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ON THE NEVANLINNA-PICK INTERPOLATION PROBLEM FOR GENERALIZED STIELTJES FUNCTIONS

D. ALPAY, V. BOLOTNIKOV and A. DIJKSMA

Dedicated to the memory of M. G. Kreĭn

The solutions of the Nevanlinna-Pick interpolation problem for generalized Stieltjes matrix functions are parametrized via a fractional linear transformation over a subset of the class of classical Stieltjes functions. The fractional linear transformation of some of these functions may have a pole in one or more of the interpolation points, hence not all Stieltjes functions can serve as a parameter. The set of excluded parameters is characterized in terms of the two related Pick matrices.

1 Introduction

The objective of this paper is to study the Nevanlinna-Pick interpolation problem in the class of generalized Stieltjes matrix functions. The Nevanlinna-Pick problem for classical Stieltjes functions was considered by M.G. Kreĭn and A.A. Nudelman [8] for the scalar case; the matrix case appeared in [6] and its tangential and two-sided generalizations in [1], [3]. In these problems the two kernels in (1.2) below are nonnegative. Scalar generalized Stieltjes functions have been treated by M.G. Kreĭn and H. Langer in [7] under the assumption that the first kernel in (1.2) below has a finite number of negative squares and the second one is nonnegative. In this paper we consider the matrix case and assume that both kernels have a finite number of negative squares. V. Derkach in [4] studied the moment problem in the same generality.

Throughout this note we work with meromorphic matrix functions, and when S is such a function, we denote the set of points of analyticity by $\rho(S)$.

Definition 1.1 *An $m \times m$ matrix function S belongs to the generalized Stieltjes class $S_{\kappa}^{\tilde{K}}$ if it is meromorphic in $\mathbb{C} \setminus \mathbb{R}$,*

$$S(\bar{z}) = S(z)^*, \quad z \in \mathbb{C}^+ \cap \rho(S), \quad (1.1)$$

and the kernels

$$K_S(z, \omega) := \frac{S(z) - S(\omega)^*}{z - \bar{\omega}}, \quad \tilde{K}_S(z, \omega) := \frac{zS(z) - \bar{\omega}S(\omega)^*}{z - \bar{\omega}} \quad (1.2)$$

have κ and $\tilde{\kappa}$ negative squares on $\mathbb{C}^+ \cap \rho(S)$, respectively; in formulas:

$$\text{sq}_-(K_S) = \kappa, \quad \text{sq}_-(\tilde{K}_S) = \tilde{\kappa}.$$

The first equality means that for every choice of an integer r and of r points $\lambda_1, \dots, \lambda_r \in \mathbb{C}^+ \cap \rho(S)$, the Hermitian $mr \times mr$ matrix $(K_S(\lambda_j, \lambda_k))_{k,j=1}^r$ has at most κ and for at least one such choice it has exactly κ negative eigenvalues counting multiplicities. The second equality is defined in the same way with K_S replaced by \tilde{K}_S .

We denote the classical Stieltjes class \mathcal{S}_0^0 by \mathcal{S} ; it consists of all functions S such that both kernels in (1.2) are nonnegative, that is, have no negative squares. It is well known that the functions in the class \mathcal{S} are analytic in $\mathbb{C} \setminus \mathbb{R}^+$ and take nonnegative values on the negative half-axis, while the functions in the class $\mathcal{S}_\kappa^{\tilde{\kappa}}$ have at most $2 \min\{\kappa, \tilde{\kappa}\}$ poles in $\mathbb{C} \setminus \mathbb{R}$ which are symmetric with respect to the real axis.

In the paper we consider the following Nevanlinna–Pick interpolation problem:

Problem 1.2 *Given are n distinct points $z_1, \dots, z_n \in \mathbb{C}^+$ and $m \times m$ matrices S_1, \dots, S_n such that the Hermitian matrices (called Pick matrices)*

$$P := \left(\frac{S_j - S_k^*}{z_j - \bar{z}_k} \right)_{k,j=1}^n, \quad \tilde{P} := \left(\frac{z_j S_j - \bar{z}_k S_k^*}{z_j - \bar{z}_k} \right)_{k,j=1}^n \tag{1.3}$$

are invertible and

$$\text{sq}_-(P) = \kappa, \quad \text{sq}_-(\tilde{P}) = \tilde{\kappa}, \tag{1.4}$$

that is, P and \tilde{P} have κ and $\tilde{\kappa}$ negative eigenvalues (counting multiplicities). Find all functions $S \in \mathcal{S}_\kappa^{\tilde{\kappa}}$ which are analytic at z_j and satisfy

$$S(z_j) = S_j, \quad j = 1, \dots, n. \tag{1.5}$$

We thank Heinz Langer for bringing this problem to our attention.

We show that there are infinitely many solutions to this problem and that the solutions can be parametrized via a fractional linear transformation over a subset of the class \mathcal{SP} of Stieltjes pairs.

Definition 1.3 *A pair $\{p, q\}$ of $m \times m$ matrix functions p and q is called a Stieltjes pair (belongs to the class \mathcal{SP}) if they are meromorphic in $\mathbb{C} \setminus \mathbb{R}_+$,*

$$q^*(\bar{z})p(z) = p^*(\bar{z})q(z), \quad \det(p(z)^*p(z) + q(z)^*q(z)) \neq 0, \quad z \in \mathbb{C}^+ \cap \rho(p, q), \tag{1.6}$$

where $\rho(p, q)$ is the set of points in which p and q are holomorphic, and

$$\frac{q(z)^*p(z) - p(z)^*q(z)}{z - \bar{z}} \geq 0, \quad \frac{zq(z)^*p(z) - \bar{z}p(z)^*q(z)}{z - \bar{z}} \geq 0, \quad z \in \mathbb{C}^+ \cap \rho(p, q). \tag{1.7}$$

Note that the inequalities (1.7) imply that the kernels

$$K_{pq}(z, \omega) := \frac{q(\omega)^*p(z) - p(\omega)^*q(z)}{z - \bar{\omega}}, \quad \tilde{K}_{pq}(z, \omega) := \frac{zq(\omega)^*p(z) - \bar{\omega}p(\omega)^*q(z)}{z - \bar{\omega}} \tag{1.8}$$

are nonnegative, that is, have no negative squares in $\mathbb{C}^+ \cap \rho(p, q)$.

Definition 1.4 *We say that a Stieltjes pair $\{p, q\}$ is strict if*

$$\frac{q(z)^*p(z) - p(z)^*q(z)}{z - \bar{z}} > 0, \quad \frac{zq(z)^*p(z) - \bar{z}p(z)^*q(z)}{z - \bar{z}} > 0, \quad z \in \mathbb{C}^+ \cap \rho(p, q). \tag{1.9}$$

We introduce an equivalence relation on the set of Stieltjes pairs \mathcal{SP} : the pair $\{p, q\}$ is said to be *equivalent* to the pair $\{p_1, q_1\}$ if there exists an $m \times m$ matrix function $X(z)$, meromorphic in $\mathbb{C} \setminus \mathbb{R}$ and satisfying $\det X(z) \neq 0$, such that $p_1(z) = p(z)X(z)$, $q_1(z) = q(z)X(z)$. It is easily seen that if $\{p, q\} \in \mathcal{SP}$ and $\det q(z) \neq 0$, then the function $S = pq^{-1}$ belongs to \mathcal{S} . Conversely, every $S \in \mathcal{S}$ generates a Stieltjes pair $\{S, I_m\}$. Thus, there is a one to one correspondence between the elements of \mathcal{S} and the equivalence classes of pairs $\{p, q\} \in \mathcal{SP}$ such that $\det q(z) \neq 0$. We denote by \mathcal{S}^+ the class of those matrix functions $S \in \mathcal{S}$ for which $\{S, I_m\}$ is a strict pair.

In Section 2 we construct from the data of Problem 1.2 a $2m \times 2m$ rational matrix function $\Theta(z)$, and we prove in Section 3 that the solutions of Problem 1.2 are obtained as a fractional linear transformation of Stieltjes pairs whose coefficients are formed by the four $m \times m$ block entries of Θ . In general not all Stieltjes pairs may serve as a parameter: the corresponding fractional linear transformation in that case does not define a solution of Problem 1.2. In Section 5 we characterize these excluded parameters. For this we use a description of the solutions of a special one point left-sided interpolation problem for Stieltjes pairs studied in Section 4. Multipoint problems of this type are studied in [3], but here we also need a description of the solutions in the degenerate cases. In Section 6 we show that Problem 1.2 has a solution. We refer the reader who wants to get an impression of our results to Section 7. There we have spelled out the scalar case and worked out two examples. Closely related to this paper are the two papers: [1], because many of the formulas in, for example, Sections 2 and 3 can be traced back to this paper, and [5], where the Nevanlinna–Pick problem for generalized Nevanlinna functions (scalar case) and in particular the excluded parameters in the parametrization of all solutions were studied. In [5] it is shown that the excluded parameters can be divided into two types: either there is only one parameter whose fractional linear transformation is not a solution and then this parameter is identically equal to a real constant, or there are infinitely many excluded parameters and then none of them is identically equal to a constant. According to Theorem 7.1 the same kinds of exclusion occur here, except that in one of the cases where the excluded parameter is unique, this parameter is not necessarily a real constant.

2 The parametrization matrix of the problem

In this section we construct the so called *parametrization matrix* formed by the four coefficients in the fractional linear transformation which describes all solutions of the Problem

1.2. Throughout the paper J denotes the signature matrix

$$J = \begin{pmatrix} 0 & -iI_m \\ iI_m & 0 \end{pmatrix}.$$

A $2m \times 2m$ matrix M is called J -unitary if $MJM^* = M$, or equivalently, $M^*JM = J$.

Definition 2.1 A $2m \times 2m$ matrix function Θ belongs to the class \mathcal{W}_κ if it is meromorphic in $\mathbb{C} \setminus \mathbb{R}$, holomorphic and J -unitary on the real axis:

$$\Theta(z)J\Theta(z)^* = J, \quad z \in \mathbb{R}, \tag{2.1}$$

and the kernel

$$K_{J,\Theta}(z, \omega) := \frac{J - \Theta(z)J\Theta(\omega)^*}{i(\bar{\omega} - z)}, \quad z, \omega \in \rho(\Theta), \quad z \neq \bar{\omega}, \tag{2.2}$$

has κ negative squares:

$$\text{sq}_-(K_{J,\Theta}) = \kappa. \tag{2.3}$$

We associate to Θ the function $\tilde{\Theta}$ defined by

$$\tilde{\Theta}(z) := P(z)\Theta(z)P^{-1}(z), \quad P(z) := \begin{pmatrix} zI_m & 0 \\ 0 & I_m \end{pmatrix}. \tag{2.4}$$

Definition 2.2 A $2m \times 2m$ matrix function Θ belongs to the class $\mathcal{W}_\kappa^{\tilde{\kappa}}$ if it belongs to \mathcal{W}_κ while the associated function $\tilde{\Theta}$ belongs to $\mathcal{W}_{\tilde{\kappa}}$.

In other words, Θ belongs to $\mathcal{W}_\kappa^{\tilde{\kappa}}$ if it satisfies (2.1), (2.3) and the following two conditions hold

$$\tilde{\Theta}(z)J\tilde{\Theta}(z)^* = J, \quad z \in \mathbb{R}; \quad \text{sq}_-(K_{J,\tilde{\Theta}}) = \tilde{\kappa}.$$

The next lemma gives an example of a function from the class \mathcal{W}_κ .

