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1 Introduction

In this contribution we want to draw the readers's attention to an open problem that concerns the existence of certain solutions to the algebraic Riccati equation. Since its introduction in control theory by Kalman in the beginning of the sixties, the algebraic Riccati equation has known an impressive range of applications, such as linear quadratic optimal control, stability theory, stochastic filtering and stochastic control, stochastic realization theory, the synthesis of linear passive networks, differential games, and H_∞ optimal control and robust stabilization. For an overview of the existing literature on the algebraic Riccati equation, we refer to [3].

In this note, we deal with the *existence* of real symmetric solutions to the algebraic Riccati equation, in particular with the existence of *negative semi-definite* solutions. It is well-known (see [9]) that the existence of a real symmetric solution to the algebraic Riccati equation is equivalent to a given frequency domain inequality along the imaginary axis. In [9], it was also stated that the existence of a *negative semi-definite solution* is equivalent to this frequency domain inequality holding for all complex numbers in the closed right half of the complex plane. Soon after the appearance of [9], a correction [10] appeared in which it was outlined that this statement is not correct, and that the frequency domain inequality in the closed right half plane is a necessary, but not sufficient condition for the existence of a negative semi-definite solution.

Since then, several attempts have been made to obtain a convenient necessary and sufficient frequency domain condition for the existence of a negative semi-definite solution. In this note we discuss some of these conditions. We also explain that we consider these conditions not to be satisfactory yet, and therefore we claim the problem of formulating a sensible frequency domain condition to be still an open problem.

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2 The algebraic Riccati equation

Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ be such that (A, B) is a controllable pair. Also, let $Q \in \mathbb{R}^{n\times n}$ be such that $Q = Q^T$, let $S \in \mathbb{R}^{m\times n}$, and let $R \in \mathbb{R}^{m\times m}$ be such that $R > 0$. We deal with the following algebraic Riccati equation:

$$A^T K + KA + Q - (KB + S^T)R^{-1}(B^T K + S) = 0 \tag{2.1}$$

In order to give a frequency domain condition for the existence of a real symmetric solution K, we define the matrix of two-variable rational functions W in the indeterminates ζ and η by

$$\begin{aligned} W(\zeta,\eta) := R + B^T(\zeta I - A^T)^{-1}S^T + S(\eta I - A)^{-1}B \\ +B^T(\zeta I - A^T)^{-1}Q(\eta I - A)^{-1}B \end{aligned} \tag{2.2}$$

which is often called the *Popov function* associated with (2.1). (see [7]). The coefficients of this matrix are quotients of real polynomials in the indeterminates ζ and η. With this two-variable rational matrix W we associate a (one-variable) rational matrix ∂W by defining $\partial W(\xi) := W(-\xi, \xi)$, i.e., obtained by taking $\zeta = -\xi$ and $\eta = \xi$. It was shown in [9] that the algebraic Riccati equation (2.1) has a real symmetric solution if and only if

$$\partial W(i\omega) \geq 0 \text{ for all } \omega \in \mathbb{R},\ i\omega \notin \sigma(A) \tag{2.3}$$

In [9] it was claimed that the two-variable rational matrix W also provides the clue to the existence of a negative semi-definite solution. Indeed, [9], Theorem 4 states that (2.1) has a real symmetric solution $K \leq 0$ if and only if

$$W(\bar{\lambda}, \lambda) \geq 0 \text{ for } \Re e(\lambda) \geq 0,\ \lambda \notin \sigma(A) \tag{2.4}$$

Unfortunately, as was noted in [10], this statement is not correct: in general the condition (2.4) is only a necessary condititition, but not a sufficient one. It is certainly interesting to note that for certain important special cases (2.4) does provide a necessary *and* sufficient condition. For one important special case this result is known as *the bounded real lemma.* This special case in concerned with the situation that $Q = -C^T C$, $S = 0$, and $R = I$. In that case, after a change of variable from K to $-K$, the Riccati equation becomes

$$A^T K + KA + C^T C + KBB^T K = 0 \tag{2.5}$$

while we have $W(\zeta, \eta) = I - G^T(\zeta)G(\eta)$. The condition (2.4) then becomes

