String theory limits and dualities

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Chapter 5

String solitons and the field theory limit

In this chapter we will study the soliton solutions appearing in the supergravity actions more carefully. Most importantly this will involve putting in the string theory parameters and a study of the near–horizon region. We will introduce a special metric frame, called the *dual* frame, in which the special properties of the near–horizon geometry are most easily detected. After that we will study a string theory limit which will leave us with the decoupled soliton worldvolume field theory on the one side, and a (well behaved) near–horizon supergravity on the other side, which are conjectured to be dual descriptions of the same system. From the outset our analysis is valid in an arbitrary number of dimensions and for very generic brane solutions. To obtain well behaved near–horizon supergravities we will need a constraint on our parameters, leading us to consider mainly Dp–branes and their intersections. We will end by presenting some examples. This chapter is based on work done in [107], which generalizes work done in [108] and [109]. Dualities between (conformal) field theories and (Anti–de Sitter) near–horizon supergravities were first discussed in [110]. Many good review articles on the subject have appeared and we refer to [111] for a nice pedagogical introduction and to [112] for an extensive overview.

5.1 String soliton geometries

We want to take a closer look at the geometries of all kind of solitons appearing as solutions to the low energy effective actions of string and M–theory. This interest is motivated by the fact that the physics of p–branes can be described in two (different) ways. On the one hand, when the target spacetime supergravity (also called the *bulk* supergravity) is decoupled, the p–brane fluctuations can be described by a (effective) worldvolume field theory living on the worldvolume of the p–brane. In another, semiclassical, regime we are also allowed to describe the physics of the p–brane by probing the p–brane background with supergravity fields. A priori one might think these two descriptions are valid in completely different string theory regimes. However, in the previous chapter we encountered an example of a worldvolume field theory of N D0–branes which was able to describe gravitational physics. This suggests that there exist string theory regimes where both descriptions describe the same physics. This is also suggested by the string interactions of Dp–branes, which from one point of view describe exchanges of *closed* strings leading to bulk supergravity physics, or from another point of view describe vacuum diagrams of *open* strings which lead to worldvolume (quantum) field theory physics (see Figure 2.6).

We want to understand this phenomenon in more generality and detail. As a first step towards that understanding we will need to analyze the p–brane geometries again. We want to study a string theory limit in which the bulk supergravity decouples and which leaves us with a non–trivial worldvolume field theory. As we will see this limit takes us into the near–horizon region of the corresponding p–brane solution. Let us therefore first discuss p–brane near–horizon geometries.

5.1.1 Near–horizon geometries of p–branes

Our starting point will be a slightly different action than the one given in (3.48). We will replace the rank $p+2$ field strength in (3.48) by its rank $D-p-2$ Hodge dual and look for p–brane solutions which are magnetically charged with respect to the Hodge dual potential (so they are electrically charged with respect to the rank $p+1$ gauge potential). This will turn out to be useful as we go along. Besides that it will also be important to keep track of all factors of g_s appearing in the action. We refer to Appendix A for the details of how to obtain the appropriate scalings with g_s, but basically these can be read off from the exponential dilaton factors in the string frame action. Our action then is

$$S_D = \int \mathrm{d}^D x\, \sqrt{g} \frac{1}{(\sqrt{\alpha'})^{D-2} g_s^2} \Big[R - \frac{4}{D-2} (\partial\Phi)^2 - \frac{g_s^{(4-2k)}}{2(\tilde{d}+1)!} \left(\frac{e^{\Phi}}{g_s}\right)^{-a} F^2_{\tilde{d}+1} \Big] \,, \tag{5.1}$$

where we introduced a parameter $\tilde{d}$ and we will also introduce a parameter d, which are defined by

$$\begin{cases} d &= p+1 \qquad\quad \text{dimension of the worldvolume}\,, \\ \tilde{d} &= D-d-2 \;\; \text{dimension of the dual brane worldvolume}\,. \end{cases} \tag{5.2}$$

We note that $d+\tilde{d}=D-2$. Also notice the change in sign in the dilaton coupling parameter a in (5.1), which is a result of performing the Hodge duality transformation (3.49). We also introduced a parameter k, which is related to a, d and D in the following way (see appendix A)

$$k = \frac{a}{2} + \frac{2d}{D-2} \tag{5.3}$$

and which determines the scaling with g_s of the rank $\tilde{d}+1$ field strength. Notice that when $k=1$ the overall scaling with g_s vanishes in front of the field strength. This is the appropriate scaling for a Ramond–Ramond field strength. For $k=2$ and $k=0$ we find the appropriate scaling of Neveu–Schwarz field strengths and their Hodge duals respectively.

We will consider the following class of "two–block" p–brane solutions of the action (5.1)

$$\begin{aligned} \mathrm{d}s_E^2 &= H^{-\frac{4\tilde{d}}{(D-2)\Delta}}\mathrm{d}x_d^2 + H^{\frac{4d}{(D-2)\Delta}}\mathrm{d}x_{\tilde{d}+2}^2\,, \\ e^{\Phi} &= g_s H^{\frac{(D-2)a}{4\Delta}}\,, \\ g_s^{(2-k)}F &= \sqrt{\frac{4}{\Delta}}\,{}^*(\mathrm{d}H\wedge \mathrm{d}x_1\wedge\cdots\wedge \mathrm{d}x_d)\,, \end{aligned} \tag{5.4}$$

where * is the Hodge operator on D–dimensional spacetime. This is the magnetically charged analog of (3.50) where we now took care of the appropriate scalings with g_s. The parameter Δ is the same as in (3.54), which expressed in terms of d and $\tilde{d}$ equals

$$\Delta = \frac{(D-2)a^2}{8} + \frac{2d\tilde{d}}{(D-2)}\,. \tag{5.5}$$

The function H is harmonic on the $\tilde{d}+2$ transverse coordinates if $\tilde{d}\neq 0,-2$ and can be expressed using $\tilde{d}$ as

$$H(r) = 1 + \left(\frac{r_0}{r}\right)^{\tilde{d}}\,, \tag{5.6}$$

where $r_0^{\tilde{d}}$ is related to the charge (and mass) of the p–brane. Looking at (5.4) it follows that the charge should scale as $g_s^{(2-k)}$[1]. We note that codimension one objects, which have $\tilde{d} = -1$ and are usually called domain–walls, are included in (5.4) as opposed to codimension 2 objects. The magnetically charged field strength belonging to a domain–wall is a rank 0 object, a cosmological constant. In fact, the solution involving linear harmonic functions presented in (5.4) is not uniquely defined for domain–walls. It will be useful to discuss these objects separately in section 4.1.2.

The Dp–branes and NS–branes in $D = 10$ and M–branes in $D = 11$ are included in (5.4), but also two–block p–branes in dimensions $D < 10$. These can arise in string theory by considering string compactifications. We will mainly be interested in two–block BPS p–branes which can be obtained from an intersection of the basic BPS p–branes in $D = 10$ or $D = 11$. When the relative transverse directions of such an intersection are all wrapped on a torus T^r with r the number of relative transverse directions, the result will be a two–block p–brane in $D = 10 - r$ (or $D = 11 - r$). Supersymmetry preserving BPS p–brane solutions in any dimension are distinguished by having $\Delta = 4/n$ with n an integer (3.55) denoting the number of participating (higher–dimensional) branes [38].