Lemma 2.3 Let $G \in \mathbb{C}^{2m \times N}$ and $Z \in \mathbb{C}^{N \times N}$ be matrices such that $\sigma(Z) \cap \mathbb{R} = \emptyset$ and

$$\bigcap_{j \geq 0} \text{Ker } GZ^j = \{0\}, \tag{2.5}$$

and assume that the Lyapunov equation

$$Z^*P - PZ = iG^*JG \tag{2.6}$$

has an invertible Hermitian solution $P \in \mathbb{C}^{N \times N}$ with $\text{sq}_-(P) = \kappa$. Then the $2m \times 2m$ matrix function

$$\hat{\Theta}(z) = I_{2m} - iG(zI - Z)^{-1}P^{-1}G^*J \tag{2.7}$$

belongs to the class \mathcal{W}_κ . Moreover, for $z, \omega \in \rho(\hat{\Theta})$ with $z \neq \bar{\omega}$,

$$K_{J,\hat{\Theta}}(z, \omega) = G(zI - Z)^{-1}P^{-1}(\bar{\omega}I - Z^*)^{-1}G^*, \tag{2.8}$$

and

$$\frac{J - \hat{\Theta}(\omega)^*J\hat{\Theta}(z)}{i(\bar{\omega} - z)} = JGP^{-1}(\bar{\omega}I - Z^*)^{-1}P(zI - Z)^{-1}P^{-1}G^*J. \tag{2.9}$$

Proof: Using (2.7) and (2.6) we get (2.8):

$$\begin{aligned}
 J - \widehat{\Theta}(z)J\widehat{\Theta}(\omega)^* &= iG(zI - Z)^{-1}P^{-1}G^* - iGJGP^{-1}(\bar{\omega}I - Z^*)^{-1}G^* \\
 &\quad - G(zI - Z)^{-1}P^{-1}G^*JGP^{-1}(\bar{\omega}I - Z^*)^{-1}G^* \\
 &= G(zI - Z)^{-1}P^{-1} \{iP(\bar{\omega}I - Z^*) - i(zI - Z)P + iZ^*P - iPZ\} \\
 &\quad \times P^{-1}(\bar{\omega}I - Z^*)^{-1}G^* \\
 &= i(\bar{\omega} - z)G(zI - Z)^{-1}P^{-1}(\bar{\omega}I - Z^*)^{-1}G^*.
 \end{aligned}$$

Let \mathcal{M} be the space of rational \mathbb{C}^N -vector functions spanned by the columns of the matrix $G(zI - Z)^{-1}$:

$$\mathcal{M} = \text{span} \{f_1, \dots, f_n\}, \quad (f_1(z), \dots, f_n(z)) = G(zI - Z)^{-1}.$$

The condition (2.5) ensures that the f_j 's form a basis for \mathcal{M} . We provide \mathcal{M} with the indefinite inner product $[\cdot, \cdot]$ such that P is the Gram matrix for this basis:

$$[f_j, f_k] = p_{kj}, \quad k, j = 1, \dots, n, \quad P = (p_{kj})_{k,j=1}^n. \tag{2.10}$$

Since \mathcal{M} is resolvent invariant, it is a reproducing kernel Pontryagin space with reproducing kernel

$$k(z, \omega) = G(zI - Z)^{-1}P^{-1}(\bar{\omega}I - Z^*)^{-1}G^*. \tag{2.11}$$

The equality $\text{sq}_-(P) = \kappa$ means that the negative index of \mathcal{M} (the dimension of any maximal negative subspace of \mathcal{M}) equals κ and therefore, the reproducing kernel (2.11) of \mathcal{M} has κ negative squares. From (2.11) and (2.8) it follows that the two kernels k and $K_{J\widehat{\Theta}}$ coincide, and it now easily follows that $\widehat{\Theta} \in \mathcal{W}_\kappa$. Finally, the identity (2.9) follows from (2.8), (2.7), and (2.6):

$$\begin{aligned}
 J - \widehat{\Theta}(\omega)^*J\widehat{\Theta}(z) &= JGP^{-1}(\bar{\omega}I - Z^*)^{-1} \{i(\bar{\omega}I - Z^*)P - iP(zI - Z) \\
 &\quad + iZ^*P - iPZ\} (zI - Z)^{-1}P^{-1}G^*J \\
 &= i(\bar{\omega} - z)JGP^{-1}(\bar{\omega}I - Z^*)^{-1}P(zI - Z)^{-1}P^{-1}G^*J. \blacksquare
 \end{aligned}$$

Using Lemma 2.3 we can construct a function Θ from the class $\mathcal{W}_\kappa^{\tilde{\kappa}}$.

Lemma 2.4 *Let $G_1, G_2 \in \mathbb{C}^{m \times N}$, $Z \in \mathbb{C}^{N \times N}$ be matrices such that $\sigma(Z) \cap \mathbb{R} = \emptyset$ and*

$$\bigcap_{j \geq 0} \text{Ker} \begin{pmatrix} G_1 \\ G_2 \end{pmatrix} Z^j = \bigcap_{j \geq 0} \text{Ker} \begin{pmatrix} G_1 Z \\ G_2 \end{pmatrix} Z^j = \{0\}, \tag{2.12}$$

and assume that there exist invertible Hermitian matrices P and \tilde{P} with $\text{sq}_-(P) = \kappa$ and $\text{sq}_-(\tilde{P}) = \tilde{\kappa}$, such that

$$\tilde{P} - PZ = G_1^*G_2. \tag{2.13}$$

Let M and \tilde{M} be the two J -unitary matrices defined by

$$M = \begin{pmatrix} I_m & 0 \\ G_2\tilde{P}^{-1}G_2^* & I_m \end{pmatrix}, \quad \tilde{M} = \begin{pmatrix} I_m & -G_1P^{-1}G_1^* \\ 0 & I_m \end{pmatrix}. \tag{2.14}$$

Then the function

$$\Theta(z) = I_{2m} + \begin{pmatrix} G_1 & G_1 Z \\ G_2 & zG_2 \end{pmatrix} \begin{pmatrix} 0 & -(zI - Z)^{-1} P^{-1} G_1^* \\ (zI - Z)^{-1} \tilde{P}^{-1} G_2^* & 0 \end{pmatrix} \quad (2.15)$$

belongs to the class $\mathcal{W}_{\kappa}^{\tilde{\kappa}}$, and for $z, \omega \in \rho(\Theta)$, $z \neq \bar{\omega}$,

$$\frac{J - \Theta(z)J\Theta(\omega)^*}{i(\bar{\omega} - z)} = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix} (zI - Z)^{-1} P^{-1} (\bar{\omega}I - Z^*)^{-1} (G_1^*, G_2^*), \quad (2.16)$$

$$\frac{J - \Theta(\omega)^*J\Theta(z)}{i(\bar{\omega} - z)} = M^* \begin{pmatrix} G_2 \\ -G_1 \end{pmatrix} P^{-1} (\bar{\omega}I - Z^*)^{-1} P (zI - Z)^{-1} P^{-1} (G_2^*, -G_1^*) M, \quad (2.17)$$

$$\frac{J - \tilde{\Theta}(z)J\tilde{\Theta}(\omega)^*}{i(\bar{\omega} - z)} = \begin{pmatrix} G_1 Z \\ G_2 \end{pmatrix} (zI - Z)^{-1} \tilde{P}^{-1} (\bar{\omega}I - Z^*)^{-1} (Z^* G_1^*, G_2^*), \quad (2.18)$$

$$\frac{J - \tilde{\Theta}(\omega)^*J\tilde{\Theta}(z)}{i(\bar{\omega} - z)} = \tilde{M}^* \begin{pmatrix} G_2 \\ -G_1 Z \end{pmatrix} \tilde{P}^{-1} (\bar{\omega}I - Z^*)^{-1} \tilde{P} (zI - Z)^{-1} \tilde{P}^{-1} (G_2^*, -Z^* G_1^*) \tilde{M}. \quad (2.19)$$

Moreover, Θ and $\tilde{\Theta}$ admit the factorizations

$$\Theta(z) = \left\{ I_{2m} + \begin{pmatrix} G_1 \\ G_2 \end{pmatrix} (zI - Z)^{-1} P^{-1} (G_2^*, -G_1^*) \right\} M, \quad (2.20)$$

$$\tilde{\Theta}(z) = \left\{ I_{2m} + \begin{pmatrix} G_1 Z \\ G_2 \end{pmatrix} (zI - Z)^{-1} \tilde{P}^{-1} (G_2^*, -Z^* G_1^*) \right\} \tilde{M}, \quad (2.21)$$

and the first factors on the right-hand sides of these identities are functions from \mathcal{W}_{κ} and $\mathcal{W}_{\tilde{\kappa}}$, respectively.

Proof: Substituting (2.15) into (2.4) we get

$$\tilde{\Theta}(z) = I_{2m} + \begin{pmatrix} zG_1 & G_1 Z \\ G_2 & G_2 \end{pmatrix} \begin{pmatrix} 0 & -(zI - Z)^{-1} P^{-1} G_1^* \\ (zI - Z)^{-1} \tilde{P}^{-1} G_2^* & 0 \end{pmatrix}. \quad (2.22)$$

Using the equalities

$$(G_2^*, -G_1^*) M = (PZ\tilde{P}^{-1}G_2^*, -G_1^*), \quad (G_2^*, -Z^*G_1^*) \tilde{M} = (G_2^*, -\tilde{P}P^{-1}G_1^*), \quad (2.23)$$

which follow from (2.13) and (2.14), it is easily checked by straightforward verifications that the formulas (2.15), (2.20) and (2.22), (2.21) define the same functions Θ and $\tilde{\Theta}$, respectively. It follows from (2.13) that

$$Z^*P - PZ = G_1^*G_2 - G_2^*G_1 = i(G_1^*, G_2^*)J \begin{pmatrix} G_1 \\ G_2 \end{pmatrix}, \quad (2.24)$$

which together with (2.12) means that the matrices

$$G = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix}, \quad (2.25)$$

Z , and P satisfy the conditions of Lemma 2.3. Therefore the function $\hat{\Theta}$ defined via (2.7) belongs to \mathcal{W}_κ . The equality (2.20) can be written as $\Theta = \hat{\Theta}M$, and since M is J -unitary, we have $K_{J,\Theta} = K_{J,\hat{\Theta}}$. Now (2.16) and (2.17) follow from (2.8), (2.9), and (2.25). In particular, $\Theta \in \mathcal{W}_\kappa$.

Similarly, it follows from (2.13) that

$$Z^* \tilde{P} - \tilde{P}Z = Z^* G_1^* G_2 - G_2^* G_1 Z = i(Z^* G_1^*, G_2^*) J \begin{pmatrix} G_1 Z \\ G_2 \end{pmatrix},$$

which together with (2.12) means that the matrices

$$G = \begin{pmatrix} ZG_1 \\ G_2 \end{pmatrix}, \tag{2.26}$$

Z , and \tilde{P} also satisfy the conditions of Lemma 2.3. Therefore the function $\tilde{\Theta}$ defined via (2.7) belongs to $\mathcal{W}_{\tilde{\kappa}}$, $\tilde{\kappa} = \text{sq}_-(\tilde{P})$. Since $\tilde{\Theta} = \tilde{\Theta} \tilde{M}$ (see (2.21)) and \tilde{M} is J -unitary, $K_{J,\tilde{\Theta}} = K_{J,\tilde{\Theta}}$. Now (2.18) and (2.19) follow from (2.8), (2.9), and (2.26). In particular, $\tilde{\Theta} \in \mathcal{W}_{\tilde{\kappa}}$. So by Definition 2.2, $\Theta \in \mathcal{W}_\kappa^{\tilde{\kappa}}$. ■

Remark 2.5 Since Θ and $\tilde{\Theta}$ are J -unitary on the real axis, the symmetry relations

$$\Theta(z)^{-1} = J\Theta(\bar{z})^*J, \quad \tilde{\Theta}(z)^{-1} = J\tilde{\Theta}(\bar{z})^*J \tag{2.27}$$

hold for $z \in \mathbb{C} \setminus \{z_1, \bar{z}_1, \dots, z_n, \bar{z}_n\}$. The right-hand sides here define holomorphic functions on $\mathbb{C} \setminus \{\bar{z}_1, \dots, \bar{z}_n\}$, which we denote by $\Theta^{-1}(z)$ and $\tilde{\Theta}^{-1}(z)$, respectively. From (2.16) and (2.18) it follows that

$$\frac{J - (\Theta^{-1}(\omega))^* J \Theta^{-1}(z)}{i(\bar{\omega} - z)} = - \begin{pmatrix} G_2 \\ -G_1 \end{pmatrix} (\bar{\omega}I - Z)^{-1} P^{-1} (zI - Z^*)^{-1} (G_2^*, -G_1^*), \tag{2.28}$$

$$\frac{J - (\tilde{\Theta}^{-1}(\omega))^* J \tilde{\Theta}^{-1}(z)}{i(\bar{\omega} - z)} = - \begin{pmatrix} G_2 \\ -G_1 Z \end{pmatrix} (\bar{\omega}I - Z)^{-1} \tilde{P}^{-1} (zI - Z^*)^{-1} (G_2^*, -Z^* G_1^*).$$

In the sequel we now take $N = mn$ and set

$$G_1 = (S_1, \dots, S_n), \quad G_2 = (I_m, \dots, I_m), \quad Z = \begin{pmatrix} z_1 I_m & & \\ & \ddots & \\ & & z_n I_m \end{pmatrix}. \tag{2.29}$$

It is easily seen that these matrices and the Pick matrices P and \tilde{P} defined by (1.3) satisfy the equality (2.13). Relations (2.12) are also in force and by Lemma 2.4, the function Θ defined by (2.15) or equivalently, by (2.20) and with this choice of the matrices, belongs to the class $\mathcal{W}_\kappa^{\tilde{\kappa}}$. We show later (see Theorem (3.8) below) that Θ is a parametrization matrix of Problem 1.2. *In what follows when we refer to Θ in (2.20), we assume that the matrices are given by (2.29) and (1.3).*

Lemma 2.6 *Let Θ be defined by (2.20). Then for every interpolation point $z_j, j = 1, \dots, n,$*

$$(I_m, -S_j^*) \Theta(\bar{z}_j) = 0, \quad \Theta^{-1}(z_j) \begin{pmatrix} S_j \\ I_m \end{pmatrix} = 0. \tag{2.30}$$

Proof: It follows from that (1.3) and (2.29) that

$$\begin{aligned} & (I_m, -S_j^*) \begin{pmatrix} G_1 \\ G_2 \end{pmatrix} (\bar{z}_j I - Z)^{-1} P^{-1} (G_2^*, -G_1^*) \\ &= - \underbrace{\left(\frac{S_1 - S_1^*}{z_1 - \bar{z}_j}, \dots, \frac{S_n - S_n^*}{z_n - \bar{z}_j} \right)}_{\text{j-th row of P}} P^{-1} \begin{pmatrix} I_m & -S_1^* \\ \vdots & \vdots \\ I_m & -S_n^* \end{pmatrix} \\ &= - (0, \dots, 0, I_m, 0, \dots, 0) \begin{pmatrix} I_m & -S_1^* \\ \vdots & \vdots \\ I_m & -S_n^* \end{pmatrix} = - (I_m, -S_j^*). \end{aligned}$$

Hence, on account of (2.20),

$$\begin{aligned} (I_m, -S_j^*) \Theta(\bar{z}_j) &= \left\{ (I_m, -S_j^*) + (I_m, -S_j^*) \begin{pmatrix} G_1 \\ G_2 \end{pmatrix} (\bar{z}_j I - Z)^{-1} P^{-1} (G_2^*, -G_1^*) \right\} M \\ &= \left\{ (I_m, -S_j^*) - (I_m, -S_j^*) \right\} M = 0. \end{aligned}$$

This implies the first equality in (2.30); the second equality follows from the first by taking adjoints. ■

3 Description of all solutions

In this section we characterize the set all solutions of Problem 1.2 in terms of a fractional linear transformation. We begin with some preliminary results.

