

(2) *$P_* H^{2r-1}(W_{\bar{k}}, \mathbb{Q}_l(r))$ is an irreducible G_k-module,*

then

$$(\mathrm{cl}_0 \circ P_*) \otimes 1 : Z^r(W)_{\mathrm{hom}} \otimes \mathbb{Q}_l \to H^1(G_k, H^{2r-1}(W_{\bar{k}}, \mathbb{Z}_l(r))) \otimes \mathbb{Q}_l$$

factors through $\mathrm{Griff}^r(W) \otimes \mathbb{Q}_l$.

Proof. The non-vanishing of the $r+1$ level of the Hodge structure implies that for each curve X/k and each correspondence $\Gamma \in Z^r(X \times W)$,

$$P_* \circ \Gamma_*(H^1(X(\mathbb{C}))) \neq P_* H^{2r-1}(W(\mathbb{C})).$$

As the comparison isomorphism between étale and Betti cohomology respects cycle classes,

$$P_* \circ \Gamma_*(H^1(X_{\bar{k}}, \mathbb{Q}_l(1))) \neq P_* H^{2r-1}(W_{\bar{k}}, \mathbb{Q}_l(r)).$$

Using either (1) or (2) we conclude $P_* \circ \Gamma_*(H^1(X_{\bar{k}}, \mathbb{Q}_l(1))) = 0$. Thus

$$\mathrm{cl}_0(P_* \Gamma_*(Z^1(X)_{\mathrm{hom}})) \otimes 1 = P_* \Gamma_*(\mathrm{cl}_0(Z^1(X)_{\mathrm{hom}})) \otimes 1 = 0,$$

which proves the lemma.

Example 2.3 (cf. [Gr-Sch]). Let $k \subset \mathbb{C}$, (E, e) an elliptic curve over k, $W = E^3$. For each subset $T \subset \{1,2,3\}$ let $p_T : E^3 \to E^{|T|}$ denote the projection obtained by omitting the factors not in T. For example, $p_{13}(x_1, x_2, x_3) = (x_1, x_3)$. $p_\emptyset : E^3 \to \mathrm{Spec}(k)$ is the structure map. Define inclusions $q_T : E^{|T|} \to E^3$ using the neutral element e to fill in the missing coordinate. For example, $q_{13}(x_1, x_2) = (x_1, e, x_2)$. Let Q_T denote the graph of the morphism $q_T \circ p_T : W \to W$, and define

$$Q = Q_{123} - Q_{12} - Q_{13} - Q_{23} + Q_1 + Q_2 + Q_3 - Q_\emptyset$$

as a 3-cycle on $W \times W$. Then Q gives rise to the endomorphism

$$(q_{123} \circ p_{123})_* - (q_{12} \circ p_{12})_* - \dots + (q_\emptyset \circ p_\emptyset)_*$$

of $Z^\cdot(W)$ respectively $H^\cdot(W, \mathbb{Z}_l(\cdot))$.

Denote by $\tau : W \to W$ the map $(x_1, x_2, x_3) \to (x_2, x_3, x_1)$. Define two new self-correspondences of W by

$$T = \mathrm{Id} + \tau + \tau^2,$$
$$P = T \circ Q.$$

Lemma 2.4. (1) *As correspondences T, Q, P are linear combinations of endomorphisms of the abelian variety W.*

(2) *As correspondences $T^2 = 3T, Q^2 = Q$, $P = Q \circ T$, $P^2 = 3P$.*

(3) $P_* H^3(W_{\bar{k}}, \mathbb{Q}_l) \simeq \mathrm{Sym}^3 H^1(E_{\bar{k}}, \mathbb{Q}_l)$.

(4) $Q_* H_2(W_{\bar{k}}, \mathbb{Z}_l) = 0$.

Proof. The proof is straightforward. In (3) note that the Künneth formula allows one to view the third symmetric product as a direct factor of $H^3(W_{\bar{k}}, \mathbb{Q}_l)$.

Remark 2.5. The pair $(E^3, \frac{1}{3}P)$ constitutes a Chow motive over $\mathbb{Q}$ [Mu], 1.2, which we denote by M. The Chow group tensored with $\mathbb{Z}[\frac{1}{3}]$ (respectively the cohomology with $\mathbb{Q}_l$-coefficients) of M is obtained by applying the projector $\frac{1}{3}P$ to $CH(E^3) \otimes \mathbb{Z}[\frac{1}{3}]$ (respectively to $H^{\cdot}(E^3_{\bar{\mathbb{Q}}}, \mathbb{Q}_l)$). It follows from 2.4 (3) that $L(H^3(M), s)$ is the L-series for the representation of the Galois group in $\mathrm{Sym}^3 H^1(E_{\bar{k}}, \mathbb{Q}_l)$.

Lemma 2.6. *Let $k \subset \mathbb{C}$. If $\mathrm{End}(E_{\bar{k}}) \simeq \mathbb{Z}$, then (W, P) satisfies the hypothesis* (1) *of* 2.2. *If k is a number field, then the hypothesis* (2) *of* 2.2 *is satisfied as well.*

Proof. (1) Write V for the Hodge structure $H^1(E_{\mathbb{C}})$. The Mumford-Tate group of V is $GL(V_{\mathbb{Q}})$. Equivalently, the Hodge torus $\phi: R_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \to GL(V_{\mathbb{R}})$ is not a subtorus defined over $\mathbb{Q}$. The third symmetric power representation on $GL(V_{\mathbb{Q}})$ has kernel $\mu_3\,\mathrm{Id}$. The inverse image of a maximal subtorus of $GL(V_{\mathbb{R}})/\mu_3$ which is defined over $\mathbb{Q}$ is again a maximal torus defined over $\mathbb{Q}$. Thus the image of the Hodge torus under this representation is not defined over $\mathbb{Q}$. Its $\mathbb{Q}$-Zariski closure is $GL(V)/\mu_3$. The assertion follows since $\mathrm{Sym}^3 V$ is an irreducible $GL(V)/\mu_3$-representation.

(2) By analogy with (1) it would suffice to know that the Lie algebra of the image of the Galois representation on the Tate module of E becomes, after tensoring with $\mathbb{Q}_l$, isomorphic to $\mathrm{gl}_2(\mathbb{Q}_l)$. That this is indeed the case is proved in [Ser], p. IV–11.

3. Preliminaries on specialization

Let K be a non-archimedean local field with integers $\mathcal{O}$ and residue field k. Let $\bar{K}$ be a separable closure of K, $\mathcal{O}^{nr}$ the integral closure of $\mathcal{O}$ in the maximal unramified extension of K, $S = \mathrm{Spec}(\mathcal{O})$ and $S^{nr} = \mathrm{Spec}(\mathcal{O}^{nr})$. Let $p: \mathcal{W} \to S$ be smooth and projective. Assume that the generic and special fibers, W_K and W_k, are geometrically integral varieties. Define $p_K := p|_{W_K}$ and $p_k := p|_{W_k}$. Write $I \subset G_K$ for the absolute Galois group of the maximal unramified extension of K. Then $G_K/I \simeq G_k$. Fix a prime l distinct from the characteristic of k. Base change isomorphisms [Mi], VI. 4.2, give rise to a sequence of isomorphisms

$$H^{2r-1}(W_{\bar{k}}, \mathbb{Z}/l^n(r)) \simeq R^{2r-1}p_* \mathbb{Z}/l^n(r)|_{\mathrm{Spec}(\bar{k})} \simeq R^{2r-1}p_* \mathbb{Z}/l^n(r)|_{\mathrm{Spec}(\bar{K})}$$
$$\simeq H^{2r-1}(W_{\bar{K}}, \mathbb{Z}/l^n(r))$$

which are equivariant with respect to the operations of G_k and G_K, since I acts trivially on the right hand side. Applying $\varprojlim_n$ and observing that cohomology commutes with inverse limit in the situation at hand [Ta], 2.2 gives a restriction homomorphism,

$$\vartheta: H^1(G_k, H^{2r-1}(W_{\bar{k}}, \mathbb{Z}_l(r))) \to H^1(G_K, H^{2r-1}(W_{\bar{K}}, \mathbb{Z}_l(r))).$$

Proposition 3.1. *Suppose that $H^{2r-1}(W_{\bar{k}}, \mathbb{Z}_l)$ is torsion free. Then ϑ is an isomorphism and there is a specialization map,* sp, *on Chow groups such that*

$$\begin{array}{ccc} CH^r(W_K)_{\mathrm{hom}} & \xrightarrow{\mathrm{sp}} & CH^r(W_k)_{\mathrm{hom}} \\ \downarrow & & \downarrow \\ H^1(G_K, H^{2r-1}(W_{\bar{K}}, \mathbb{Z}_l(r))) & \xrightarrow{\vartheta^{-1}} & H^1(G_k, H^{2r-1}(W_{\bar{k}}, \mathbb{Z}_l(r))) \end{array}$$

commutes.

Proof. To see that ϑ is an isomorphism consider the exact sequence

$$0 \to H^1(G_k, H^{2r-1}(W_{\bar{k}}, \mathbb{Z}/l^n(r))) \to H^1(G_K, H^{2r-1}(W_{\bar{K}}, \mathbb{Z}/l^n(r))) \to \mathrm{Hom}(I, H^{2r-1}(W_{\bar{K}}, \mathbb{Z}/l^n(r))^{G_k}).$$

The sequence remains exact after applying $\varprojlim_n$. As cohomology and $\varprojlim_n$ commute here [Ta], 2.2, ϑ will be an isomorphism if $H^{2r-1}(W_{\bar{k}}, \mathbb{Z}_l(r))^{G_k} = 0$. Since $H^{2r-1}(W_{\bar{k}}, \mathbb{Z}_l(r))$ is torsion free, the assertion follows from Deligne's theorem which implies that 1 is not an eigenvalue of the action of Frobenius on $H^{2r-1}(W_{\bar{k}}, \mathbb{Q}_l(r))$.

Consider the diagram

$$CH^r(W_K) \xleftarrow{j^*} CH^r(\mathscr{W}) \xrightarrow{i^!} CH^r(W_k)$$

in which j^* is flat pullback and $i^!$ is intersection with the closed fiber. The map

$$\mathrm{sp} := i^! \circ \xi : CH^r(W_K) \to CH^r(W_k)$$

is independent of the choice of a left inverse, ξ, of j^* [Fu], 6.3.7, and is called the specialization homomorphism. In [Gro-De], 2.2–3 a cycle class map

$$\mathrm{cl}_S^n : Z^r(\mathscr{W})_f \to H^{2r}(\mathscr{W}, \mathbb{Z}/l^n(r))$$

is defined, where $Z^r(\mathscr{W})_f \subset Z^r(\mathscr{W})$ is the subgroup generated by integral subschemes which are flat over S. The diagram

$$\begin{array}{ccc} Z^r(\mathscr{W})_f & \xrightarrow{i^!} & CH^r(W_k) \\ \mathrm{cl}_S^n \downarrow & & \mathrm{cl}_k^n \downarrow \\ H^{2r}(\mathscr{W}, \mathbb{Z}/l^n(r)) & \longrightarrow & H^{2r}(W_k, \mathbb{Z}/l^n(r)) \end{array}$$

commutes [Gro-De], 2.3.8 (ii), and the map on cohomology is an isomorphism [Mi], VI. 2.7. Thus $\mathrm{cl}_S^n(z) = 0$, if $z \sim_{\mathrm{rat}} 0$ on $\mathscr{W}$. We may extend cl_S^n to all of $Z^r(\mathscr{W})$ by sending cycles supported on the special fiber to zero. This gives rise to a commutative diagram

$$(3.2) \quad \begin{array}{ccccc} CH^r(W_K) & \xleftarrow{j^*} & CH^r(\mathscr{W}) & \xrightarrow{i^!} & CH^r(W_k) \\ \mathrm{cl}_K^n \downarrow & & \mathrm{cl}_S^n \downarrow & & \mathrm{cl}_k^n \downarrow \\ H^{2r}(W_K, \mathbb{Z}/l^n(r)) & \longleftarrow & H^{2r}(\mathscr{W}, \mathbb{Z}/l^n(r)) & \xrightarrow{\sim} & H^{2r}(W_k, \mathbb{Z}/l^n(r)) \\ \gamma_K \downarrow & & \gamma_S \downarrow & & \gamma_k \downarrow \\ H^{2r}(W_{\bar{K}}, \mathbb{Z}/l^n(r))^{G_K} & \xleftarrow{\sim} & H^{2r}(\mathscr{W} \times_S S^{nr}, \mathbb{Z}/l^n(r))^{G_k} & \xrightarrow{\sim} & H^{2r}(W_{\bar{k}}, \mathbb{Z}/l^n(r))^{G_k}, \end{array}$$

where the isomorphisms are consequences of the standard base change theorems [Mi], VI. 2.7 and 4.2. We may view the maps γ_K, γ_S, and γ_k as coming from the Leray spectral sequences for p_K, p, and p_k. Since $E_\infty^{1,2r-1} \subset E_2^{1,2r-1}$ for these spectral sequences, (3.2) gives rise to a diagram

$$
(3.3) \qquad \begin{array}{ccccc}
CH^r(W_K)_{\text{hom}} & \xleftarrow{j^*} & (j^*)^{-1}(CH^r(W_K)_{\text{hom}}) & \xrightarrow{i!} & CH^r(W_k)_{\text{hom}} \\
\downarrow & & \downarrow & & \downarrow \\
H^1(\operatorname{Spec}(K), R^{2r-1}p_{K*}\mathbb{Z}/l^n(r)) & \xleftarrow{j_n^*} & H^1(S, R^{2r-1}p_*\mathbb{Z}/l^n(r)) & \underset{\sim}{\xrightarrow{\varsigma_n}} & H^1(\operatorname{Spec}(k), R^{2r-1}p_{k*}\mathbb{Z}/l^n(r)) .
\end{array}
$$

The left hand square commutes because the Leray spectral sequence is functorial for base change by an open immersion. It follows from the proper base change theorem [Mi], VI. 2.7 that the individual terms in the spectral sequences for p and p_k are canonically isomorphic. Thus the right hand square commutes as well. By Hensel's lemma [Mi], I. 4.4, etale covers of Spec(k) lift uniquely to etale covers of S. This leads to an explicit description of the map ς_n^{-1} on Cech cohomology. The standard isomorphism between the Cech cohomology of an etale sheaf over the spectrum of a field and Galois cohomology [Mi], III. 2.6, allows us to identify $\varprojlim_n (\varsigma_n^{-1} \circ j_n^*)$ with ϑ. Finally we remark that the left and right vertical maps in (3.3) may be identified with cl_0^n by [J], 9.4.