To discuss the near–horizon geometry of these p–branes, we want to consider a limit in which the constant part in the harmonic function (5.6) is negligible, which means

$$\begin{aligned} r &\gg r_0 \quad \text{for } \tilde{d} = -1\,, \\ r &\ll r_0 \quad \text{all other cases}\,. \end{aligned} \tag{5.7}$$

The p–branes are positioned at $r = 0$ so this limit brings us close to the brane when $\tilde{d} > 0$. When $\tilde{d} = -1$, so for domain–walls, this limit actually takes us far away from the brane. We will still refer to this limit as a near–horizon limit. Strictly speaking, because we are considering extremal BPS p–branes, near–"horizon" is not good terminology even in those cases where $\tilde{d} \neq -1$. This is because the p–brane Einstein frame metric in (5.4) is singular at $r = 0$, except for some special cases where the dilaton is constant (e.g. the D3–brane in $D = 10$ Type IIB supergravity). Therefore it would perhaps be more suitable to call this a near–core limit. We will soon see however that this singularity in the metric at $r = 0$ can be removed by a conformal transformation (3.31) to a special frame called the dual frame, in which the hypersurface at $r = 0$ has become a non–singular horizon.

[1]To obtain natural units we still have to divide with the gravitational constant $\kappa_D \propto g_s^2$ (3.29), giving the expected scaling of the charges and tensions of the different p–branes $\propto g_s^{-k}$.

This can be understood by noting that $e^{\pm\Phi}$ is singular at $r=0$ as well and therefore we can perform a conformal transformation "canceling" the singularity in the metric. Of course we are still left with a singularity in $e^{\pm\Phi}$, but the limit (5.7) can in this sense be referred to as a near–horizon limit.

In the limit (5.7) the Einstein metric and the dilaton can be written as

$$\mathrm{d}s_E^2 = \left(\frac{r_0}{r}\right)^{-\frac{4\tilde{d}^2}{\Delta(D-2)}} \mathrm{d}x_d^2 + \left(\frac{r_0}{r}\right)^{\frac{4d\tilde{d}}{\Delta(D-2)}} \mathrm{d}x_{\tilde{d}+2}^2 \quad , \quad e^{\Phi} = g_s \left(\frac{r_0}{r}\right)^{\frac{(D-2)a\tilde{d}}{4\Delta}} . \tag{5.8}$$

Let us now introduce the conformal transformation which will factor off the singularities in the above metric. The following conformal transformation will do exactly that

$$g_D^{\mu\nu} = \left(\frac{e^{\Phi}}{g_s}\right)^{a/\tilde{d}} g_E^{\mu\nu} , \tag{5.9}$$

where we divided by g_s to not introduce (extra) g_s dependence in the metric. This conformal transformation will have the following effect on the action (5.1)

$$S_D = \int \mathrm{d}^D x \sqrt{g_D} \frac{1}{(\sqrt{\alpha'})^{D-2} g_s^2} \left(\frac{e^{\Phi}}{g_s}\right)^{\delta} \left[R_D + \gamma(\partial\Phi)^2 - \frac{g_s^{(4-2k)}}{2(\tilde{d}+1)!} F_{\tilde{d}+1}^2\right] , \tag{5.10}$$

with

$$\delta = -\frac{(D-2)a}{2\tilde{d}} , \qquad \gamma = \frac{D-1}{D-2}\delta^2 - \frac{4}{D-2} . \tag{5.11}$$

So the dual frame can be characterized by saying that all fields in the action are multiplied with the same e^{Φ} factor. We note that this would not have been true if we had used electrically charged potentials. Another special feature of this frame, which explains the name *dual* frame, is that Hodge dual $(D-p-4)$–branes probing the p–brane background solution couple naturally to the dual frame metric without a dilaton term.

The regular dual frame metric is given by

$$\mathrm{d}s_D^2 = \left(\frac{r_0}{r}\right)^{2(1-\frac{2\tilde{d}}{\Delta})} \mathrm{d}x_d^2 + \left(\frac{r_0}{r}\right)^2 \mathrm{d}r^2 + r_0^2 \mathrm{d}\Omega_{\tilde{d}+1}^2 . \tag{5.12}$$

Notice that the size of the transverse sphere $S^{\tilde{d}+1}$ no longer depends on r, it has become constant with radius r_0. Because the charge can be calculated by integrating the flux over the transverse sphere, we conclude that the (dualized) field strength in (5.4) can no longer depend on r either. Therefore we will not consider the solution for the field strength and just consider the metric and the dilaton expression. The metric in (5.12) generically describes a $d+1$–dimensional Anti–de Sitter spacetime times a $\tilde{d}+1$–dimensional sphere. Let us next consider coordinate transformations to connect the metric (5.12) to more standard and familiar parameterizations of Anti–de Sitter spacetimes.

Consider the following coordinate transformation redefining the radius r as

$$\left(\frac{r_0}{r}\right) = e^{-\lambda/r_0}\,, \tag{5.13}$$

which transforms the metric and dilaton into

$$\begin{aligned} \mathrm{d}s_D^2 &= e^{-2(1-\frac{2\tilde{d}}{\Delta})\lambda/r_0}\mathrm{d}x_d^2 + \mathrm{d}\lambda^2 + r_0^2\mathrm{d}\Omega_{\tilde{d}+1}^2 \\ \Phi &= \ln(g_s) - \frac{(D-2)a\tilde{d}}{4\Delta r_0}\lambda\,. \end{aligned} \tag{5.14}$$

As already mentioned the metric in (5.14) (generically) is a parameterization of a $d+1$–dimensional Anti–de Sitter spacetime times a $\tilde{d}+1$–dimensional sphere, in shorthand notation $AdS_{d+1}\times S^{\tilde{d}+1}$. The full p–brane geometry can therefore be described as interpolating between an asymptotic flat Minkowski and a near–horizon curved $AdS_{d+1}\times S^{\tilde{d}+1}$ geometry, connected by a throat as shown in Figure 5.1. This interpolating property of two–block p–branes (5.4) was first discussed in [41, 42] and generalized in [113, 114]. The only exception occurs when $1-2\tilde{d}/\Delta = 0$, because then the metric describes a $d+1$–dimensional Minkowski spacetime times the same sphere. Another special case is when $a=0$, giving a constant dilaton background. A nice feature of the stereographic coordinates (5.13), as they are called, is that the dilaton depends linearly on the radial coordinate λ.

We can also use horospherical coordinates to parametrise the $AdS_{d+1}\times S^{\tilde{d}+1}$ spacetime, which are defined as

$$u\beta = \frac{r^\beta}{r_0^{\beta+1}}\,, \tag{5.15}$$

with the dimensionless parameter β given by

$$\beta = \frac{2\tilde{d}}{\Delta} - 1\,. \tag{5.16}$$

These coordinates clearly do not make sense when $\beta = 0$. Comparing with (5.14) we see that this is exactly when the near–horizon spacetime becomes Minkowski

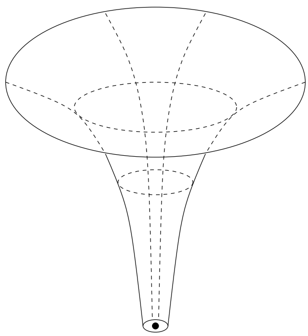

Figure 5.1: The p–brane solutions interpolating between Minkowski and Anti–de Sitter geometries.

and that is why the horospherical coordinates can only be used to describe AdS spacetimes.

