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Dissipative eigenvalue problems for a Sturm–Liouville operator with a singular potential*

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In this paper we consider the Sturm–Liouville operator $d^2/dx^2 - 1/x$ on the interval $[a, b]$, $a < 0 < b$, with Dirichlet boundary conditions at a and b, for which $x = 0$ is a singular point. In the two components $\mathcal{L}^2(a, 0)$ and $\mathcal{L}^2(0, b)$ of the space $\mathcal{L}^2(a, b) = \mathcal{L}^2(a, 0) \oplus \mathcal{L}^2(0, b)$ we define minimal symmetric operators and describe all the maximal dissipative and self-adjoint extensions of their orthogonal sum in $\mathcal{L}^2(a, b)$ by interface conditions at $x = 0$. We prove that the maximal dissipative extensions whose domain contains only continuous functions f are characterized by the interface condition $\lim_{x\to 0+}(f'(x) - f'(-x)) = \gamma f(0)$ with $\gamma \in \mathcal{C}^+ \cup \mathbb{R}$ or by the Dirichlet condition $f(0+) = f(0-) = 0$. We also show that the corresponding operators can be obtained by norm resolvent approximation from operators where the potential $1/x$ is replaced by a continuous function, and that their eigen and associated functions can be chosen to form a Bari basis in $\mathcal{L}^2(a, b)$.

1. Introduction

In this paper we consider the differential expression

$$l[f](x) := -f''(x) - \frac{f(x)}{x} \tag{1.1}$$

and the corresponding differential equation

$$-f''(x) - \frac{f(x)}{x} - \lambda f(x) = 0 \tag{1.2}$$

*Dedicated to Professor Boele Braaksma on the occasion of his 65th birthday, in friendship.

on the interval $[a, b]$, where $a < 0 < b$, with the boundary conditions

$$f(a) = f(b) = 0. \tag{1.3}$$

Since the potential is not summable at $x = 0$, it is not a classical Sturm–Liouville problem. We associate with this boundary eigenvalue problem two minimal operators in the spaces $\mathcal{L}^2([a, 0))$ and $\mathcal{L}^2((0, b])$. Since these operators are in the limit case at $x = 0$, they are not self-adjoint and their direct sum operator S in the space $\mathcal{L}^2([a, b])$ is symmetric with defect index $(2, 2)$. It is the aim of this paper to describe all self-adjoint and maximal dissipative extensions of S in $\mathcal{L}^2([a, b])$. Recall that an operator A in some Hilbert space $\mathcal{H}$ is called *dissipative* if $\operatorname{Im}(Af, f) \geqslant 0$ for all $f \in \mathcal{H}$ and *maximal dissipative* if it does not have a proper dissipative extension. In particular, we also describe those extensions among them for which the domain consists only of continuous functions. This set turns out to be a one-parameter family of operators T_γ, $\gamma \in \mathbb{C}^+ \cup \{\infty\}$, which are defined by the interface condition

$$\lim_{x \to 0+} (f'(x) - f'(-x)) = \gamma f(0) \quad \text{if } \gamma \in \mathbb{C},$$

and by

$$f(0+) = f(0-) = 0 \quad \text{if } \gamma = \infty.$$

The problem (1.1) has been studied by several authors [4,8,12]. In [4] the potential $-x^{-1}$ is replaced by the regular potential $-(x - \mathrm{i}\varepsilon)^{-1}$ and the resulting operator for $\varepsilon \to 0$ is considered. This operator is the extension T_γ with $\gamma = -\mathrm{i}\pi$ (see Remark 5.2). In [8] the operator T_∞ is studied: it is the direct sum of two self-adjoint operators on $[a, 0)$ and $(0, b]$, respectively, with Dirichlet boundary conditions. Gunson treats the operators $T_{-\mathrm{i}\pi}$ [12, theorem 2.6 and eqn (2.13)] and T_∞ [12, theorem 2.2 and eqn (2.1)] as well as T_0, where the potential $-x^{-1}$ is considered in the distributional sense as the Cauchy principal value [12, theorem 2.4 and eqn (2.9)]. This self-adjoint operator is also studied in [1] from the viewpoint of quasi-derivatives. We mention that the operators T_γ considered here have discrete spectrum. The case where the interval $[a, b]$ is replaced by the real axis is also considered in [12]. In this case the corresponding operators $T_{\mathrm{i}\theta}$ with $0 < \theta < \pi$ also have an absolutely continuous spectrum and $T_{\mathrm{i}\pi}$ has only absolutely continuous spectrum. For a more recent discussion about the potential $-x^{-1}$ in the physics literature, we refer to [14,17,18,20], and the references therein.

In § 2 we introduce the symmetric operator S. In § 3 all self-adjoint and maximal dissipative extensions of S are described by an interface condition at 0. Here we use essentially the fact that all these extensions are contained in S^*. There also exist extensions of S in $\mathcal{L}^2([a, b])$ with a non-empty resolvent set which are not contained in S^* [3]. The extensions T_γ, $\gamma \in \mathbb{C} \cup \{\infty\}$, are described in § 4. By a method already used in [12] it is shown that the extensions T_γ for $\gamma \in \mathbb{C}$ can be obtained as norm resolvent limits of operators generated by regular potentials. An analogous result for the case $\gamma = \infty$ can be found in [3]. In § 5 we express the solutions of equation (1.2) by Whittaker functions in order to get information about the characteristic determinant and the asymptotics of the eigenvalues. This is used in § 6, where we prove that the system of root vectors of the operator T_γ forms a Bari basis in $\mathcal{L}^2([a, b])$. Finally, the Fourier coefficients of the corresponding expansions are expressed by inner products in $\mathcal{L}^2([a,b])$ with the complex conjugate functions of the root functions (which are the root functions of the adjoint operator).

2. The symmetric operator S

Let $a < 0 < b$. We consider the differential expression $l[f]$ from (1.1) on the intervals $I := [a,b]$, $I_- := [a,0)$ and $I_+ := (0,b]$; at the endpoints a and b we always impose the Dirichlet boundary conditions (1.3). In the space $\mathcal{L}^2(I_\pm)$ a minimal operator $L_\pm$ and a maximal operator $L_\pm^*$, which is the adjoint of the minimal operator in $\mathcal{L}^2(I_\pm)$, are associated with the differential expression l. The domain of the maximal operator L_+^* is

$$\mathcal{D}(L_+^*) := \{f \in \mathcal{L}^2(I_+)\colon f, f' \in \mathcal{AC}_{\mathrm{loc}}(I_+),\ f(b) = 0,\ l[f] \in \mathcal{L}^2(I_+)\}$$

and $L_+^* f = l[f]$ if $f \in \mathcal{D}(L_+^*)$. Here, for example, $\mathcal{AC}_{\mathrm{loc}}(I_+)$ is the set of locally absolutely continuous functions on I_+. The set $\mathcal{D}(L_-^*)$ and the operator L_-^* are defined correspondingly. To describe the domains of the minimal operators $L_\pm$, we introduce for $f, g \in \mathcal{D}(L_\pm^*)$ and $x, x_1, x_2 \in I_\pm$ the sesquilinear forms

$$[f,g]_x := f(x)\overline{g'(x)} - f'(x)\overline{g(x)}, \quad [f,g]_{x_1}^{x_2} := [f,g]_{x_2} - [f,g]_{x_1}. \tag{2.1}$$

Then Green's formula becomes

$$[f,g]_{x_1}^{x_2} = \int_{x_1}^{x_2} (l[f](x)\overline{g(x)} - f(x)\overline{l[g](x)}\,\mathrm{d}x. \tag{2.2}$$

It implies that the limits $\lim_{x\to 0\pm}[f,g]_x =: [f,g]_{0\pm}$ exist and are finite and that the sesquilinear forms $[\cdot,\cdot]_{x_1}^{x_2}$ are continuous on $\mathcal{D}(L_\pm^*)$ with respect to the $L_\pm^*$-graph norms. The domains of the minimal operators can be described as follows [7, theorem 2.3]:

$$\mathcal{D}(L_-) = \{f \in \mathcal{D}(L_-^*)\colon [f,g]_a^{0-} = 0 \text{ for all } g \in \mathcal{D}(L_-^*)\}, \tag{2.3}$$

$$\mathcal{D}(L_+) = \{f \in \mathcal{D}(L_+^*)\colon [f,g]_{0+}^{b} = 0 \text{ for all } g \in \mathcal{D}(L_+^*)\}, \tag{2.4}$$

and Green's formula (2.2) implies that the operators $L_\pm$ are symmetric.