4. The test varieties and their cycles

In this section we apply the methods described above to certain cycles on threefold self-products of certain elliptic curves defined over $\mathbb{Q}$. Our main result, Theorem 4.8, is a criterion for when such a cycle has infinite order in the Griffiths group in terms of the rational equivalence class of a divisor on the curve reduced modulo a prime of good reduction.

We work with the product of elliptic curves, $W_a := E_a^3$, where $a \in \mathbb{Q}$, $a \notin \{-1, \pm 2\}$,

$$(4.1) \qquad E_a: \quad zy^2 = (a^2-4)x^3 + (2a^2-4a)zx^2 + (a^2-4)z^2x .$$

The neutral element in the group law on $E_a(\mathbb{Q})$ will be the point $e = (0:1:0)$. Following ideas in [T], the non-singular genus 3 curve

$$C_a: \quad x^4 + y^4 + z^4 + a(x^2y^2 + y^2z^2 + z^2x^2) = 0$$

may be used to construct an element $\Xi_a \in Z^2(W_{a\mathbb{Q}})_{\text{hom}}$. In fact consider the composition of degree two maps

$$(4.2) \qquad \pi : C_a \xrightarrow{\mu_1} E_a' \xrightarrow{\mu_2} E_a ,$$

where E_a' is the smooth, projective model of

$$(4.3) \qquad w^2 = (a^2-4)v^4 + (2a^2-4a)v^2 + (a^2-4) .$$

The map μ_1 is the canonical quotient morphism associated to the involution

$$(x:y:z) \to (-x:y:z) .$$

In terms of coordinate functions

$$(4.4)\qquad w\circ\mu_1=y/z,\quad v\circ\mu_1=(2x^2+ay^2+az^2)/z^2\,;$$

and

$$x/z\circ\mu_2=v^2,\quad y/z\circ\mu_2=vw\,.$$

The permutation of the coordinates $\sigma\in \mathrm{Aut}(k[x,y,z])$ defined by

$$x\circ\sigma=y,\quad y\circ\sigma=z,\quad z\circ\sigma=x$$

gives rise to an automorphism of $\mathbb{P}^2$ which stabilizes the curve C_a. Define

$$(4.5)\qquad \varrho: C_a\to W_a,\quad\text{by}\quad \varrho=(\pi,\pi\circ\sigma,\pi\circ\sigma^2)\,,$$

and

$$(4.6)\qquad \Xi_a:=\varrho(C_a)-[-1]_*\varrho(C_a)\,.$$

Note that $\Xi_a\in Z^2(W_a)_{\mathrm{hom}}$ since $-1\in\mathrm{Aut}(E_a^3)$ acts trivially on $H^4(W_a(\mathbb{C}),\mathbb{Z}_l(2))$.

Let P be the correspondence introduced in 2.3. In order to state the main theorem about the class of $P_*\Xi_a$ in the Griffiths group $\mathrm{Griff}^2(W_a)$ we introduce some notation. Fix a rational prime $p>3$ where E_a and C_a have good reduction (i.e., $a\notin\{-1,\pm2\}\bmod p$). Write $S=\mathrm{Spec}(\mathbb{Z}_p)$, and let $\mathscr{W},\mathscr{E},\mathscr{C}$ be smooth, projective models for W_a, E_a, and C_a over S. As $a\in\mathbb{Q}$ will not vary, we frequently drop it from the notation. The special fibers will be denoted $W_{\mathbb{F}_p}$, $E_{\mathbb{F}_p}$, $C_{\mathbb{F}_p}$. Let $F\in\mathrm{End}(E_{\mathbb{F}_p})$ denote the Frobenius endomorphism. Write $D\subset W_{\mathbb{F}_p}$ (respectively $\Gamma\subset(W\times E)_{\mathbb{F}_p}$) for the image of

$$E^2_{\mathbb{F}_p}\to E^3_{\mathbb{F}_p},\quad (a,b)\mapsto(a,F(a),b)\quad(\text{respectively}\quad E^2\to E^4,\quad (a,b)\mapsto(a,F(a),b,b))\,.$$

The map $\Gamma_*: CH^2(W_{\mathbb{F}_p})\to CH^1(E_{\mathbb{F}_p})$ satisfies

$$(4.7)\qquad \Gamma_*(z)=p_{3*}(D\cdot z)\,.$$

Given an abelian variety A with neutral element 0 over a field k, define

$$\alpha: CH_0(A_k)\to CH_0(A_k)_{\mathrm{hom}},\quad\text{by}\quad \alpha(c)=c-\deg(c)0\,.$$

If k is finite, $CH_0(A_k)_{\mathrm{hom}}$ is a finite group and we define for each prime l, α_l to be the composition of α with projection to the l-primary component.

If E_a has good reduction at a prime q, write $a_q\in\mathbb{Z}$ for the trace of the Frobenius endomorphism, $F\in\mathrm{End}(E_{a\mathbb{F}_q})$.

Theorem 4.8. (1) *Let $l>3$ be a rational prime with $(l,p)=1$. If*

$$\alpha_l(\Gamma_*\varrho(C_{a\mathbb{F}_p}))\in CH_0(E^3_{a\mathbb{F}_p})_{\mathrm{hom}}$$

is not zero, then

$$\mathrm{cl}_0(P_* \Xi_a) \in H^1(G_{\mathbb{Q}}, P_* H^3(W_{a\,\bar{\mathbb{Q}}}, \mathbb{Z}_l(2))) \simeq H^1(G_{\mathbb{Q}}, \mathrm{Sym}^3 H^1(E_{a\,\bar{\mathbb{Q}}}, \mathbb{Z}_l)(2))$$

is not zero.

(2) *Let q be a rational prime, distinct from l and p, such that E has good reduction at q. If l does not divide $(q-a_q+1)(q^2+q-a_q^3-3a_q q)$, then $H^1(G_{\mathbb{Q}}, \mathrm{Sym}^3 H^1(E_{a\,\bar{\mathbb{Q}}}, \mathbb{Z}_l)(2))$ is torsion free.*

(3) *If $a \notin \{\pm 2, -1, 0, -3, -3/2\}$ and the hypotheses in* (1) *and* (2) *are satisfied, then the class of $P_* \Xi_a$ in* $\mathrm{Griff}^2(W_{a\mathbb{Q}})$ *is of infinite order.*

Proof. (1) To show $\mathrm{cl}_0(P_* \Xi) \neq 0$ we may extend scalars to $\mathbb{Q}_p$ and show 1.9 (5)

$$\mathrm{cl}_0(P_* \Xi|_{\mathbb{Q}_p}) \neq 0 \in H^1(G_{\mathbb{Q}_p}, H^3(W_{\bar{\mathbb{Q}}_p}, \mathbb{Z}_l(2))).$$

Since $P_* \Xi|_{\mathbb{Q}_p}$ specializes to $P_* \Xi_{\mathbb{F}_p}$ it suffices by 3.1 to show

$$\mathrm{cl}_0(P_* \Xi_{\mathbb{F}_p}) \neq 0 \in H^1(G_{\mathbb{F}_p}, H^3(W_{\bar{\mathbb{F}}_p}, \mathbb{Z}_l(2))).$$

By 1.9 (3) we further reduce to showing

$$\Gamma_* \mathrm{cl}_0(P_* \Xi_{\mathbb{F}_p}) = \mathrm{cl}_0(\Gamma_* P_* \Xi_{\mathbb{F}_p}) \neq 0 \in H^1(G_{\mathbb{F}_p}, H^1(E_{\bar{\mathbb{F}}_p}, \mathbb{Z}_l(2))).$$

By 1.12 it suffices to show that the l-primary component of $\Gamma_* P_* \Xi_{\mathbb{F}_p} \in CH^1(E_{\mathbb{F}_p})_{\mathrm{hom}}$ is not zero. The assertion (1) thus follows from the following

Lemma 4.9. $\alpha(6\Gamma_* \varrho(C_{\mathbb{F}_p})) = \Gamma_* P_* \Xi_{\mathbb{F}_p}$.

Proof. As the argument involves only computations over the fixed base field, $\mathbb{F}_p$, we will frequently drop the subscript $\mathbb{F}_p$ to simplify the notation. First note

$$\begin{aligned}
\Gamma_* P_* \Xi_{\mathbb{F}_p} &= p_{3*}(D \cdot P_* \Xi) \\
&= p_{3*}(D \cdot (P_* \varrho(C) - [-1]_* P_* \varrho(C))) \\
&= p_{3*}(D \cdot P_* \varrho(C) - [-1]_* (D \cdot P_* \varrho(C))) \\
&= 2 p_{3*}(D \cdot P_* \varrho(C)) \\
&= 2 p_{3*}(D \cdot Q_* T_* \varrho(C)) = p_{3*}(6D \cdot Q_* \varrho(C)).
\end{aligned}$$

In the fourth equality we are using the fact that $Q_* \varrho(C)$ and hence $P_* \varrho(C)$ is homologous to zero (2.4 (4)). To prove the lemma we need only show

$$p_{3*}(D \cdot Q_* \varrho(C)) = \alpha(\Gamma_* \varrho(C_{\mathbb{F}_p})), \tag{4.10}$$

which we do by considering the action of the individual components of the correspondence Q separately.

First note $D \cdot Q_{\emptyset *}\varrho(C) = D \cdot Q_{3*}\varrho(C) = 0$ in $CH_0(W_{\mathbb{F}_p})$. Also,

$$D \cap Q_{1*}\varrho(C) = D \cap Q_{2*}\varrho(C) = e \times e \times e$$

whence, $\alpha(D \cdot Q_{1*}\varrho(C)) = \alpha(D \cdot Q_{2*}\varrho(C)) = 0$.

To show $\alpha(D \cdot Q_{23*}\varrho(C)) = 0$ observe that

$$\begin{aligned} D \cdot Q_{23*}\varrho(C) &= q_{23*}(q_{23}^* D \cdot p_{23}(\varrho(C))) \\ &= q_{23*}(e \times E \cdot p_{23} \circ \varrho(C)) \\ &= q_{23*}(e \times (\pi \circ \sigma^2)_*(\pi \circ \sigma)^*(e)) . \end{aligned}$$

Now the desired vanishing follows from

Lemma 4.11. *For arbitrary i and j,* $\alpha((\pi \circ \sigma^i)_*(\pi \circ \sigma^j)^*(e)) = 0$.

Proof. When $i = j$ we must show $\alpha(4e) = 0$ which is obvious. Now $\pi^*(e)$ is the degree 4 zero cycle on C cut out by $z = 0$. As σ fixes the rational equivalence class of $\pi^*(e) \sim \mathcal{O}_{\mathbb{P}^2}(1)|_C$ we are reduced to the case $i = j$.

To show $\alpha(D \cdot Q_{13*}\varrho(C)) = 0$ observe that

$$\begin{aligned} D \cdot Q_{13*}\varrho(C) &= q_{13*}(q_{13}^* D \cdot p_{13}(\varrho(C))) \\ &= q_{13*}(p(E \times e) \cdot p_{13} \circ \varrho(C)) \\ &= q_{13*}(\pi_*(\pi \circ \sigma^2)^*(e) \times e) . \end{aligned}$$

The desired vanishing follows from the previous lemma.