We note that u carries dimensions of $[l]^{-1} = [m]$, which defines an energy scale. This will have interesting consequences. Rewriting the dual frame solution (5.12) using the horospherical coordinates we obtain

$$\begin{aligned} \mathrm{d}s_D^2 &= r_0^2 \left[(u\beta)^2 \mathrm{d}x_d^2 + \left(\frac{1}{u\beta} \right)^2 \mathrm{d}u^2 + \mathrm{d}\Omega_{\tilde{d}+1}^2 \right] \\ e^{\Phi} &= g_s r_0^{-\frac{(D-2)a}{8}\left(\frac{\beta+1}{\beta}\right)} (u\beta)^{-\frac{(D-2)a}{8}\left(\frac{\beta+1}{\beta}\right)} . \end{aligned} \tag{5.17}$$

We will prefer these coordinates when analyzing the string limit in which the worldvolume field theory decouples. In the metric (5.17) the $u = 0$ hypersurface is a non–singular horizon and $u \to \infty$ corresponds to the boundary of AdS. We will say more about some of the properties of AdS spacetimes in section 4.1.2.

Summarizing, we showed that in the dual frame, defined by (5.9), all p–branes solutions in (5.4) generically have a $AdS_{d+1} \times S^{\tilde{d}+1}$ near–horizon geometry and a non–trivial dilaton. Special cases arise when $\beta = 0$ or when $a = 0$. The constant sphere allows for a reduction of the D–dimensional fields. All the indices of the non–trivial field strength lie on the sphere, so this will give rise to a cosmological constant in $d+1$ dimensions and so does the curvature of the sphere. Therefore the reduced solution (throwing away the sphere part), generically consisting of an AdS metric and a non–trivial dilaton, solves the equations of motion of an action with a cosmological constant. The reduced object indeed has $p = (d+1) - 2$

spatial extended directions and this is what we called a domain–wall. Before actually performing a (truncated) reduction [115], we would first like to discuss domain–wall solutions in general.

5.1.2 Domain–walls and Anti–de Sitter spacetimes

Domain–wall spacetimes [116] solve the equations of motion obtained by varying a (super)gravity action with a cosmological constant Λ and a dilaton. They correspond to p–branes with worldvolume dimension $d = p+1$ which is one less than the dimension D of the target spacetime they live in (this also means that $\tilde{d} = -1$). Although domain–wall solutions do appear in (5.4), it turns out that these are not the most general solutions one can write down. Because all p–brane near–horizon solutions are described by a domain–wall in $d+1$ dimensions (when reduced over the sphere), it will be useful to study domain–wall solutions more carefully to make a connection with p–brane near–horizon solutions.

Again performing a Hodge dualization, which replaces the cosmological constant Λ by a rank $d+1$ field strength F_{d+1} in (5.1), we can naturally discuss objects of codimension one coupling to a d-form potential, defining a domain–wall. In terms of the field strength F_{d+1} the action is now given by

$$S^E_{d+1} = \int \mathrm{d}^{d+1}x\, \sqrt{g}\frac{1}{(\sqrt{\alpha'})^{d-1}g_s^2}\left[R - \frac{4}{d-1}(\partial\Phi)^2 - \frac{g_s^{(4-2\tilde{k})}}{2(d+1)!}\left(\frac{e^{\Phi}}{g_s}\right)^b F^2_{d+1}\right]. \tag{5.18}$$

We introduced a different dilaton coupling parameter b to stress the difference with (5.1) and $\tilde{k}$ equals

$$\tilde{k} = \frac{-b}{2} + \frac{-2}{d-1}, \tag{5.19}$$

which is just the appropriate modification of (5.3) to this domain–wall case using an electrically charged potential. The equations of motion following from the action (5.18) can be solved using the general p–brane Ansatz (3.50) involving harmonic functions, but not uniquely. The solutions are

$$\begin{aligned}
\mathrm{d}s^2_E &= H^{-\frac{4\varepsilon}{(d-1)\Delta_{DW}}}\mathrm{d}x_d^2 + H^{\frac{-4\varepsilon d}{(d-1)\Delta_{DW}}-2(\varepsilon+1)}\mathrm{d}y^2\,, \\
e^{\Phi} &= g_s H^{\frac{-(d-1)b\varepsilon}{4\Delta_{DW}}}\,, \\
g_s^{(2-\tilde{k})}F_{01\ldots d-1y} &= \sqrt{\frac{4}{\Delta_{DW}}}\,\partial_y H^{\varepsilon}\,,
\end{aligned} \tag{5.20}$$

where ε is now an arbitrary parameter as opposed to $\varepsilon = -1$ for ordinary p–branes (when using electrically charged potentials). The parameter Δ_{DW} is defined by

$$\Delta_{DW} = \frac{(d-1)b^2}{8} - \frac{2d}{d-1}\,, \tag{5.21}$$

which is just (5.5) with $\tilde{d} = -1$ and $a \to b$.

The function H is harmonic on the 1-dimensional transverse space with coordinate y and equals

$$\begin{aligned} H(y) &= 1 + Q_+ y \quad \forall y > 0\,, \\ H(y) &= 1 + Q_- y \quad \forall y < 0\,, \end{aligned} \tag{5.22}$$

with $Q_\pm$ constants and we fixed an arbitrary integration constant c to equal 1. The equations of motion allow for a discontinuity and so Q_+ and Q_- do not have to be equal[2]. It is understood that the domain–wall is positioned at the discontinuity $y = 0$. The value of $Q_\pm$ on any side of the domain–wall can be expressed in terms of a mass parameter $m_\pm$ in the following way

$$Q_\pm \varepsilon = m_\pm\,, \tag{5.23}$$

where $m_\pm$ is related to the cosmological constant through the equation

$$\Lambda_\pm = \frac{-2m_\pm^2}{\Delta_{DW}}\,. \tag{5.24}$$

So a domain–wall is an object which interpolates between two different cosmological constant vacua. The charge $Q_\pm$ should not be associated with the physical charge or mass of the domain–wall because it cannot be measured, which follows from the dependence on the arbitrary parameter ε. The physical mass and charge of a domain–wall have to be proportional to the (discontinuous) change in the cosmological constant. This is the only way to detect such an object.

We saw that use of the Ansatz (3.50) allows for an undetermined parameter ε in the domain–wall solution. The origin of this parameter is the fact that there are coordinate transformations, labeled by ε, that keep the solution within the same Ansatz. The explicit form of these coordinate transformations is given in [20]. Another way of understanding this is that the Ansatz (3.50) is not a suitable one

[2] Strictly speaking this is only true when using the rank $d+1$ field strength formulation as we did.

in the domain–wall case because it does not uniquely specify the solution. This also means that it should be possible to consider coordinate transformations that get rid of the free parameter ε.