Consider on the interval $[a,b]$ the functions

$$u(x) = x \quad \text{and} \quad v(x) = 1 - x\ln|x|.$$

We choose numbers $\varepsilon_1, \varepsilon_2\colon 0 < \varepsilon_1 < \varepsilon_2 < \min\{-a, b\}$ and twice continuously differentiable functions $u_\pm$ on $I_\pm$ with the properties

$$u_+(x) := \begin{cases} u(x) & \text{if } 0 < x < \varepsilon_1, \\ 0 & \text{if } \varepsilon_2 < x < b, \end{cases} \qquad u_-(x) := \begin{cases} 0 & \text{if } a < x < -\varepsilon_2, \\ u(x) & \text{if } -\varepsilon_1 < x < 0, \end{cases}$$

and, analogously, functions $v_\pm$. For x in a neighbourhood of 0,

$$l[u_\pm](x) = -1, \qquad l[v_\pm](x) = \ln|x|,$$

hence $l[u_\pm], l[v_\pm] \in \mathcal{L}^2(I_\pm)$ and $u_\pm, v_\pm \in \mathcal{D}(L_\pm^*)$. Further,

$$[v_-, v_-]_a^{0-} = \lim_{x\to 0-} (v_-(x)\overline{v_-'(x)} - v_-'(x)\overline{v_-(x)}) = 0, \tag{2.5}$$

$$[u_-, v_-]_a^{0-} = \lim_{x\to 0-} (u_-(x)\overline{v_-'(x)} - u_-'(x)\overline{v_-(x)}) = -1, \tag{2.6}$$

and, analogously,

$$\left.\begin{aligned} [u_-,u_-]_a^{0-} &= [v_+,v_+]_{0+}^b = [u_+,u_+]_{0+}^b = 0,\\ [v_-,u_-]_a^{0-} &= -[v_+,u_+]_{0+}^b = [u_+,v_+]_{0+}^b = 1. \end{aligned}\right\} \tag{2.7}$$

The sesqilinear forms $[\cdot,\cdot]_a^{0-}$ and $[\cdot,\cdot]_{0+}^b$ vanish on $\mathcal{D}(L_-)$ and $\mathcal{D}(L_+)$, respectively; see equations (2.3) and (2.4). Therefore, the functions $u_\pm$ and $v_\pm$ are linearly independent modulo $\mathcal{D}(L_\pm)$. Since l is a second-order differential operator and boundary conditions at a and b have been fixed, the dimension of the factor space $\mathcal{D}(L_\pm^*)/\mathcal{D}(L_\pm)$ is at most 2, and we find

$$\mathcal{D}(L_-^*) = \mathcal{D}(L_-) \dot{+} \operatorname{span}\{u_-,v_-\}, \qquad \mathcal{D}(L_+^*) = \mathcal{D}(L_+) \dot{+} \operatorname{span}\{u_+,v_+\}. \tag{2.8}$$

Now we consider in the Hilbert space

$$\mathcal{L}^2(I) = \mathcal{L}^2(I_-) \oplus \mathcal{L}^2(I_+) \tag{2.9}$$

the operator $S := L_- \oplus L_+$. Evidently, $S^* = L_-^* \oplus L_+^*$ and on $\mathcal{D}(S^*)$ we define the sesquilinear form

$$[f,g] := [f_-,g_-]_a^{0-} + [f_+,g_+]_{0+}^b, \qquad f,g \in \mathcal{D}(S^*), \tag{2.10}$$

where $f = f_- + f_+$ and $g = g_- + g_+$ are the decompositions of the elements f and g with respect to (2.9). Relation (2.2) implies the Green's formula

$$[f,g] = (S^*f,g) - (f,S^*g), \qquad f,g \in \mathcal{D}(S^*), \tag{2.11}$$

and the sesquilinear form on the left-hand side is again continuous in the S^*-graph norm on $\mathcal{D}(S^*)$.

We extend the functions $u_\pm$ and $v_\pm$ to the whole interval $[a,b]$ as follows:

$$\tilde{u}_-(x) := \begin{cases} u_-(x) & \text{if } x \in [a,0),\\ 0 & \text{if } x \in (0,b], \end{cases} \qquad \tilde{u}_+(x) := \begin{cases} 0 & \text{if } x \in [a,0),\\ u_+(x) & \text{if } x \in (0,b], \end{cases}$$

and $\tilde{v}_\pm$ are defined analogously. All these extended functions belong to $\mathcal{D}(S^*)$. On $f \in \mathcal{D}(S^*)$ the following functionals $\mathsf{u}_\pm, \mathsf{v}_\pm$ are defined:

$$\mathsf{u}_- f := [f,\tilde{u}_-], \quad \mathsf{u}_+ f := [f,\tilde{u}_+], \quad \mathsf{v}_- f := [f,\tilde{v}_-], \quad \mathsf{v}_+ f := [f,\tilde{v}_+]. \tag{2.12}$$

From (2.3) and (2.4) it follows that the functionals $\mathsf{u}_\pm, \mathsf{v}_\pm$ vanish on $\mathcal{D}(S)$, and the definition of the functions $\tilde{u}_\pm, \tilde{v}_\pm$ yields for $f \in \mathcal{D}(S^*)$ the relations

$$\mathsf{u}_\pm f = \mp f(0\pm), \qquad \mathsf{v}_\pm f = \pm \lim_{x\to 0\pm} (f'(x) + f(x)(1 + \ln|x|)), \tag{2.13}$$

where we have used that the functions $f \in \mathcal{D}(S^*)$ satisfy the relation

$$f'(x) = O(\ln|x|) \quad \text{for } x \to 0; \tag{2.14}$$

see [8, lemma 2.2]. Since the operators $L_\pm$ are symmetric, also S is a symmetric operator and we have

$$\mathcal{D}(S) = \{f \in \mathcal{D}(S^*) \colon \mathsf{u}_- f = \mathsf{u}_+ f = \mathsf{v}_- f = \mathsf{v}_+ f = 0\} \tag{2.15}$$

and

$$\mathcal{D}(S^*) = \mathcal{D}(S) + \operatorname{span}\{\tilde{u}_-, \tilde{u}_+, \tilde{v}_-, \tilde{v}_+\}. \tag{2.16}$$

Therefore, the defect index of the operator S is $(2,2)$.

LEMMA 2.1. *If $f \in \mathcal{D}(S)$, it holds that*

$$f(x) = o(x), \;\; f'(x) = o(1) \textit{ for } x \to 0, \tag{2.17}$$

and

$$\mathcal{D}(S) = \{f \in \mathcal{D}(S^*)\colon f, f' \textit{ are continuous in } 0 \textit{ and } f(0) = f'(0) = 0\}. \tag{2.18}$$

Proof. If $f \in \mathcal{D}(S)$, then (2.15) and the first relation in (2.13) imply, for $x \to 0$,

$$f(x) = o(1). \tag{2.19}$$

Now relation (2.14) yields the sharper estimate

$$f(x) = \int_0^x f'(t)\,\mathrm{d}t = O(x \ln |x|), \tag{2.20}$$

and if we observe that $v_\pm f = 0$, it follows by (2.15) and the second relation in (2.13) that

$$f'(x) = -(1 + \ln |x|) O(x \ln |x|) + o(1) = o(1)$$

and finally

$$f(x) = \int_0^x f'(t)\,\mathrm{d}t = o(x).$$

Thus the relations (2.17) and the inclusion

$$\mathcal{D}(S) \subset \{f \in \mathcal{D}(S^*)\colon f, f' \text{ are continuous in } 0 \text{ and } f(0) = f'(0) = 0\}$$

are proved. The equality sign in (2.18) follows now from (2.16) and the fact that no linear combination f of the functions $u_\pm, v_\pm$, except the trivial one, has the property that f and f' are continuous and fulfil $f(0) = f'(0) = 0$. □

3. The self-adjoint and the maximal dissipative extensions of *S*

The symmetric operator S in $\mathcal{L}^2(I)$ with defect index $(2,2)$, which was associated with the differential expression l from (1.1) and the Dirichlet boundary conditions (1.3), has self-adjoint and maximal dissipative canonical extensions; here *canonical* means that these extensions act in the originally given space $\mathcal{L}^2(I)$. We shall characterize these extensions by interface conditions at 0.

To this end, we first observe that all symmetric and dissipative canonical extensions of S are restrictions of the adjoint S^* (see [11, theorem 3.1.3] and [15, theorem 1.3.7]). Relation (2.15) implies that such an extension is determined by a linear relation between the functionals $u_\pm, v_\pm$, which are defined on $\mathcal{D}(S^*)$. Let $b\colon \mathcal{D}(S^*) \to \mathbb{C}^4$ be the mapping

$$b := (u_- \;\; v_- \;\; u_+ \;\; v_+)^{\mathrm{T}}, \tag{3.1}$$

by J_0 we denote the 2×2 matrix

$$J_0 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \tag{3.2}$$

and by J the 4×4 matrix

$$J = \begin{pmatrix} J_0 & 0 \\ 0 & -J_0 \end{pmatrix}.$$

Proposition 3.1. *The linear mapping* b *from (3.1) has these properties:*

(i) $\mathcal{R}(b) = \mathbb{C}^4$,

(ii) $\ker b = \mathcal{D}(S)$,

(iii) $\dfrac{(S^*f, g) - (f, S^*g)}{\mathrm{i}} = (bg)^* J bf, \quad f, g \in \mathcal{D}(S^*)$.