By (4.7) equation (4.10) is a consequence of

$$p_{3*}(D \cdot (\varrho(C) - Q_{12*}\varrho(C))) = \alpha(p_{3*}(D \cdot \varrho(C))) . \tag{4.12}$$

To check that this holds rewrite the left hand side

$$D \cdot (\varrho(C) - Q_{12*}\varrho(C)) = D \cdot \varrho(C) - D \cdot (q_{12} \circ p_{12})_* \varrho(C) . \tag{4.13}$$

As $D = (q_{12} \circ p_{12})^* D$, the projection formula allows us to rewrite the above expression as

$$D \cdot \varrho(C) - (q_{12} \circ p_{12})_*(D \cdot \varrho(C)) = (\mathrm{Id} - (q_{12} \circ p_{12})_*)(D \cdot \varrho(C)) . \tag{4.14}$$

Now

$$\alpha((\mathrm{Id} - (q_{12} \circ p_{12})_*)(D \cdot \varrho(C))) = \alpha(e \times e \times ((q_3 \circ p_3)_*(D \cdot \varrho(C)) - \deg.(D \cdot \varrho(C))e)) ,$$

which implies (4.12). This completes the proof of 4.9 and hence of 4.8 (1).

(2) Write γ and γ' for the roots of the minimal polynomial of the Frobenius endomorphism, $F \in \mathrm{End}(E_{a\mathbb{F}_q})$. The arithmetic Frobenius acts on $H^1(E_{\bar{\mathbb{F}}_q}, \mathbb{Q}_l)$ with eigenvalues $\gamma^{-1}, (\gamma')^{-1}$. The criterion of Zelinsky [Ze], 6.1, for torsion freeness of

$$H^1(G_{\mathbb{Q}}, \mathrm{Sym}^3 H^1(E_{\bar{\mathbb{Q}}}, \mathbb{Z}_l)(2))$$

is that $\det(f_a - 1) \in \mathbb{Z}_l^*$, where f_a is the arithmetic Frobenius acting on $\mathrm{Sym}^3 H^1(E_{\bar{\mathbb{F}}_q}, \mathbb{Z}_l)(2)$. Now

$$\begin{aligned} \det(f_a - 1) &= (\gamma^{-3} q^2 - 1)((\gamma')^{-3} q^2 - 1)(\gamma^{-1} q - 1)((\gamma')^{-1} q - 1) \\ &= q^{-5}(q - a_q + 1)(q^2 + q - a_q^3 - 3 a_q q). \end{aligned}$$

(3) The hypothesis $a \notin \{\pm 2, -1, 0, -3, -3/2\}$ implies that $E_{a\bar{\mathbb{Q}}}$ is an elliptic curve without complex multiplication [T], 3.4.1. By parts (1) and (2) of the theorem

$$\mathrm{cl}_0(P_* \Xi_a) \in H^1(G_{\mathbb{Q}}, \mathrm{Sym}^3 H^1(E_{\bar{\mathbb{Q}}}, \mathbb{Z}_l)(2)) \otimes \mathbb{Q}_l$$

is not zero. The assertion follows from 2.2 and 2.6.

Remark 4.15. The case that E_a is of CM-type was investigated in [T] using a map $\varrho_2 : C_a \to W_a$ [T], 3.4. This map is related to our map ϱ as follows: Let

$$\beta : E_a \to E_a: \quad \beta(x : y : z) = (xz : yz : x^2)$$

denote the map which sends a point P to $T - P$, where T is the point of order 2 on E_a with coordinates $(0 : 0 : 1)$. Then

$$\varrho_2 = (\pi, \beta\pi\sigma, \beta\pi\sigma^2).$$

The following lemma shows that it makes essentially no difference whether we study the cycle Ξ_a or the cycle $\varrho_2(C_a) - [-1]_* \varrho_2(C_a)$ used in [T].

Lemma 4.16. *Let l be an odd prime. Then $\mathrm{cl}_0(\Xi_a)$ and $\mathrm{cl}_0(\varrho_2(C_a) - [-1]_* \varrho_2(C_a))$ span the same submodule of $H^1(G_{\mathbb{Q}}, H^3(W_{a\bar{\mathbb{Q}}}, \mathbb{Z}_l(2)))$.*

Proof. As cycles

$$[2]_*(\varrho_2(C_a) - [-1]_* \varrho_2(C_a)) = (\mathrm{Id}|_{E_a}, \beta, \beta)_* [2]_* \Xi_a.$$

On cohomology $[2]_*$ acts invertibly and $(\mathrm{Id}|_{E_a}, \beta, \beta) \in \mathrm{Aut}(E^3)$ acts as the identity.

5. Computation of the cycle class

This section describes the final step in the procedure for showing that the nullhomologous cycle $P_* \Xi_a$ on $W_{a\mathbb{Q}} = E_{a\mathbb{Q}}^3$ has infinite order in the Griffiths group for certain values $a \in \mathbb{Q}$. Theorem 4.8 tells us to look for a prime $l > 3$ such that $\alpha_l(\Gamma_* \varrho(C_{a\mathbb{F}_p}))$ is not zero in $CH_0(E_{a\mathbb{F}_p}^3)_{\mathrm{hom}}$. By (4.7), $\alpha_l(\Gamma_* \varrho(C_{a\mathbb{F}_p}))$ will be non-zero if the projections of the points in the intersection $D \cap \varrho(C_{a\mathbb{F}_p})$ to $E_{a\bar{\mathbb{F}}_p}$ have a non-zero sum. Although the intersection cycle is defined over $\mathbb{F}_p$ the individual intersection points may be defined over extensions of large degree making computer computations difficult. Following [Bl], § 3 and [T], § 3.4, we replace $p_{*3}(D \cap \varrho(C_{a\mathbb{F}_p}))$ by a rationally equivalent divisor each of whose points is defined over $\mathbb{F}_{p^3}$. This makes computations feasible.

Assume that p is an odd prime. Define a polynomial,

$$H(t) := t^{2p+2} + at^{2p+1} + t^{2p} + at^{p+1} + at^p + 1,$$

and the related sets,

$$\mathscr{H} = \{t \in \overline{\mathbb{F}}_p : H(t) = 0\} \quad \text{and} \quad \mathscr{S} = \{t \in \mathscr{H} : t^{p^2+p+1} = 1\}.$$

Evidently $\mathscr{S}$ is a subset of the norm one elements in $\mathbb{F}_{p^3}$. Also define a morphism

$$\varphi : \mathbb{P}^1 \to \mathbb{P}^2,$$

$$\varphi((t:1)) = (t^{(p+1)/2} : 2t^p + a + at^{p+1} : t^{(3p+3)/2}).$$

Theorem 5.1. *If $l \neq 2$ and $p \neq l$ is an odd prime where C_a and E_a have good reduction, then*

$$\varphi((\mathscr{S}:1)) \subset E_a \quad \text{and} \quad \alpha_l(\Gamma_* \varrho(C_{a\mathbb{F}_p})) = 4\alpha_l\Big(\sum_{t \in \mathscr{S}} \varphi((t:1))\Big).$$

Remark 5.2. The sum involves only points defined over $\mathbb{F}_{p^3}$. Our computer program can compute this expression in a reasonable amount of time when $p < 200$. In the event that there is no prime in this range for which $\alpha_l(\sum_{t \in \mathscr{S}} \varphi((t:1))) \neq 0$, then we cannot determine whether or not $\mathrm{cl}_0(P_* \Xi_a) = 0$. Fortunately, in the course of our investigations this situation never occurred. (See however the discussion following Table 9.2 for a related situation which did occur.)

The remainder of this section is devoted to the proof of 5.1. By (4.7)

$$\alpha_l(\Gamma_* \varrho(C_{a\mathbb{F}_p})) = \alpha_l(p_{3*}(D \cdot \varrho(C_{a\mathbb{F}_p}))).$$

A main step in the proof of 5.1 is to establish

Proposition 5.3. *As divisors on $E_{a\overline{\mathbb{F}}_p}$,*

$$p_{3*}(D \cdot \varrho(C_{a\mathbb{F}_p})) = 2 \sum_{t \in \mathscr{H}} \mathfrak{p}_{t,\kappa(t)},$$

where

$$\mathfrak{p}_{t,\kappa(t)} = (t^{(p+1)/2} : \kappa(t)(2t^p + a + at^{p+1}) : t^{(3p+3)/2}) \in E_a(\overline{\mathbb{F}}_p),$$

and $\kappa(t) = 1$ if $t^{p^2+p+1} = 1$ and $\kappa(t) = -1$ otherwise.

For the proof we need

Lemma 5.4. *The intersection $\varrho(C_{a\mathbb{F}_p}) \cap D$ is transversal.*

Proof. From (4.4) and (4.5) we have that $\varrho(x:y:z)\in E_a^3$ is given by

$$(5.5)\qquad ((y^2z:2x^2y+ay^3+ayz^2:z^3),\ (xz^2:2y^2z+az^3+ax^2z:x^3),\ (x^2y:2xz^2+ax^3+axy^2:y^3)).$$

It is easy to check that no point $(x:y:z)\in C_a$, where one of the coordinates x, y, or z vanish, maps to D. A point $(x:y:1)\in\mathbb{P}^2$ lies on C_a when

$$(5.6)\qquad x^4+y^4+1+a(x^2y^2+y^2+x^2)=0,$$

and by (5.5) it maps to D precisely when

$$(5.7)\qquad y^{2p}=x^{-2}\quad\text{and}\quad 2x^{2p}y^p+ay^{3p}+ay^p=(2y^2+a+ax^2)x^{-3}.$$

Now given $(x:y:1)$ such that (5.6) and (5.7) hold, plug in $(x+\varepsilon)$ for x and $(y+\delta)$ for y. Assuming $\varepsilon^2=\delta^2=0$, the first equation in (5.7) yields $y^{2p}=(x-2\varepsilon)x^{-3}$. This implies $\varepsilon=0$. Use this fact when plugging into the second equation of (5.7) to get $4y\delta=0$, whence $\delta=0$. The transversality of the intersection follows.

Proof of 5.3. By (5.5)

$$(5.8)\qquad p_3\circ\varrho(x:y:1)=(x^2y:2x+ax^3+axy^2:y^3).$$

By 5.4 the divisor $p_{3*}(D\cdot\varrho(C_{a\bar{\mathbb{F}}_p}))$ is obtained by summing $p_3\circ\varrho(x:y:1)$ over all x and y satisfying (5.6) and (5.7). Use the first equation in (5.7) to write $x=\kappa y^{-p}$ with $\kappa=\pm1$. Also set $t=y^2$ and define $\mathfrak{p}_{t,\kappa}=(t^{(p+1)/2}:\kappa(2t^p+a+at^{p+1}):t^{(3p+3)/2})\in E_a(\bar{\mathbb{F}}_p)$. In this notation

$$(5.9)\qquad p_{3*}(D\cdot\varrho(C_{a\bar{\mathbb{F}}_p}))=2\sum_{(t,\kappa)}\mathfrak{p}_{t,\kappa},$$

where the sum is taken over all simultaneous solutions (t,κ) of the equations

$$(5.6')\qquad H(t)=0,$$

$$(5.7')\qquad \kappa=\pm1,\quad\text{and}\quad 2+at^{2p+p}+at^{p^2}=\kappa(2t^{p^2+p+1}+at^{p^2+p}+at^{p^2}).$$

Lemma 5.10. *If* (t,κ) *is a solution to the system* (5.6′), (5.7′) *then* $(t,-\kappa)$ *is not.*

Proof. If $(t,1)$ is a solution, then (5.7′) implies $t^{p^2+p+1}=1$. If $(t,-1)$ were also a solution, then $at^{p^2+p}+at^{p^2}+2=0$. Multiplying this by t^{p+1} and using $t^{p^2+p+1}=1$ gives the first equation in

$$(5.11)\qquad at^p+a+2t^{p+1}=0\quad\text{and}\quad a+at^{p+1}+2t^p=0.$$

The second equation arises from the first by raising to the p^{th}-power and then multiplying by t^{p+1}. Subtracting the two equations in (5.11) yields $t^p(a-2)(t-1)=0$. We may assume

$a \neq 2$, since the curve E_2 is singular. The only possible solution is $t=1$. Plugging back into (5.11) yields $a=-1$. As E_{-1} is also singular, the lemma follows.

In order to prove 5.3 it will suffice to show that every root of $H(t)=0$ also satisfies (5.7′). This will follow if we find $2p+2=\deg(H)$ distinct solutions (t,κ) to the system (5.6′), (5.7′). Equivalently, we may check that there are $4p+4$ distinct solutions to the system (5.6), (5.7). Since D meets $\varrho(C_{a\mathbb{F}_p})$ transversally, this is a consequence of

Lemma 5.12. *The intersection number* $D \cdot \varrho(C_{a\mathbb{F}_p}) = 4p+4$.