We will now focus on one side of the domain–wall, let us say $y > 0$ and define $Q \equiv Q_+$ (from here on we will also drop the $+$ subscript on all parameters related to Q). Let us also assume that we are far away from the domain wall discontinuity[3]. This is what we called a "near–horizon" limit in the previous section and allows us to neglect the constant 1 in the harmonic function (5.22). We can get rid of the free parameter ε by making the following $y \to \lambda$ coordinate transformation

$$Qy = e^{-Q\lambda} . \tag{5.25}$$

The domain-wall solution in the new stereographic λ coordinate reads

$$\begin{aligned} \mathrm{d}s_E^2 &= e^{m\lambda\left(\frac{(d-1)b^2}{4\Delta_{DW}}\right)}\left(e^{-2m\lambda\left(\frac{2+\Delta_{DW}}{\Delta_{DW}}\right)}\mathrm{d}x_d^2 + \mathrm{d}\lambda^2\right) \\ \Phi &= \ln(g_s) + \frac{(d-1)bm}{4\Delta_{DW}}\lambda . \end{aligned} \tag{5.26}$$

This is a solution of the action (5.18) with Λ given by (5.24). The overall term in the metric can be removed by performing a conformal transformation to the dual frame, which is now defined as

$$g_D^{\mu\nu} = \left(\frac{e^\Phi}{g_s}\right)^{-b} g_E^{\mu\nu} , \tag{5.27}$$

which is just (5.9) with $\tilde{d} = -1$ and $a \to b$. The solution in the dual frame becomes

$$\begin{aligned} \mathrm{d}s_D^2 &= e^{-2m\lambda\left(\frac{2+\Delta_{DW}}{\Delta_{DW}}\right)}\mathrm{d}x_d^2 + \mathrm{d}\lambda^2 \\ \Phi &= \ln(g_s) + \frac{(d-1)bm}{4\Delta_{DW}}\lambda . \end{aligned} \tag{5.28}$$

This is just (5.14) with $\tilde{d} = -1$, $m = 1/r_0$ and $a \to b$ and generically the dual frame domain–wall metric describes an AdS_{d+1} spacetime [117]. When $\Delta_{DW} = -2$ the metric becomes flat Minkowski spacetime, which is equivalent to taking $\beta = 0$, $\tilde{d} = -1$ and $a \to b$ in (5.14).

[3] We could also perform a shift coordinate transformation, but this would also shift the position of the discontinuity changing the range of the transversal coordinate y.

Near–horizon spacetimes of p–branes should fall in this category of domain–wall solutions after the reduction over the sphere. To make this connection we have to relate the original parameters of the p–brane solution $a, d, \tilde{d}$ and r_0 to the parameters of the $d+1$–dimensional domain–wall which are just b, d and m. Reducing just the fields participating in the solution (5.4) in the dual frame will only replace the field strength by a cosmological constant whose value is determined by the Ricci curvature of the sphere and the charge of the original p–brane

$$S^R_{d+1} = \int \mathrm{d}^{d+1}x\, \sqrt{g} \frac{1}{(\sqrt{\alpha'})^{d-1} g_s^2} \left(\frac{e^{\Phi}}{g_s}\right)^{\delta} \left[R + \gamma(\partial\Phi)^2 + g_s^{(4-2k)}\Lambda\right]. \qquad (5.29)$$

This action should be a truncation of a gauged supergravity action which presumably can be obtained by reducing the complete higher–dimensional supergravity action on a sphere [115]. To compare with (5.18) we have to perform a conformal transformation to the Einstein frame and rescale the dilaton $\Phi \to \Phi/c$ to obtain the standard normalization of the dilaton kinetic term. The scale factor c equals

$$c^2 = \frac{2\tilde{d}^2}{\Delta(\tilde{d}+1) - 2\tilde{d}}. \qquad (5.30)$$

We can then read off the domain–wall dilaton coupling parameter b

$$b = a\,\frac{(d+\tilde{d})}{(d-1)\tilde{d}}\,c. \qquad (5.31)$$

This is all we need to express Δ_{DW} (5.21) in terms of the parameters of the original p–brane[4]. We find

$$\Delta_{DW} = \frac{-2\tilde{d}\Delta}{\Delta(\tilde{d}+1) - 2\tilde{d}}. \qquad (5.32)$$

Comparing the reduced p–brane solution in the Einstein frame with (5.26) we conclude that m and r_0 are related as follows

$$m = \frac{-\tilde{d}}{r_0}. \qquad (5.33)$$

As a consistency check we should find $c^2 = 1, a = b, \Delta_{DW} = \Delta$ and $m = 1/r_0$ when $\tilde{d} = -1$, in which case the original p–brane is already a domain–wall. The relations (5.30), (5.31), (5.32) and (5.33) indeed satisfy this requirement.

[4]The parameter Δ is only invariant under toroidal reductions and not under reductions on spheres. This explains why $\Delta_{DW} \neq \Delta$.

Using these relations we can express the value of the cosmological constant Λ in terms of the original p–brane parameters. We just use (5.24) and plug in (5.33) and (5.32). This gives

$$g_s^{4-2k}\Lambda = \frac{\tilde{d}}{2r_0^2}\left[2(\tilde{d}+1) - \frac{4\tilde{d}}{\Delta}\right]. \tag{5.34}$$

The first term in this expression originated in the reduction of the D–dimensional Ricci scalar and the second term comes from the reduction of the magnetically charged rank $\tilde{d}+1$ field strength curvature. Analyzing this expression we find that all p–brane near-horizon geometries give a $\Lambda > 0$ (with $1 \le \tilde{d} \le (D-3)$) except for the domain-walls, which have $\tilde{d} = -1$ and a sign change occurs, giving $\Lambda < 0$. We note that this is not in contradiction with the fact that all p–branes (including the domain–walls) have AdS geometries, which are defined by having $\Lambda > 0$, in the near–horizon limit. The dilaton kinetic term in (5.18) will contribute to an *effective* cosmological constant which is always positive, as can be most easily seen using stereographic coordinates when the dilaton is a linear function of λ (5.28)[5].

So we have now related all near–horizon geometries of p–branes in (5.4) to domain–wall solutions. In the dual frame the generic domain–wall metric described an AdS_{d+1} spacetime, with one Minkowski spacetime exception when $\Delta_{DW} = -2$. Let us now discuss some of the special properties of AdS spacetimes, for a more extensive discussion we refer to [118].