$$G^T(\bar{\lambda})G(\lambda) \leq I \text{ for } \Re e(\lambda) \geq 0,\ \lambda \notin \sigma(A)$$

which indeed is well-known to be equivalent to the existence of a solution $K \geq 0$ to the algebraic Riccati equation (2.5) (see, for example [2]). In [5], a number of additional special cases in which the frequency domain inequality (2.4) is a necessary and sufficient condition were established. We will not go into these special cases here. Instead, we want to discuss an alternative condition on the two-variable rational matrix W that was proven in [4] to be equivalent to the existence of a negative semi-definite solution to (2.1). In order to rederive this condition here, we make use of recent results in [12] on quadratic differential forms and dissipativity of linear differential systems

3 Dissipativity of linear differential systems

As was also noted in [9], the study of the algebraic Riccati equation can be put into the more general framework of studying quadratic storage functions for linear systems with quadratic supply rates. In that context, an important role is played by the *linear matrix inequality* $L(K) \geq 0$, where

$$L(K) := \begin{pmatrix} A^T K + KA + Q & KB + S^T \\ B^T K + S & R \end{pmatrix} \tag{3.1}$$

It can be shown that $L(K) \geq 0$ if and only if the quadratic function $V(x) := -x^T K x$ satisfies the inequality

$$\frac{d}{dt} V(x(t)) \leq x^T(t) Q x(t) + 2x^T(t) S^T u(t) + u^T(t) R u(t) \tag{3.2}$$

for all $t \in \mathbb{R}$, for all x and u satisfying the differential equation $\dot{x} = Ax + Bu$. The inequality (3.2) is called *the dissipation inequality* for the system $\dot{x} = Ax + Bu$ with *supply rate* $x^T Q x + 2x^T S^T u + u^T R u$. If a function $V(x)$ satisfies this inequality, it is called a *storage function*. The dissipation inequality expresses the property that along trajectories x and u of the system the increase in internal storage cannot exceed the rate at which storage is supplied to the system. If such a function $V(x)$ exists, we call the system *dissipative* with respect to the given supply rate. It can be shown that the system $\dot{x} = Ax + Bu$ with supply rate $x^T Q x + 2x^T S^T u + u^T R u$ is dissipative if and only if the frequency domain condition (2.3) (so along the imaginary axis) holds. Thus, (2.3) is equivalent to the existence of a real symmetric solution K to the linear matrix inequality $L(K) \geq 0$. Moreover, if this condition holds, there exist real symmetric solutions K^- and K^+ such that any real symmetric solution K satisfies $K^- \leq K \leq K^+$. The function $V_1(x) := -x^T K^+ x$ is then the *smallest*, and the function $V_2(x) := -x^T K^- x$ is the *largest* storage function. Also, K^- and K^+ are solutions of the algebraic Riccati equation (2.1). These considerations show that the existence of a negative semi-definite solution to (2.1) is equivalent to the existence of a negative semi-definite solution to the linear matrix inequality (3.1), equivalently, to the existence of a positive semi-definite storage function of the system $\dot{x} = Ax + Bu$ with supply rate $x^T Q x + 2x^T S^T u + u^T R u$.

The general problem of the existence of storage functions was recently put in the framework of quadratic differential forms for linear differential systems [12], [8]). An important role in these references is played by *two-variable polynomial matrices*, i.e., matrices whose coefficients are real polynomials in two indeterminates, say ζ and η. A two-variable polynomial matrix Φ can be represented as

$$\Phi(\zeta, \eta) = \sum_{k,j} \Phi_{k,j} \zeta^k \eta^j$$

where the $\Phi_{k,j}$ are matrices with real coefficients, $k, j \in \mathbb{N}$, and the sum is a finite one. Φ is called symmetric if $\Phi(\zeta, \eta)^T = \Phi(\eta, \zeta)$. Each symmetric $q \times q$

two-variable poynomial matrix Φ induces a *quadratic differential form* (QDF), i.e., a map $Q_\Phi : \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^q) \to \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R})$, defined by

$$Q_\Phi(\ell) := \sum_{k,j} (\frac{d^k \ell}{dt^k})^T \Phi_{k,j} \frac{d^j \ell}{dt^j}$$