Proof. Set $\xi = (\pi, \pi \circ \sigma): C_a \to E_a^2$ and write $\Gamma_F \subset E_a^2$ for the graph of Frobenius. Evidently,

$$\Gamma_F \cdot \xi_*(C_a) = D \cdot \varrho(C_a).$$

To prove the lemma it clearly suffices to verify the following equality of cohomology classes:

$$\xi_*[C_a] = 4[E_a \times e] + 4[e \times E_a] \in H^2(E_a^2, \mathbb{Q}_l(1)). \qquad (5.13)$$

For this consider the automorphisms $(1,-1)$ of E_a^2 and $\mu_1 \in \operatorname{Aut}(C_a)$ defined by $\mu_1(x:y:z) = (-x:y:z)$. Now the diagram

$$\begin{array}{ccc} C_a & \xrightarrow{\xi} & E_a^2 \\ {\scriptstyle \mu_1}\downarrow & & \downarrow{\scriptstyle (1,-1)} \\ C_a & \xrightarrow{\xi} & E_a^2 \end{array}$$

commutes and $(1,-1)_*$ acts by -1 on $H^1(E_a) \otimes H^1(E_a)$. It follows that

$$\xi_*[C_a] \in H^2(E_a) \otimes H^0(E_a) \oplus H^0(E_a) \otimes H^2(E_a).$$

Since both π and $\pi \circ \sigma$ have degree 4, (5.13) follows. This completes the proofs of 5.12 and 5.3.

To prove 5.1 we need only establish

Lemma 5.14. *If* $l \neq 2$, *then*

$$2\alpha_l\Big(\sum_{t \in \mathscr{H}} \mathfrak{p}_{t,\kappa(t)}\Big) = 4\alpha_l\Big(\sum_{t \in \mathscr{S}} \varphi((t:1))\Big).$$

Proof. Let $\mathfrak{d}$ be a divisor on E_a. Then $\alpha_l(\mathfrak{d}) = 0$ in each of the following circumstances:

(1) $\mathfrak{d} = T \cdot E_a$, where $T \subset \mathbb{P}^2$ is a divisor, since $T \cdot E_a \sim_{\text{rat}} 3\deg(T)e$;

(2) $\mathfrak{d} = m_1(0:0:1) - m_2 e$, since $(0:0:1)$ is a two torsion point on E_a and $l \neq 2$;

(3) $\mathfrak{d} = [-1]_* \mathfrak{d}$.

By (1)

$$(5.15)\qquad 2\alpha_l\Big(\sum_{t\in\mathscr{H}} \mathfrak{p}_{t,\kappa(t)}\Big) = 2\alpha_l\Big(\sum_{t\in\mathscr{H}} \mathfrak{p}_{t,\kappa(t)}\Big) + 2\alpha_l(\varphi_*[\mathbb{P}^1]\cdot E_a)\,.$$

One checks immediately that

$$\varphi((1:0)) = (0:0:1),\quad \varphi((0:1)) = e \ \text{ if } \ a \neq 0, \ \text{ and } \ \varphi((0:1)) \notin E_a \ \text{ if } \ a = 0\,,$$

and that the ideal in $k[t, t^{-1}]$ defining the subscheme,

$$(\mathbb{P}^1 - \{(1:0), (0:1)\}) \times_{\mathbb{P}^2} E_a \subset (\mathbb{P}^1 - \{(1:0), (0:1)\})$$

obtained by pulling E_a back along φ, is generated by the polynomial $H(t)$. We have seen in the course of the proof of 5.3 that all $2p+2$ roots of $H(t)$ are simple. By (2)

$$\alpha_l(\varphi_*[\mathbb{P}^1]\cdot E_a) = \alpha_l\Big(\sum_{t\in\mathscr{H}} \varphi((t:1))\Big)\,.$$

Now $\mathfrak{p}_{t,\kappa(t)} = \varphi((t:1))$ if $\kappa(t) = 1$ and $\mathfrak{p}_{t,\kappa(t)} = [-1]_*\varphi((t:1))$ if $\kappa(t) = -1$. By (3) above, (5.15) is equal to

$$4\alpha_l\Big(\sum_{t\in\mathscr{S}} \varphi((t:1))\Big)\,.$$

This proves 5.14 and hence 5.1.

6. Griffiths group of rank ≧ 2

Up to this point we have been concerned with the problem of how to show that the cycle class $P_*\Xi_a \in \mathrm{Gr}^2(E^3_{a\mathbb{Q}})$ has infinite order. In this section it is shown how to use isogenies between elliptic curves to produce more cycles and how to show that the resulting cycle classes are linearly independent. We have been inspired by [N] and [Ba].

Theorem 6.1. *Let K be a field of characteristic $\neq 2$ or 7. Given*

$$a \in K,\quad a \notin \{-1, \pm 2, -5\}\,,$$

set $K' = K(\sqrt{(a-2)(a+5)})$.

(1) *There is a degree 4 isogeny of elliptic curves,* $\phi: E_{-3-a} \to E_a$, *defined over* K'.

(2) *The conic curve* $b^2 = (a-2)(a+5)$ *has the parameterization*

$$(a,b) = \big((2t^2+5)(t^2-1)^{-1}, 7t(t^2-1)^{-1}\big),\quad t \notin \{\pm 1\}\,.$$

Proof. Since (2) is clear, it suffices to verify (1). Set $\gamma = 2a(a-2)$, $\varepsilon = a^2 - 4$, $\delta = \varepsilon^2$. Substituting $X = \varepsilon x/z$, $Y = \varepsilon y/z$ into (4.1) yields the affine model

$$E_a: Y^2 = X^3 + \gamma X^2 + \delta X\,.$$

Define

$$\hat{E}_a : Y^2 = X(X-\gamma+2\varepsilon)(X-\gamma-2\varepsilon),$$

and observe that

$$E_a \to \hat{E}_a: \quad (X,Y) \to (Y^2/X^2, Y(\delta - X^2)/X^2)$$

is a degree 2 isogeny with kernel generated by $(0,0)$. Introduce the new variable $U = X - \gamma + 2\varepsilon$ so that the equation for $\hat{E}_a$ becomes

$$\hat{E}_a : Y^2 = U^3 + (\gamma - 6\varepsilon) U^2 - 4\varepsilon(\gamma - 2\varepsilon) U.$$

Once again take the quotient by the order two subgroup generated by $(0,0)$. This gives

$$\hat{E}_a \to \hat{\hat{E}}_a \quad \text{with} \quad \hat{\hat{E}}_a : Y^2 = X^3 - 2(\gamma - 6\varepsilon) X^2 + (\gamma + 2\varepsilon)^2 X.$$

The substitution

$$y = \frac{1}{8}\left(\frac{a+5}{a-2}\right)^{3/2} Y, \quad x = \frac{1}{4}\left(\frac{a+5}{a-2}\right) X$$

transforms the equation for $\hat{\hat{E}}_a$ into the equation for E_{-3-a}. Thus we have described an isogeny $E_a \to E_{-3-a}$. Take $\phi : E_{-3-a} \to E_a$ to be the dual isogeny.

Define $\psi := \phi^3 : E^3_{-3-a} \to E^3_a$. Fix a pair of primes, p, l, each greater than 3, such that E_a has good reduction at p. Let $u_{a,p}$ denote the order of the element $\Gamma_* \varrho(C_{a\mathbb{F}_p})$ in the finite group $CH^1(E_{a\mathbb{F}_p})_{\text{hom}}$. Write $\mu_{a,p} = \text{ord}_l(u_{a,p})$ for the valuation of $u_{a,p}$ at l.

If q is a prime of good reduction for E_a write $a_q \in \mathbb{Z}$ for the trace of the Frobenius endomorphism, $F \in \text{End}(E_{a\mathbb{F}_q})$.

Independence criterion 6.2. *Let $t \in \mathbb{Q} - \{0, \pm 1\}$ and define $a = (2t^2+5)(t^2-1)^{-1}$. Choose distinct primes l, p_1, p_2, q, each greater than 3. Suppose that the following conditions are satisfied:*

(1) *l does not divide $(q - a_q + 1)(q^2 + q - a_q^3 - 3a_q q)$.*

(2) *$\text{ord}_{p_i}(a) \geqq 0$ and $a \neq -1, \pm 2$, or $-5 \bmod p_i$, $i \in \{1,2\}$.*

(3) *$\mu_{a,p_i} > 0$ for some $i \in \{1,2\}$ and $\mu_{-3-a,p_i} > 0$ for some $i \in \{1,2\}$.*

(4) *$\mu_{a,p_1} - \mu_{-3-a,p_1} \neq \mu_{a,p_2} - \mu_{-3-a,p_2}$.*

(5) *If $\mu_{a,p_i} = 0$, then $\mu_{a,p_j} \geqq \mu_{-3-a,p_j}$ for $i \neq j$. The same holds when a is replaced by $-3-a$.*

Then $P_* \Xi_a$ and $P_* \psi_* \Xi_{-3-a}$ span a rank two subgroup of $\text{Griff}^2(E^3_{a\mathbb{Q}})$.

Proof. The hypotheses on t imply that E_a and E_{-3-a} are (non-singular) elliptic curves which are not of CM-type [T], 3.4.1. Now (1) implies that

$$H^1(G_{\mathbb{Q}}, \mathrm{Sym}^3 H^1(E_{a\bar{\mathbb{Q}}}, \mathbb{Z}_l(2)))$$

is a torsion free $\mathbb{Z}_l$-module (4.8 (2)). The same is true when a is replaced by $-3-a$, since the eigenvalues of the arithmetic Frobenius acting on isogenous elliptic curves are equal. By § 2 it will suffice to show that the equation

$$c_1 \mathrm{cl}_0(P_* \Xi_a) + c_2 \mathrm{cl}_0(P_* \psi_* \Xi_{-3-a}) = 0$$

has no solutions in coprime l-adic integers c_1 and c_2. By (2), C_a, C_{-3-a}, E_a and E_{-3-a} have good reduction modulo both p_1 and p_2. According to 3.1 it will suffice to show that the pair of equations

$$(6.3) \qquad c_1 \mathrm{cl}_0(P_* \Xi_{a\mathbb{F}_{p_i}}) + c_2 \mathrm{cl}_0(P_* \psi_* \Xi_{-3-a\mathbb{F}_{p_i}}) = 0, \quad i \in \{1,2\},$$

has no common solution in coprime l-adic integers c_1 and c_2. Apply the correspondence Γ of (4.7) to (6.3). Using the equality of correspondences, $P_a \circ \psi = \psi \circ P_{-3-a}$, we get

$$(6.4) \qquad c_1 \Gamma_{a*} \mathrm{cl}_0(P_{a*} \Xi_{a\mathbb{F}_{p_i}}) + c_2 \Gamma_{a*} \psi_* \mathrm{cl}_0(P_{-3-a*} \Xi_{-3-a\mathbb{F}_{p_i}}) = 0.$$

Recall from (4.7) the divisor D for which $\Gamma_*(z) = p_{3*}(D \cdot z)$ holds for all 1-cycles, z. It is straightforward to check $\psi^* D_a = 4D_{-3-a} \in CH^1(E^3_{-3-a\mathbb{F}_{p_i}})$, whence

$$\begin{aligned} \Gamma_{a*}\psi_*(z) &= p_{3*}(D_a \cdot \psi_*(z)) = p_{3*}\psi_*(\psi^* D_a \cdot z) \\ &= 4\phi_* p_{3*}(D_{-3-a} \cdot z) = 4\phi_* \Gamma_{-3-a*}(z). \end{aligned}$$

Use 4.9 and 1.12 to deduce from (6.4)

$$(6.5) \qquad 6c_1 \alpha_l(\Gamma_{a*} \varrho(C_{a\mathbb{F}_{p_i}})) = -24 c_2 \phi_* \alpha_l(\Gamma_{-3-a*} \varrho(C_{-3-a\mathbb{F}_{p_i}})).$$

Since l is prime to $\deg(\phi)$, $\phi_* : CH^1(E_{-3-a\mathbb{F}_{p_i}})_{\mathrm{hom}} \otimes \mathbb{Z}_l \to CH^1(E_{a\mathbb{F}_{p_i}})_{\mathrm{hom}} \otimes \mathbb{Z}_l$ is an isomorphism. Taking the valuations of the orders of the elements in (6.5) yields either

$$(6.6) \qquad \mu_{a,p_i} - \mathrm{ord}_l(c_1) = \mu_{-3-a,p_i} - \mathrm{ord}_l(c_2) > 0$$

or

$$(6.7) \qquad \mu_{a,p_i} - \mathrm{ord}_l(c_1) \leqq 0 \quad \text{and} \quad \mu_{-3-a,p_i} - \mathrm{ord}_l(c_2) \leqq 0.$$

Now $\gcd(c_1, c_2) = 1$ implies $\mathrm{ord}_l(c_j) = 0$ for some j. By (3), (6.7) does not hold for both $i = 1$ and $i = 2$. Thus (6.6) holds for some i, say $i = 1$. By (4), (6.6) does not hold for $i = 2$, so (6.7) holds for $i = 2$. Suppose $\mu_{a,p_1} \geqq \mu_{-3-a,p_1}$. Then $\mathrm{ord}_l(c_2) = 0$. By (6.7), $\mu_{-3-a,p_2} = 0$. Now (5) implies, $\mu_{a,p_1} = \mu_{-3-a,p_1}$. Thus $\mathrm{ord}_l(c_1) = 0$ and $\mu_{a,p_2} = 0$. This contradicts (3) and shows that (6.5) has no solution in coprime integers when $\mu_{a,p_1} \geqq \mu_{-3-a,p_1}$. If $\mu_{a,p_1} < \mu_{-3-a,p_1}$, (6.7) implies $\mu_{a,p_2} = 0$, which is incompatible with (5). Thus (6.5) and hence (6.3) has no solutions in coprime integers.