Lemma 3.1 *Let $\Psi(z, \omega)$ be an $(N + m) \times (N + m)$ matrix kernel for z, ω in an open subset Ω of \mathbb{C} defined by*

$$\Psi(z, \omega) = \begin{pmatrix} A & B(z) \\ B(\omega)^* & K(z, \omega) \end{pmatrix},$$

where $A \in \mathbb{C}^{N \times N}$ is a Hermitian matrix, $B(z)$ is an $N \times m$ matrix function on Ω and $K(z, \omega) = K(\omega, z)^*$ is an $m \times m$ matrix kernel with a finite number of negative squares in Ω . If A is invertible then

$$\text{sq}_-(\Psi(z, \omega)) = \text{sq}_-(A) + \text{sq}_-(K(z, \omega) - B(\omega)^* A^{-1} B(z)).$$

On the other hand if

$$A = \begin{pmatrix} K(z_1, z_1) & \cdots & K(z_n, z_1) \\ \vdots & & \vdots \\ K(z_1, z_n) & \cdots & K(z_n, z_n) \end{pmatrix}, \quad B(z) = \begin{pmatrix} K(z, z_1) \\ \vdots \\ K(z, z_n) \end{pmatrix}, \tag{3.1}$$

then

$$\text{sq}_-(\Psi(z, \omega)) = \text{sq}_-(K(z, \omega)).$$

Proof: The first assertion follows from the factorization

$$\Psi(z, \omega) = \begin{pmatrix} I_N & 0 \\ B(\omega)^*A^{-1} & I_m \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & K(z, \omega) - B(\omega)^*A^{-1}B(z) \end{pmatrix} \begin{pmatrix} I_N & A^{-1}B(z) \\ 0 & I_m \end{pmatrix}.$$

The second assertion follows from the definition of a kernel having κ negative squares. ■

Corollary 3.2 *Assume that Problem 1.2 has a solution S . Define the kernels K_S and \widetilde{K}_S by (1.2) and the matrices G_1, G_2 and Z by (2.29), and let the Pick matrices P and \widetilde{P} be given by (1.3). Then the two difference kernels*

$$D_S(z, \omega) := K_S(z, \omega) - (S(\omega)^*G_2 - G_1)(\bar{\omega}I - Z)^{-1}P^{-1}(zI - Z^*)^{-1}(G_2^*S(z) - G_1^*), \tag{3.2}$$

$$\widetilde{D}_S(z, \omega) := \widetilde{K}_S(z, \omega) - (\bar{\omega}S(\omega)^*G_2 - G_1Z)(\bar{\omega}I - Z)^{-1}\widetilde{P}^{-1}(zI - Z^*)^{-1}(zG_2^*S(z) - Z^*G_1^*) \tag{3.3}$$

are nonnegative in $\mathbb{C} \setminus \mathbb{R}$.

Proof: Since S is a solution of Problem 1.2, it belongs to $\mathcal{S}_\kappa^{\tilde{\kappa}}$ and therefore,

$$\text{sq}_-(K_S(z, \omega)) = \kappa, \quad \text{sq}_-(\widetilde{K}_S(z, \omega)) = \tilde{\kappa}.$$

We first apply the second part of Lemma 3.1 with $K(z, \omega) := K_S(z, \omega)$. Then, see (3.1),

$$A = P, \quad B(z) = \begin{pmatrix} \frac{S(z) - S_1^*}{z - \bar{z}_1} \\ \vdots \\ \frac{S(z) - S_n^*}{z - \bar{z}_n} \end{pmatrix} = (zI - Z^*)^{-1}(G_2^*S(z) - G_1^*),$$

and hence

$$\text{sq}_- \left(\begin{pmatrix} P & (zI - Z^*)^{-1}(G_2^*S(z) - G_1^*) \\ (S(\omega)^*G_2 - G_1)(\bar{\omega}I - Z)^{-1} & K_S(z, \omega) \end{pmatrix} \right) = \kappa. \tag{3.4}$$

Similarly, if $K(z, \omega) := \widetilde{K}_S(z, \omega)$ then

$$A = \widetilde{P}, \quad B(z) = \begin{pmatrix} \frac{zS(z) - \bar{z}_1 S_1^*}{z - \bar{z}_1} \\ \vdots \\ \frac{zS(z) - \bar{z}_n S_n^*}{z - \bar{z}_n} \end{pmatrix} = (zI - Z^*)^{-1}(zG_2^*S(z) - Z^*G_1^*),$$

and the second part of Lemma 3.1 yields

$$\text{sq}_- \left(\begin{pmatrix} \widetilde{P} & (zI - Z^*)^{-1}(zG_2^*S(z) - Z^*G_1^*) \\ (\bar{\omega}S(\omega)^*G_2 - G_1Z)(\bar{\omega}I - Z)^{-1} & \widetilde{K}_S(z, \omega) \end{pmatrix} \right) = \tilde{\kappa}. \tag{3.5}$$

Applying the first assertion of Lemma 3.1 to the matrix kernels in (3.4), (3.5) and taking into account (1.4) we find that the difference kernels in (3.2) and (3.3) have no negative squares. ■

Let

$$\Theta(z) = \begin{pmatrix} \theta_{11}(z) & \theta_{12}(z) \\ \theta_{21}(z) & \theta_{22}(z) \end{pmatrix} \tag{3.6}$$

be a partition of the function $\Theta(z)$ given by (2.20) into four $m \times m$ blocks. The following two theorems imply that $\Theta(z)$ is a parametrization matrix. In the first one we use that $\Theta(z)$ is defined in $z = \bar{z}_j, j = 1, \dots, n$, that is, in the complex conjugates of the interpolation points.

Theorem 3.3 *All solutions S of Problem 1.2 are parametrized by the fractional linear transformation*

$$S(z) = (\theta_{11}(z)p(z) + \theta_{12}(z)q(z)) (\theta_{21}(z)p(z) + \theta_{22}(z)q(z))^{-1}, \tag{3.7}$$

where the parameter runs through the set of those pairs $\{p, q\} \in \mathcal{SP}$ which satisfy

$$\det (\theta_{21}(\bar{z}_j)p(\bar{z}_j) + \theta_{22}(\bar{z}_j)q(\bar{z}_j)) \neq 0, \quad \text{for } j = 1, \dots, n. \tag{3.8}$$

Under the transformation (3.7), two pairs correspond to the same function S if and only if they are equivalent.

Proof: Since Θ is J -unitary on the real axis, the function S of the form (3.7) satisfies the symmetry relation (1.1) if and only if the corresponding parameter $\{p, q\}$ is subject to symmetry relation in (1.6): setting

$$V(z) = \theta_{21}(z)p(z) + \theta_{22}(z)q(z) \tag{3.9}$$

we get

$$\begin{aligned} S(z) - S(\bar{z})^* &= -i(S(\bar{z})^*, I_m) J \begin{pmatrix} S(z) \\ I_m \end{pmatrix} \\ &= -iV(\bar{z})^{-*} (p(\bar{z})^*, q(\bar{z})^*) \Theta(\bar{z})^* J \Theta(z) \begin{pmatrix} p(z) \\ q(z) \end{pmatrix} V^{-1}(z) \\ &= -iV(\bar{z})^{-*} (p(\bar{z})^*, q(\bar{z})^*) J \begin{pmatrix} p(z) \\ q(z) \end{pmatrix} V^{-1}(z) \\ &= V(\bar{z})^{-*} \{q^*(\bar{z})p(z) - p^*(\bar{z})q(z)\} V^{-1}(z). \end{aligned}$$

Let S be a solution of the Problem 1.2 and let $\{p, q\}$ be the pair defined by

$$\begin{pmatrix} p(z) \\ q(z) \end{pmatrix} = \Theta^{-1}(z) \begin{pmatrix} S(z) \\ I_m \end{pmatrix}. \tag{3.10}$$

Then by (2.4),

$$\begin{pmatrix} zp(z) \\ q(z) \end{pmatrix} = \tilde{\Theta}^{-1}(z) \begin{pmatrix} zS(z) \\ I_m \end{pmatrix}. \tag{3.11}$$

Using the partition (3.6) of Θ we get from (3.10)

$$\theta_{11}(z)p(z) + \theta_{12}(z)q(z) = S(z) \quad \text{and} \quad \theta_{21}(z)p(z) + \theta_{22}(z)q(z) \equiv I_m. \tag{3.12}$$

In particular, S admits the representation (3.7) and the second condition in (1.6) is in force. It remains to show that $\{p, q\} \in \mathcal{SP}$.

The kernels K_S and \widetilde{K}_S can be written as

$$K_S(z, \omega) = (S(\omega)^*, I_m) \frac{J}{i(z - \bar{\omega})} \begin{pmatrix} S(z) \\ I_m \end{pmatrix}, \tag{3.13}$$

$$\widetilde{K}_S(z, \omega) = (\bar{\omega}S(\omega)^*, I_m) \frac{J}{i(z - \bar{\omega})} \begin{pmatrix} zS(z) \\ I_m \end{pmatrix}. \tag{3.14}$$

Hence by the first relation in (2.28), (3.10) and (1.8), the difference kernel in (3.2) takes the form

$$\begin{aligned} D_S(z, \omega) &= (S(\omega)^*, I_m) \times \\ &\quad \left\{ \frac{J}{i(z - \bar{\omega})} - \begin{pmatrix} G_2 \\ -G_1 \end{pmatrix} (\bar{\omega}I - Z)^{-1} P^{-1} (zI - Z^*)^{-1} (G_2^*, -G_1^*) \right\} \begin{pmatrix} S(z) \\ I_m \end{pmatrix} \\ &= (S(\omega)^*, I_m) \frac{(\Theta^{-1}(\omega))^* J \Theta^{-1}(z)}{z - \bar{\omega}} \begin{pmatrix} S(z) \\ I_m \end{pmatrix} \\ &= (p(\omega)^*, q(\omega)^*) \frac{J}{i(z - \bar{\omega})} \begin{pmatrix} p(z) \\ q(z) \end{pmatrix} \\ &= \frac{p(z)q(\omega)^* - q(z)p(\omega)^*}{z - \bar{\omega}} = K_{pq}(z, \omega). \end{aligned} \tag{3.15}$$

Similarly, by the second relation in (2.28), (3.11) and (1.8), the difference kernel in (3.3) can be written as

$$\begin{aligned} \widetilde{D}_S(z, \omega) &= (\bar{z}S(z)^*, I_m) \times \\ &\quad \left\{ \frac{J}{i(\bar{z} - z)} - \begin{pmatrix} G_2 \\ -G_1 Z \end{pmatrix} (\bar{z}I - Z)^{-1} \widetilde{P}^{-1} (zI - Z^*)^{-1} (G_2^*, -Z^*G_1^*) \right\} \begin{pmatrix} zS(z) \\ I_m \end{pmatrix} \\ &= (\bar{\omega}S(\omega)^*, I_m) \frac{(\Theta^{-1}(\omega))^* J \Theta^{-1}(z)}{z - \bar{\omega}} \begin{pmatrix} zS(z) \\ I_m \end{pmatrix} \\ &= (\bar{\omega}p(\omega)^*, q(\omega)^*) \frac{J}{i(z - \bar{\omega})} \begin{pmatrix} zp(z) \\ q(z) \end{pmatrix} \\ &= \frac{zp(z)q(\omega)^* - \bar{\omega}p(\omega)^*q(z)}{z - \bar{\omega}} = \widetilde{K}_{pq}(z, \omega). \end{aligned} \tag{3.16}$$

Corollary 3.2 states that these kernels are nonnegative, and therefore $\{p, q\} \in \mathcal{SP}$.