Proof. The definitions (2.12) and the relations (2.5), (2.6) and (2.7) imply

$$b\tilde{u}_- = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \quad b\tilde{v}_- = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad b\tilde{u}_+ = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad b\tilde{v}_+ = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \tag{3.3}$$

and (i) follows. Statement (ii) is a consequence of (2.15).

In order to prove (iii), we observe that, according to (2.16), each $f \in \mathcal{D}(S^*)$ is a linear combination of an element $f_0 \in \mathcal{D}(S)$ and $\tilde{u}_\pm, \tilde{v}_\pm$. Relations (2.5), (2.6) and (2.7) imply that $f = f_0 + f_1$ with $f_0 \in \mathcal{D}(S)$ and

$$f_1 := (u_- f)\tilde{v}_- - (v_- f)\tilde{u}_- - (u_+ f)\tilde{v}_+ + (v_+ f)\tilde{u}_+.$$

With an analogous decomposition of $g \in \mathcal{D}(S^*)$ it follows from (2.11), (2.3) and (2.4) that

$$\frac{(S^*f, g) - (f, S^*g)}{\mathrm{i}} = \frac{[f_1, g_1]}{\mathrm{i}}.$$

By means of (2.11), (2.5), (2.6) and (2.7) we find for the expression on the right-hand side the form

$$(bg)^* J bf,$$

and relation (iii) is proved. □

We equip the space $\mathbb{C}^4$ with the inner product generated by J: $(Jx, y) := y^* Jx$. Then a subspace $\mathcal{U}$ of $\mathbb{C}^4$ is called *J-non-negative* (*J-neutral*, respectively) if $(Jx, x) \geqslant 0$ ($= 0$, respectively) for all $x \in \mathcal{U}$.

Corollary 3.2. *The operator T is a (maximal) dissipative canonical extension of S if and only if $\mathcal{U} = \{bf : f \in \mathcal{D}(T)\}$ is a (maximal) J-non-negative subspaces of $\mathbb{C}^4$, and T is a (maximal) symmetric canonical extension of S if and only if this subspace is (maximal) J-neutral.*

Indeed, it follows from statement (iii) of proposition 3.1 that the operator $T \subset S^*$ is, for example, dissipative if and only if, for all $f \in \mathcal{D}(T)$, it holds that

$$0 \leqslant 2\,\mathrm{Im}(Tf, f) = \frac{(Tf, f) - (f, Tf)}{\mathrm{i}} = \frac{(S^*f, f) - (f, S^*f)}{\mathrm{i}} = ({}^{b}f)^* J\, {}^{b}f.$$

The other claims follow in the same way.

In the sequel, B denotes a complex 2×4 matrix, which we write also as a block matrix

$$B = (C \quad D)$$

with two 2×2 matrices C and D; J_0 is the matrix defined in (3.2). Since the eigenvalues of the matrix J are ± 1, each of multiplicity 2, the maximal J-non-negative subspaces of $\mathbb{C}^4$ are of dimension 2.

Theorem 3.3. *The operator T is a maximal dissipative canonical extension of S if and only if*

$$\mathcal{D}(T) = \{f \in \mathcal{D}(S^*) \colon B\,{}^{b}f = 0\}, \tag{3.4}$$

where the 2×4 matrix $B = (C \quad D)$ is such that its rank is 2 and the inequality

$$CJ_0C^* \leqslant DJ_0D^* \tag{3.5}$$

holds; T is a self-adjoint canonical extension of S if and only if the rank of the matrix B in (3.4) is 2 and the relation

$$CJ_0C^* = DJ_0D^* \tag{3.6}$$

holds.

Proof. By corollary 3.2, T is maximal dissipative if and only if $\mathcal{U} = \{{}^{b}f \colon f \in \mathcal{D}(T)\} = \ker B$ is maximal J-non-negative. This is the case if and only if $\mathcal{U}^{\perp} = \mathcal{R}(B^*)$ is maximal J-nonpositive, which is equivalent to (3.5) and rank $B = 2$. The proof of the second statement of the theorem is similar. □

If we write the matrices C and D in the form

$$C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}, \qquad D = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix},$$

the interface condition $B\,{}^{b}f = 0$ in (3.4) becomes

$$\begin{aligned} c_{11}f(0-) - c_{12}\lim_{x\to 0-}(f'(x) + (1 + \ln|x|)f(x)) \qquad & \\ - d_{11}f(0+) + d_{12}\lim_{x\to 0+}(f'(x) + (1 + \ln|x|)f(x)) &= 0, \\ c_{21}f(0-) - c_{22}\lim_{x\to 0-}(f'(x) + (1 + \ln|x|)f(x)) \qquad & \\ - d_{21}f(0+) + d_{22}\lim_{x\to 0+}(f'(x) + (1 + \ln|x|)f(x)) &= 0. \end{aligned} \tag{3.7}$$

4. Continuity at the origin

In this section we consider those maximal dissipative canonical extensions T of the symmetric operator S for which the functions $f \in \mathcal{D}(T)$ are continuous at zero. Continuity of f at zero means that $f(0-) = f(0+)$, which according to (2.13) is equivalent to ${}^{u}{}_{-}f + {}^{u}{}_{+}f = 0$. Therefore, these extensions are described by a matrix B with the property

$$c_{11} = d_{11} \neq 0, \quad c_{12} = d_{12} = 0,$$

and we can assume that

$$C = \begin{pmatrix} 1 & 0 \\ c_{21} & c_{22} \end{pmatrix}, \qquad D = \begin{pmatrix} 1 & 0 \\ d_{21} & d_{22} \end{pmatrix}.$$

Condition (3.5) is equivalent to

$$c_{22} = d_{22} \quad \text{and} \quad \frac{c_{21}\overline{c_{22}} - c_{22}\overline{c_{21}}}{\mathrm{i}} \leqslant \frac{d_{21}\overline{d_{22}} - d_{22}\overline{d_{21}}}{\mathrm{i}}. \tag{4.1}$$

If $c_{22} = d_{22} = 0$, matrix B can be supposed to have the form

$$B = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

If $c_{22} = d_{22} \neq 0$ we can assume that this number is 1, and inequality (4.1) becomes $\operatorname{Im} c_{21} \leqslant \operatorname{Im} d_{21}$. By subtracting a multiple of the first row of B from the second row, we arrive at the following result.

Theorem 4.1. *The functions in the domain of the maximal dissipative canonical extension T of S are continuous in* 0 *if and only if the matrix B in (3.4) can be chosen as*

$$B_\gamma = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & \gamma & 1 \end{pmatrix} \quad \textit{with} \quad \operatorname{Im}\gamma \geqslant 0, \tag{4.2}$$

or as

$$B_\infty = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{4.3}$$

This extension T is self-adjoint if and only if in (4.2) $\operatorname{Im}\gamma = 0$ *or if B is of the form (4.3).*

The extension T of S having the form (3.4) with $B = B_\gamma$ is denoted by T_γ, $\gamma \in \mathbb{C}^+ \cup \{\infty\}$. It is easy to see that also for $\gamma \in \mathbb{C}^-$ an extension T_γ is defined by the same interface conditions; then the operator $-T_\gamma$ is maximal dissipative.

In order to write the boundary conditions for the extension T_γ in a more explicit form than (3.7), we need a lemma.

Lemma 4.2. *If $f \in \mathcal{D}(S^*)$ and $f(0+) = f(0-)$, then*

$$\lim_{x \to 0+} (f(x) - f(-x))(1 + \ln|x|) = 0. \tag{4.4}$$

Proof. If $f \in \mathcal{D}(S)$, the claim follows from (2.17). So it remains to consider linear combinations

$$f = \alpha_- \tilde{u}_- + \beta_- \tilde{v}_- + \alpha_+ \tilde{u}_+ + \beta_+ \tilde{v}_+,$$

for which, because of the continuity of f at 0, also $\beta_- = \beta_+ =: \beta$. Hence f has the form

$$f = \alpha_- \tilde{u}_- + \alpha_+ \tilde{u}_+ + \beta v,$$

and relation (4.4) follows easily from the definition of functions $\tilde{u}_\pm$ and v. □

THEOREM 4.3. *The extension T_γ, $\gamma \in \mathbb{C} \cup \{\infty\}$, of S is given by interface conditions of the form*

$$f(0-) = f(0+), \quad \lim_{x \to 0+} (f'(x) - f'(-x)) = \gamma f(0) \quad \text{if } \gamma \in \mathbb{C}, \tag{4.5}$$

and by the Dirichlet interface conditions

$$f(0+) = f(0-) = 0 \quad \text{if } \gamma = \infty. \tag{4.6}$$

T_γ is self-adjoint if and only if $\gamma \in \mathbb{R} \cup \{\infty\}$.