Example 6.8. *Let $t \in \mathbb{Q}$ be such that $t - 4/3$ has positive valuation at* 5, 73 *and* 137. *Set $a = (2t^2 + 5)(t^2 - 1)^{-1}$. Then $\mathrm{Griff}^2(E^3_{a\mathbb{Q}})$ has rank at least* 2.

Proof. Let $l = 11$, $q = 5$, $p_1 = 73$, $p_2 = 137$. Then $a \equiv -3 - a \equiv 1 \bmod 5$. One computes easily that $M_5(T) = T^2 - a_5 T + 5$ with $a_5 = 2$. Clearly hypotheses (1) and (2) of 6.2 are satisfied. Our assumptions imply $a \equiv 11 \bmod 73$ and $\bmod 137$. The entry for $t = 4/3$ in Table 9.3 indicates that we have performed an intersection computation in characteristics 73 and 137. In fact, this computation revealed $\mu_{a,73} = \mu_{a,137} = \mu_{-3-a,73} = 1$ and $\mu_{-3-a,137} = 0$. By 6.2, the rank of $\mathrm{Griff}^2(E^3_{a\mathbb{Q}}) \geqq 2$.

Remark 6.9. The isogeny ϕ is closely related to the Atkin-Lehner operator for the universal elliptic curve with a point of order 4. In fact with γ, δ, and ε as in the proof of 6.1

$$E_a : Y^2 = X^3 + \gamma X^2 + \delta X$$

has a point of order 4 at $(-\varepsilon, 2\varepsilon\sqrt{2-a})$. Twisting by $\sqrt{2-a}$ gives an elliptic curve E'_a with a point of order 4 and j-invariant $j = -16\dfrac{(a^2 - 12a - 12)^3}{(a+2)^4(a+1)}$. There is a fine moduli space, $X_1(4)$, for elliptic curves with a point of order 4. The tautological map from this moduli space to the j-line has degree 6. It follows easily that we may identify E'_a with the universal family of elliptic curves with a point of order 4. The Atkin-Lehner involution is $a \to -3 - a$ and corresponds to a degree 4 isogeny $\tilde{w} : E'_a \to E''_{-3-a}$, where E''_{-3-a} is the twist of E'_{-3-a} by $\sqrt{-1}$. Twisting E''_{-3-a} by $\sqrt{-5-a}$ gives E_{-3-a}. Over $\mathbb{Q}(a, \sqrt{2-a}, \sqrt{-a-5})$, $\tilde{w}$ gives an isogeny $E_a \to E_{-3-a}$, which is in fact already defined over $\mathbb{Q}(a, \sqrt{(a-2)(a+5)})$.

7. Computation of the leading term of the L-series

The L-series $L_E(s)$ of an elliptic curve E over $\mathbb{Q}$ is defined by an Euler product

$$L_E(s) := \prod_p L_{E,p}(p^{-s})^{-1}.$$

If E has good reduction at p then $L_{E,p}(x)$ is a quadratic polynomial determined by the number of points of E over $\mathbb{F}_p$:

$$L_{E,p}(x) = 1 - a_{E,p}x + px^2 = (1 - \alpha_p x)(1 - \beta_p x), \quad a_{E,p} = 1 - |E(\mathbb{F}_p)| + p.$$

If E has additive reduction at p then $L_{E,p}(x) = 1$, and if E has multiplicative reduction at p then $L_{E,p} = 1 \pm x$. It is conjectured (and now, by results of Wiles, Taylor, and Diamond, known for many curves) that there is a positive integer N_E such that the function

$$\Lambda_E(s) = N_E^{s/2}(2\pi)^{-s}\Gamma(s)L_E(s)$$

is an entire function on the complex plane and satisfies a functional equation

$$\Lambda_E(2-s) = w_E \Lambda_E(s)$$

for some $w_E \in \{\pm 1\}$.

We are interested in the "symmetric cube" of the L-series of E, which we will denote

$$(7.1)\qquad L(s) := L(\mathrm{Sym}^3 H^1(E_{\bar{\mathbb{Q}}}), s) = L(H^3(M), s) = \prod_p L_p(p^{-s})^{-1} = \sum_{n=1}^{\infty} \frac{a_n}{n^s}.$$

In order to render this computable, we need to specify the Euler factors $L_p(x)$. It turns out that this can be done easily in terms of the α_p and β_p above. If E has good reduction at p then $L_p(x)$ is the quartic polynomial

$$(1-\alpha_p^3 x)(1-\alpha_p^2\beta_p x)(1-\alpha_p\beta_p^2 x)(1-\beta_p^3 x).$$

If the reduction of E at p is multiplicative then $L_p(x)$ is equal to $L_{E,p}(x)$. If the reduction of E at p is additive and is not of type IV or IV* (in the standard Néron-Tate notation) then $L_p(x) = 1$. For the two exceptional types of reduction $L_p(x)$ is a quadratic polynomial that can be computed locally by considering a larger field over which the curve has good reduction; we omit the details since our curves E_a never have that type of reduction.

Using this description of the Euler product for $L(s)$ it isn't too hard to see that $a_n = O(n^{3/2+\varepsilon})$ so that $L(s)$ converges for $\mathrm{Re}(s) > 5/2$. It is conjectured that $L(s)$ can be extended to an entire function defined on the complex plane, satisfying a functional equation. If the level N_E of the elliptic curve is square-free (so that, in particular, by Wiles/Taylor, the curve is modular) then Gross and Kudla, building on earlier work of Garrett, Piatetski-Shapiro, and Rallis, proved this (see [Gr-Ku]).

The shape of the conjectured functional equation is as follows. There is a positive integer N, called the conductor of $\mathrm{Sym}^3 H^1(E_{\bar{\mathbb{Q}}})$, such that

$$(7.2)\qquad \Lambda(s) := N^{s/2}(2\pi)^{-2s}\Gamma(s)\Gamma(s-1)L(s)$$

is an entire function that satisfies

$$(7.3)\qquad \Lambda(s) = w\Lambda(4-s), \quad w \in \{\pm 1\}.$$

The conductor N is closely related to the familiar conductor N_E of the elliptic curve; for instance, in the case in which all primes have multiplicative reduction $N = N_E^3$. In §8 we will consider these conductors more generally, and in particular will give a succinct formula in the case of our curves $E = E_a$.

As one easily verifies (see for instance most of the examples in the tables at the end of this paper), our curves E_a in general do *not* have square-free conductor. Hence the results of Gross and Kudla do not apply. However, we do know that the curves E_a are modular, as the following result explains.

Proposition 7.4. *For $a \in \mathbb{Q}$ with $a \neq -1, \pm 2$ the elliptic curve E_a given by*

$$y^2 = (a^2-4)x^3 + (2a^2-4a)x^2 + (a^2-4)x$$

is modular.

Proof. This follows using the celebrated results of Wiles and Taylor [Wi], [T-W]. As is explained in the proof of Theorem 6.1 above, the curve E_a is isogenous (over $\mathbb{Q}$ in our case) to an elliptic curve which has all its points of order 2 rational. As was first noted by Rubin and Silverberg, a quadratic twist of such a curve has good or multiplicative reduction at every odd prime, so in particular at the primes 3 and 5. An extension of the result of Wiles obtained by Diamond then implies that the curve is modular. In fact, modularity can also be concluded directly from Wiles' paper, as is explained in [D-K].

Remark 7.5. It is not hard to compute that, for instance when $a \in \mathbb{Z}$ such that $a \neq -1$ and $a \equiv -1 \bmod 16$, the curve $E_a/\mathbb{Q}$ is, up to possibly a quadratic twist, a curve with square-free conductor. Hence knowledge on how the symmetric cube L-series behave under quadratic twists would suffice to conclude the functional equation in such cases.

In the remainder of this section we summarize the methods that we used to compute the order of vanishing of, and the leading term of the Taylor series of, $L(s)$ at the central point $s = 2$.

Convergent expansion for $L(2)$. We assume throughout that $L(s)$ has an analytic continuation and that it satisfies the functional equation (7.3). This implies that there are "rapidly converging" expressions for the leading nonzero term of the Taylor series of $\Lambda(s)$ at any complex number s. These formulas take a simpler form at the central point $s = 2$, and for simplicity we start by stating the formula for $L(2)$.

For positive real x let

$$F(x) = 2x K_2(2\sqrt{x})$$

where $K_2(x)$ is the usual K-Bessel function. Standard power series and asymptotic expansions for the Bessel function imply that $F(0) = 1$ and $F(x) = O(\sqrt{x}e^{-2\sqrt{x}})$ as x goes to infinity. We now define a function $S(t)$ in which the terms of the infinite series $L(2)$ have been multiplied by a "convergence factor" defined in terms of the special function $F(x)$:

$$S(t) = \sum_{n=1}^{\infty} \frac{a_n}{n^2} F\left(\frac{16\pi^2 nt}{\sqrt{N}}\right), \quad t \in \mathbb{R}^+.$$

The asymptotics for $F(t)$ imply that this series converges for all positive t. Note also that formally $S(0)$ is just $L(2)$, though we have no reason to expect that the infinite series for $L(s)$ converges at $s = 2$.

With this notation, a special case of our next theorem asserts that for any positive t

$$L(2) = S(t) + wS(t^{-1}). \tag{7.6}$$

The general idea of expansions like this can be traced back (at least) to Hecke, and has been used in a number of theoretical and computational contexts, see [Fr] for a fairly general description and, e.g., [Lo] for a recent example. We will soon prove a somewhat more general statement; see [Fr] for another proof.

In practice one can hope to use (7.6) to "compute" $L(2)$ even if the conjectured analytic continuation and functional equation are not known to hold, and the precise values of N or w are unknown. The primes dividing N are the same as the primes dividing the conductor N_E of the curve, and the exponent on primes larger than 3 is easily determined, so we have a fair amount of information about N. Thus we can take a good guess at N and for several values of t we compute both $S(t)$ and $S(t^{-1})$. Each value of t gives a linear equation for w and $L(2)$, and we get an overdetermined system of linear equations if we have tried more than two values of t. If the candidate N is not the true value, then the system of equations for w will be inconsistent. In this case we try another guess for N. If the equations are consistent and w is equal to ± 1 (to within a small error), then we have strong evidence that N and w have been correctly chosen and that the functional equation (7.3) does indeed hold. Notice that one needs to guess N, but that w is determined by the computation.

The formula (7.6) is obviously easiest to compute for $t = 1$, and in general it is convenient to not let t stray too far away from 1, so that $S(t)$ and $S(t^{-1})$ require roughly the same number of terms. We ended up, for instance, using $t = 1, 1.2, 1.4, 1.6$ to check the correctness of the functional equation, and the conductor, by the above method.

We applied this process to a number of elliptic curves E_a of the form (4.1) before we noticed that the conductor always obeyed the following simple formula:

$$N = N_{\mathrm{mult}}^3 N_{\mathrm{add}}^2 , \tag{7.7}$$

where $N_{E_a} = N_{\mathrm{mult}} N_{\mathrm{add}}$ and N_{mult} (resp. N_{add}) is the product of the prime powers corresponding to primes of multiplicative (resp. additive) reduction for E_a. In the next section we will in fact *prove* a general formula for N which reduces to (7.7) for elliptic curves of the form (4.1). We do not know a formula for w, unless E has no places of additive reduction, in which case w is equal to the sign w_E in the functional equation for E. In fact, the computation of the value of the L-series $L_E(s)$ of E at $s = 1$ is a much simpler version of the above computation (Bessel functions are replaced by e^{-x}, and many fewer terms are needed for convergence). The coefficients a_n of $L(s)$ can be described in terms of the coefficients of $L_E(s)$, so we checked the correctness of the $a_{E,p}$ by first computing $L_E(1)$. As a byproduct we computed w_E. When E has multiplicative reduction at all primes we could also verify that (the empirical) values of w and w_E were equal.

Leading Taylor series terms. Set $c = \sqrt{N}/16\pi^2$ so that we are assuming

$$\Lambda(s) = c^s \Gamma(s) \Gamma(s-1) L(s) = w \Lambda(4-s)$$

as above. We will suppose that $L(s)$ vanishes to order at least m at the central points $s = 2$.

Let b be a positive real number and let

$$F_m(x) := \frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} \frac{\Gamma(s+2)\Gamma(s)}{s^m} x^{-s} ds .$$

The integrand vanishes rapidly in vertical directions, and the fact that the integrand is entire in the right half plane implies that the vertical line of integration can be shifted at will, so that $F_m(x)$ is independent of the choice of b.

The usual Mellin-Barnes integral representation for $K_2(x)$ can be used to show that the earlier function $F(x)$ is just $F_0(x)$. By differentiating the contour integral definition of $F_m(x)$ one finds that $F_m'(x) = -x^{-1}F_{m-1}(x)$ so that the function $F_m(x)$ is, very roughly, an iterated integral of $K_2(x)$.