Associated with Φ, we define the (one-variable) polynomial matrix $\partial\Phi$ by $\partial\Phi(\xi) := \Phi(-\xi, \xi)$. Note that this polynomial matrix is para-hermitian, i.e., $(\partial\Phi(-\xi))^T = \partial\Phi(\xi)$. A QDF Q_Φ is called *average non-negative* if for all $\ell \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^q)$ of compact support we have

$$\int_{-\infty}^{\infty} Q_\Phi(\ell) dt \geq 0$$

In [12] this property was shown to be equivalent with the existence of a symmetric two-variable polynomial matrix Ψ such that

$$\frac{d}{dt} Q_\Psi(\ell) \leq Q_\Phi(\ell) \text{ for all } \ell \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^q) \tag{3.3}$$

It was also shown that Q_Φ is average non-negative if and only if $\partial\Phi$ is non-negative along the imaginary axis, i.e., $\partial\Phi(i\omega) \geq 0$ for all $\omega \in \mathbb{R}$. We now explain how this result can be used to rederive the condition 2.3 for the existence of a real symmetric solution to the algebraic Riccati equation. Consider the controllable system $\dot{x} = Ax + Bu$, more precisely, the system $\Sigma = (\mathbb{R}, \mathbb{R}^{n+m}, \mathfrak{B})$, with time axis $\mathbb{R}$, signal space $\mathbb{R}^{n+m}$ and behavior $\mathfrak{B} := \{(x, u) \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{n+m}) \mid \dot{x} = Ax + Bu\}$. Any controllable system also admits an image representation (see ([6, 11]). Consider the image representation

$$\begin{pmatrix} x \\ u \end{pmatrix} = \begin{pmatrix} C(\frac{d}{dt})B \\ p(\frac{d}{dt})I \end{pmatrix} \ell \tag{3.4}$$

Here, $p(\xi)$ is the characteristic polynomial of A, i.e., $p(\xi) := \det(\xi I - A)$, and $C(\xi)$ is the polynomial matrix defined by $C(\xi) := p(\xi)(\xi I - A)^{-1}$, that is, the classical adjoint of $\xi I - A$, appearing when one applies Cramer's rule to compute the inverse $(\xi I - A)^{-1}$. We have

$$\mathfrak{B} = \{(x, u) \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{n\times m}) \mid \text{ there exists } \ell \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^m) \text{ such that 3.4}\}$$

so (3.4) indeed defines an image representation of our system Σ. Now define the symmetric two-variable polynomial matrix Φ by

$$\Phi(\zeta, \eta) := \begin{pmatrix} C(\zeta)B \\ p(\zeta)I \end{pmatrix}^T \begin{pmatrix} Q & S^T \\ S & R \end{pmatrix} \begin{pmatrix} C(\eta)B \\ p(\eta)I \end{pmatrix} \tag{3.5}$$

It is immediate that if x, u and ℓ are related by (3.4), then for the QDF Q_Φ associated with Φ we have $Q_\Phi(\ell) = x^T Q x + 2x^T S^T u + u^T R u$. Assume now that Q_Φ is average non-negative, equivalently, there exists Ψ such that $\frac{d}{dt} Q_\Psi \leq Q_\Phi$. It was proven in ([8]) that any such Q_Ψ can be represented as a (static) quadratic function of any state of the underlying system. In our case, a (minimal) state is given by $x = C(\frac{d}{dt})B\ell$, so there exists a real symmetric matrix $K \in \mathbb{R}^{n\times n}$ such that if x and ℓ are related by $x = C(\frac{d}{dt})B\ell$, then $Q_\Psi(\ell) = x^T K x$. Collecting these facts we find that the following three statements are equivalent:

1. $\int_{-\infty}^{\infty}(x^TQx + 2x^TS^Tu + u^TRu)dt \geq 0$ for all $x \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^n)$ and $u \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^m)$ of compact support, satisfying $\dot{x} = Ax + Bu$

2. there exists a real symmetric matrix $K \in \mathbb{R}^{n\times n}$ such that $\frac{d}{dt}x^TKx \leq x^TQx + 2x^TS^Tu + u^TRu$ for all $x \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^n)$ and $u \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^m)$ satisfying $\dot{x} = Ax + Bu$