Conversely, let $\{p, q\}$ be a Stieltjes pair which is analytic at the interpolation points $z_j, j = 1, \dots, n$, and satisfies (3.8), and let S be defined by (3.7). Using (3.9) we rewrite (3.7) as

$$\begin{pmatrix} S(z) \\ I_m \end{pmatrix} V(z) = \Theta(z) \begin{pmatrix} p(z) \\ q(z) \end{pmatrix}. \tag{3.17}$$

By (2.30),

$$(I_m, -S_j^*) \begin{pmatrix} S(\bar{z}_j) \\ I_m \end{pmatrix} V(\bar{z}_j) = (I_m, -S_j^*) \Theta(\bar{z}_j) \begin{pmatrix} p(\bar{z}_j) \\ q(\bar{z}_j) \end{pmatrix} = 0, \quad j = 1, \dots, n,$$

and since $\det V(\bar{z}_j) \neq 0$, it follows that S is subject to the conditions

$$S(\bar{z}_j) = S_j^*, \quad j = 1, \dots, n,$$

which in view of (1.1), are equivalent to interpolation conditions (1.5).

Let K_S and \widetilde{K}_S be the kernels as in (1.2). In view of (1.3) and the interpolation conditions (1.5),

$$(K_S(z_j, z_k))_{k,j=1}^n = P, \quad (\widetilde{K}_S(z_j, z_k))_{k,j=1}^n = \widetilde{P},$$

and the relations (1.4) mean that K_S and \widetilde{K}_S have at least κ and $\tilde{\kappa}$ negative squares, respectively. Substituting (3.17) into (3.13) and using the notation (1.8) we get

$$\begin{aligned} K_S(z, \omega) &= V(\omega)^{-*} (p(\omega)^*, q(\omega)^*) \frac{\Theta(\omega)^* J \Theta(z)}{i(\bar{\omega} - z)} \begin{pmatrix} p(z) \\ q(z) \end{pmatrix} V^{-1}(z) \\ &= V(\omega)^{-*} K_{pq}(z, \omega) V^{-1}(z) \\ &\quad + V(\omega)^{-*} (p(\omega)^*, q(\omega)^*) \frac{\Theta(\omega)^* J \Theta(z) - J}{i(\bar{\omega} - z)} \begin{pmatrix} p(z) \\ q(z) \end{pmatrix} V^{-1}(z), \end{aligned}$$

where the first kernel on the right-hand side is nonnegative and the second one has at most κ negative squares in view of (2.17). Therefore K_S has at most κ negative squares; hence $\text{sq}_-(K_S) = \kappa$. Similarly,

$$\begin{aligned} \widetilde{K}_S(z, \omega) &= V(\omega)^{-*} (\bar{\omega}p(\omega)^*, q(\omega)^*) \frac{\widetilde{\Theta}(\omega)^* J \widetilde{\Theta}(z)}{i(\bar{\omega} - z)} \begin{pmatrix} zp(z) \\ q(z) \end{pmatrix} V^{-1}(z) \\ &= V(\omega)^{-*} \widetilde{K}_{pq}(z, \omega) V^{-1}(z) \\ &\quad + V(\omega)^{-*} (\bar{\omega}p(\omega)^*, q(\omega)^*) \frac{\widetilde{\Theta}(\omega)^* J \widetilde{\Theta}(z) - J}{i(\bar{\omega} - z)} \begin{pmatrix} zp(z) \\ q(z) \end{pmatrix} V^{-1}(z), \end{aligned}$$

and in view of (2.19), \widetilde{K}_S has at most $\tilde{\kappa}$ negative squares and therefore $\text{sq}_-(\widetilde{K}_S) = \tilde{\kappa}$. By Definition 1.1, S belongs to $S_{\kappa}^{\tilde{\kappa}}$ and thus, is a solution of Problem 1.2.

Obviously, the transformation (3.7) applied to equivalent pairs give the same function S . Conversely, assume that S has two different representations $S(z) = U_\ell(z) V_\ell^{-1}(z)$ with

$$U_\ell(z) = \theta_{11}(z)p_\ell(z) + \theta_{12}(z)q_\ell(z), \quad V_\ell(z) = \theta_{21}(z)p_\ell(z) + \theta_{22}(z)q_\ell(z),$$

and such that $\det V_\ell(z) \neq 0, \ell = 1, 2$. Then

$$\Theta^{-1}(z) \begin{pmatrix} S(z) \\ I_m \end{pmatrix} = \begin{pmatrix} p_1(z) \\ q_1(z) \end{pmatrix} V_1^{-1}(z) = \begin{pmatrix} p_2(z) \\ q_2(z) \end{pmatrix} V_2^{-1}(z).$$

Hence $p_1 = p_2 X, q_1 = q_2 X$ with $X = V_2^{-1} V_1$, that is, the pairs $\{p_1, q_1\}$ and $\{p_2, q_2\}$ are equivalent. ■

In the following parametrization theorem we replace the conditions (3.8) in Theorem 3.3 by limit conditions in the interpolation points (see (3.19) below). In these points $\Theta(z)$ has simple poles.

Theorem 3.4 *All solutions S of Problem 1.2 are parametrized by the fractional linear transformation*

$$S(z) = (\theta_{11}(z)\tilde{p}(z) + \theta_{12}(z)\tilde{q}(z)) (\theta_{21}(z)\tilde{p}(z) + \theta_{22}(z)\tilde{q}(z))^{-1}, \tag{3.18}$$

where the parameter runs through the set of those $\{\tilde{p}, \tilde{q}\} \in \mathcal{SP}$ which satisfies

$$\det \left\{ \lim_{z \rightarrow z_j} (z - z_j) (\theta_{21}(z)\tilde{p}(z) + \theta_{22}(z)\tilde{q}(z)) \right\} \neq 0 \quad \text{for } j = 1, \dots, n. \tag{3.19}$$

Under the transformation (3.18), two pairs correspond to the same function S if and only if they are equivalent.

A pair $\{p, q\} \in \mathcal{SP}$ is called an *excluded parameter* in the parametrization (3.18) of the solutions of the Nevanlinna–Pick problem if it does *not* satisfy (3.19).

Proof: Let S be a solution of Problem 1.2. Without loss of generality we assume that S is of the form (3.7) with $\{p, q\} \in \mathcal{SP}$ as in (3.10), hence the relations (3.12) hold. By the second formula in (2.30),

$$\begin{pmatrix} p(z_j) \\ q(z_j) \end{pmatrix} = \Theta^{-1}(z_j) \begin{pmatrix} S(z_j) \\ I_m \end{pmatrix} = \Theta^{-1}(z_j) \begin{pmatrix} S_j \\ I_m \end{pmatrix} = 0, \quad j = 1, \dots, n,$$

and therefore

$$\begin{pmatrix} p(z) \\ q(z) \end{pmatrix} = \prod_{j=1}^n (z - z_j) \begin{pmatrix} \tilde{p}(z) \\ \tilde{q}(z) \end{pmatrix} \tag{3.20}$$

for some pair $\{\tilde{p}, \tilde{q}\} \in \mathcal{SP}$ which is analytic at the interpolation points z_j . The representation (3.18) follows from (3.7), since the pairs $\{p, q\}$ and $\{\tilde{p}, \tilde{q}\}$ are equivalent. In view of (3.12) and (3.20),

$$(z - z_j) (\theta_{21}(z)\tilde{p}(z) + \theta_{22}(z)\tilde{q}(z)) = \frac{\theta_{21}(z)p(z) + \theta_{22}(z)q(z)}{\prod_{\ell \neq j} (z - z_\ell)} = \frac{I_m}{\prod_{\ell \neq j} (z - z_\ell)}.$$

Therefore the conditions (3.19) are fulfilled.

Conversely, let $\{\tilde{p}, \tilde{q}\}$ be a Stieltjes pair which is analytic at the interpolation points z_j and satisfies (3.19), and let S be defined by (3.18). Multiplying the numerator and the denominator in the right-hand side of (3.18) by $(z - z_j)$ and taking into account (3.19) we conclude that S is analytic at all z_j 's. To show that S satisfies the interpolation conditions (1.5) we note that on account of (2.20) and (2.29),

$$\lim_{z \rightarrow z_j} (z - z_j)\Theta(z) = \begin{pmatrix} 0 & \cdots & S_j & \cdots & 0 \\ 0 & \cdots & I_m & \cdots & 0 \end{pmatrix} P^{-1} (G_2^*, -G_1^*) M, \tag{3.21}$$

and therefore

$$(I_m, -S_j) \lim_{z \rightarrow z_j} (z - z_j)\Theta(z) = 0. \tag{3.22}$$

We write (3.18) in the equivalent form

$$\begin{pmatrix} S(z) \\ I_m \end{pmatrix} = (z - z_j)\Theta(z) \begin{pmatrix} \tilde{p}(z) \\ \tilde{q}(z) \end{pmatrix} ((z - z_j)(\theta_{21}(z)\tilde{p}(z) + \theta_{22}(z)\tilde{q}(z)))^{-1},$$

and multiply both sides from the left by $(I_m, -S_j)$. If we then let $z \rightarrow z_j$ we obtain, on account of (3.19) and (3.22),

$$\begin{aligned} (I_m, -S_j) \begin{pmatrix} S(z_j) \\ I_m \end{pmatrix} &= \\ &= (I_m, -S_j) \lim_{z \rightarrow z_j} (z - z_j)\Theta(z) \begin{pmatrix} \tilde{p}(z) \\ \tilde{q}(z) \end{pmatrix} ((z - z_j)(\theta_{21}(z)\tilde{p}(z) + \theta_{22}(z)\tilde{q}(z)))^{-1} = 0. \end{aligned}$$

Hence S satisfies (1.5). Since Θ belongs to $\mathcal{W}_\kappa^{\tilde{\kappa}}$, we conclude as in Theorem 3.3 that the kernels K_S and \tilde{K}_S in (3.18) have at most κ and $\tilde{\kappa}$ negative squares, respectively. Due to the interpolation conditions (1.5) and the equalities (1.4) these kernels have exactly κ and $\tilde{\kappa}$ negative squares, respectively, and therefore $S \in \mathcal{S}_\kappa^{\tilde{\kappa}}$. ■

4 A one point interpolation problem for Stieltjes pairs

This section is auxiliary. We consider a left-sided one point interpolation problem for Stieltjes pairs which will be the main tool in the investigation of the excluded parameters of the parametrization (3.7) in the next section. The multipoint version of this problem is discussed in [1] and [3] for the nondegenerate case (the precise meaning will be given below). Here we consider also the degenerate cases and we present them in a form which is convenient for later use.

Problem 4.1 *Given two row vectors $a, b \in \mathbb{C}^m$ and a point $\omega \in \mathbb{C}^+$, find all pairs $\{p, q\} \in \mathcal{SP}$ which are analytic at ω and satisfy the interpolation condition*

$$ap(\omega) = bq(\omega). \tag{4.1}$$

Note that condition (4.1) is invariant with respect to the equivalence relation in the class \mathcal{SP} : if $\{p, q\}$ satisfies (4.1) then every equivalent pair $\{p_1, q_1\}$ which is analytic at ω satisfies (4.1) as well.

Problem 4.1 is solvable if and only if the corresponding Pick matrices are nonnegative, that is,

$$k := \frac{ba^* - ab^*}{\omega - \bar{\omega}} \geq 0, \quad \tilde{k} := \frac{\omega ba^* - \bar{\omega} ab^*}{\omega - \bar{\omega}} \geq 0. \tag{4.2}$$

The description of all solutions of Problem 4.1 relies on the following theorem.