Proof. If the matrix $B = B_\gamma$ given by (4.2), then the interface conditions at 0 for $f \in \mathcal{D}(T) \subset \mathcal{D}(S^*)$ are $f(0-) = f(0+)$ and

$$-\lim_{x \to 0-} (f'(x) + (1 + \ln|x|) f(x)) + \lim_{x \to 0+} (f'(x) + (1 + \ln|x|) f(x)) = \gamma f(0+). \tag{4.7}$$

By lemma 4.2, relation (4.7) is equivalent to relation (4.5). If the matrix $B = B_\infty$ given by (4.3), we obtain the Dirichlet interface conditions. □

For the canonical extensions of S which were considered in [12], it was shown there that they are norm resolvent limits of Sturm–Liouville operators with a regular potential. We show by the same method as in [12] that this is true for all the operators T_γ, $\gamma \in \mathbb{C}$. To this end, we define for $\gamma \in \mathbb{C}$ and $\varepsilon > 0$ the Sturm–Liouville operators $T_{\gamma,\varepsilon}$ as follows:

$$\mathcal{D}(T_{\gamma,\varepsilon}) := \{f \in \mathcal{L}^2(I) \colon f, f' \in \mathcal{AC}_{\text{loc}}(I),\ f'' \in \mathcal{L}^2(I),\ f(a) = f(b) = 0\},$$
$$(T_{\gamma,\varepsilon} f)(x) := -f''(x) - \frac{1}{2}\left(\frac{1 + \gamma/\mathrm{i}\pi}{x + \mathrm{i}\varepsilon} + \frac{1 - \gamma/\mathrm{i}\pi}{x - \mathrm{i}\varepsilon}\right) f(x).$$

THEOREM 4.4. *For $\gamma \in \mathbb{C}$, the operator T_γ is the norm resolvent limit of the operators $T_{\gamma,\varepsilon}$ if $\varepsilon \to 0+$.*

Proof. On the set

$$\mathcal{D} := \{f \in \mathcal{AC}_{\text{loc}}(I) \colon f' \in \mathcal{L}^2(I), f(a) = f(b) = 0\}$$

we consider the following sesquilinear forms:

$$\begin{aligned}
\mathfrak{l}^0[f,g] &:= \int_a^b f'(x)\overline{g'(x)}\,\mathrm{d}x,\\
\mathfrak{q}_\varepsilon[f,g] &:= -\int_a^b \frac{f(x)\overline{g(x)}}{x+\mathrm{i}\varepsilon}\,\mathrm{d}x, \text{ if } \varepsilon \neq 0, \quad \mathfrak{q}_0[f,g] := -P\int_a^b \frac{f(x)\overline{g(x)}}{x}\,\mathrm{d}x,\\
\mathfrak{b}[f,g] &:= f(0)\overline{g(0)},
\end{aligned}$$

where P denotes the Cauchy principal value. The form $\mathfrak{l}^0$ is closed and non-negative; the forms $\mathfrak{q}_0$ and $\mathfrak{b}$ are symmetric and $\mathfrak{l}^0$-bounded with relative bound zero [12, lemmas 2.3 and 2.5]. Hence, according to [13, theorem VI.1.33],

$$\mathfrak{t}_\gamma := \mathfrak{l}^0 + \mathfrak{q}_0 + \gamma\mathfrak{b}$$

is a closed sectorial form on $\mathcal{D}$. By the second representation theorem [13, theorem VI.2.1], there exists an m-sectorial operator T_{t_γ} such that

1. $\mathcal{D}(T_{t_\gamma}) \subset \mathcal{D}$;
2. $\mathfrak{t}_\gamma[f,g] = (T_{t_\gamma} f, g),\ f \in \mathcal{D}(T_{t_\gamma}),\ g \in \mathcal{D}$;
3. $\mathcal{D}(T_{t_\gamma})$ is a core of $\mathfrak{t}_\gamma$;
4. if $f \in \mathcal{D},\ y \in \mathcal{L}^2(I)$ such that the equality $\mathfrak{t}_\gamma[f,g] = (y,g)$ holds for all g in a core of $\mathfrak{t}_\gamma$, then $f \in \mathcal{D}(T_{t_\gamma})$ and $T_{t_\gamma} f = y$.

We shall show that $T_{t_\gamma} = T_\gamma$. Theorem 4.3 implies $\mathcal{D}(T_\gamma) \subset \mathcal{D}$, and for $f \in \mathcal{D}(T_\gamma)$ and $g \in \mathcal{D}$ it holds that

$$\begin{aligned}
(T_\gamma f, g) &= \left(\int_a^0 + \int_0^b\right)\left(-f''(x) - \frac{f(x)}{x}\right)\overline{g(x)}\,\mathrm{d}x\\
&= \lim_{\varepsilon\to 0+}\left(\int_a^{-\varepsilon} + \int_\varepsilon^b\right)\left(-f''(x) - \frac{f(x)}{x}\right)\overline{g(x)}\,\mathrm{d}x\\
&= P\int_a^b\left(f'(x)\overline{g'(x)} - \frac{f(x)\overline{g(x)}}{x}\right)\mathrm{d}x + \lim_{\varepsilon\to 0+}(f'(\varepsilon)\overline{g(\varepsilon)} - f'(-\varepsilon)\overline{g(-\varepsilon)})\\
&= \mathfrak{l}^0[f,g] + \mathfrak{q}_0[f,g] + \lim_{\varepsilon\to 0+}(f'(\varepsilon)\overline{g(\varepsilon)} - f'(-\varepsilon)\overline{g(-\varepsilon)})\\
&= \mathfrak{t}_\gamma[f,g] + \lim_{\varepsilon\to 0+}(f'(\varepsilon)\overline{g(\varepsilon)} - f'(-\varepsilon)\overline{g(-\varepsilon)}) - \gamma f(0)\overline{g(0)}.
\end{aligned} \tag{4.8}$$

If $g \in \mathcal{D}$, we have

$$|g(x) - g(0)| \leqslant \int_0^x |g'(s)|\,\mathrm{d}s \leqslant \sqrt{|x|}\|g'(s)\|.$$

Therefore, relation (2.14) yields, for $f \in \mathcal{D}(T_\gamma)$,

$$\begin{aligned}
&\lim_{x\to 0+}(f'(x)\overline{g(x)} - f'(-x)\overline{g(-x)}) - \gamma f(0)\overline{g(0)}\\
&\qquad = \lim_{x\to 0+}(f'(x) - f'(-x) - \gamma f(0))\overline{g(0)} = 0.
\end{aligned}$$

Hence (4.8) becomes

$$(T_\gamma f, g) = \mathfrak{t}_\gamma[f, g], \qquad f \in \mathcal{D}(T_\gamma), \quad g \in \mathcal{D},$$

which implies $T_\gamma \subset T_{t_\gamma}$. Since, on the other hand, T_γ or $-T_\gamma$ is a maximal dissipative operator, in this inclusion the equality sign must prevail.

The differential operator $T_{\gamma,\varepsilon}$ is associated with the sesquilinear form

$$\mathfrak{t}_{\gamma,\varepsilon} = \mathfrak{l}^0 + \frac{\pi\mathrm{i} + \gamma}{2\pi\mathrm{i}}\mathfrak{q}_\varepsilon + \frac{\pi\mathrm{i} - \gamma}{2\pi\mathrm{i}}\mathfrak{q}_{-\varepsilon},$$

which is also defined on $\mathcal{D}$. As in the proof of [12, theorem 3.3], for $f, g \in \mathcal{D}$ it follows that

$$|\mathfrak{q}_{\pm\varepsilon}[f, g] - (\mathfrak{q}_0[f, g] \mp \pi\mathrm{i}\mathfrak{b}[f, g])| = o(1)\mathfrak{l}^0[f, g] + o(1)(f, g), \qquad \varepsilon \to 0+,$$

and we get

$$\mathfrak{t}_{\gamma,\varepsilon}[f, g] - \mathfrak{t}_\gamma[f, g] = o(1)\mathfrak{l}^0[f, g] + o(1)(f, g), \qquad \varepsilon \to 0+.$$

Now the resolvent convergence of the operators $T_{\gamma,\varepsilon}$ to T_γ follows from [13, theorem VI.3.6]. □

5. Representation of the solutions by Whittaker functions

In this section we express the resolvents of the extensions T_γ from § 4 by means of Whittaker functions. We first recall Whittaker's differential equation [2, 5, 16, 21]:

$$\frac{\mathrm{d}^2 f(z)}{\mathrm{d}z^2} + \left(-\frac{1}{4} + \frac{\kappa}{z} + \frac{1-\mu^2}{4z^2}\right) f(z) = 0. \tag{5.1}$$

Two linearly independent solutions of this differential equation are the Whittaker functions

$$M_{\kappa,\mu(z)/2} = z^{(1+\mu)/2}\mathrm{e}^{-z/2}\Phi(\tfrac{1}{2}(1+\mu) - \kappa, 1+\mu, z),$$
$$W_{\kappa,\mu(z)/2} = z^{(1+\mu)/2}\mathrm{e}^{-z/2}\Psi(\tfrac{1}{2}(1+\mu) - \kappa, 1+\mu, z),$$

where Φ is the confluent hypergeometric function. In the following we use the function $\psi(z) := \Gamma'(z)/\Gamma(z)$, and for complex numbers α and β and an integer k the symbols

$$(\alpha)_k := \alpha(\alpha+1)\cdots(\alpha+k-1), \qquad d_k(\alpha, \beta) := \psi(\alpha+k) - \psi(1+k) - \psi(\beta+k).$$