Apply Cauchy's Theorem to the function $\Lambda(2+s)/t^2 s^{m+1}$ on a very tall rectangle with vertical sides $\mathrm{Re}(s) = b$ and $\mathrm{Re}(s) = -b$. Since $L(s)$ and therefore $\Lambda(s)$ vanish to order at least m the residue at $s = 0$ is easy to compute and we have, in the limit of an infinitely tall rectangle,

$$\Lambda^{(m)}(2)/m! = c^2 L^{(m)}(2)/m! = \frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} \frac{\Lambda(2+s)t^{-s}}{s^{m+1}}\,ds - \frac{1}{2\pi i}\int_{-b-i\infty}^{-b+i\infty} \frac{\Lambda(2+s)t^{-s}}{s^{m+1}}\,ds$$

where t is an arbitrary positive real number. In the second integral replace s by $-s$ and use the functional equation for $\Lambda(s)$ to get

$$c^2 L^{(m)}(2)/m! = \frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} \frac{\Lambda(2+s)t^{-s}}{s^{m+1}}\,ds + \frac{(-1)^m w}{2\pi i}\int_{b-i\infty}^{b+i\infty} \frac{\Lambda(2+s)t^{s}}{s^{m+1}}\,ds\,.$$

The second integral is now the same as the first except that t has been replaced by t^{-1}, so we will confine our attention to the first integral

$$I_1 = \frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} \frac{c^{2+s}\Gamma(s+2)\Gamma(s+1)L(2+s)t^{-s}}{s^{m+1}}\,ds\,.$$

By substituting $L(s) = \sum a_n/n^s$ and rearranging we get

$$I_1 = c^2 \sum_{n=1}^{\infty} \frac{a_n}{n^2}\frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} \frac{\Gamma(s+2)\Gamma(s)(nt/c)^{-s}}{s^m}\,dx\,.$$

Recalling the definition of $F_m(x)$ we find that we have proved the following theorem.

Theorem 7.8. *With the above notation,*

(7.9) $$L^{(m)}(2) = S_m(t) + w(-1)^m S_m(t^{-1})\,,$$

where

$$S_m(t) = m! \sum_{n=1}^{\infty} \frac{a_n}{n^2} F_m\left(\frac{16\pi^2 nt}{\sqrt{N}}\right).$$

Using this formula, the empirical order of vanishing can be computed quickly (at least if the conductor isn't too large); in fact the series converges slightly more quickly as m increases. In all of our computations there was never any uncertainty on the order of vanishing: all values that we judged to be non-zero were at least .3 in absolute value (and also provably non-zero) and all values that we judged to be 0 were zero to as many decimal places as we cared to accurately compute them. For some of the smaller conductors the series was carried far enough out that we could prove that the value was less than 10^{-14} in absolute value. For larger conductors the series in (7.9) converge more slowly. We finally settled on $\varepsilon = 10^{-6}$ as the error tolerance for our computations; this represented a compromise between the desire to carry the computations through for a large number of curves, and the desire to maximize our subjective probability that the L-series values less than ε were actually equal to 0.

Computing the L-series coefficients. The desired accuracy, 10^{-6}, of the computation and the size of the conductor N (often above 10^{10}) required that the series be summed to occasionally as many as several million terms. A profile of an early version of the program showed that as much as 70% of the entire computing time was consumed by the calculation of the coefficient $a_{E,p}$ of the elliptic curve itself. The initial algorithm was an optimized version of a straightforward $O(p)$ algorithm. This was replaced with a version of the baby-step giant-step algorithm [Co] which cut the running time; we found that the crossover between the naive algorithm and the baby-step giant-step algorithm was around $p \approx 1000$.

Once the $a_{p,E}$ were in hand, the evaluation of the coefficients of the polynomials $L_p(x)$ was easy, and the computation of the a_n was also quick but somewhat intricate. Although it probably didn't really matter (given the amount of memory available on large workstations nowadays and the rapidity of any sensible algorithm) we implemented a generalization of the parsimonious algorithm in [BuGr], p. 27 or [Cr], p. 29 which saves time and (especially) space. As in [BuGr], the basic idea is to efficiently traverse the tree whose vertices are integers less than or equal to (say) 10^6 with edges joining n and np for primes p.

Computing $F_m(x)$. The contour integral expression for $F_m(x)$ allows one to obtain a power series for $F_m(x)$ by the standard device of successively shifting the line of integration to the left. The poles at the negative integers are simple, and the contribution of those residues can be readily computed. The poles at $s = 0$ are a bit messy to work out by hand for $m > 1$, but they can be handled by a symbolic algebra package which supports the calculation of residues. The result is

$$F_m(x) = r_0 + (-1)^{m+1} x + (-1)^m \sum_{k=2}^{\infty} \frac{b_k x^k}{k^m k!(k-2)!} \tag{7.10}$$

where

$$r_0 = \mathrm{Res}_{s=0}\, \Gamma(s)^2 s^{1-m} (1+s) e^{-s\log(x)}$$

and

$$b_k = H_k + H_{k-2} - 2\gamma - \log(x) + m/k\,.$$

Here γ is Euler's constant, and the $H_k = 1 + 1/2 + \cdots + 1/k$ are the harmonic numbers (and $H_0 = 0$). This series converges for all x although for large x, say $x > 10$, the series becomes numerically ill-conditioned since the terms grow considerably in value before they decrease.

For large x there is an asymptotic expansion

$$F_m(x) \simeq \sqrt{\pi}\, x^{3/4-m/2} e^{-2\sqrt{x}} \left(1 + \sum_{k=1}^{\infty} \frac{c_{k,m}}{x^{k/2}}\right) \tag{7.11}$$

where the coefficients $c_{k,m}$ satisfy a 3-term recurrence relation. Unfortunately, this did not give an accurate and fast enough evaluation for $F_m(x)$ in the range (roughly) $10 < x < 100$.

There is a considerable literature on evaluating special functions of this type ([Lu]). After trying various techniques, we finally settled on using the above power series to compute $F_m(x)$ for $x < 4$ and using a Chebyshev expansion for $x > 4$.

The Chebyshev scheme that seemed to give the most accuracy was as follows. For the sake of getting a roughly constant function we divide $F_m(x)$ by its leading asymptotic term to get a new function $f_m(x)$ defined by

$$f_m(x) := F_m(x)\, \pi^{-1/2} x^{m/2-3/4} e^{2\sqrt{x}}.$$

Then we considered the function

$$g_m(x) := f_m(16/(x+1)^2),$$

defined on $[-1,1]$, and computed about 50 very high precision values of $g_m(x)$ (using the power series expansion for $F_m(x)$) and used these to compute the coefficients of the Chebyshev expansion. The error of each step can be precisely bounded. There are a number of other ways to compute these coefficients ([Lu]), including complicated closed-form recursions, but it seemed easiest to rely on a handful of high precision calculations performed using arbitrary precision arithmetic.

In addition to the power series and Chebyshev expansions, we implemented asymptotic expansions, rational function approximations derived from the power series and asymptotic expansions, and numerical solutions to differential equations. This variety of algorithms provided useful checks for the calculations. For the 6-digit accuracy that we settled on, the power series and Chebyshev expansions were fine, but the question of the asymptotically best methods, or the best methods if many more coefficients were wanted, is interesting and deserves further study.

In any event, this technique produced rapidly decreasing d_k^m such that

$$g_m(x) = \sum_{k=0}^{\infty} d_k^m T_k(x)$$

where $T_k(x)$ is the usual Chebyshev polynomial. In practice d_k^m was less than 10^{-17} for $k \geqq 15$ so we computed (at most) 15 terms of the expansion. Note that our computation needed absolute, rather than relative, accuracy of 10^{-6} so for large x the leading term in $F_m(x)$ is already small and even fewer terms of the Chebyshev expansion are required.

8. The conductor

Given any smooth projective variety, $W/\mathbb{Q}$, and a positive integer m, there is conjecturally a conductor associated to the cohomology $H^m(W)$ which may be expressed as a product of local factors [Ser2], 4.1:

$$N = \prod_p p^{f(p)}. \tag{8.1}$$

The term $f(p)$ is defined by fixing an auxiliary prime $l \neq p$ and considering the representation

$$\varrho : I := \mathrm{Gal}(\bar{\mathbb{Q}}_p/K) \to GL\big(H^m(W_{\bar{\mathbb{Q}}_p}, \mathbb{Q}_l)\big), \tag{8.2}$$

where K is the maximal unramified extension of $\mathbb{Q}_p$. Then $f(p) = \varepsilon(p) + \delta(p)$, where

$$\varepsilon(p) := \dim H^m(W_{\bar{\mathbb{Q}}_p}, \mathbb{Q}_l) - \dim H^m(W_{\bar{\mathbb{Q}}_p}, \mathbb{Q}_l)^I.$$

To define $\delta(p)$ one constructs in a natural way an I-stable filtration $V_\bullet$ on $H^m(W_{\bar{\mathbb{Q}}_p}, \mathbb{Q}_l)$ with the property that I acts on the associated graded vector space through a finite quotient group, J:

$$\bar{\varrho} : J \to GL(\mathrm{gr}(V_\bullet)).$$

Identify J with the Galois group of a finite extension L/K. Let $b : J \to \mathbb{Z}$ denote the Swan character and define

$$\delta(p) = \langle \mathrm{Tr}(\bar{\varrho}), b \rangle = \frac{1}{|J|} \sum_{g \in J} \mathrm{Tr}(\bar{\varrho}(g)) \cdot b(g).$$

For a more complete discussion of $\delta(p)$ we refer to [Ser2], 2.1. One conjectures that $\varepsilon(p)$ and $\delta(p)$ are independent of l, in which case the conductor N is well defined. When $m = 1$ and $W = E$ is an elliptic curve this is known to hold [Ogg].

If the variety W is replaced by a motive over $\mathbb{Q}$, then the same approach leads to a conjectural definition of the conductor of the motive in degree m: Simply replace the cohomology of the variety in (8.2) with the cohomology of the motive [Mu], 1.4. Given an elliptic curve $E/\mathbb{Q}$, let $M = (E^3, \frac{1}{3}P)$ denote the motive defined in 2.4. Write $\varepsilon_1(p), \delta_1(p), f_1(p)$ (respectively $\varepsilon_3(p), \delta_3(p), f_3(p)$) for the local terms corresponding to $H^1(E)$ (respectively $H^3(M)$). For $i \in \{1, 3\}$ set $N_i := \prod_p p^{f_i(p)}$. The following proposition shows that the conductor N_3 is well defined and describes how to compute it.

Proposition 8.3. *The terms $\varepsilon_3(p), \delta_3(p), f_3(p)$ are independent of the choice of auxiliary prime $l \neq p$. Furthermore we have the following formulas which depend on the reduction type of E at p and, in the case of potential good reduction, on the group J:*

$\varepsilon_3(p) = \varepsilon_1(p) = 0$ *when the reduction at p is good;*

$\varepsilon_3(p) = 3$, $\varepsilon_1(p) = 1$ *when the reduction at p is multiplicative;*

$\varepsilon_3(p) = 4$, $\varepsilon_1(p) = 2$ *in the case of potential multiplicative reduction, or potential good reduction, except when* $J = \mathbb{Z}/3$;

$\varepsilon_3(p) = \varepsilon_1(p) = 2$ *in the case of potential good reduction with* $J = \mathbb{Z}/3$;

$\delta_3(p) = \delta_1(p) = 0$ *whenever* $p \geqq 5$;

$\delta_3(2) = 2 \cdot \delta_1(2)$;

$\delta_3(3) = \delta_1(3)$.

Proof. Set $\mathbb{V}_l = H^1(E_{\bar{\mathbb{Q}}_p}, \mathbb{Q}_l)$. The representation ϱ_3 associated to the third cohomology of the motive M is obtained from the representation

$$\varrho_1 : I \to GL(\mathbb{V}_l)$$

associated to E by composing with the third symmetric power representation

$$GL(\mathbb{V}_l) \to GL(\mathrm{Sym}^3(\mathbb{V}_l)).$$

If E has good reduction at p, then ϱ_1 and ϱ_3 are trivial representations and all invariants $\varepsilon_1(p)$, $\varepsilon_3(p)$, $\delta_1(p)$, and $\delta_3(p)$ are zero.