Theorem 4.2 *A pair $\{p, q\}$ is a solution of Problem 4.1 if and only if it satisfies the following inequalities for all $z \in \mathbb{C}^+$,*

$$\begin{pmatrix} k & \frac{ap(z) - bq(z)}{z - \bar{z}} \\ \frac{p(z)^*a^* - q(z)^*b^*}{\bar{z} - \bar{\omega}} & \frac{q(z)^*p(z) - \bar{p}(z)^*q(z)}{z - \bar{z}} \end{pmatrix} \geq 0, \tag{4.3}$$

$$\begin{pmatrix} \tilde{k} & \frac{z\alpha p(z) - \omega\beta q(z)}{z - \bar{z}} \\ \frac{\bar{z}p(z)^*a^* - \bar{\omega}q(z)^*b^*}{\bar{z} - \bar{\omega}} & \frac{zq(z)^*p(z) - \bar{z}p(z)^*q(z)}{z - \bar{z}} \end{pmatrix} \geq 0. \tag{4.4}$$

Proof: Let $\{p, q\}$ be a Stieltjes pair satisfying (4.1). From

$$(p + iq)(p + iq)^* = pp^* + qq^* + \frac{q^*p - p^*q}{i},$$

it follows that $\det(p(z) + iq(z)) \neq 0$ for all $z \in \mathbb{C}^+$ outside some set of isolated points. The pair $\{\hat{p}, \hat{q}\}$ defined by

$$\hat{p}(z) = p(z)(p(z) + iq(z))^{-1}, \quad \hat{q}(z) = q(z)(p(z) + iq(z))^{-1}$$

is equivalent to $\{p, q\}$ and satisfies (4.1) and $\hat{p}(z) + i\hat{q}(z) \equiv I_m$. Therefore it can be assumed without loss of generality that $\{p, q\}$ a priori satisfies

$$p(z) + iq(z) \equiv I_m. \tag{4.5}$$

By assumption (4.5), the interpolation condition (4.1) is equivalent to

$$(b + ia)p(\omega) = b, \quad (b + ia)q(\omega) = a. \tag{4.6}$$

Moreover, the matrix function

$$R(z) := p(z) - iq(z) \tag{4.7}$$

is a contraction in \mathbb{C}^+ :

$$\begin{aligned} I_m - R(z)^*R(z) &= (p(z) + iq(z))^*(p(z) + iq(z)) - (p(z) - iq(z))^*(p(z) - iq(z)) \\ &= 2\frac{q(z)^*p(z) - p(z)^*q(z)}{i} \geq 0 \end{aligned} \tag{4.8}$$

and satisfies, on account of (4.6),

$$(b + ia)R(\omega) = (b - ia). \tag{4.9}$$

As an analytic contraction R satisfies the inequality

$$\begin{pmatrix} \frac{I_m - R(\omega)R(\omega)^*}{i(\bar{\omega} - \omega)} & \frac{R(z) - R(\omega)}{i(z - \omega)} \\ \frac{R(z)^* - R(\omega)^*}{i(\bar{\omega} - \bar{z})} & \frac{I_m - R(z)^*R(z)}{i(\bar{z} - z)} \end{pmatrix} \geq 0, \quad z \in \mathbb{C}^+ \setminus \{\omega\}, \tag{4.10}$$

(see e.g., [2, p.77]). We multiply the matrix in (4.10) from the left by $\begin{pmatrix} b+ia & 0 \\ 0 & I_m \end{pmatrix}$, and from the right by its adjoint $\begin{pmatrix} b^*-ia^* & 0 \\ 0 & I_m \end{pmatrix}$ and we obtain, on account of (4.2) and (4.9),

$$\left(\begin{array}{cc} 2k & \frac{(b+ia)R(z) - b + ia}{i(z-\omega)} \\ \frac{R(z)^*(b^* - ia^*) - b^* - ia^*}{i(\bar{\omega} - \bar{z})} & \frac{I_m - R(z)^*R(z)}{i(\bar{z} - z)} \end{array} \right) \geq 0. \tag{4.11}$$

By (4.5) and (4.7),

$$\begin{aligned} (b+ia)R(z) - b + ia &= (b+ia)(p(z) - iq(z)) - (b-ia)(p(z) + iq(z)) \\ &= 2i(ap(z) - bq(z)), \end{aligned}$$

and we substitute this and (4.8) into (4.11), we get (4.3). Applying the same arguments to the pair $\{zp(z), q(z)\}$ we come to (4.4).

Conversely, assume that $\{p, q\}$ satisfies the conditions (1.6) and the inequalities (4.3) and (4.4) for all $z \in \mathbb{C}^+$ outside some finite set of isolated points. Then, in particular, the inequalities (1.7) hold and therefore $\{p, q\}$ is a Stieltjes pair. Since p and q are analytic at ω , the diagonal blocks in (4.3) are bounded in a neighbourhood of ω . Therefore, the nondiagonal block $\frac{ap(z)-bq(z)}{z-\omega}$ is bounded for $z \rightarrow \omega$ and thus (4.1) is fulfilled. ■

For the multipoint interpolation problem the inequalities (4.3) and (4.4) were considered in [3] to construct a recursive Schur process associated with a Nevanlinna–Pick problem for Stieltjes functions.

To describe the solutions of Problem 4.1 we consider four cases: the *nondegenerate* case $k > 0$ and $\tilde{k} > 0$, and the three *degenerate* cases in which one of numbers k and \tilde{k} is zero or both are zero.

I. The nondegenerate case: $k > 0$ and $\tilde{k} > 0$. In this case the inequalities (4.3) and (4.4) are equivalent to

$$(p(z)^*, q(z)^*) \left\{ \frac{J}{i(\bar{z} - z)} - \frac{1}{|z - \omega|^2} \begin{pmatrix} a^* \\ -b^* \end{pmatrix} k^{-1} (a, -b) \right\} \begin{pmatrix} p(z) \\ q(z) \end{pmatrix} \geq 0 \tag{4.12}$$

and

$$(\bar{z}p(z)^*, q(z)^*) \left\{ \frac{J}{i(\bar{z} - z)} - \frac{1}{|z - \omega|^2} \begin{pmatrix} a^* \\ -\bar{\omega}b^* \end{pmatrix} \tilde{k}^{-1} (a, -\omega b) \right\} \begin{pmatrix} zp(z) \\ q(z) \end{pmatrix} \geq 0, \tag{4.13}$$

respectively. Using the function

$$\Psi(z) = I_{2m} + \frac{1}{z - \bar{\omega}} \begin{pmatrix} b^* & \bar{\omega}b^* \\ a^* & za^* \end{pmatrix} \begin{pmatrix} 0 & -k^{-1}b \\ \tilde{k}^{-1}a & 0 \end{pmatrix} \tag{4.14}$$

and the associated function

$$\tilde{\Psi}(z) = P(z)\Psi(z)P^{-1}(z), \quad P(z) = \begin{pmatrix} zI_m & 0 \\ 0 & I_m \end{pmatrix},$$

we can rewrite the inequalities (4.12) and (4.13) as follows:

$$\begin{aligned} (p(z)^*, q(z)^*) \frac{\Psi(z)^{-*} J \Psi(z)^{-1}}{i(\bar{z} - z)} \begin{pmatrix} p(z) \\ q(z) \end{pmatrix} &\geq 0, \\ (\bar{z}p(z)^*, q(z)^*) \frac{\tilde{\Psi}(z)^{-*} J \tilde{\Psi}(z)^{-1}}{i(\bar{z} - z)} \begin{pmatrix} zp(z) \\ q(z) \end{pmatrix} &\geq 0. \end{aligned}$$

The kernels in these inequalities are "projective" analogues of the kernels (3.15) and (3.16). The inequalities can easily be solved. The solutions are parametrized via a linear (rather than fractional linear) transformation, and the parametrization matrix is given by Ψ .

Theorem 4.3 *If $k > 0$ and $\tilde{k} > 0$, then all solutions $\{p, q\}$ of Problem 4.1 can be parametrized by the linear transformation*

$$\begin{pmatrix} p(z) \\ q(z) \end{pmatrix} = \Psi(z) \begin{pmatrix} p_1(z) \\ q_1(z) \end{pmatrix}, \tag{4.15}$$

where $\Psi(z)$ is the $2m \times 2m$ matrix function defined by (4.14) and the parameter $\{p_1, q_1\}$ runs through all of \mathcal{SP} .

II. The case $k = 0$ and $\tilde{k} > 0$. It follows from (4.2) that $ab^* = ba^*$ and therefore

$$\tilde{k} = ab^* > 0. \tag{4.16}$$

Theorem 4.4 *If $k = 0$ and $\tilde{k} > 0$, then all solutions $\{p, q\}$ of Problem 4.1 can be parametrized by*

$$\begin{pmatrix} p(z) \\ q(z) \end{pmatrix} = \begin{pmatrix} I_m & 0 \\ a^* \tilde{k}^{-1} a & I_m \end{pmatrix} \begin{pmatrix} p_1(z) \\ q_1(z) \end{pmatrix}, \tag{4.17}$$

where the parameter runs through the set of pairs $\{p_1, q_1\} \in \mathcal{SP}$ for which

$$bq_1(z) \equiv 0. \tag{4.18}$$

Proof: Let $\{p, q\}$ be a solution of Problem 4.1. By Theorem 4.2, it satisfies the inequalities (4.3) and (4.4). Since $k = 0$, (4.3) implies

$$ap(z) \equiv bq(z), \tag{4.19}$$

and therefore, (4.3) and (4.4) are equivalent to

$$\frac{q(z)^* p(z) - p(z)^* q(z)}{z - \bar{z}} \geq 0, \quad \left(\begin{matrix} \tilde{k} \\ p(z)^* a^* \frac{ap(z)}{zq(z)^* p(z) - \bar{z}p(z)^* q(z)} \end{matrix} \right) \geq 0. \tag{4.20}$$

Since $\tilde{k} > 0$, the second inequality in (4.20) is equivalent to

$$\frac{zq(z)^* p(z) - \bar{z}p(z)^* q(z)}{z - \bar{z}} - p(z)^* a^* \tilde{k}^{-1} ap(z) \geq 0. \tag{4.21}$$

Define the pair $\{p_1, q_1\}$ by

$$\begin{pmatrix} p_1(z) \\ q_1(z) \end{pmatrix} = \begin{pmatrix} I_m & 0 \\ -a^* \tilde{k}^{-1} a & I_m \end{pmatrix} \begin{pmatrix} p(z) \\ q(z) \end{pmatrix}. \tag{4.22}$$

Then the first inequality in (4.20) and the inequality (4.21) can be expressed in terms of $\{p_1, q_1\}$:

$$\frac{q_1(z)^* p_1(z) - p_1(z)^* q_1(z)}{z - \bar{z}} \geq 0, \quad \frac{z_1 q(z)^* p_1(z) - \bar{z} p_1(z)^* q_1(z)}{z - \bar{z}} \geq 0, \tag{4.23}$$

respectively. Thus, the inequalities (4.20) are equivalent to the fact that $\{p_1, q_1\}$ of the form (4.22) is a Stieltjes pair, or equivalently, that $\{p, q\}$ admits a representation (4.17) for some $\{p_1, q_1\} \in \mathcal{SP}$. Finally, it follows from (4.16) and (4.22) that the identities (4.18) and (4.19) are equivalent:

$$bq_1(z) = bq(z) - ba^* \tilde{k}^{-1} ap(z) = bq(z) - ap(z). \blacksquare$$

III. The case $k > 0$ and $\tilde{k} = 0$. Now it follows from (4.2) that $\omega ba^* = \bar{\omega} ab^*$ and

$$k = -\omega^{-1} ab^* > 0. \tag{4.24}$$

Theorem 4.5 *If $k > 0$ and $\tilde{k} = 0$, then all solutions $\{p, q\}$ of Problem 4.1 can be parametrized by*

$$\begin{pmatrix} p(z) \\ q(z) \end{pmatrix} = \begin{pmatrix} I_m & -\frac{1}{z} b^* k^{-1} b \\ 0 & I_m \end{pmatrix} \begin{pmatrix} p_1(z) \\ q_1(z) \end{pmatrix}, \tag{4.25}$$

where the parameter runs through the set of $\{p_1, q_1\} \in \mathcal{SP}$ for which

$$ap_1(z) \equiv 0. \tag{4.26}$$

Proof: Let $\{p, q\}$ satisfy (4.3) and (4.4). Since $\tilde{k} = 0$, (4.4) implies

$$ap(z) \equiv \frac{\omega}{z} bq(z). \tag{4.27}$$

Hence (4.3) and (4.4) are equivalent to

$$\begin{pmatrix} k & -\frac{1}{z} bq(z) \\ -\frac{1}{z} q(z)^* b^* & \frac{q(z)^* p(z) - p(z)^* q(z)}{z - \bar{z}} \end{pmatrix} \geq 0, \quad \frac{zq(z)^* p(z) - \bar{z} p(z)^* q(z)}{z - \bar{z}} \geq 0. \tag{4.28}$$

Since $k > 0$, the first inequality in (4.28) can be written as

$$\frac{q(z)^* p(z) - p(z)^* q(z)}{z - \bar{z}} - \frac{1}{|z|^2} q(z)^* b^* k^{-1} bq(z) \geq 0. \tag{4.29}$$

Let $\{p_1, q_1\}$ be defined by

$$\begin{pmatrix} p_1(z) \\ q_1(z) \end{pmatrix} = \begin{pmatrix} I_m & \frac{1}{z} b^* k^{-1} b \\ 0 & I_m \end{pmatrix} \begin{pmatrix} p(z) \\ q(z) \end{pmatrix}. \tag{4.30}$$

In terms of this pair, the inequality (4.29) and the second inequality in (4.28) take the same form as in (4.23). By the arguments following (4.23) we get the representation (4.25) with $\{p_1, q_1\} \in \mathcal{SP}$. The equivalence of the identities (4.26) and (4.27) follows from (4.24) and (4.30):

$$ap_1(z) = ap(z) + \frac{1}{z}ab^*k^{-1}bq(z) = ap(z) - \frac{\omega}{z}bq(z). \blacksquare$$

IV. The case $k = \tilde{k} = 0$. It follows from (4.2) that $ba^* = 0$, that is the row vectors a and b are orthogonal. Now the inequalities (4.3) and (4.4) are equivalent to the fact that $\{p, q\}$ is a Stieltjes pair such that

$$ap(z) \equiv bq(z), \quad ap(z) \equiv \frac{\omega}{z}bq(z),$$

or equivalently,

$$ap(z) \equiv 0, \quad bq(z) \equiv 0. \tag{4.31}$$

Theorem 4.6 *If $k = \tilde{k} = 0$, then a pair $\{p, q\} \in \mathcal{SP}$ is a solution of Problem 4.1 if and only if it satisfies (4.31).*

Note that of the four cases only in this case it can happen that a or b is zero; then in fact we have in (4.31) only one condition. If $a = b = 0$ then every pair $\{p, q\} \in \mathcal{SP}$ is a solution of Problem 4.1.