Then the function Φ is given by the relation

$$\Phi(\alpha, \beta, z) = \sum_{k=0}^{\infty} \frac{(\alpha)_k}{(\beta)_k}\frac{z^k}{k!},$$

and in the case that β is a positive integer, $\Psi(\alpha,\beta,z)$ admits the following representation [2, § 6.1, § 6.7, formula (13)]:

$$\begin{aligned}\Psi(\alpha,\beta,z) = {} & \frac{(-1)^\beta}{\Gamma(\beta)\Gamma(\alpha-\beta+1)}\left(\Phi(\alpha,\beta,z)\ln z + \sum_{k=0}^{\infty}\frac{(\alpha)_k d_k(\alpha,\beta)z^k}{(\beta)_k k!}\right) \\ & + \frac{\Gamma(\beta-1)}{\Gamma(\alpha)}\sum_{k=0}^{\beta-2}\frac{(\alpha-\beta+1)_k z^{k-\beta+1}}{(2-\beta)_k k!}. \end{aligned} \tag{5.2}$$

If we make the substitution

$$\mu = 1, \qquad \kappa = \frac{\mathrm{i}}{2\sqrt{\lambda}}, \qquad z = \frac{x}{\kappa} = -2\mathrm{i}\sqrt{\lambda}x,$$

equation (5.1) becomes equation (1.2): $l[f] - \lambda f = 0$. Therefore, two linearly independent solutions of (1.2) are the functions

$$\left.\begin{aligned} f_M(x,\lambda) &= M_{\mathrm{i}/2\sqrt{\lambda},1/2}(-2\mathrm{i}\sqrt{\lambda}x), \\ f_W(x,\lambda) &= \Gamma(1-\mathrm{i}/2\sqrt{\lambda})W_{\mathrm{i}/2\sqrt{\lambda},1/2}(-2\mathrm{i}\sqrt{\lambda}x); \end{aligned}\right\} \tag{5.3}$$

see also [4,8]. The function f_M is entire in x, whereas f_W has a logarithmic branch point at $x = 0$. The function f_W is understood as the principal branch, which is obtained from the principal branch of the logarithm in (5.2).

With the functions $f_M(x,\lambda)$ and $f_W(x,\lambda)$ we form for $\lambda \neq 0$ the solutions

$$f_-(x,\lambda) := \begin{cases} \dfrac{f_M(a,\lambda)f_W(x,\lambda) - f_W(a,\lambda)f_M(x,\lambda)}{f_M(a,\lambda)f_W'(a,\lambda) - f_W(a,\lambda)f_M'(a,\lambda)} & \text{if } x < 0, \\ 0 & \text{if } x > 0, \end{cases} \tag{5.4}$$

$$f_+(x,\lambda) := \begin{cases} 0 & \text{if } x < 0, \\ \dfrac{f_M(b,\lambda)f_W(x,\lambda) - f_W(b,\lambda)f_M(x,\lambda)}{f_M(b,\lambda)f_W'(b,\lambda) - f_W(b,\lambda)f_M'(b,\lambda)} & \text{if } x > 0. \end{cases} \tag{5.5}$$

They satisfy for $x \neq 0$ the differential equation $l[f] - \lambda f = 0$ and the boundary conditions

$$\begin{aligned} f_-(a,\lambda) &= 0, & f_-'(a,\lambda) &= 1, \\ f_+(b,\lambda) &= 0, & f_+'(b,\lambda) &= 1. \end{aligned}$$

If $x \neq 0$ is fixed, $f_\pm(x,\lambda)$ are entire functions in λ. Further, we introduce the kernel

$$K(x,\xi;\lambda) := \begin{cases} \dfrac{f_M(\xi,\lambda)f_W(x,\lambda) - f_W(\xi,\lambda)f_M(x,\lambda)}{f_M(\xi,\lambda)f_W'(\xi,\lambda) - f_W(\xi,\lambda)f_M'(\xi,\lambda)} & \text{if } \xi \leqslant x < 0, \\ -\dfrac{f_M(\xi,\lambda)f_W(x,\lambda) - f_W(\xi,\lambda)f_M(x,\lambda)}{f_M(\xi,\lambda)f_W'(\xi,\lambda) - f_W(\xi,\lambda)f_M'(\xi,\lambda)} & \text{if } 0 < x \leqslant \xi, \\ 0 & \text{otherwise.} \end{cases}$$

It satisfies for $x \neq 0$ and $x \neq \xi$ the differential equation

$$-\frac{\partial^2 K}{\partial x^2}(x,\xi;\lambda) - \frac{K(x,\xi;\lambda)}{x} = \lambda K(x,\xi;\lambda)$$

and the boundary conditions

$$\frac{\partial K}{\partial x}(\xi+,\xi;\lambda) = 1 \text{ if } \xi < 0, \qquad \frac{\partial K}{\partial x}(\xi-,\xi;\lambda) = -1 \text{ if } \xi > 0.$$

We introduce the following operators K_λ, $\lambda \in \mathbb{C}$, in $\mathcal{L}^2(I)$:

$$(K_\lambda f)(x) := \int_a^b K(x,\xi;\lambda) f(\xi)\,\mathrm{d}\xi, \qquad f \in \mathcal{L}^2(I).$$

Then $K_\lambda f \in \mathcal{D}(S^*)$ and $(S^* - \lambda)K_\lambda f = f$ for arbitrary $f \in \mathcal{L}^2(I)$. This implies for functions $f \in \mathcal{D}(S^*)$ that $K_\lambda(S^* - \lambda)f = f + g$ with $g \in \ker(S^* - \lambda)$. If f vanishes identically near a and b, then also $K_\lambda(S^* - \lambda)f$ does. In this case $g = 0$, and $K_\lambda(S^* - \lambda)f = f$, which yields $\tilde{u}_\pm, \tilde{v}_\pm \in \mathcal{R}(K_\lambda)$ and further

$$\mathcal{R}({}^{b}K_\lambda) = \mathbb{C}^4. \tag{5.6}$$

The functions $f_-(\cdot,\lambda)$ and $f_+(\cdot,\lambda)$ span the kernel $\ker(S^* - \lambda)$. For given $f \in \mathcal{L}^2(I)$ the equation

$$(T_\gamma - \lambda)f = y \tag{5.7}$$

is satisfied if and only if $f = c_- f_- + c_+ f_+ + K_\lambda y$ with numbers c_- and c_+ such that $B_\gamma{}^{b}(c_- f_- + c_+ f_+ + K_\lambda y) = 0$. Relation (5.6) implies that the latter equation has a unique solution for arbitrary $y \in \mathcal{L}^2(I)$ if and only if the 2×2 matrix

$$M_\gamma(\lambda) := \begin{pmatrix} B_\gamma{}^{b} f_-(\cdot\,;\lambda) & B_\gamma{}^{b} f_+(\cdot;\lambda) \end{pmatrix} \tag{5.8}$$

is invertible, and the solution of equation (5.7) can be written as

$$f(x) = (K_\lambda y)(x) - \begin{pmatrix} f_-(x,\lambda) & f_+(x,\lambda) \end{pmatrix} M_\gamma(\lambda)^{-1} B_\gamma{}^{b}(K_\lambda y). \tag{5.9}$$

For the following theorem see [19, I § 2].

Theorem 5.1. *Suppose $\gamma \in \mathbb{C}$ and let $M_\gamma(\lambda)$ be the matrix function from (5.8). Then $\lambda \in \rho(T_\gamma)$ if and only if* $\det M_\gamma(\lambda) \neq 0$, *and in this case the resolvent $(T_\gamma - \lambda)^{-1}$ is given by (5.9): $(T_\gamma - \lambda)^{-1} y = f$. The eigenvalues of T_γ are geometrically simple, and the length of the Jordan chain of T_γ at an eigenvalue λ equals the order of the zero $\zeta = \lambda$ of the function* $\det M_\gamma(\zeta)$.

Proof. If $\det M_\gamma(\lambda) \neq 0$, the resolvent $(T_\gamma - \lambda)^{-1}$ exists and is given by (5.9). Now suppose that $\det M_\gamma(\lambda) = 0$. Then the non-zero 2-vector $(c_-, c_+)^{\mathrm{T}}$ belongs to $\ker M_\gamma(\lambda)$ if and only if the function $f(x) := c_- f_-(x,\lambda) + c_+ f_+(x,\lambda)$ fulfils the interface condition $B_\gamma{}^{b} f = 0$ and hence is an eigenfunction of T_γ at λ. Since all eigenfunctions of T_γ at λ are of this form and the matrix $M_\gamma(\lambda)$ is not the zero matrix, the geometric multiplicity of the eigenvalue λ equals one.