Anti–de Sitter metrics describe spacetimes of constant negative curvatures. By considering a $d+1$–dimensional Einstein–Hilbert action with a cosmological constant term we find

$$\begin{aligned} R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R &= \Lambda g_{\mu\nu} \Rightarrow \\ R &= -\frac{d+1}{d-1}\Lambda \Rightarrow \\ R_{\mu\nu} &= -\frac{\Lambda}{d-1}g_{\mu\nu}\,. \end{aligned} \tag{5.35}$$

So these spaces have the property that the Ricci tensor is proportional to the metric tensor, which is the definition of Einstein spacetimes. When $\Lambda > 0$ and $d > 1$ the solutions describe spacetimes of constant negative curvature and to obtain AdS we

[5] As one might expect one obtains a flat Minkowski near–horizon geometry when the *effective* cosmological constant vanishes.

need maximal symmetry implied by demanding

$$R_{\mu\nu\rho\sigma} = \frac{R}{d(d+1)}(g_{\nu\sigma}g_{\mu\rho} - g_{\nu\rho}g_{\mu\sigma})\,. \tag{5.36}$$

It is possible to embed AdS_{d+1} in a $d+2$–dimensional flat space. The metric of this $d+2$–dimensional flat space is

$$\eta_{ab} = \mathrm{diag}(-,+,+,\dots,+,-)\,. \tag{5.37}$$

The $d+2$–dimensional spacetime therefore has two times, or signature $(2,d)$. The invariant distance or length (positive for timelike worldlines) is defined as

$$-l^2 \equiv \sum_{i=1}^{i=d}(y^i)^2 - (y^0)^2 - (y^{d+2})^2\,. \tag{5.38}$$

We note that this length is preserved by a generalization of the Lorentz group rotations into $SO(2,d)$. An AdS_{d+1} embedded surface is then defined as a hyperboloid with $l^2 = \mathscr{R}^2 =$ constant

$$-\mathscr{R}^2 = \sum_{i=1}^{i=d}(y^i)^2 - (y^0)^2 - (y^{d+2})^2\,. \tag{5.39}$$

The length scale $\mathscr{R}$ can be interpreted as the embedding radius of the AdS_{d+1} surface. Through this embedding equation (5.39) the isometry group of an AdS_{d+1} spacetime obviously is $SO(2,d)$, which has $\frac{1}{2}(d+1)(d+2)$ generators[6]. Quantum theories on AdS_{d+1} should therefore have an $SO(2,d)$ invariance. We note that the group of conformal transformations in d dimensions is also $SO(2,d)$. The AdS_{d+1} embedding equation (5.39) also implies the existence of closed timelike curves in the embedded surface. This can be avoided by considering the universal cover of the AdS_{d+1} geometry (which means we introduce an infinite set of AdS_{d+1} geometries allowing timelike curves to pass through different AdS_{d+1} geometries avoiding closed timelike curves).

We can now choose suitable coordinates on AdS_{d+1} satisfying the embedding constraint (5.39) and defining an induced AdS_{d+1} metric. For example let us define

$$u' \equiv y^{d+2} + y^d \quad , \quad v \equiv y^{d+2} - y^d \tag{5.40}$$

[6]Notice that the number of generators is the same as the Poincaré group in $d+1$ dimensions.

where we picked out y^d as one of the spacelike coordinates in (5.39). Also define the left–over coordinates as

$$y^\mu \equiv \frac{u'}{\mathscr{R}} x^\mu \quad \mu \in [0,1,\ldots,d-1] \tag{5.41}$$

and introduce a flat d–dimensional metric $\eta_{\mu\nu}$ with usual Minkowski signature to lower the Greek indices on the x coordinates. The induced (mostly plus) AdS_{d+1} metric is just

$$ds^2 = dy^\mu dy_\mu - du' dv. \tag{5.42}$$

Working out the differentials and expressing v in terms of u and x^μ through the embedding equation (5.39) we obtain

$$ds^2 = \left(\frac{u'}{\mathscr{R}}\right)^2 dx^\mu dx_\mu + \left(\frac{\mathscr{R}}{u'}\right)^2 du'^2. \tag{5.43}$$

This can be recognized as the horospherical parameterization of AdS_{d+1} (5.17) if we identify $u' \equiv ur_0^2$ and

$$\mathscr{R} \equiv \frac{r_0}{\beta} = \frac{\Delta r_0}{2\tilde{d} - \Delta}. \tag{5.44}$$

An important property of AdS_{d+1} is that it has a "projective boundary". This has the effect that in many (physical) situations AdS_{d+1} spacetime acts as a finite volume box. Lightlike trajectories can reach this AdS boundary in finite time as opposed to timelike trajectories. Considering the embedding (5.39) and defining new coordinates Ry' with R very large, the boundary can be parametrized (approximately) as

$$-\left(\frac{\mathscr{R}}{R}\right)^2 \to 0 = \sum_{i=1}^{i=d} (y'^i)^2 - (y'^0)^2 - (y'^{d+2})^2. \tag{5.45}$$

Since tR with $t \in \mathbb{R}$ is just as good as R, we should consider the boundary as a projective equivalence class defined as

$$\begin{aligned} 0 &= \sum_{i=1}^{i=d} (y'^i)^2 - (y'^0)^2 - (y'^{d+2})^2 \\ y &\sim ty. \end{aligned} \tag{5.46}$$

We can use the scaling equivalence to fix one of the coordinates and this means the boundary is a d–dimensional surface, as it should be. For example we can fix

$y'^{d+2} \equiv 1$. In that case we find that

$$1 = \sum_{i=1}^{i=d} (y'^i)^2 - (y'^0)^2 \,, \tag{5.47}$$

which means the topology of the boundary is $S^1 \times S^{d-1}$ [7]. Considering the universal cover of AdS_{d+1} decompactifies the S^1, avoiding closed timelike curves. An important property of the boundary of AdS_{d+1} is that the isometry group $SO(2,d)$ acts precisely as the conformal group on Minkowski space. The conformal group consists of the usual Poincaré group together with the following conformal transformations

- Dilations or scale transformations acting as

$$x^\mu \to \lambda x^\mu \,, \quad \lambda \in \mathbb{R} \,. \tag{5.48}$$

- Special conformal transformations acting as

$$\begin{aligned} x^\mu &\to x'^\mu \,, \quad \text{such that} \\ \frac{x'^\mu}{x'^2} &= \frac{x^\mu}{x^2} + a^\mu \,. \end{aligned} \tag{5.49}$$

Together with the Poincaré group these make up the group $SO(2,d)$. It is not very hard to see that some infinitesimal $SO(2,d)$ isometries, namely infinitesimal translations $u' \to u' + a$ (using horospherical coordinates (5.40)), indeed reproduce dilations on the boundary, which follows from the equivalence class condition (5.46) of coordinates on the boundary. For a more extensive discussion on this point we refer to [111]. We conclude that $SO(2,d)$ AdS isometries can be identified with conformal transformations from the boundary point of view.

We found that generic p–brane near–horizon geometries have AdS metrics, but the complete generic solution is also described by a non–trivial dilaton. A non–trivial dilaton breaks the $SO(2,d)$ isometries of the complete solution. For example infinitesimal $SO(2,d)$ translations will not leave the value of e^Φ invariant, breaking the symmetry. From the boundary point of view this has to correspond to broken conformal or scale invariance. Only when the dilaton background is

[7] This result only refers to the topology of the boundary and does not mean that the boundary is a curved geometry. Rather the boundary is a flat Minkowski geometry which can be thought of as the infinite radius limit of a sphere.

constant do we expect the complete background solution to be invariant under the $SO(2,d)$ isometries of AdS_{d+1}. In that case we also find supersymmetry enhancement in the near–horizon limit [41, 42, 119], meaning that a pure AdS background solution preserves all of the supersymmetries, just like flat Minkowski space.

Let us end this section by discussing the two special near–horizon cases in the dual frame.