If a and b are orthogonal, it is shown in, for example, [3] that up to equivalence, a pair $\{p, q\} \in \mathcal{SP}$ which satisfies the conditions (4.31) has the block diagonal form

$$p(z) = U_{ab} \begin{pmatrix} 1 & & \\ & 0 & \\ & & \hat{p}(z) \end{pmatrix}, \quad q(z) = U_{ab} \begin{pmatrix} 0 & & \\ & 1 & \\ & & \hat{q}(z) \end{pmatrix}, \tag{4.32}$$

where $U_{ab} \in \mathbb{C}^{m \times m}$ is a unitary matrix which depends only on a and b and $\{\hat{p}, \hat{q}\}$ is a Stieltjes pair of smaller size. If in particular $a = 0$ or $b = 0$, than we have in (4.31) at most one condition: $bq(z) \equiv 0$ or $ap(z) \equiv 0$, and a pair $\{p, q\} \in \mathcal{SP}$ which satisfies the remaining condition has, up to equivalence, the form

$$p(z) = U_b \begin{pmatrix} 0 & \\ & \hat{p}(z) \end{pmatrix}, \quad q(z) = U_b \begin{pmatrix} 1 & \\ & \hat{q}(z) \end{pmatrix}, \quad \text{if } a = 0, \tag{4.33}$$

or

$$p(z) = U_a \begin{pmatrix} 1 & \\ & \hat{p}(z) \end{pmatrix}, \quad q(z) = U_a \begin{pmatrix} 0 & \\ & \hat{q}(z) \end{pmatrix}, \quad \text{if } b = 0. \tag{4.34}$$

Here U_b and U_a are unitary matrices which depend only on b and a , respectively. In particular, the representations (4.33) and (4.34) give a complete characterization of all admissible parameters $\{p_1, q_1\}$ in the linear transformations (4.17) and (4.25).

Remark 4.7 Using (4.14) one can easily show that the linear transformation (4.15) maps a strict Stieltjes pair $\{p_1, q_1\}$ into a strict Stieltjes pair $\{p, q\}$. Hence in the nondegenerate case Problem 4.1 always has strict Stieltjes solutions. From the representations (4.32) and (4.33), it follows that in the three degenerate cases, none of the solutions of Problem 4.1 is strict.

5 Excluded parameters

In this section we describe more explicitly the set of excluded parameters, that is, the set all pairs $\{p, q\} \in \mathcal{SP}$ which satisfy together with all equivalent pairs at least one of the following n conditions

$$\det \left\{ \lim_{z \rightarrow z_j} (z - z_j) (\theta_{21}(z)p(z) + \theta_{22}(z)q(z)) \right\} = 0, \tag{5.1}$$

where the z_j 's, $j = 1, \dots, n$, are interpolation points of Problem 1.2. Set

$$E_j = \underbrace{(0_m, \dots, 0_m, I_m, 0_m, \dots, 0_m)}_j \in \mathbb{C}^{m \times mn}. \tag{5.2}$$

Then from the block decomposition (3.6) of Θ and (3.21) we get

$$(\alpha_j, \beta_j) := \lim_{z \rightarrow z_j} (z - z_j) (\theta_{21}(z), \theta_{22}(z)) = E_j P^{-1} (G_2^*, -G_1^*) M,$$

where by (2.23),

$$\alpha_j = E_j Z \tilde{P}^{-1} G_2^* \in \mathbb{C}^{m \times m}, \quad \beta_j = -E_j P^{-1} G_1^* \in \mathbb{C}^{m \times m}. \tag{5.3}$$

Thus (5.1) can be written as

$$\det (\alpha_j p(z_j) + \beta_j q(z_j)) = 0 \tag{5.4}$$

with α_j and β_j as in (5.3). Substituting (3.6) into (2.16) we obtain in particular,

$$\frac{\theta_{22}(z)\theta_{21}(\omega)^* - \theta_{21}(z)\theta_{22}(\omega)^*}{z - \bar{\omega}} = G_2(zI - Z)^{-1} P^{-1} (\bar{\omega}I - Z^*)^{-1} G_2^*. \tag{5.5}$$

The block decomposition

$$\tilde{\Theta}(z) = \begin{pmatrix} \theta_{11}(z) & z\theta_{12}(z) \\ z^{-1}\theta_{21}(z) & \theta_{22}(z) \end{pmatrix}$$

follows from (2.4) and (3.6), and if it is substituted into (2.18) we get

$$\frac{z\theta_{22}(z)\theta_{21}(\omega)^* - \bar{\omega}\theta_{21}(z)\theta_{22}(\omega)^*}{z - \bar{\omega}} = z\bar{\omega}G_2(zI - Z)^{-1} \tilde{P}^{-1} (\bar{\omega}I - Z^*)^{-1} G_2^*. \tag{5.6}$$

Let

$$P^{-1} = (\pi_{j\ell})_{j,\ell=1}^n, \quad \tilde{P}^{-1} = (\tilde{\pi}_{j\ell})_{j,\ell=1}^n, \quad \pi_{j\ell}, \tilde{\pi}_{j\ell} \in \mathbb{C}^{m \times m}, \tag{5.7}$$

be the block decompositions of P^{-1} and \tilde{P}^{-1} of the inverses of the $mn \times mn$ Pick matrices in (1.3). Consider (5.5) and (5.6) for $\omega = z$, multiply by $|z - z_j|^2$ and let $z \rightarrow z_j$, then, on account of (2.29) and (5.3), we obtain

$$\frac{\beta_j \alpha_j^* - \alpha_j \beta_j^*}{z_j - \bar{z}_j} = E_j P^{-1} E_j^* = \pi_{jj}, \quad \frac{z_j \beta_j \alpha_j^* - \bar{z}_j \alpha_j \beta_j^*}{z_j - \bar{z}_j} = |z_j|^2 E_j \tilde{P}^{-1} E_j^* = |z_j|^2 \tilde{\pi}_{jj}. \tag{5.8}$$

Of interest here is that π_{jj} and $\tilde{\pi}_{jj}$ are the diagonal entries in the decompositions (5.7); see below. Returning to (5.4) we conclude that a pair $\{p, q\} \in \mathcal{SP}$ is an excluded parameter in the fractional linear transformation (3.18) if and only if it satisfies together with all equivalent pairs which are also analytic at z_j , the condition

$$h(\alpha_j p(z_j) + \beta_j q(z_j)) = 0 \tag{5.9}$$

at least for one $j \in \{1, \dots, n\}$ and for some nonzero row vector $h \in \mathbb{C}^{1 \times m}$. To show that Problem 1.2 has a solution we first verify that the kernels of α_j and β_j have a zero intersection, otherwise each pair $\{p, q\}$, which is analytic at z_j , satisfies (5.9) for $h \in \text{Ker } \alpha_j \cap \text{Ker } \beta_j$ and is therefore, an excluded parameter in the transformation (3.18).

Lemma 5.1 *Let α_j and β_j be defined by (5.3). Then*

$$\alpha_j \alpha_j^* + \beta_j \beta_j^* > 0. \tag{5.10}$$

Proof: Let $h \in \mathbb{C}^m$ be a row vector such that

$$h\alpha_j = hE_j Z \tilde{P}^{-1} G_2^* = 0, \quad h\beta_j = hE_j P^{-1} G_1^* = 0. \tag{5.11}$$

We show that $h = 0$, which implies (5.10). If we multiply both sides of (2.13) by $hE_j P^{-1}$ from the left and by $\tilde{P}^{-1} G_2^*$ from the right, we get

$$hE_j P^{-1} G_2^* - hE_j Z \tilde{P}^{-1} G_2^* = hE_j P^{-1} G_1^* G_2 \tilde{P}^{-1} G_2^*,$$

hence in view of (5.11),

$$hE_j P^{-1} G_2^* = 0. \tag{5.12}$$

If we multiply both sides of (2.24) by $hE_j P^{-1}$ from the left, by $P^{-1} E_\ell^*$ from the right and invoke (5.11) and (5.12), we obtain

$$hE_j P^{-1} Z^* E_\ell^* - hE_j Z P^{-1} E_\ell^* = 0, \quad \ell = 1, \dots, n.$$

From the formulas (2.29), (5.7), and (5.2) of the matrices Z , P^{-1} , and E_j (and E_ℓ) the last equality simply reads:

$$h\pi_{j\ell} \bar{z}_\ell = h z_j \pi_{j\ell}, \quad \ell = 1, \dots, n.$$

As all interpolation points belong to \mathbb{C}^+ , so in particular $z_j \neq \bar{z}_\ell$,

$$h\pi_{j\ell} = 0, \quad \ell = 1, \dots, n,$$

that is, $hE_j P^{-1} = 0$, hence $h = 0$. ■

We now apply the results from the previous section to describe the set of pairs $\{p, q\}$ which satisfy the left-sided interpolation condition (5.9) for each $j \in \{1, \dots, n\}$ and nonzero row vector $h \in \mathbb{C}^m$ separately. The union of all these sets (taken over j and h) will

form the set of excluded parameters. For fixed j and h the condition (5.9) is of the form (4.1) of Problem 4.1 with

$$a = h\alpha_j, \quad b = -h\beta_j, \quad \omega = z_j. \tag{5.13}$$

If we substitute (5.13) into the defining expressions (4.2) for the Pick matrices k and \tilde{k} and use (5.8), we get

$$k = h \frac{\alpha_j \beta_j^* - \beta_j \alpha_j^*}{z_j - \bar{z}_j} h^* = -h\pi_{jj} h^*, \quad \tilde{k} = h \frac{\bar{z}_j \alpha_j \beta_j^* - z_j \beta_j \alpha_j^*}{z_j - \bar{z}_j} h^* = -|z_j|^2 h \tilde{\pi}_{jj} h^*. \tag{5.14}$$

From (5.14) and the solvability criteria (4.2) of Problem 4.1, we conclude that there exists a pair $\{p, q\}$ satisfying (5.9) if and only if $h\pi_{jj}h^* \leq 0$ and $h\tilde{\pi}_{jj}h^* \leq 0$. In particular, if $\pi_{jj} > 0$ or $\tilde{\pi}_{jj} > 0$, then there is no pair satisfying (5.9) and therefore, every pair $\{p, q\} \in \mathcal{SP}$ is admissible at z_j . According to Remark 4.7, if $\pi_{jj} \geq 0$ or $\tilde{\pi}_{jj} \geq 0$, then there is no strict Stieltjes pair satisfying (5.9) and therefore, every strict Stieltjes pair $\{p, q\}$ is admissible at z_j . We see that the signs of the $m \times m$ Hermitian matrices π_{jj} and $\tilde{\pi}_{jj}$ on the diagonal of the inverse of the Pick matrices P and \tilde{P} play an important role in the description of the excluded parameters. To describe this role we partition \mathbb{C}^m in two different ways:

$$\mathbb{C}^m = \mathcal{L}_j^+ \cup \mathcal{L}_j^- \cup \mathcal{L}_j^0 = \tilde{\mathcal{L}}_j^+ \cup \tilde{\mathcal{L}}_j^- \cup \tilde{\mathcal{L}}_j^0,$$

where $\mathcal{L}_j^+, \mathcal{L}_j^-$ and \mathcal{L}_j^0 consist of all row vectors $h \in \mathbb{C}^m$ for which $h\pi_{jj}h^*$ is positive, negative and zero, respectively, and where $\tilde{\mathcal{L}}_j^+, \tilde{\mathcal{L}}_j^-$ and $\tilde{\mathcal{L}}_j^0$ are defined similarly with respect to $\tilde{\pi}_{jj}$. In other words, the indicated spaces consist of positive, negative and neutral vectors with respect to the indefinite inner product in \mathbb{C}^m induced by Hermitian matrices π_{jj} and $\tilde{\pi}_{jj}$, respectively. From the previous section we get the following description of all excluded parameters of the transformation (3.18). That is, every excluded parameter belongs to one of the sets described by the following four corollaries to Theorems 4.3 – 4.6, respectively.

Corollary 5.2 Assume $\mathcal{L}_j^- \cap \tilde{\mathcal{L}}_j^- \neq \{0\}$. Define for $h \in \mathcal{L}_j^- \cap \tilde{\mathcal{L}}_j^-$, $h \neq 0$,

$$\Psi_{j,h}(z) = I_{2m} + \frac{1}{z - \bar{z}_j} \begin{pmatrix} -\beta_j^* & -\bar{z}_j \beta_j^* \\ \alpha_j^* & z \alpha_j^* \end{pmatrix} \begin{pmatrix} 0 & h^*(h\pi_{jj}h^*)^{-1}h\beta_j \\ -\frac{1}{|z_j|^2} h^*(h\tilde{\pi}_{jj}h^*)^{-1}h\alpha_j & 0 \end{pmatrix}, \tag{5.15}$$

where α_j, β_j are given by (5.3). Then all pairs $\{p, q\}$ of the form

$$\begin{pmatrix} p(z) \\ q(z) \end{pmatrix} = \Psi_{j,h}(z) \begin{pmatrix} p_1(z) \\ q_1(z) \end{pmatrix}, \quad \{p_1, q_1\} \in \mathcal{SP}, \tag{5.16}$$

are excluded parameters.