Suppose now that λ is a zero of order m of the function $\det M_\gamma(\zeta)$. Then (5.9) implies that the length of the Jordan chain of T_γ at λ is at most m. A chain of length m can be obtained as follows. Since

$$M_\gamma(\zeta) = \begin{pmatrix} m_{\gamma,11}(\zeta) & m_{\gamma,12}(\zeta) \\ m_{\gamma,21}(\zeta) & m_{\gamma,22}(\zeta) \end{pmatrix}$$

is not the zero matrix, at least one entry does not vanish. Suppose, for example, that this is $m_{\gamma,11}(\lambda)$; the other cases can be treated similarly. With the matrices

$$E(\zeta) = \begin{pmatrix} 1 & 0 \\ -\dfrac{m_{\gamma,21}(\zeta)}{m_{\gamma,11}(\zeta)} & 1 \end{pmatrix}, \qquad F(\zeta) = \begin{pmatrix} 1 & -\dfrac{m_{\gamma,12}(\zeta)}{m_{\gamma,11}(\zeta)} \\ 0 & 1 \end{pmatrix}$$

we get

$$E(\zeta)M_\gamma(\zeta)F(\zeta) = \begin{pmatrix} m_{\gamma,11}(\zeta) & 0 \\ 0 & \dfrac{\det M_\gamma(\zeta)}{m_{\gamma,11}(\zeta)} \end{pmatrix}.$$

Therefore, the analytic family of vectors

$$\begin{pmatrix} c_-(\zeta) \\ c_+(\zeta) \end{pmatrix} = F(\zeta)\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

fulfils for $\zeta \to \lambda$ the relations

$$\begin{pmatrix} c_-(\zeta) \\ c_+(\zeta) \end{pmatrix} \not\to 0, \qquad \begin{pmatrix} d_-(\zeta) \\ d_+(\zeta) \end{pmatrix} = M_\gamma(\zeta)\begin{pmatrix} c_-(\zeta) \\ c_+(\zeta) \end{pmatrix} = O((\zeta-\lambda)^m).$$

Then (3.3) and (5.8) give

$$f(\cdot,\zeta) = c_-(\zeta)f_-(\cdot,\zeta) + c_+(\zeta)f_+(\cdot,\zeta) - d_-(\zeta)\tilde{v}_-(\cdot) - d_+(\zeta)\tilde{u}_+(\cdot) \in \mathcal{D}(T_\gamma),$$

and the relation $(T_\gamma - \zeta)f(\cdot,\zeta) = O((\zeta-\lambda)^m)$ implies that the functions

$$f_i(\cdot,\lambda) := \frac{\partial f(\cdot,\lambda)}{\partial \lambda^i}, \qquad i = 0,1,\ldots,m-1,$$

form a Jordan chain at λ. □

In the following we need some asymptotic properties of the eigenvalues of the operators T_γ. To this end, we study the asymptotic behaviour of the functions f_M and f_W. The relations (5.3) imply the following asymptotics. If $\lambda \in \mathbb{C}\backslash\{0\}$ is fixed, then for $x \to 0$,

$$f_M(x,\lambda) = -2\mathrm{i}\sqrt{\lambda}x + O(x^2), \tag{5.10}$$

$$\begin{aligned} f_W(x,\lambda) &= \mathrm{e}^{-z/2} - \kappa z \mathrm{e}^{-z/2}((1+O(z)\ln z + d_0(1-\kappa,2) + O(z)) \\ &= 1 + \mathrm{i}\sqrt{\lambda}x - \ln z - d_0(1-\kappa,2)x + O(x^2\ln x) \\ &= 1 - x\ln|x| + c_\lambda(x)x + O(x^2\ln x), \end{aligned} \tag{5.11}$$

where

$$c_\lambda(x) := \mathrm{i}\sqrt{\lambda} - d_0\left(1 - \frac{\mathrm{i}}{2\sqrt{\lambda}}, 2\right) + \ln|x| - \ln(-2\mathrm{i}\sqrt{\lambda}x). \tag{5.12}$$

Note that $c_\lambda(x)$ does not depend on $|x|$, hence it is bounded if $x \to \pm 0$. Further, it holds that

$$c_\lambda(+1) - c_\lambda(-1) = \ln(2\mathrm{i}\sqrt{\lambda}) - \ln(-2\mathrm{i}\sqrt{\lambda}) = \mathrm{i}\pi. \tag{5.13}$$

Relations (5.10) and (5.11) imply

$$f_W(0-,\lambda) = f_W(0+,\lambda) = 1, \qquad f_M(0-,\lambda) = f_M(0+,\lambda) = 0 \tag{5.14}$$

and

$$\left.\begin{aligned} \lim_{x\to 0}(f_M'(x,\lambda) + (1+\ln|x|)f_M(x,\lambda)) &= -2\mathrm{i}\sqrt{\lambda}, \\ \lim_{x\to 0-}(f_W'(x,\lambda) + (1+\ln|x|)f_W(x,\lambda)) &= c_\lambda(-1), \\ \lim_{x\to 0+}(f_W'(x,\lambda) + (1+\ln|x|)f_W(x,\lambda)) &= c_\lambda(+1), \end{aligned}\right\} \tag{5.15}$$

where $c_\lambda(x)$ is given by (5.12).

Remark 5.2. Boyd [4] considered the boundary value problem (1.1) with boundary conditions (1.3), replacing the potential $-x^{-1}$ first by $-(x-\mathrm{i}\varepsilon)^{-1}$ with $\varepsilon > 0$ and letting $\varepsilon \to 0$. He required the eigenfunctions to admit an analytic continuation onto the lower half-plane. This requirement specifies an interface condition in $x = 0$, which, however, turns out not to be self-adjoint. Indeed, the solutions of (1.1) which admit an analytic continuation onto the lower half-plane are linear combinations of the functions $f_M(x,\lambda)$ and $\tilde{f}_W(x,\lambda)$, where $\tilde{f}_W(x,\lambda)$ equals the function $f_W(x,\lambda)$ for positive real x, and with the branch cut at $\arg x = \pi/2$. This corresponds to a branch cut in the logarithm in the definition of the function Ψ in (5.2) at $\arg z = \arg\sqrt{\lambda}$. For real x and $-\pi < \arg\lambda \leqslant \pi$, this means

$$\tilde{f}_W(x,\lambda) = \begin{cases} f_W(x,\lambda) & \text{if } x > 0, \\ f_W(x,\lambda) - \dfrac{\pi}{\sqrt{\lambda}} f_M(x,\lambda) & \text{if } x < 0. \end{cases}$$

Now it follows from (5.13), (5.14) and (5.15) that

$$b_{f_M(\cdot,\lambda)} = \begin{pmatrix} 0 \\ 2\mathrm{i}\sqrt{\lambda} \\ 0 \\ -2\mathrm{i}\sqrt{\lambda} \end{pmatrix}, \qquad b_{\tilde{f}_W(\cdot,\lambda)} = \begin{pmatrix} 1 \\ -c_\lambda(1) - \mathrm{i}\pi \\ -1 \\ c_\lambda(1) \end{pmatrix}.$$

These vectors span the kernel of the 2×4 matrix $B_{-\mathrm{i}\pi}$. Therefore, the operator which was considered in [4] is (up to its sign) $T_{-\mathrm{i}\pi}$.

In order to study the asymptotic behaviour of the functions f_M and f_W for $\lambda\to\infty$, we use the following relations [2, 6.13(1) and (2)] [16, 4.7(2)–(4)]:

$$\begin{aligned} \Phi(\alpha,\beta,z) &= \frac{\Gamma(\beta)\mathrm{e}^{\alpha\mathrm{i}\pi\,\mathrm{sgn}\,\mathrm{Im}\,z}}{\Gamma(\beta-\alpha)} z^{-\alpha} + \frac{\Gamma(\beta)}{\Gamma(\alpha)}\mathrm{e}^z z^{\alpha-\beta} \\ &\qquad + O(z^{-\alpha-1}) + O(\mathrm{e}^z z^{\alpha-\beta-1}), \end{aligned} \tag{5.16}$$

$$\Psi(\alpha,\beta,z) = z^{-\alpha} + O(z^{-\alpha-1}) \tag{5.17}$$

if $z\to\infty$. The expansion (5.16) holds in the sector $-\pi < \arg z < \pi$, the expansion (5.17) in the sector $-3\pi/2 < \arg z < 3\pi/2$. If $x\in\mathbb{R}\backslash\{0\}$ is fixed, then for $\kappa\to 0$,

$$\Gamma(1\pm\kappa) = 1 + O(\kappa), \quad \mathrm{e}^{\pm\mathrm{i}\pi(1-\kappa)} = -1 + O(\kappa), \quad z^\kappa = \mathrm{e}^{\kappa(\ln x - \ln\kappa)} = 1 + O(\kappa\ln\kappa),$$

and we find the following asymptotics for $\lambda \to \infty$ in the sector $-\pi < \arg\lambda < \pi$:

$$f_M(x,\lambda) = \mathrm{e}^{-\mathrm{i}\sqrt{\lambda}x} - \mathrm{e}^{\mathrm{i}\sqrt{\lambda}x} + O\biggl(\frac{\ln\lambda}{\sqrt{\lambda}}\mathrm{e}^{|x\,\mathrm{Im}\sqrt{\lambda}|}\biggr), \tag{5.18}$$

$$f_W(x,\lambda) = \mathrm{e}^{\mathrm{i}\sqrt{\lambda}x} + O\biggl(\frac{\ln\lambda}{\sqrt{\lambda}}\mathrm{e}^{\mathrm{i}\sqrt{\lambda}x}\biggr). \tag{5.19}$$

Theorem 5.3. *If $\gamma \in \mathbb{C}$, then the spectrum $\sigma(T_\gamma)$ consists of isolated normal eigenvalues λ_n, $n \in \mathbb{N}$, of geometric multiplicity one, and all but finitely many of them are simple. If they are numbered according to non-decreasing absolute value, then the following asymptotic formula holds:*

$$\lambda_n = \frac{\pi^2 n^2}{(b-a)^2} + O(\ln n) \quad \textit{for } n \to \infty. \tag{5.20}$$

In the proof of the theorem we use the following lemma.