- Flat Minkowski near–horizon spacetime. This requires

$$\beta = 0 \quad \text{or} \quad 2\tilde{d} = \Delta . \tag{5.50}$$

  We will only consider supersymmetry preserving cases, which means $\Delta = 4/n$ with n an integer. It is important to note that the parameter $\tilde{d}$ is invariant under double dimensional reductions. This means that once we found a p–brane satisfying the constraint (5.50), p–branes with r legs compactified on a T^r giving a $p-r$–brane in $D-r$ dimensions, will also satisfy the constraint (5.50). So we find families of solutions. Because $\tilde{d}$ has to be an integer solutions can only be found for $n=1$ and $n=2$. Relating our results to existing branes in string– or M–theory we find the 10–dimensional $p=5$–branes for $n=1$. When $n=2$ we find $p=5$–branes in $D=9$, which can be obtained from reduction of $D=10$ Kaluza–Klein monopoles in the $N=1$ supergravity theories.

- Pure Anti–de Sitter backgrounds. This requires

$$a = 0 \quad \text{or} \quad \frac{2d\tilde{d}}{\Delta} = (d+\tilde{d}) . \tag{5.51}$$

  This condition can be satisfied for the cases where we preserve some supersymmetry or equivalently $\Delta = 4/n$. We summarized the results [41, 42, 120, 119] in Table 5.1.

  The p–brane listed in Table 5.1 with $\Delta = 2$ in $D=6$ can be traced back to an intersection of 2 Dp–branes in $D=10$, hence the terminology. The same holds for the p–branes in $D=5$ with $\Delta = 4/3$, which are related to intersections of 3 M–branes. Finally, the 0–brane or extreme Reissner–Nordström black hole in $D=4$ is related to an intersection of 4 Dp–branes in $D=10$. Notice that only three possible values of β occur: $\frac{1}{2}, 1$ and 2. Remember that β defines the ratio of the radius r_0 of the transverse sphere $S^{\tilde{d}+1}$ to the radius r_0/β of the embedded AdS_{d+1}. Considering $D=10$ Dp–branes

D	β	$\Delta_{\rm DW}$	Δ	Name
11	$\frac{1}{2}$	–12/5	4	M5–brane
	2	–3	4	M2–brane
10	1	–8/3	4	D3–brane
6	1	–4	2	d1–brane
5	$\frac{1}{2}$	–4	4/3	m1–brane
	2	∞	4/3	m0–brane
4	1	∞	1	RN black hole

Table 5.1: The Table indicates the values of β, $\Delta_{\rm DW}$ and Δ for all p–branes that have a pure AdS near–horizon background.

or their reduced intersections we always find this ratio to be 1. Considering $D = 11$ M2–branes or M5–branes and their reduced intersections we always find 2 and $\frac{1}{2}$ respectively.

Finally notice that $p = 0$ is special because Δ_{DW} blows up and using (5.35) we find that $\Lambda \equiv 0$. However we can still consider maximally symmetric Einstein spaces of constant negative curvature and the AdS_2 metric satisfies these requirements.

This finishes our discussion on p–brane near–horizon geometries and their relation to domain–wall and Anti–de Sitter spaces. We will next introduce a string theory low energy limit which leads us into the p–brane near–horizon region, decoupled from the asymptotic Minkowski supergravity. From the p–brane worldvolume theory point of view the same low energy limit also decouples bulk Minkowski supergravity and leaves us with the non–trivial field theory living on the p–brane. This will lead to the surprising and interesting conjecture that p–brane field theories can be mapped to closed superstring theories on (p–brane near–horizon) domain–wall backgrounds. By now the best understood and checked example is that of N D3–branes in Type IIB string theory, which was first discussed by Maldacena, together with the other pure AdS backgrounds, in [110]. This was generalized to the other $D = 10$ Dp–branes in [108] and somewhat later in [109] using the dual frame metric.

5.2 The field theory limit

In this section we will set up the limit taking us into the near–horizon region, fixing the worldvolume field theory coupling constant and energy scale. We will work out this limit for the general class of p–branes described by (5.4). To obtain well–behaved near–horizon background solutions we need a constraint on our p–brane parameters. The result will be that dualities relating domain–wall supergravities having $a \neq 0$ to large N worldvolume field theories are only well–behaved for Dp–branes and their reduced intersections. We will first try to be as general as possible, only excluding the flat Minkowski near–horizon spacetimes, which will not be treated in this thesis.

5.2.1 The general setup

A string low energy[8] limit will always involve

$$u^2\,\alpha' \to 0\,, \tag{5.52}$$

as explained in section 2.2, where we substituted u for U to denote the natural energy scale. There are two ways to interpret this limit (5.52). One usually considers $u \to 0$ and keeps α' fixed. However one can equivalently consider fixed energies u and consider the limit $\alpha' \to 0$. We will use the last option and therefore consider the limit

$$\alpha' \to 0\,, \tag{5.53}$$

keeping fixed a natural energy scale u. We will also assume that from the outset g_s is small, in order for the p–brane soliton solution to make sense in a string theory low energy limit.

In one regime the system we want to analyze consists of N p–branes described by a worldvolume theory, coupled perturbatively to a Minkowski bulk supergravity theory (so we neglect the back reaction of the N p–branes on the spacetime geometry). The dynamics of the (effective) field theory on the corresponding p–brane should be non–trivial. This means that at least one field theory coupling constant should be fixed in the limit (5.53). The field content of the p–brane worldvolume field theory determines the dimensions of the different coupling constants (scalars have dimensions different from vectors), which can be easily read off from the different kinetic terms. The dependence on g_s is fixed through

[8] It should be clear that when we consider M–theory we just replace the string length scale by the Planck length scale and we lose the string theory coupling constant.

the scaling of the effective tension of the p–brane under consideration, which is denoted by the parameter k (5.3). A p–brane soliton solution must be thought of as a stack of N microscopic single p–branes. We will fix (generalized) 't Hooft coupling constants[9], which involve this integer N. In general we will assume the following structure of a p–brane worldvolume ('t Hooft) coupling constant

$$g_f^2 = c_p N g_s^k (\sqrt{\alpha'})^x , \tag{5.54}$$

with c_p some (dimensionless) constant. When considering M–theory branes, the g_s dependence is of course absent[10]. We introduced the parameter x to denote the dimension of the coupling constant, which is unconstrained (for now). Both c_p and x depend on the specific p–brane worldvolume theory fields under consideration. When considering Yang–Mills coupling constants of Dp–branes, x is equal to $p-3$. On the other hand, when considering scalar coupling constants $x = p-1$. We want the coupling constant (5.54) to stay fixed in the limit (5.53). Depending on the sign of x this has the following consequences

$$\begin{aligned} x < 0 \quad &\rightarrow \quad N g_s^k \sim \sqrt{\alpha'}^{-x} \rightarrow 0 \\ x > 0 \quad &\rightarrow \quad N g_s^k \sim \sqrt{\alpha'}^{-x} \rightarrow \infty . \end{aligned} \tag{5.55}$$

We will only be considering p–branes with $k > 0$. This is reasonable because otherwise the effective tension would scale with a positive power of g_s, saying that in a weak string coupling limit the tension of such an object would vanish. This implies the absence of solitonic solutions to the string effective equations of motion which are only defined in a weak coupling limit, and so we arrive at a contradiction because we do want to consider the existence of p–brane soliton solutions. Positive k and $g_s \ll 1$ imply that in order to keep g_f fixed we need $g_s \rightarrow 0$ when $x < 0$ and $N \rightarrow \infty$ when $x > 0$.