Corollary 5.3 Assume $\mathcal{L}_j^0 \cap \tilde{\mathcal{L}}_j^- \neq \{0\}$. For each $h \in \mathcal{L}_j^0 \cap \tilde{\mathcal{L}}_j^-$, $h \neq 0$, all pairs $\{p, q\}$ of the form

$$\begin{pmatrix} p(z) \\ q(z) \end{pmatrix} = \begin{pmatrix} I_m & 0 \\ -\frac{1}{|z_j|^2} \alpha_j^* h^*(h\tilde{\pi}_{jj}h^*)^{-1}h\alpha_j & I_m \end{pmatrix} \begin{pmatrix} p_1(z) \\ q_1(z) \end{pmatrix}, \tag{5.17}$$

where $\{p_1, q_1\}$ is an arbitrary Stieltjes pair such that

$$h\beta_j q_1(z) \equiv 0, \tag{5.18}$$

are excluded parameters.

Corollary 5.4 Assume $\mathcal{L}_j^- \cap \tilde{\mathcal{L}}_j^\circ \neq \{0\}$. For each $h \in \mathcal{L}_j^- \cap \tilde{\mathcal{L}}_j^\circ$, $h \neq 0$, all pairs $\{p, q\}$ of the form

$$\begin{pmatrix} p(z) \\ q(z) \end{pmatrix} = \begin{pmatrix} I_m & \frac{1}{z} \beta_j^* h^* (h \pi_{jj} h^*)^{-1} h \beta_j \\ 0 & I_m \end{pmatrix} \begin{pmatrix} p_1(z) \\ q_1(z) \end{pmatrix}, \tag{5.19}$$

where $\{p_1, q_1\}$ is an arbitrary Stieltjes pair such that

$$h \alpha_j p_1(z) \equiv 0, \tag{5.20}$$

are excluded parameters.

Corollary 5.5 Assume $\mathcal{L}_j^\circ \cap \tilde{\mathcal{L}}_j^\circ \neq \{0\}$. For each $h \in \mathcal{L}_j^\circ \cap \tilde{\mathcal{L}}_j^\circ$, $h \neq 0$, all Stieltjes pairs $\{p, q\}$ satisfying

$$h \alpha_j p(z) \equiv h \beta_j q(z) \equiv 0$$

are excluded parameters. All other pairs $\{p, q\} \in \mathcal{SP}$ are admissible: the function S defined by (3.18) is a solution of Problem 1.2.

6 The existence of a solution

In this section we prove that Problem 1.2 is always solvable.

Lemma 6.1 *The Pick matrices*

$$P_\Phi = \left(\frac{\Phi_j - \Phi_\ell^*}{z_j - \bar{z}_\ell} \right)_{\ell, j=1}^n, \quad \tilde{P}_\Phi = \left(\frac{z_j \Phi_j - \bar{z}_\ell \Phi_\ell^*}{z_j - \bar{z}_\ell} \right)_{\ell, j=1}^n$$

with

$$\Phi_j = \ln \left(1 - \frac{1}{z_j} \right) I_m = \left(\ln \left| 1 - \frac{1}{z_j} \right| + i \arg \left(1 - \frac{1}{z_j} \right) \right) I_m, \quad j = 1, \dots, n, \tag{6.1}$$

are strictly positive.

Proof: It is easily seen that

$$\begin{aligned} P_\Phi &= \int_0^1 \begin{pmatrix} (t - \bar{z}_1)^{-1} I_m \\ \vdots \\ (t - \bar{z}_n)^{-1} I_m \end{pmatrix} dt \left((t - z_1)^{-1} I_m, \dots, (t - z_n)^{-1} I_m \right) \geq 0, \\ \tilde{P}_\Phi &= \int_0^1 \begin{pmatrix} (t - \bar{z}_1)^{-1} I_m \\ \vdots \\ (t - \bar{z}_n)^{-1} I_m \end{pmatrix} t dt \left((t - z_1)^{-1} I_m, \dots, (t - z_n)^{-1} I_m \right) \geq 0. \end{aligned}$$

If $g P_\Phi = 0$ for some row vector $g = (g_1, \dots, g_n) \in \mathbb{C}^{nm}$ where each g_j is a row vector in \mathbb{C}^m , we get

$$0 = g P_\Phi g^* = \int_0^1 \left(\sum_{j=1}^n \frac{g_j}{t - \bar{z}_j} \right) \left(\sum_{j=1}^n \frac{g_j^*}{t - z_j} \right) dt,$$

and therefore

$$\sum_{j=1}^n \frac{g_j^*}{t - z_j} \equiv 0.$$

Since the functions $\frac{1}{t - z_j}$ are linearly independent, each $g_j = 0$, hence P_Φ has a zero kernel. A similar argument proves the strict positivity of \tilde{P}_Φ . ■

Theorem 6.2 *Problem 1.2 has a solution.*

Proof: By Theorem 3.4, it suffices to establish the existence of a pair $\{p, q\} \in \mathcal{SP}$ which satisfies the conditions (3.19), or equivalently (see (5.4)), the conditions

$$\det (\alpha_j p(z_j) + \beta_j q(z_j)) \neq 0, \quad j = 1, \dots, n, \tag{6.2}$$

where $\alpha_j, \beta_j \in \mathbb{C}^{m \times m}$ are defined by (5.3). As observed in the previous section, after the formulas (5.14), if $\pi_{jj} \geq 0$ or $\tilde{\pi}_{jj} \geq 0$, or equivalently, $\mathcal{L}_j^- \cap \tilde{\mathcal{L}}_j^- = \{0\}$, then every pair $\{p, q\} \in \mathcal{SP}$ is admissible at z_j , that is, the fractional linear transformation (3.18) applied to this pair yields a Stieltjes function $S(z)$ which satisfies the interpolation condition at $z = z_j$ in (1.5). Thus the Problem 1.2 is solvable if $\mathcal{L}_j^- \cap \tilde{\mathcal{L}}_j^- = \{0\}$ for all $j \in \{1, \dots, n\}$.

Now we assume that not all these intersections are trivial: let J_1 be the set of those $j \in \{1, \dots, n\}$ for which $\mathcal{L}_j^- \cap \tilde{\mathcal{L}}_j^- \neq \{0\}$, then $J_1 \neq \emptyset$. For $j \in J_1$, let $\mathbf{h}_j \in \mathbb{C}^{r_j \times m}$ be a matrix whose rows form a basis in $\mathcal{L}_j^- \cap \tilde{\mathcal{L}}_j^-$. For $j \in J_1$ and every nonzero row vector $f \in \mathbb{C}^{r_j}$,

$$f \mathbf{h}_j \pi_{jj} \mathbf{h}_j^* f^* < 0, \quad f \mathbf{h}_j \tilde{\pi}_{jj} \mathbf{h}_j^* f^* < 0. \tag{6.3}$$

Therefore, the left-sided interpolation problem

$$T(z) \in \mathcal{S}^+, \quad \mathbf{h}_j \alpha_j T(z_j) = -\mathbf{h}_j \beta_j \tag{6.4}$$

has a solution $T_j(z)$, say, in the strict Stieltjes class \mathcal{S}^+ , because by (5.8) and (6.3), the Pick matrices corresponding to this problem are positive:

$$\begin{aligned} k &= \mathbf{h}_j \frac{\alpha_j \beta_j^* - \beta_j \alpha_j^*}{z_j - \bar{z}_j} \mathbf{h}_j^* = -\mathbf{h}_j \pi_{jj} \mathbf{h}_j^* > 0, \\ \tilde{k} &= \mathbf{h}_j \frac{\bar{z}_j \alpha_j \beta_j^* - z_j \beta_j \alpha_j^*}{z_j - \bar{z}_j} \mathbf{h}_j^* = -\mathbf{h}_j \tilde{\pi}_{jj} \mathbf{h}_j^* > 0. \end{aligned}$$

The matrices

$$\Upsilon_j := T_j(z_j) \in \mathbb{C}^{m \times m}, \quad j \in J_1,$$

satisfy

$$\mathbf{h}_j \alpha_j \Upsilon_j = -\mathbf{h}_j \beta_j, \quad \frac{\Upsilon_j - \Upsilon_j^*}{z_j - \bar{z}_j} > 0, \quad \frac{z_j \Upsilon_j - \bar{z}_j \Upsilon_j^*}{z_j - \bar{z}_j} > 0. \tag{6.5}$$

We set for $j \in \{1, \dots, n\} \setminus J_1$, $\Upsilon_j := \Phi_j$ given by (6.1) (the choice is not important, for example, $\Upsilon_j = 0$ would also work). The Pick matrices

$$P_\Upsilon = \left(\frac{\Upsilon_j - \Upsilon_\ell^*}{z_j - \bar{z}_\ell} \right)_{\ell, j=1}^n, \quad \tilde{P}_\Upsilon = \left(\frac{z_j \Upsilon_j - \bar{z}_\ell \Upsilon_\ell^*}{z_j - \bar{z}_\ell} \right)_{\ell, j=1}^n$$

have positive diagonal blocks but in general, they are not positive. But by Lemma 6.1, there exists a $\delta > 0$ such that

$$P_\Upsilon + \delta P_\Phi > 0, \quad \tilde{P}_\Upsilon + \delta \tilde{P}_\Phi > 0. \tag{6.6}$$

The matrices on the left-hand side are the Pick matrices associated with the following Nevanlinna–Pick problem in the strict Stieltjes class:

$$T(z) \in \mathcal{S}^+, \quad T(z_j) = \Upsilon_j + \delta \Phi_j, \quad j = 1, \dots, n. \tag{6.7}$$

The inequalities in (6.6) are the solvability criteria for this problem, and hence this problem has a solution $T(z)$, say. We show that the strict Stieltjes pair $\{p, q\}$ defined by

$$p(z) = T(z), \quad q(z) \equiv I_m, \tag{6.8}$$

satisfies the conditions (6.2). By Theorem 3.4, the fractional linear transformation (3.18) applied to this pair gives a solution $S(z)$ of Problem 1.2, which completes the proof.

We argue by contradiction. Assume that for some $j \in \{1, \dots, n\}$, there is a nonzero row vector $f_j \in \mathbb{C}^m$ such that

$$f_j (\alpha_j p(z_j) + \beta_j q(z_j)) = f_j (\alpha_j T(z_j) + \beta_j) = 0. \tag{6.9}$$

The space \mathbb{C}^m is the union of the three (not necessarily disjoint) sets $\mathcal{L}_j^+ \cup \tilde{\mathcal{L}}_j^+$, $\mathcal{L}_j^\circ \cup \tilde{\mathcal{L}}_j^\circ$, and $\mathcal{L}_j^- \cap \tilde{\mathcal{L}}_j^-$, and accordingly we prove three noninclusions which contradict $f_j \in \mathbb{C}^m$:

(i) $f_j \notin \mathcal{L}_j^+ \cup \tilde{\mathcal{L}}_j^+$: A Stieltjes pair $\{p, q\}$ cannot satisfy (6.9) if f_j does belong to this set; see the discussion after (5.14).

(ii) $f_j \notin \mathcal{L}_j^\circ \cup \tilde{\mathcal{L}}_j^\circ$: By Remark 4.7, a strict Stieltjes pair $\{p, q\}$ cannot satisfy (6.9) if f_j does belong to this set.