Lemma 5.4. *An entire function $F(z)$ of the form*

$$F(z) = \sin z + O\biggl(\frac{\ln z}{z}\exp|\mathrm{Im}\,z|\biggr) \quad \textit{for } |z| \to \infty$$

has infinitely many zeros and all but finitely many of them are simple. For $n \in \mathbb{Z}$ with $|n|$ sufficiently large, there is a disc of radius

$$\rho_n = O\biggl(\frac{\ln|n|}{n}\biggr) \quad \textit{for } |n| \to \infty,$$

around the point $n\pi$ which contains exactly one zero of $F(z)$; outside these discs lie only finitely many zeros of $F(z)$.

Proof. Since $F(z)$ is entire and does not vanish identically, its zeros are countable and have no accumulation point in $\mathbb{C}$. We consider the zeros only in the right half-plane; the zeros in the left half-plane can be treated similarly. There exist positive real numbers r, C_1, C_2 such that for the zeros $\zeta = s + \mathrm{i}t$, with $|\zeta| > r$,

$$|\sinh t| \leqslant |\sin\zeta| \leqslant C_1\biggl|\frac{\ln\zeta}{\zeta}\biggr|\exp|t| \leqslant 2C_1\biggl|\frac{\ln\zeta}{\zeta}\biggr|(|\sinh t| + 1).$$

Hence

$$1 + \frac{1}{|\sinh t|} \geqslant \frac{1}{C_2}\biggl|\frac{\zeta}{\ln\zeta}\biggr|, \qquad |\zeta| > r,$$

which implies that all zeros ζ lie in a strip $|t| \leqslant C$ with $C > 0$, and with $C_3 = C_1\exp C$,

$$|\sin\zeta| \leqslant C_3\biggl|\frac{\ln\zeta}{\zeta}\biggr|, \qquad |\zeta| > r.$$

Denote by R_n, $n \in \mathbb{N}$, the rectangle

$$n\pi - \pi/2 \leqslant \mathrm{Re}\,z \leqslant n\pi + \pi/2, \qquad -C \leqslant \mathrm{Im}\,z \leqslant C.$$

Then

$$m := \min_{z\in\partial R_n} |\sin z| > 0$$

and for sufficiently large n, say $n \geqslant n_0$,

$$|F(z) - \sin z| < m \leqslant |\sin z|, \qquad z \in \partial R_n.$$

Rouché's theorem implies that $F(z)$, like $\sin z$, has exactly one zero in R_n for $n \geqslant n_0$ and that this zero is simple. We now claim that for n sufficiently large, the zero of $F(z)$ in R_n lies in a circle of radius $\rho_n = O(n^{-1}\ln n)$ around the zero $z = n\pi$ of $\sin z$. To prove the claim, first choose $\rho > 0$ such that the inequality

$$|\sin z| \geqslant \tfrac{1}{2}|z - \pi n|$$

holds for all $n \in \mathbb{N}$ and $|z - \pi n| \leqslant \rho$. Then choose C_4 such that

$$|F(z) - \sin z| < \frac{C_4}{2}\left(\frac{\ln n}{n}\right), \qquad z \in R_n,\ n \geqslant 2.$$

Finally, choose $n_1 \geqslant \max(2, n_0)$ so large that $\rho_n := C_4(n^{-1}\ln n) < \rho$ for all $n \geqslant n_1$. Then, for $n \geqslant n_1$ and $|z - \pi n| = \rho_n$,

$$|F(z) - \sin z| < \tfrac{1}{2}\rho_n = \tfrac{1}{2}|z - \pi n| \leqslant |\sin z|.$$

The claim now follows again from Rouché's theorem. □

Proof of theorem 5.3. We consider the matrix B_γ from (4.2) and the corresponding 2×2 matrix function M_γ defined by (5.8). Since $M_\gamma(\lambda)$, $\lambda \in \mathbb{C}$, is never the zero matrix, the geometric multiplicity of the eigenvalues of T_γ is one; see theorem 5.1. A straightforward calculation shows that, up to a non-zero factor, the determinant $\det M_\gamma(\lambda)$ equals

$$\begin{aligned}
&\begin{vmatrix} f_M(a,\lambda) & f_M(b,\lambda) \\ -2\mathrm{i}\sqrt{\lambda} f_W(a,\lambda) + (\mathrm{i}\pi - c_\lambda(1)) f_M(a,\lambda) & -2\mathrm{i}\sqrt{\lambda} f_W(b,\lambda) + (\gamma - c_\lambda(1)) f_M(b,\lambda) \end{vmatrix} \\
&\qquad = \begin{vmatrix} f_M(a,\lambda) & f_M(b,\lambda) \\ -2\mathrm{i}\sqrt{\lambda} f_W(a,\lambda) + \mathrm{i}\pi f_M(a,\lambda) & -2\mathrm{i}\sqrt{\lambda} f_W(b,\lambda) + \gamma f_M(b,\lambda). \end{vmatrix}
\end{aligned}$$

Now, relations (5.18) and (5.19) imply that this determinant for $\lambda \to \infty$ asymptotically behaves like

$$\begin{aligned}
&-2\mathrm{i}\sqrt{\lambda}\begin{vmatrix} f_M(a,\lambda) & f_M(b,\lambda) \\ f_W(a,\lambda) & f_W(b,\lambda) \end{vmatrix} + O(\mathrm{e}^{|(b-a)\operatorname{Im}\sqrt{\lambda}|}) \\
&\qquad = -2\mathrm{i}\sqrt{\lambda}\begin{vmatrix} \mathrm{e}^{-\mathrm{i}\sqrt{\lambda}a} - \mathrm{e}^{\mathrm{i}\sqrt{\lambda}a} & \mathrm{e}^{-\mathrm{i}\sqrt{\lambda}b} - \mathrm{e}^{\mathrm{i}\sqrt{\lambda}b} \\ \mathrm{e}^{-\mathrm{i}\sqrt{\lambda}a} & \mathrm{e}^{-\mathrm{i}\sqrt{\lambda}b} \end{vmatrix} + O(\mathrm{e}^{|(b-a)\operatorname{Im}\sqrt{\lambda}|}\ln\lambda) \\
&\qquad = 4\sqrt{\lambda}((b-a)\sqrt{\lambda}) + O(\mathrm{e}^{|(b-a)\operatorname{Im}\sqrt{\lambda}|}\ln\lambda).
\end{aligned}$$

If we put $\zeta = (b-a)\sqrt{\lambda}$, apply lemma 5.4, and observe again theorem 5.1, then the claim follows. □

6. Basis properties of the root vectors of T_γ

Recall that a sequence (f_n), $n \in \mathbb{N}$, of elements of a separable Hilbert space $\mathcal{H}$ is called a *basis* of $\mathcal{H}$ if each $y \in \mathcal{H}$ has a unique representation

$$y = \sum_{n=1}^{\infty} c_n f_n, \quad \text{with } c_n \in \mathbb{C},\ n \in \mathbb{N},$$

where the sum converges in the norm of $\mathcal{H}$. The basis (f_n), $n \in \mathbb{N}$, of $\mathcal{H}$ is called a *Bari basis* if it is quadratically close to an orthonormal basis $\{e_n : n \in \mathbb{N}\}$ of $\mathcal{H}$, which means that

$$\sum_{n=1}^{\infty} \|f_n - e_n\|^2 < \infty.$$

For this notion and its properties, see, for example, [9, ch. VI]. We use the following criterion about the existence of a Bari basis [9, theorem VI.4.1]:

CRITERION. *Let T be a bounded dissipative operator in a Hilbert space such that $T - T^*$ is compact. Denote by μ_n, $n \in \mathbb{N}$, the mutually different eigenvalues of T and by l_n the geometric multiplicity of μ_n, and suppose that*

$$\sum \min(l_n, l_m) \frac{\operatorname{Im} \mu_n \operatorname{Im} \mu_m}{|\mu_n - \overline{\mu_m}|^2} < \infty, \tag{6.1}$$

where the sum runs over all $n, m \in N$ such that $n \neq m$ and $\operatorname{Im} \mu_n \neq 0$, $\operatorname{Im} \mu_m \neq 0$. If we choose in each eigenspace of T an orthonormal basis, then the sequence of all these basis elements forms a Bari basis in its closed linear hull.