The non–trivial worldvolume field theory will be decoupled from the bulk Minkowski supergravity theory when the gravitational coupling constant vanishes in the limit (5.53). The gravitational coupling constant in D spacetime dimensions is proportional to

$$G_D \propto (\sqrt{\alpha'})^{D-2} g_s^2 , \tag{5.56}$$

[9]These are called 't Hooft coupling constants after 't Hooft's idea to treat $U(N)$ Yang–Mills theories in a $1/N$ expansion. The suggestion was that these theories might simplify when the number of colors N is large, because only planar Feynman diagrams contribute in a large N limit. If one could solve the theory with $N = \infty$ exactly, the hope was that one could analyze $SU(3)$ QCD beyond the perturbative weak coupling expansion by doing an expansion in $1/N = 1/3 < 1$.

[10]In fact the α' dependence also disappears in that case because the worldvolume field theories are scale invariant.

which clearly vanishes in the limit (5.53) and u and g_f fixed, as long as we do not consider taking $g_s \to \infty$. We conclude that in the limit described above, which we will refer to as the field theory limit, we end up with a bulk Minkowski supergravity theory decoupled from a non–trivial p–brane low energy effective worldvolume theory.

Now consider the p–brane supergravity soliton solution (5.4). We have to define a natural energy scale with respect to the p–brane solution which coincides with the one in the worldvolume field theory and which should be kept fixed in the limit (5.53). We already encountered an energy scale when we discussed the AdS_{d+1} horospherical coordinates, where we defined a parameter u with the dimensions of mass (5.15). This can be shown to be the natural energy scale associated to a massless supergravity field probing the p–brane near–horizon geometry [121]. That energy scale could only be defined when $\beta \neq 0$ (5.16), so we will only be discussing p–branes with AdS_{d+1} near–horizon geometries (in the dual frame). The cases with $\beta = 0$, resulting in a dual frame flat Minkowski near–horizon geometry, can be treated but we refer to [108, 109] and [122] to learn more about the holographic duality conjectures in these special cases.

We want to replace all quantities appearing in the harmonic function by the fixed parameters in the field theory limit. All r dependence will be replaced by u and we also need to express r_0 in terms of the appropriate string theory parameters. In Appendix A we deduce that

$$r_0^{\tilde{d}} = (d_p N g_s^{2-k}) \sqrt{\alpha'}^{\tilde{d}}\,, \tag{5.57}$$

where we introduced a dimensionless constant d_p. We can now rewrite the harmonic function $H(r)$ in terms of the fixed quantities u and g_f^2, extracting powers of α' and g_s which are left–over. This gives

$$H = 1 + (\sqrt{\alpha'})^{\frac{x-\tilde{d}}{\beta}} (g_s)^{\frac{2(k-1)}{\beta}} \left[g_f^2 \, (u\beta)^{\tilde{d}} \left(\frac{d_p}{c_p} \right) \right]^{-1/\beta} . \tag{5.58}$$

The field theory limit will take us into the p–brane near–horizon region when

$$\frac{x-\tilde{d}}{\beta} < 0\,. \tag{5.59}$$

The power of g_s could still spoil this behavior, but we will soon constrain our parameters in such a way that this possibility is excluded.

Let us for the moment assume that the constraint (5.59) is fulfilled and the field theory limit takes us into the p–brane near–horizon geometry. We will use the dual frame metric solution written in terms of horospherical coordinates and we will not give the expression for the field strength, which is of less importance. Expressing the p–brane near–horizon solution in terms of the fixed quantities, removing all the α' dependence in the dilaton, we find

$$\begin{aligned} ds^2 &= (d_p N g_s^{2-k})^{2/\tilde{d}}\,\alpha' \left[(u\beta)^2 dx_d^2 + \left(\frac{1}{u\beta}\right)^2 du^2 + d\Omega_{\tilde{d}+1}^2 \right] \\ e^{\Phi} &= g_s^{1+\frac{(D-2)a}{2\Delta\beta}(k-1)} \left(N g_s^k\right)^{\frac{a(D-2)(\tilde{d}-x)}{4\Delta\beta x}} \left[(g_f^2)^{1/x} (u\beta) \left(\frac{d_p^{1/\tilde{d}}}{c_p^{1/x}}\right)\right]^{\frac{-(D-2)a}{8}\left(\frac{\beta+1}{\beta}\right)} \end{aligned} \quad (5.60)$$

We can rescale the metric to lose the factors $(d_p N g_s^{2-k})^{2/\tilde{d}}$ and α'. This will introduce (extra) α' and $(d_p N g_s^{2-k})^{2/\tilde{d}}$ dependence in the dual frame action (5.10). Collecting all α' dependence, we find the important result that all α''s drop out. All g_s dependence nicely combines into the $e^{\delta\Phi}$ in front of the dual frame action. After the rescaling our action (5.10) becomes

$$S_D = \int d^D x\,\sqrt{g_D}(d_p N)^{(D-2)/\tilde{d}} e^{\delta\Phi}\left[R_D + \gamma(\partial\Phi)^2 - \frac{1}{2(d_p N)^2(\tilde{d}+1)!}F_{\tilde{d}+1}^2\right]. \quad (5.61)$$

This means that if the dilaton expression is non–singular in the field theory limit, we are left with a near–horizon supergravity theory with a finite Planck length and a finite string coupling constant defined by e^{Φ}! This strongly suggests that in the field theory limit on the supergravity soliton side, we end up with a superstring theory on the $AdS_{d+1} \times S^{\tilde{d}+1}$ p–brane near–horizon background with new (finite) parameters $\tilde{\alpha}'$ and $\tilde{g}_s$. The field theory limit decouples this near–horizon superstring theory from the asymptotic Minkowski superstring theory we started with.

The special cases $a = 0$ imply a constant dilaton and the supergravity background carries an unbroken $SO(2,d)$ isometry. This should match with a conformal symmetry group in the worldvolume field theory description, meaning the fixed coupling constant (5.54) should be dimensionless or $x = 0$. Those cases (which include the M–branes) do not require a parameter restriction and will be discussed separately in 4.2.2. When $a \neq 0$ non–singular dilaton expressions in the field theory limit require a restriction on our p–brane parameters. We will now constrain our p–brane parameters such that the near–horizon limit (5.59) is guaranteed and the new string coupling constant e^{Φ} is finite or independent of the old string coupling constant g_s.