(iii) $f_j \notin \mathcal{L}_j^- \cap \tilde{\mathcal{L}}_j^-$. Assume that

$$f_j \in \mathcal{L}_j^- \cap \tilde{\mathcal{L}}_j^-. \tag{6.10}$$

We show that (6.9) implies that then also $f_j \in \mathcal{L}_j^\circ \cap \tilde{\mathcal{L}}_j^\circ$, which contradicts the inclusion (6.10). We use the special structure of the chosen pair (6.8). By (6.10), $f_j = g_j h_j$ for some nonzero row vector $g_j \in \mathbb{C}^{r_j}$. Because of (6.7), the first relation in (6.5) and (6.9), we have

$$0 = f_j (\alpha_j T(z_j) + \beta_j) = g_j (\mathbf{h}_j \alpha_j \Upsilon_j + \delta \mathbf{h}_j \alpha_j \Phi_j + \mathbf{h}_j \beta_j) = \delta g_j \mathbf{h}_j \alpha_j \Phi_j = \delta f_j \alpha_j \Phi_j.$$

Since Φ_j is invertible (see (6.1)), $f_j \alpha_j = 0$. In view of (5.8),

$$f_j \pi_{jj} f_j^* = f_j \frac{\beta_j \alpha_j^* - \alpha_j \beta_j^*}{z_j - \bar{z}_j} f_j^* = 0, \quad |z_j|^2 f_j \tilde{\pi}_{jj} f_j^* = f_j \frac{z_j \mathbf{e} t a_j \alpha_j^* - \bar{z}_j \alpha_j \beta_j^*}{z_j - \bar{z}_j} f_j^* = 0,$$

that is, $f_j \in \mathcal{L}_j^\circ \cap \tilde{\mathcal{L}}_j^\circ$, which contradicts (6.10). ■

7 The scalar case

In this section we apply the preceding analysis to the scalar case: $m = 1$. Then the characterization of the excluded parameters in the parametrization of all solutions of the scalar Problem 1.2 is especially explicit and elegant. For the scalar case the rule

$$\{p, q\} \leftrightarrow pq^{-1} \quad (q \neq 0) \quad \text{and} \quad \{p, 0\} \leftrightarrow \infty,$$

establishes a one to one correspondence between the equivalence classes on \mathcal{SP} and the elements of the extended Stieltjes class $\tilde{\mathcal{S}} = \mathcal{S} \cup \{\infty\}$. With the data (the interpolation points z_1, \dots, z_n and the values S_1, \dots, S_n) and the $n \times n$ Pick matrices P and \tilde{P} in (1.3) we associate the matrices (see (2.29)):

$$G_1 = (S_1, \dots, S_n), \quad G_2 = (1, \dots, 1), \quad Z = \text{diag}(z_1, \dots, z_n),$$

and the parametrization matrix (2.20):

$$\begin{aligned} \Theta(z) &= \begin{pmatrix} \theta_{11}(z) & \theta_{12}(z) \\ \theta_{21}(z) & \theta_{22}(z) \end{pmatrix} = \\ &= \left\{ I_{2m} + \begin{pmatrix} G_1 \\ G_2 \end{pmatrix} (zI - Z)^{-1} P^{-1} (G_2^*, -G_1^*) \right\} \begin{pmatrix} I_m & 0 \\ G_2 \tilde{P}^{-1} G_2^* & I_m \end{pmatrix}. \end{aligned}$$

According to Theorem 3.4, all solutions $S(z)$ of the scalar Problem 1.2 are parametrized by the fractional linear transformation

$$S(z) = \frac{\theta_{11}(z)T(z) + \theta_{12}(z)}{\theta_{21}(z)T(z) + \theta_{22}(z)}, \tag{7.1}$$

where the parameter $T(z)$ runs through the extended Stieltjes class $\tilde{\mathcal{S}}$ restricted only by the n conditions

$$\lim_{z \rightarrow z_j} (z - z_j) (\theta_{21}(z)T(z) + \theta_{22}(z)) \neq 0, \quad j = 1, \dots, n.$$

For $T = \infty$, $S = \theta_{11}/\theta_{21}$. These restrictions are equivalent to the conditions (see (5.9))

$$\alpha_j T(z_j) + \beta_j \neq 0, \quad j = 1, \dots, n,$$

where the complex coefficients are defined by

$$\alpha_j = E_j Z \tilde{P}^{-1} G_2^*, \quad \beta_j = -E_j P^{-1} G_1^*,$$

and E_j is the j -th unit row vector in \mathbb{C}^m :

$$E_j = \underbrace{(0, \dots, 0, 1, 0, \dots, 0)}_j$$

(see (5.2), (5.3)). A function $T(z) \in \tilde{\mathcal{S}}$ is called an excluded parameter for the interpolation problem if the corresponding function $S(z)$ in (7.1) is not a solution of problem 1.2, or equivalently, for at least one $j \in \{1, \dots, n\}$ $T(z)$ satisfies the condition

$$\alpha_j T(z_j) + \beta_j = 0. \tag{7.2}$$

The excluded parameters can be classified according to the sign of the diagonal entries π_{jj} and $\tilde{\pi}_{jj}$ (they are real numbers) of the inverses of the Pick matrices P and \tilde{P} , respectively.

Theorem 7.1 (1). Assume at least one of the two numbers π_{jj} and $\tilde{\pi}_{jj}$ is positive. Then the relation (7.2) is satisfied for no $T \in \tilde{\mathcal{S}}$. Therefore for all parameters, the function S defined by (7.1) is analytic at z_j and satisfies $S(z_j) = S_j$.

(2). Assume π_{jj} and $\tilde{\pi}_{jj}$ are both negative. Then the relation (7.2) is satisfied for infinitely many $T \in \tilde{\mathcal{S}}$: they are parametrized by the formula

$$T(z) = \frac{\psi_{11}(z)\hat{T}(z) + \psi_{12}(z)}{\psi_{21}(z)\hat{T}(z) + \psi_{22}(z)}, \tag{7.3}$$

where

$$\begin{aligned} \Psi(z) &= \begin{pmatrix} \psi_{11}(z) & \psi_{12}(z) \\ \psi_{21}(z) & \psi_{22}(z) \end{pmatrix} = \\ &= I_{2m} + \frac{1}{z - \bar{z}_j} \begin{pmatrix} -\beta_j^* & -\bar{z}_j\beta_j^* \\ \alpha_j^* & z\alpha_j^* \end{pmatrix} \begin{pmatrix} 0 & \pi_{jj}^{-1}\beta_j \\ -\frac{1}{|z_j|^2}\tilde{\pi}_{jj}^{-1}\alpha_j & 0 \end{pmatrix} \end{aligned}$$

and $\hat{T}(z)$ is a free parameter from $\tilde{\mathcal{S}}$. For all other parameters, S is analytic at z_j and satisfies $S(z_j) = S_j$.

(3). Assume $\pi_{jj} = 0$ and $\tilde{\pi}_{jj} < 0$. Then the relation (7.2) is satisfied for exactly one parameter $T \in \tilde{\mathcal{S}}$: it is given by

$$T(z) \equiv -\frac{1}{|z_j|^2} (\alpha_j^* \tilde{\pi}_{jj}^{-1} \alpha_j)^{-1} > 0.$$

For all other parameters, S is analytic at z_j and satisfies $S(z_j) = S_j$.

(4). Assume $\pi_{jj} < 0$ and $\tilde{\pi}_{jj} = 0$. Then the relation (7.2) is satisfied for exactly one parameter $T \in \tilde{\mathcal{S}}$: it is given by

$$T(z) = \frac{1}{z} \beta_j^* \pi_{jj}^{-1} \beta_j.$$

For all other parameters, S is analytic at z_j and satisfies $S(z_j) = S_j$.

(5). Assume $\pi_{jj} = 0$ and $\tilde{\pi}_{jj} = 0$. Then $\alpha_j\beta_j = 0$ and the only excluded parameter is

$$T \equiv 0 \text{ (if } \beta_j = 0) \text{ or } T \equiv \infty \text{ (if } \alpha_j = 0).$$

Proof: (1) follows from the solvability criterion of the Nevanlinna–Pick problem for Stieltjes pairs, (2) is a consequence of (5.16), (5.15). It follows from (5.8) that if at least one of the numbers π_{jj} and $\tilde{\pi}_{jj}$ is not zero, then both α_j, β_j differ from zero. (3) follows from (5.17), since (up to equivalence) the only pair $\{p_1, q_1\}$ satisfying (5.18) is $\{1, 0\}$. Similarly, (4) follows from (5.19), since (up to equivalence) the only pair $\{p_1, q_1\}$ satisfying (5.18) is $\{0, 1\}$. (5) is clear. By Lemma 5.1, the equalities $\alpha_j = 0, \beta_j = 0$ do not hold simultaneously. ■

Example 7.2 Take $z_1 = 1 + i, z_2 = 2 + i, S_1 = 1 + i$ and $S_2 = 1 - i$. Then

$$P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \tilde{P} = \begin{pmatrix} 2 & 1 - i \\ 1 + i & -1 \end{pmatrix},$$

$\text{sq}_-(P) = \text{sq}_-(\tilde{P}) = 1$, and

$$P^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \tilde{P}^{-1} = \frac{1}{4} \begin{pmatrix} 1 & 1-i \\ 1+i & -2 \end{pmatrix}.$$

According to (2.14),

$$M = \begin{pmatrix} 1 & 0 \\ \frac{1}{4}(1, 1) \begin{pmatrix} 1 & 1-i \\ 1+i & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{1}{4} & 1 \end{pmatrix},$$

and by (2.20),

$$\begin{aligned} \Theta(z) &= \left\{ I_2 + \begin{pmatrix} S_1 & S_2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} (z-z_1)^{-1} & 0 \\ 0 & (z-z_2)^{-1} \end{pmatrix} \begin{pmatrix} 1 & -S_1^* \\ 1 & -S_2^* \end{pmatrix} \right\} \begin{pmatrix} 1 & 0 \\ \frac{1}{4} & 1 \end{pmatrix} \\ &= \frac{1}{(z-1-i)(z-2-i)} \begin{pmatrix} z^2 - 3z + \frac{5}{2} & 2 \\ \frac{z^2-3z}{4} & z^2 - 3z + 4 \end{pmatrix}. \end{aligned}$$

The solutions of the interpolation problem in the class \mathcal{S}_1^1 are given by the formula

$$S(z) = \frac{(z^2 - 3z + \frac{5}{2})T(z) + 2}{\frac{z^2-3z}{4}T(z) + (z^2 - 3z + 4)}.$$

Since

$$\left. \frac{z^2 - 3z + 4}{z^2 - 3z} \right|_{z=1+i} = \frac{-1 + 2i}{5} \quad \text{and} \quad \text{Im} \frac{-1 + 2i}{5} > 0,$$

the function $S(z)$ is analytic at $z = 1 + i$ for every parameter $T(z)$. Since

$$\left. \frac{z^2 - 3z + 4}{z^2 - 3z} \right|_{z=2+i} = \frac{1 - 2i}{3}, \quad \left. z \frac{z^2 - 3z + 4}{z^2 - 3z} \right|_{z=2+i} = \frac{4 - 3i}{3}$$

and

$$\text{Im} \frac{1 - 2i}{3} < 0, \quad \text{Im} \frac{4 - 3i}{3} < 0$$

there are infinitely many excluded parameters $T(z)$ which solve the interpolation problem

$$T(2 + i) = \frac{4}{3}(-1 + 2i);$$

they are described by the fractional linear transformation (7.3).

Example 7.3 Take $z_1 = 1 + i$, $z_2 = 2i$, $S_1 = 2$ and $S_2 = 2i$. Then

$$P = \frac{1}{5} \begin{pmatrix} 0 & 4 + 2i \\ 4 - 2i & 5 \end{pmatrix}, \quad \tilde{P} = \frac{2}{5} \begin{pmatrix} 5 & 3 + 4i \\ 3 - 4i & 0 \end{pmatrix},$$

$\text{sq}_-(P) = \text{sq}_-(\tilde{P}) = 1$, and

$$P^{-1} = \frac{1}{4} \begin{pmatrix} -5 & 4 + 2i \\ 4 - 2i & 0 \end{pmatrix}, \quad \tilde{P}^{-1} = \frac{1}{10} \begin{pmatrix} 0 & 3 + 4i \\ 3 - 4i & -5 \end{pmatrix}.$$

According to (2.14) and (2.20),

$$\begin{aligned} \Theta(z) &= \left\{ I_2 + \begin{pmatrix} 2 & 2i \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{z-1-i} & 0 \\ 0 & \frac{1}{z-2i} \end{pmatrix} \begin{pmatrix} -\frac{5}{4} & 1 + \frac{i}{2} \\ 1 - \frac{i}{2} & 0 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 1 & 2i \end{pmatrix} \right\} \begin{pmatrix} \frac{1}{10} & 0 \\ 0 & 1 \end{pmatrix} \\ &= \frac{1}{(z-1+i)(z-2i)} \begin{pmatrix} z^2 - \frac{2}{5}z + \frac{8}{5} & z + 6 \\ \frac{1}{10}(z^2 + 6z) & z^2 - \frac{3}{2}z + 5 \end{pmatrix}. \end{aligned}$$

The solutions of the interpolation problem in the class \mathcal{S}_1^1 are given by the formula

$$S(z) = \frac{(z^2 - \frac{2}{5}z + \frac{8}{5})T(z) + z + 6}{\frac{1}{10}(z^2 + 6z)T(z) + z^2 - \frac{3}{2}z + 5}.$$

where $T(z)$ runs over $\tilde{\mathcal{S}}$ except for $T_1(z) = -\frac{5}{z}$ (an excluded parameter at z_1) and $T_2(z) \equiv \frac{5}{2}$ (an excluded parameter at z_2). For these exceptional values of $T(z)$ we have that

$$S_1(z) = -\frac{4}{z} \quad \text{and} \quad S_2(z) \equiv 2,$$

respectively. These functions are analytic at the interpolation points, but they are only partial solutions of the problem:

$$S_1(2i) = 2i, \quad S_1(1-i) = -2 - 2i \neq 2, \quad S_2(1-i) = 2, \quad S_2(2i) = 2 \neq 2i,$$

and moreover, they belong to \mathcal{S}_0^0 and not to \mathcal{S}_1^1 .

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