We also use the well-known result of Lidskiĭ [9, theorem V.2.3]:

RESULT. *A dissipative trace class operator has a complete system of root vectors.*

If γ is real or ∞, then the operator T_γ is self-adjoint. By an argument as in the proof of the following theorem, it follows that its resolvent is a trace class operator. Hence T_γ, $\gamma \in \mathbb{R} \cup \{\infty\}$, has an orthonormal basis of eigenfunctions. The main result of this section is the following theorem.

THEOREM 6.1. *If $\gamma \in \mathbb{C}^+ \cup \mathbb{C}^-$, then the root vectors of T_γ can be chosen to form a Bari basis of $\mathcal{L}^2(I)$.*

Proof. Let $l \in \rho(T_\gamma) \cap \rho(T_0)$ be a real number. The spectral mapping theorem and theorem 5.3 imply that the eigenvalues η_n, $n \in \mathbb{N}$, of $(T_\gamma - l)^{-1}$ satisfy the relation

$$\eta_n = \frac{1}{cn^2 + O(\ln n)} = \frac{1}{cn^2} + O\left(\frac{\ln n}{n^4}\right) \quad \text{for } n \to \infty \tag{6.2}$$

with $c := \pi^2(b-a)^{-2}$. By theorem 4.3, T_0 is self-adjoint, hence also $(T_0 - l)^{-1}$ is self-adjoint, and since its eigenvalues satisfy relation (6.2), it is a trace class operator. If $\gamma \neq 0$, the difference $(T_\gamma - l)^{-1} - (T_0 - l)^{-1}$ is one-dimensional and therefore also $(T_\gamma - 1)^{-1}$ is a trace class operator.

In order to prove that the root vectors of T_γ form a Bari basis, we suppose that $\gamma \in \mathbb{C}^+$; the case $\gamma \in \mathbb{C}^-$ can be treated analogously. The operator $-(T_\gamma - l)^{-1}$ is dissipative and a trace class operator. Therefore, the closed linear span of its root vectors is the whole space $\mathcal{L}^2(I)$. Next we verify that the eigenvalues of $(T_\gamma - l)^{-1}$, which we denote by η_n, satisfy condition (6.1). Since the algebraic multiplicity of all but finitely many eigenvalues is one by theorem 5.3, this condition simplifies to

$$\sum_{1\leqslant m<n} \frac{\operatorname{Im}\eta_m \operatorname{Im}\eta_n}{|\eta_m - \overline{\eta_n}|^2} < \infty. \tag{6.3}$$

Relation (6.2) implies for $1 \leqslant m < n$ and suitable constants C_1, C_2, C_3 that

$$\frac{\operatorname{Im}\eta_m \operatorname{Im}\eta_n}{|\eta_m - \overline{\eta_n}|^2} \leqslant C_1 \frac{\ln m \ln n}{|(n-m)(n+m) - C_1(\ln m + \ln n)|^2} \leqslant C_3 \frac{(\ln(m+n))^2}{(n-m)^2(n+m)^2};$$

here we have used the inequalities $\ln n, \ln m \leqslant \ln(n+m)$ and the fact that

$$(n-m)^{-1}(n+m)^{-1}(\ln m + \ln n) \to 0 \quad \text{if } m < n,\ n \to \infty.$$

Since for sufficiently large x the function $x^{-1}\ln x$ is decreasing, then with $k = n-m$ and some constant C_4, we finally obtain

$$\sum_{m=1}^{\infty}\sum_{k=1}^{\infty} \frac{(\ln(2m+k))^2}{k^2(2m+k)^2} \leqslant C_4 \sum_{n=1}^{\infty} \frac{(\ln 2m)^2}{(2n)^2} \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty.$$

□

If $\gamma \in \mathbb{C}^+ \cup \mathbb{C}^-$, then T_γ or $-T_\gamma$ is dissipative and it is easy to see that the relation

$$T_\gamma^* = T_{\overline{\gamma}}$$

holds. Denote by $(\lambda)_n$, $n \in \mathbb{N}$, the sequence of (mutually different) eigenvalues of T_γ, and denote by

$$g_{n,1}, g_{n,2}, \ldots, g_{n,m_n}$$

a basis of the root subspace of T_γ corresponding to λ_n, such that the system of all elements $g_{n,k}$, $k = 1, 2, \ldots, m_n$, $n \in \mathbb{N}$, is a Bari basis of $\mathcal{L}^2(I)$. Then the complex conjugate functions

$$\overline{g_{n,1}}, \overline{g_{n,2}}, \ldots, \overline{g_{n,m_n}}$$

form a basis of the root subspace of $T_{\overline{\gamma}} = T_\gamma^*$ corresponding to $\overline{\lambda_n}$. We introduce for $n \in \mathbb{N}$ the $m_n \times m_n$ matrix

$$G_n := \begin{pmatrix} (g_{n,1}, \overline{g_{n,1}}) & \cdots & (g_{n,m_n}, \overline{g_{n,1}}) \\ \vdots & & \vdots \\ (g_{n,1}, \overline{g_{n,m_n}}) & \cdots & (g_{n,m_n}, \overline{g_{n,m_n}}) \end{pmatrix}.$$

The root subspaces of T_γ at λ_n and of T_γ^* at $\overline{\lambda_m}$ are orthogonal if $m \neq n$, and are in duality if $m = n$. Hence the matrix G_n is invertible. For $y \in \mathcal{L}^2(I)$ we define numbers $c_{n,k}$, $k = 1, 2, \ldots, m_n$, $n \in \mathbb{N}$, by the relation

$$\begin{pmatrix} c_{n,1}(y) \\ \vdots \\ c_{n,m_n}(y) \end{pmatrix} := G_n^{-1} \begin{pmatrix} (y, \overline{g_{n,1}}) \\ \vdots \\ (y, \overline{g_{n,m_n}}) \end{pmatrix}. \tag{6.4}$$

THEOREM 6.2. *If $\gamma \in \mathbb{C}\backslash\mathbb{R}$, then each element $y \in \mathcal{L}^2(I)$ admits the following unique expansion,*

$$y = \sum_{n=1}^{\infty} \sum_{k=1}^{l_n} c_{n,k}(y) g_{n,k}, \tag{6.5}$$

where the left sum converges in the norm of $\mathcal{L}^2(I)$.

Proof. For $y = g_{n_0,l}$ with $1 \leqslant l \leqslant m_{n_0}$, the expansion (6.5) follows from the definitions of the matrix G_n and of the coefficients $c_{n,k}(y)$ and from the fact that $c_{n,k}(g_{n_0,l}) = 0$ if $n \neq n_0$. For arbitrary $y \in \mathcal{L}^2(I)$ it is now a consequence of the properties of a Bari basis. □

If the elements $g_{n,k}$, $k = 1, 2, \dots, m_k$, which span the root subspace of T_γ at λ_n are chosen to form a Jordan chain:

$$(T_\gamma - \lambda_n) g_{n,1} = 0, \quad (T_\gamma - \lambda_n) g_{n,2} = g_{n,1}, \quad (T_\gamma - \lambda_n) g_{n,m_n} = g_{n,m_{n-1}},$$

then the elements $\overline{g_{n,k}}$, $k = 1, 2, \dots, m_n$, form a Jordan chain of T_γ^* at $\overline{\lambda_n}$ and we get

$$\begin{aligned}(g_{n,k}, \overline{g_{n,l}}) &= ((T_\gamma - \lambda_n)^{m_n - k} g_{n,m_n}, (T_\gamma^* - \overline{\lambda_n})^{m_n - l} \overline{g_{n,m_n}}) \\ &= ((T_\gamma - \lambda_n)^{2m_n - (k+l)} g_{n,m_n}, \overline{g_{n,m_n}}).\end{aligned}$$

Therefore, the matrix G_n is now a Hankel matrix and right lower triangular. Since G_n is invertible, the numbers $(g_{n,k}, \overline{g_{n,l}})$ with $k + l = m_n$ are not zero. Now it is easy to see that the Jordan chain $g_{n,k}$, $k = 1, 2, \dots, m_n$, can be modified such that the matrix G_n becomes $(g_{n,1}, \overline{g_{n,n_m}})$ times the m_n-sip matrix $(\delta_{k,m_n-l+1})_{k,l=1}^{m_n}$ [10, theorem I.3.3]. Indeed, replace the Jordan chain $g_{n,k}$ by a Jordan chain $g'_{n,k}$, the last element of which has the form $g'_{n,k} = \sum_{k=1}^{m_n} \alpha_k g_{n,k}$, and determine the α_k such that $(g'_{n,k}, \overline{g'_{n,l}}) = \delta_{k,m_n-l+1}$, $k, l = 1, 2, \dots, m_n$. With this choice of the Jordan chains at all the eigenvalues λ_n of T_γ, expansion (6.5) simplifies to

$$y = \sum_{n=1}^{\infty} \sum_{k=1}^{m_n} (y, \overline{g_{n,m_n-k+1}}) g_{n,k}.$$

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