If E has multiplicative reduction at p, then E_K is isomorphic to a Tate curve. Standard arguments show that there is a non-trivial I-stable filtration $V_\bullet : 0 = V_0 \subset V_1 \subset V_2 = \mathbb{V}_l$ such that I acts trivially on $\mathrm{gr}(V_\bullet)$, while it acts non-trivially on $\mathbb{V}_l$. Write $\mathrm{Sym}^3 V_\bullet$ for the induced filtration

$$\mathrm{Sym}^3 V_\bullet : 0 \subset V_1^3 \subset V_1^2 \cdot V_2 \subset V_1 \cdot V_2^2 \subset \mathrm{Sym}^3(\mathbb{V}_l).$$

Then $V_1^3 = \mathrm{Sym}^3(\mathbb{V}_l)^I$ and I acts trivially on $\mathrm{gr}(\mathrm{Sym}^3 V_\bullet)$. These filtrations may be used in the definition of $\delta(p)$ outlined above. Thus,

$$\varepsilon_1(p) = 1, \quad \varepsilon_3(p) = 3, \quad \delta_1(p) = 0, \quad \delta_3(p) = 0.$$

If E has potential multiplicative reduction at p, it becomes isomorphic to a Tate curve over some quadratic extension L/K. The previous discussion gives a $\mathrm{Gal}(\bar{\mathbb{Q}}_p/L)$-stable filtration $V_\bullet$, which is in fact I-stable, since $\mathrm{Gal}(\bar{\mathbb{Q}}_p/L)$ is a normal subgroup. The quotient $J = I/\mathrm{Gal}(\bar{\mathbb{Q}}_p/L) \simeq \mathbb{Z}/2$ acts non-trivially on $\mathrm{gr}(V_\bullet)$, since the Tate module has no non-zero invariants when there is additive reduction. The determinant applied to ϱ_1 is identically 1, since ϱ_1 respects the Weil pairing. It follows that a generator of J acts by $-\mathrm{Id}$ on $\mathrm{gr}(V_\bullet)$. Furthermore, as J-modules

$$\mathrm{gr}(\mathrm{Sym}^3 V_\bullet) \simeq \mathrm{gr}(V_\bullet) \oplus \mathrm{gr}(V_\bullet).$$

Thus

$$\varepsilon_1(p) = 2, \quad \varepsilon_3(p) = 4, \quad \delta_3(p) = 2\delta_1(p).$$

When $p > 2$, L/K is tamely ramified and the Swan character is identically 0. In this case $\delta_1(p) = 0$.

Finally consider the case of potential good reduction. By the criterion of Néron, Ogg, and Shafarevich [Si], VII. 7.1, there is a unique minimal extension L/K over which E acquires good reduction. L/K is finite and Galois. Furthermore, $J := \mathrm{Gal}(L/K)$ acts on the Néron model of E_L and acts faithfully on the Tate module. As the Néron model is an abelian scheme, the Tate module of E_L may be identified with the Tate module of the special fiber E_0. It follows that J acts faithfully on E_0. Write $J \subset A := \mathrm{Aut}(E_0)$. It is known that $A \subset SL(2, \mathbb{Z}/3)$ when $p = 2$, $A \subset \mathbb{Z}/3 \rtimes \mathbb{Z}/4$ when $p = 3$, and $A \subset \mathbb{Z}/6$ or $A \subset \mathbb{Z}/4$ when $p \geqq 5$ [Si], Appendix A. This shows that J must belong to a short list of well known groups. It is easy to check that every group on this list has, up to isomorphism, exactly one faithful two dimensional representation with determinant 1 over an algebraically closed field of characteristic 0. Thus there is at most one choice for the representation

$$\bar{\varrho}_1 : J \to GL(\mathbb{V}_l)$$

and its character, $\mathrm{Tr}(\bar{\varrho}_1)$, must take values in $\mathbb{Z}$ independent of l. The same is true for the character of the third symmetric power which is given by the formula, $\mathrm{Tr}(\bar{\varrho}_1)^3 - 4\,\mathrm{Tr}(\bar{\varrho}_1)$. Since the dimension of the invariant subspace may be computed from the character and since one may take the trivial filtration in the definitions of $\delta_1(p)$ and $\delta_3(p)$, we see that $\varepsilon_1(p)$, $\varepsilon_3(p)$, $\delta_1(p)$, $\delta_3(p)$ are independent of l.

It is easy to run through the list of possible J's to deduce that $\varepsilon_1(p) = 2$ and $\varepsilon_3(p) = 4$ except when $J \simeq \mathbb{Z}/3$, in which case $\varepsilon_1(p) = \varepsilon_3(p) = 2$. Write $J_p \subset J$ for the (unique) Sylow-p-subgroup (or the empty set if $\gcd(p, |J|) = 1$) and recall that $b|_{J-J_p} \equiv 0$ [Ser 2], 2.1. If $p = 3$ and $J_p \neq \emptyset$, then $J_p \simeq \mathbb{Z}/3$ and

$$\mathrm{Sym}^3(\mathbb{V}_l) \simeq \mathbb{V}_l \oplus 1 \oplus 1$$

as J_p-modules. In this case $\delta_3(p) = \delta_1(p)$. If $p = 2$ and $J_p \neq \emptyset$, then J_p is a subgroup of the quaternions. Independent of the choice of subgroup and the choice of l we find

$$\mathrm{Sym}^3(\mathbb{V}_l) \simeq \mathbb{V}_l \oplus \mathbb{V}_l$$

as J_p-modules, so $\delta_3(2) = 2\delta_1(2)$, which completes the proof of the proposition.

Given an elliptic curve E, factor the conductor $N_E = N_{\mathrm{mult}} N_{\mathrm{add}}$, where N_{mult} is divisible only by the primes of multiplicative reduction and N_{add} only by the primes of additive reduction.

Corollary 8.4. *If E has tame reduction at 3 and $J \simeq \mathbb{Z}/3$ does not occur at places of potential good reduction, then*

$$N = N_{\mathrm{mult}}^3 N_{\mathrm{add}}^2 .$$

Proof. Immediate from 8.3.

The next lemma shows that the corollary applies to all curves E_a defined by (4.1).

Lemma 8.5. *If E_a has additive reduction for $p \geqq 3$, then the reduction is of Kodaira type I_v^* for some $v \geqq 0$. Furthermore, the reduction at 3 is tame and potential good reduction at 2 with $J \simeq \mathbb{Z}/3$ does not occur.*

Proof. The elliptic curve defined by equation (4.1) has a $\mathbb{Q}$-rational two torsion point at $(x:y:z)=(0:0:1)$ which may be used to restrict the types of bad reduction. Thus E has no fibers in characteristic $p>2$ of Kodaira type IV or IV*, since the Néron models of such reductions have no non-trivial two torsion [Si], p. 359. This rules out potential good reduction at primes $p \geqq 5$ with $J \simeq \mathbb{Z}/3$.

To complete the proof, recall from 6.9 that there is a quadratic extension of $\mathbb{Q}$ over which E acquires a rational point of order 4. This allows us to rule out reduction types for which the Néron model over a quadratic extension does not have a 4-torsion point. The lemma follows from the list of singular fibers in Néron models [Si], p. 359.

9. Results

For convenience, we divide the curves that we considered into three categories:

(1) tabulated curves E of small conductor;

(2) curves E_a for which the isogeny $\phi: E_a \to E_{-3-a}$ of § 6 is not defined over $\mathbb{Q}$;

(3) curves E_a for which the isogeny $\phi: E_a \to E_{-3-a}$ of § 6 is defined over $\mathbb{Q}$.

Curves of the first type are easy to find: the Antwerp table [MF] contains all curves with conductor less than 200, and the more recent table [Cr] goes much further. Curves of the second type with small enough N were easy to produce, and we could have found more if we liked. Curves of the third type tended to have significantly larger conductors, and were considerably harder to come by. A number of reasonable parameter values produced conductors that were far too large for us to carry out the L-series computations (given our self-imposed bound on the error of 10^{-6}).

In Table 9.1 the curves of the first type are given. They are a small sampling chosen from the Antwerp tables [MF] to make sure that our programs worked properly, to investigate hypotheses on the symmetric cube conductor, and to insure that it was possible for the symmetric cube L-series to be non-zero at the central point. We have no information concerning $\mathrm{Griff}^2(E^3_{\bar{\mathbb{Q}}})$ for these curves. The columns contain:

(1) the empirical order **Ord** of vanishing of the symmetric cube L-series at the central point $s=2$;

(2) the name of the curve as in the Antwerp tables;

(3) the conductor, N_E, of the curve, in factored form;

(4) the Kodaira symbol for the reduction of the curve at the successive primes dividing the conductor; if the curve has multiplicative reduction R then $R_{:s}$, resp. $R_{:n}$ indicates that the reduction is split, resp. nonsplit;

(5) the conductor, N, of the symmetric cube L-series;

(6) the leading term $L^{(\mathbf{Ord})}(2)$ of the L-series at $s=2$.

The conductor N of the symmetric cube is included in this table because it cannot be inferred from N_E as easily as in the case of the curves E_a (cf. 8.4).

Table 9.1. Curves of small conductor.

Ord	curve	N_E	reduction	N	$L^{(\mathbf{Ord})}(2)$
2	73*B*	73	$I_{2:s}$	73^3	6.23132
2	109*A*	109	$I_{1:s}$	109^3	7.28745
2	139*A*	139	$I_{1:s}$	139^3	10.47970
1	37*A*	37	$I_{1:n}$	37^3	2.46864
1	121*D*	11^2	III	11^4	2.29176
1	162*A*	$2 \cdot 3^4$	$I_{1:n}$, IV	$2^3 \cdot 3^6$	1.47250
1	162*B*	$2 \cdot 3^4$	$I_{3:n}$, IV*	$2^3 \cdot 3^6$	1.47250
0	11*A*	11	$I_{1:s}$	11^3	1.14023
0	50*E*	$2 \cdot 5^2$	$I_{1:n}$, IV	$2^3 \cdot 5^2$	0.36156
0	54*B*	$2 \cdot 3^3$	$I_{9:s}$, IV	$2^3 \cdot 3^5$	1.32432
0	67*A*	67	$I_{1:s}$	67^3	3.03007
0	88*A*	$2^3 \cdot 11$	$I_1^*, I_{1:n}$	$2^6 \cdot 11^3$	0.83738
0	128*B*	2^7	I_2^*	2^{14}	1.79729
0	135*B*	$3^3 \cdot 5$	II*, $I_{2:s}$	$3^5 \cdot 5^3$	1.95428
0	174*A*	$2 \cdot 3 \cdot 29$	$I_{4:n}, I_{1:s}, I_{1:n}$	$2^3 \cdot 3^3 \cdot 29^3$	1.88156

Table 9.2 contains curves E_a for which the isogeny $\phi : E_a \to E_{-3-a}$ of § 6 is not defined over $\mathbb{Q}$. For each value of the parameter a which we tested we were able to find primes l, p, and q so that the hypotheses of Theorem 4.8 held. Thus $P_* \Xi$ gives an element of infinite order in the Griffiths group. The lower bound for the rank (**Rnk** in the notation of 0.4) is in each case 1. The columns contain:

(1) the empirical order of vanishing of the symmetric cube L-series at the central point $s = 2$;

(2) the value of the parameter a;

(3) the conductor of the curve, in factored form;

(4) the Kodaira symbol for the reduction of the curve at the successive primes dividing the conductor;

(5) the leading term of the L-series.

(6) two prime numbers p, ℓ which satisfy the hypotheses of Theorem 4.8. In fact, we verified in all cases that the cycle defined in Section 4 has infinite order. The additional prime q which is needed to deduce this from Theorem 4.8 is not listed, since in virtually all cases the smallest possible prime of good reduction different from p and ℓ works.

Table 9.2. Curves with Rnk equal to one.