3. $\partial\Phi(i\omega) \geq 0$ for all $\omega \in \mathbb{R}$

As noted before, condition (2) is equivalent to the existence of a real symmetric solution of the algebraic Riccati equation. Note that for Φ given by (3.5) we have $\Phi(\zeta, \eta) = p(\zeta)p(\eta)W(\zeta, \eta)$ with W the two-variable rational matrix given by (2.2). Hence, along the imaginary axis we have $\partial\Phi(i\omega) = |p(i\omega)|^2\partial W(i\omega)$. Therefore, condition (3) is equivalent to the frequency domain inequality (2.3). Thus we have re-established the fact that the existence of a real symmetric solution to the algebraic Riccati equation is equivalent to the frequency domain inequality (2.3).

4 The existence of negative semi-definite solutions

Let us now study what the set-up of QDF's for linear differential systems can tell us on the existence of negative semi-definite solutions to the algebraic Riccati equation. To start with, let Φ be an arbitrary $q \times q$ symmetric two-variable polynomial matrix. The associated QDF Q_Φ is called *half-line non-negative* if for all $\ell \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^q)$ of compact support we have

$$\int_{-\infty}^{0} Q_\Phi(\ell)dt \geq 0$$

It was proven in [12] that a QDF Q_Φ is half-line non-negative if and only if there exists a symmetric two-variable polynomial matric Ψ such that $Q_\Psi \geq 0$ and such that (3.3) holds. Applying this fact to our system Σ, again using that it has an image representation given by (3.4), we this time find that the following two statements are equivalent:

1. $\int_{-\infty}^{0}(x^TQx + 2x^TS^Tu + u^TRu)dt \geq 0$ for all $x \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^n)$ and $u \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^m)$ of compact support, satisfying $\dot{x} = Ax + Bu$

2. there exists a real symmetric matrix $K \in \mathbb{R}^{n\times n}$, $K \leq 0$, such that $\frac{d}{dt}(-x^TKx) \leq x^TQx + 2x^TS^Tu + u^TRu$ for all $x \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^n)$ and $u \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^m)$ satisfying $\dot{x} = Ax + Bu$

As before, condition (2) is equivalent to the existence of a negative semi-definite real symmetric solution to the algebraic Riccati equation. In the following, we establish a condition on the two-variable polynomial matrix Φ that is equivalent to condition (1). It turns out that in this way we reobtain the frequency domain condition for the existence of a negative semi-definite solution to the algebraic Riccati equation that was obtained before in [4]. First note that condition (1) is equivalent to the condition that for all $\ell \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^m)$ of compact support we have

$$\int_{-\infty}^{0} \begin{pmatrix} A(\frac{d}{dt})B\ell \\ p(\frac{d}{dt})\ell \end{pmatrix}^T \begin{pmatrix} Q & S^T \\ S & R \end{pmatrix} \begin{pmatrix} A(\frac{d}{dt})B\ell \\ p(\frac{d}{dt})\ell \end{pmatrix} dt \geq 0 \tag{4.1}$$

By an approximation argument, this condition is equivalent to the same inequality holding for all $\ell \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^m)$ such that $\int_{-\infty}^{0} \|\ell\|^2 dt$ is finite. Now, let $N \in \mathbb{N}$, and let $\lambda_1, \lambda_2, \ldots, \lambda_N$ be N distinct complex numbers in $\Re e(\lambda) > 0$. Also, let $v_1, v_2, \ldots v_N$ be arbitrary vectors in $\mathbb{C}^m$. Consider the function

$$\ell(t) := \sum_{i=1}^{N} e^{\lambda_i t} v_i \tag{4.2}$$

Applying (4.1) to this function ℓ (silently moving from real valued functions to complex valued functions), we obtain

$$\sum_{i=1}^{N}\sum_{j=1}^{N} \int_{-\infty}^{0} (e^{(\bar{\lambda_i}+\lambda_j)t} v_i^* \begin{pmatrix} A(\bar{\lambda_i})B \\ p(\bar{\lambda_i})I \end{pmatrix}^T \begin{pmatrix} Q & S^T \\ S & R \end{pmatrix} \begin{pmatrix} A(\lambda_j)B \\ p(\lambda_j)I \end{pmatrix} v_j) dt \geq 0$$

which, after integration, yields

$$\sum_{i=1}^{N}\sum_{j=1}^{N} v_i^* \frac{\Phi(\bar{\lambda_i}, \lambda_j)}{\bar{\lambda_i} + \lambda_j} v_j \geq 0 \tag{4.3}$$