Ord	a	N_E	reduction	$L^{(\mathrm{Ord})}(2)$	p, ℓ
3	7/3	$2^5 \cdot 3 \cdot 5 \cdot 13$	$I_0^*, I_{1:n}, I_{1:n}, I_{4:s}$	609.95270	29, 7
3	−18/5	$2^4 \cdot 5 \cdot 7^2 \cdot 13$	$I_0^*, I_{1:s}, I_0^*, I_{1:s}$	965.32579	37, 11
3	−16	$2^6 \cdot 3 \cdot 5 \cdot 7$	$II, I_{1:s}, I_{1:n}, I_{4:s}$	607.71508	13, 5
3	−11/2	$2^5 \cdot 3^2 \cdot 5^2 \cdot 7$	$I_0^*, I_2^*, I_0^*, I_{4:s}$	874.51073	37, 11
3	−11	$3^5 \cdot 3 \cdot 5 \cdot 13^2$	$I_0^*, I_{8:s}, I_{1:n}, I_0^*$	953.96251	19, 5
2	8/3	$2^6 \cdot 3 \cdot 7 \cdot 11$	$II, I_{1:n}, I_{4:s}, I_{1:n}$	54.16555	17, 5
2	8	$2^6 \cdot 3^2 \cdot 5$	$II, I_2^*, I_{4:s}$	43.80381	19, 5
2	6/5	$5 \cdot 11$	$I_{1:s}, I_{1:n}$	6.92373	23, 5
2	6	$2^4 \cdot 7$	$I_0^*, I_{1:s}$	8.54671	19, 7
2	5/2	$2^5 \cdot 3 \cdot 7$	$I_0^*, I_{8:s}, I_{1:s}$	30.87214	43, 13
2	47	$3 \cdot 5^2 \cdot 7$	$I_{1:n}, I_0^*, I_{8:s}$	14.38118	73, 17
2	4	$2^6 \cdot 3 \cdot 5$	$II, I_{4:s}, I_{1:n}$	42.30223	29, 7
2	26/3	$2 \cdot 3 \cdot 5^2 \cdot 29$	$I_{4:s}, I_{1:n}, I_0^*, I_{1:n}$	80.66035	47, 5
2	26	$2^6 \cdot 3^2 \cdot 7$	$I_0^*, I_3^*, I_{4:s}$	35.05693	29, 5
2	2/5	$2^6 \cdot 3 \cdot 5 \cdot 7$	$I_0^*, I_{4:s}, I_{1:s}, I_{1:s}$	133.68636	17, 5
2	18/7	$2 \cdot 5 \cdot 7$	$I_{4:s}, I_{2:n}, I_{1:n}$	16.36461	13, 5
2	14/3	$2^6 \cdot 3 \cdot 5 \cdot 17$	$I_0^*, I_{1:n}, I_{4:s}, I_{1:n}$	88.34283	37, 5
2	15	$13^2 \cdot 17$	$I_0^*, I_{4:s}$	8.92900	23, 5
2	10	$2^6 \cdot 3 \cdot 11$	$I_0^*, I_{4:s}, I_{1:n}$	34.50741	13, 5
2	−9/8	$2^4 \cdot 7$	$I_3^*, I_{4:s}$	8.54671	19, 7
2	−8/5	$2^6 \cdot 3 \cdot 5$	$II, I_{1:n}, I_{1:s}$	38.87475	17, 5
2	−7/3	$2^3 \cdot 3 \cdot 13^2$	$III^*, I_{1:n}, I_0^*$	33.17891	31, 5
2	−34	$2 \cdot 3 \cdot 11$	$I_{4:s}, I_{1:n}, I_{1:n}$	14.03100	13, 5
2	−26	$2^4 \cdot 3 \cdot 5 \cdot 7^2$	$I_0^*, I_{4:s}, I_{2:n}, I_0^*$	76.31136	31, 5
2	−2/7	$2^4 \cdot 3 \cdot 5 \cdot 7$	$II, I_{4:s}, I_{1:n}, I_{1:n}$	56.47924	17, 5
2	−19/3	$3 \cdot 13$	$I_{1:n}, I_{4:s}$	4.54546	19, 5
2	−14/5	$2^6 \cdot 3^2 \cdot 5$	$I_0^*, I_2^*, I_{1:s}$	44.91142	13, 5

2	−13/7	$2^5 \cdot 3^2 \cdot 7$	$I_0^*, I_1^*, I_{1:n}$	24.31534	29, 7
2	−10/3	$2^4 \cdot 3 \cdot 7$	$II, I_{1:n}, I_{1:s}$	21.37720	23, 5
2	−11/8	$2^4 \cdot 3^2 \cdot 5$	$I_3^*, I_1^*, I_{4:s}$	24.99702	13, 5
2	−1/4	$2^4 \cdot 3 \cdot 7$	$I_2^*, I_{1:n}, I_{4:s}$	21.37720	23, 5
2	−2/5	$2^4 \cdot 3^2 \cdot 5$	$I_0^*, I_1^*, I_{1:s}$	24.99702	13, 5
1	7/2	$2^5 \cdot 3^2 \cdot 11$	$I_0^*, I_2^*, I_{4:s}$	7.37280	13, 5
1	7	$2^3 \cdot 3 \cdot 5^2$	$II^*, I_{8:s}, I_0^*$	6.75735	17, 5
1	5/3	$2^3 \cdot 3 \cdot 11$	$II^*, I_{1:n}, I_{4:s}$	3.75056	19, 5
1	5	$2^5 \cdot 3^2 \cdot 7$	$I_0^*, I_1^*, I_{4:s}$	5.50251	13, 5
1	4/5	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	$II, I_2^*, I_{1:s}, I_{4:s}$	9.81988	19, 5
1	30	$2 \cdot 7^2 \cdot 31$	$I_{4:s}, I_0^*, I_{1:s}$	12.67917	29, 7
1	23	$2^3 \cdot 3^2 \cdot 5 \cdot 7^2$	$II^*, I_1^*, I_{8:s}, I_0^*$	6.71263	13, 5
1	22/5	$2 \cdot 3^2 \cdot 5$	$I_{4:s}, I_3^*, I_{1:s}$	4.44140	101, 5
1	2/3	$2^4 \cdot 3 \cdot 5$	$I_0^*, I_{1:n}, I_{1:n}$	5.47620	13, 5
1	2/7	$3^2 \cdot 7$	$I_2^*, I_{1:n}$	1.43468	41, 11
1	18	$2^4 \cdot 5 \cdot 19$	$II, I_{4:s}, I_{1:s}$	4.92283	13, 5
1	1/3	$2^3 \cdot 3 \cdot 5^2 \cdot 7$	$III^*, I_{1:n}, I_0^*, I_{4:s}$	5.62784	31, 5
1	1/2	$2^5 \cdot 3^2 \cdot 5$	$I_0^*, I_1^*, I_{4:s}$	2.96196	19, 7
1	14	$3^2 \cdot 5$	$I_1^*, I_{1:n}$	1.28488	17, 5
1	1	$2^5 \cdot 3$	$I_0^*, I_{4:s}$	2.63783	29, 7
1	−9/5	$2^3 \cdot 5 \cdot 19^2$	$III^*, I_{1:s}, I_0^*$	3.32994	23, 5
1	−8/3	$2^6 \cdot 3 \cdot 5 \cdot 7^2$	$II, I_{1:n}, I_{1:n}, I_0^*$	3.15527	19, 7
1	−7	$2^5 \cdot 3 \cdot 5$	$I_0^*, I_{1:n}, I_{4:s}$	7.22978	31, 7
1	−6/7	$2^4 \cdot 5^2 \cdot 7$	$I_0^*, I_0^*, I_{1:n}$	4.82130	19, 7
1	−6/5	$2^3 \cdot 5$	$III, I_{1:s}$	1.88137	31, 5
1	−6	$2^6 \cdot 5$	$I_0^*, I_{1:n}$	3.79533	23, 5
1	−5	$2^3 \cdot 3 \cdot 7^2$	$III^*, I_{4:s}, I_0^*$	3.87849	17, 5
1	−5/2	$2^5 \cdot 3$	$I_0^*, I_{1:s}$	2.63783	29, 7
1	−4/7	$2^6 \cdot 3 \cdot 5 \cdot 7$	$II, I_{1:s}, I_{4:s}, I_{1:n}$	17.39415	13, 5
1	−4	$2^6 \cdot 3^2$	II, I_1^*	3.18279	29, 7
1	−25	$2^3 \cdot 3^2 \cdot 23$	$II^*, I_1^*, I_{4:s}$	7.76286	29, 7
1	−22	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	$I_0^*, I_1^*, I_{4:s}, I_{1:s}$	25.41468	17, 5

Ord	a	N_E	reduction	$L^{(\mathrm{Ord})}(2)$	p, ℓ
1	$-2/3$	$2^6 \cdot 3$	$\mathrm{I}_0^*, \mathrm{I}_{1:n}$	3.01850	31, 5
1	$-14/3$	$2^4 \cdot 3 \cdot 5^2 \cdot 11$	$\mathrm{I}_0^*, \mathrm{I}_{1:n}, \mathrm{I}_0^*, \mathrm{I}_{1:n}$	2.82070	19, 5
1	$-11/5$	$2^5 \cdot 3^2 \cdot 5 \cdot 7^2$	$\mathrm{I}_0^*, \mathrm{I}_1^*, \mathrm{I}_{1:s}, \mathrm{I}_0^*$	17.74720	31, 7
1	$-10/7$	$2^6 \cdot 3^2 \cdot 7$	$\mathrm{I}_0^*, \mathrm{I}_1^*, \mathrm{I}_{1:n}$	0.96707	23, 5
1	-10	$2^4 \cdot 3^2$	$\mathrm{I}_0^*, \mathrm{I}_2^*$	2.38506	17, 5
1	3	$2^3 \cdot 5$	$\mathrm{III}^*, \mathrm{I}_{4:s}$	1.88137	31, 5

It should be clear that the data in Table 9.2 is consistent with recurring fantasy 0.2: in all cases, the (numerical) order of vanishing of the L-series is at least 1, which is the lower bound on the rank in the current situation. Note that in many cases the order of vanishing in fact exceeds 1. When this happens, 0.2 predicts that cycles exist which are independent of the one we found. We have not produced a single example of such an independent cycle.

There are two instances in Table 9.2 where the computed order of vanishing is 1, but where we know of two cycles. Namely, it turns out that the curves E_3 and $E_{-6/5}$ are isogenous. A similar isogeny occurs between E_1 and $E_{-5/2}$. Analogous to the construction in Section 6 one can use the isogeny to transport a cycle from one triple product to the other. Since the computed order of vanishing equals 1, recurring fantasy 0.2 predicts in these two cases that the two cycles are dependent. Using Theorem 4.8, we have checked that all cycles involved here have infinite order. Although we have not been able to prove dependence, we did check that (notations as in independence criterion 6.2) $\mu_{3,p} = \mu_{-6/5,p}$ and $\mu_{1,p} = \mu_{-5/2,p}$ for all primes p of good reduction with $5 \leqq p < 200$, and all $\ell > 3$. Hence also this is consistent with 0.2.

Table 9.3 contains information concerning curves E_a for which the isogeny

$$\phi : E_a \to E_{-3-a}$$

of §6 is defined over $\mathbb{Q}$. In each case we were able to apply 6.2 to show that $P_* \Xi$ and $P_* \psi_* \Xi_{-3-a}$ span a rank two subgroup of $\mathrm{Griff}^2(E^3_{a\mathbb{Q}})$. Thus the lower bound **Rnk** on the rank of the Griffiths group is in each case 2. The columns contain:

(1) the empirical order of vanishing of the symmetric cube L-series at the central point $s = 2$;

(2) the value of the parameter t, where $a = (2t^2 + 5)/(t^2 - 1)$;

(3) the conductor of the curve, in factored form;

(4) the Kodaira symbol for the reduction of the curve at the successive primes dividing the conductor;

(5) the leading term of the L-series;

(6) two primes p_1, p_2 which, together with suitable l and q satisfy the conditions of independence criterion 6.2. The primes l, q are not listed since l is one of the primes larger than 3 which divides the order of $E_a(\mathbb{F}_{p_1})$ and is therefore easily found, and a q satisfying 6.2 (1) is always very easily found.

Table 9.3. Curves E_a with Rnk equal to 2.

Ord	t	N_E	reduction	$L^{(\mathrm{Ord})}(2)$	p_1, p_2
4	3/2	$3 \cdot 5 \cdot 7^2 \cdot 43$	$\mathrm{I}_{4:s}, \mathrm{I}_{1:s}, \mathrm{I}_0^*, \mathrm{I}_{1:s}$	5089.76245	13, 23
4	10/3	$3 \cdot 13 \cdot 61$	$\mathrm{I}_{1:s}, \mathrm{I}_{1:s}, \mathrm{I}_{4:s}$	2326.85819	23, 47
3	9/2	$3 \cdot 11 \cdot 37$	$\mathrm{I}_{4:s}, \mathrm{I}_{1:n}, \mathrm{I}_{1:s}$	226.26037	13, 17
3	11/10	$3 \cdot 7 \cdot 109$	$\mathrm{I}_{1:n}, \mathrm{I}_{4:s}, \mathrm{I}_{1:s}$	175.29295	89, 97
3	10/17	$3 \cdot 13 \cdot 181$	$\mathrm{I}_{3:n}, \mathrm{I}_{1:s}, \mathrm{I}_{4:s}$	460.66348	31, 43
2	4/3	$2^3 \cdot 3 \cdot 13$	$\mathrm{III}^*, \mathrm{I}_{1:s}, \mathrm{I}_{4:s}$	26.67519	73, 137
2	1/8	$2^3 \cdot 3 \cdot 7 \cdot 37$	$\mathrm{III}, \mathrm{I}_{2:n}, \mathrm{I}_{4:s}, \mathrm{I}_{1:s}$	63.83898	13, 43
2	5/9	$2^4 \cdot 3 \cdot 7 \cdot 19$	$\mathrm{I}_3^*, \mathrm{I}_{1:s}, \mathrm{I}_{8:s}, \mathrm{I}_{1:s}$	70.32110	13, 23
2	5/2	$3 \cdot 13$	$\mathrm{I}_{1:n}, \mathrm{I}_{1:s}$	4.54546	19, 83
2	6	$3 \cdot 5 \cdot 7$	$\mathrm{I}_{4:s}, \mathrm{I}_{1:s}, \mathrm{I}_{4:n}$	12.14358	13, 37
2	2	$3 \cdot 7^2 \cdot 19$	$\mathrm{I}_{1:n}, \mathrm{I}_0^*, \mathrm{I}_{4:s}$	17.86944	13, 23

Remark 9.4. The tables indicate that the leading term $L^{(\mathrm{Ord})}(2)$ is positive in each case tested. This is predicted by the generalized Riemann hypothesis since $L(s) > 0$ for real s in the domain of convergence of the Euler product ($s > 5/2$). In the case that N_E is square-free, Gross and Kudla prove that $L(2) \geqq 0$.

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