Since, for fixed complex numbers $\lambda_1, \lambda_2, \ldots, \lambda_N$, this holds for *all* $v_1, v_2, \ldots v_N$, this implies that the hermitian $mN \times mN$ matrix whose (i,j)th block is equal to the $m \times m$ matrix $\frac{\Phi(\bar{\lambda_i}, \lambda_j)}{\bar{\lambda_i}+\lambda_j}$ is positive semi-definite. If, conversely, (4.3) holds for any choice of N, any choice of distinct complex numbers $\lambda_1, \lambda_2, \ldots, \lambda_N$ in $\Re e(\lambda) > 0$, and any choice of vectors $v_1, v_2, \ldots v_N$ in $\mathbb{C}^m$, then clearly (4.1) holds for any function ℓ of the form (4.2). Again by an approximation argument, the inequality (4.1) must then hold for all $\ell \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^m)$ of compact support. This proves that any of the statements (1) and (2) is equivalent with the following condition (3):

3. for all $N \in \mathbb{N}$, and for any choice $\lambda_1, \lambda_2, \ldots, \lambda_N$ of distinct complex numbers in $\Re e(\lambda) > 0$, the hermitian $mN \times mN$ matrix $\left(\frac{\Phi(\bar{\lambda_i}, \lambda_j)}{\bar{\lambda_i}+\lambda_j}\right)_{i,j=1,2,\ldots N}$ is positive semi-definite.

We now express condition (3) in terms of the two-variable rational matrix W. It is easily verified that

$$\left(\frac{\Phi(\bar{\lambda_i}, \lambda_j)}{\bar{\lambda_i} + \lambda_j}\right)_{i,j=1,2,\ldots,N} = D^* \left(\frac{W(\bar{\lambda_i}, \lambda_j)}{\bar{\lambda_i} + \lambda_j}\right)_{i,j=1,2,\ldots,N} D$$

Here D is the blockdiagonal matrix whose ith diagonal block is equal to $p(\lambda_i)I$ (with I the $m \times m$ identity matrix). Thus we claim that condition (3) is equivalent to the following condition (4):

4. for all $N \in \mathbb{N}$, and for any choice $\lambda_1, \lambda_2, \ldots, \lambda_N$ of distinct complex numbers in $\Re e(\lambda) > 0$ such that $\lambda_i \notin \sigma(A)$, $i = 1, 2, \ldots, N$, the hermitian $mN \times mN$ matrix $\left(\frac{W(\bar{\lambda}_i, \lambda_j)}{\bar{\lambda}_i + \lambda_j}\right)_{i,j=1,2,\ldots N}$ is positive semi-definite.

Indeed, if $\lambda_i \notin \sigma(A)$, $i = 1, 2, \ldots, N$, then D is nonsingular. From this the proof of the implication (3) $\Rightarrow$ (4) is obvious. The proof of the converse implication then follows by using a continuity argument.

We have now shown that the existence of a negative semi-definite real symmetric solution of the algebraic Riccati equation is equivalent to condition (4) on the two-variable rational matrix W. Thus we have re-established the condition that was proposed in [4] as a correct alternative for the errorous frequency domain condition (2.4). A similar condition was obtained in [1] in related work on spectral factorization.

We remind the reader that the title of this note is: "When does the algebraic Riccati equation have a negative semi-definite solution?" We have re-derived the necessary and sufficient frequency domain condition (4) for this to hold. Yet, in our opinion, this condition is not a satisfactory one, since not only it requires one to check non-negativity of an infinite number of hermitian matrices, but also there is no upper bound to the dimension of these matrices. In view of this, we formulate the following open problem:

> **Find a reasonable necessary and sufficient frequency domain condition, i.e., a condition in terms of the rational matrix ∂W, or possibly in terms of the two-variable rational matrix W, for the existence of a real symmetric negative semi-definite solution of the algebraic Riccati equation (2.1).**

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