In the analysis of the effect of the field theory limit on the supergravity soliton, we defined a fixed energy scale u. A priori there is no reason for this fixed energy scale to be the same as the natural energy scale in the worldvolume field theory. To be able to compare both descriptions we need related or better, equivalent fixed energy scales. To determine these relations we need to probe the system under consideration in both descriptions by the same objects or fields. Suppose we are dealing with N Dp–branes probed by a single Dp–brane. Then we know how to relate a natural energy scale in the Dp–brane Yang–Mills worldvolume field theory to the length of stretched strings, defining a distance scale in the bulk. Open strings stretching from the probe brane to the system of N Dp–branes correspond to energy scales equal to

$$U = \frac{r}{\alpha'}\,, \tag{5.62}$$

which is just the distance between the probe and the system of N Dp–branes times the open string tension (see 1.1.5). This is obviously not the same as the definition of the energy scale u (5.17), which we kept fixed when considering the field theory limit in the supergravity soliton description. The two energy scales are related in the following way

$$u\beta = \alpha'^{\frac{x+\tilde{d}-\Delta}{\Delta}}\, g_s^{\frac{4(k-1)}{\Delta}} \left(\frac{d_p}{c_p} g_f^2\right)^{\frac{-2}{\Delta}} U^{\beta}\,. \tag{5.63}$$

We would like u and U to be related through fixed quantities only. Otherwise fixed energy scales U in the worldvolume field theory would correspond to diverging energy scales u in the supergravity soliton description and vice versa. Looking at (5.63) this is only possible when

$$k = 1 \quad , \quad x = \Delta - \tilde{d}\,. \tag{5.64}$$

The first constraint $k = 1$ just confirms our restriction to Dp–branes in any dimension. The second constraint is interesting, because it tells us which coupling constant in the worldvolume theory we should keep fixed. Until now we kept x as a free parameter, but now we see we have to fix it in order to connect the Dp–brane supergravity soliton and Dp–brane field theory energy scales. When $D = 10$ and $\Delta = 4$ we find $x = p - 3$, telling us that we should keep fixed the 't Hooft Yang–Mills coupling constant (5.54). The connection between the two energy scales was discussed extensively in [121]. There it was observed that u is the natural energy scale for supergravity probes (instead of Dp–brane probes), which could also be obtained in the worldvolume field theory by considering the self–energy of a point charge. The point charge is the interpretation of the stretched string from the worldvolume gauge theory point of view, which has energy U. However, the self–energy is also proportional to the effective strength of the Coulomb interaction and this will reproduce (5.63). To get the correct holographic relation between the number of degrees of freedom on both sides of the duality [123], the energy scale u should be used and therefore this parameter is also called the holographic energy scale. This also means that from the holographic point of view, the supergravity fields are the natural holographic probes of the AdS geometry. In [109] it was noted for the first time that the holographic energy scale u is the natural energy scale coordinate for AdS spacetimes obtained as the near–horizon geometries of Dp–branes in $D = 10$ in the dual frame. We have seen that this phenomenon extends to p–branes in arbitrary dimensions.

Using the restriction (5.64) first of all drops all g_s dependence in (5.58), and the power of α' (5.59) becomes $-\Delta$. Because we only consider $\Delta > 0$, it is guaranteed that the field theory limit takes us into the near–horizon region. Importantly, using the constraints (5.64) we find the following expression for e^{Φ} in the field theory limit

$$e^{\Phi} = \frac{1}{N}\left[(g_f^2)^{1/x}(u\beta)\left(\frac{d_p^{1/\tilde{d}}}{c_p^{1/x}}\right)\right]^{\frac{-(D-2)a}{8}\left(\frac{\beta+1}{\beta}\right)} . \tag{5.65}$$

Remember that $x = \Delta - \tilde{d}$ in this expression. This is finite (at least when we do not consider $N \to \infty$) and defines a new string coupling constant independent of the old Minkowski string coupling constant g_s. From (5.65) we conclude that this new coupling constant is proportional to $1/N$. So in a large N limit we obtain a weakly coupled string theory and string theory quantum corrections are $1/N$ effects.

We are now ready to state the conjecture naturally following from the above analysis, restricting the parameters as in (5.64) when $a \neq 0$. Fixing an energy scale u and a ’t Hooft coupling constant g_f, the low energy limit $\alpha' \to 0$ decouples Minkowski supergravity in both descriptions. We are left with a p–brane worldvolume field theory on one side and a well defined domain–wall supergravity solution on the other side. Choosing a fixed energy scale and ’t Hooft coupling constant we can now either describe the system of p–branes by a closed superstring theory on a domain–wall background, or by the p–brane worldvolume field theory and both descriptions are conjectured to give the same results.

This automatically leads to the conjecture that *closed superstring theory on a $DW_{d+1} \times S^{\tilde{d}+1}$ p–brane near–horizon background is dual to the d–dimensional p–brane worldvolume field theory*. These kind of relations we will very often call domain–wall/quantum field theory dualities (or in short DW/QFT dualities). The suggestion to go beyond the supergravity approximation is based on the fact that we found finite Planck length and string coupling in the supergravity analysis. Of course when considering M–branes we loose the dilaton (and thus the string coupling) and we should replace "*closed superstring theory on a $DW_{d+1} \times S^{\tilde{d}+1}$ p–brane near–horizon background*" by "*M–theory on a pure $AdS_{d+1} \times S^{\tilde{d}+1}$ M–brane near–horizon background*".

Let us explain what we mean when we say that the two descriptions are *dual* to each other. This will become clear when we start analyzing the regions of the fixed quantities u and g_f (and usually N) where the different descriptions are in their perturbative, calculable, regime. We will first analyze the perturbative regime of the worldvolume field theory giving a restriction on the quantities u and g_f and after that deduce the restriction needed on the quantities u and g_f (and N) to be in a perturbative closed string regime.

Although the field theory description is in principle defined non–perturbatively, in practice we (almost always) need a perturbative expansion which is only defined for small effective dimensionless coupling constant. This effective dimensionless coupling constant in the worldvolume field theory can be constructed from the energy scale u and the coupling constant g_f^2 and should be much smaller than one, giving

$$g_{\rm eff}^2 = g_f^2 u^x \ll 1 \,. \tag{5.66}$$

Depending on the sign of x the perturbative field theory description will either be valid when $u \ll 1$ or $u \gg 1$. This effective coupling constant determines the (classical) scaling of the supersymmetric d–dimensional quantum field theory under consideration. We note that this is the combination of u and g_f that appears in the dilaton background (5.65).

The situation is different for the string theory (as explained in 1.1.6), which first of all is only *defined* for weak string coupling. Also the domain–wall background solution on which the string theory is defined can only be trusted as long as the spacetime curvatures are small and finite size corrections can be neglected. We can use a supergravity approximation (neglecting string loop diagrams) when the string coupling and the curvature of the background are both small. Let us investigate the regions in the p–brane near–horizon background solution where we can trust this supergravity approximation.

Small curvature (as seen by closed strings) can be translated in demanding that the effective tension in the dual frame times the characteristic spacetime length is large. The characteristic spacetime length is determined by the dual frame $AdS_{d+1} \times S^{\tilde{d}+1}$ metric and is of order one (in $\tilde{\alpha}'$ units). Calculating the effective string tension in the dual frame (using (5.9) and (2.1) we find small curvature when

$$\tau_s = \left(d_p N e^{(2-k)\Phi}\right)^{2/\tilde{d}} \gg 1\,. \tag{5.67}$$

We note that we did not yet use the constraint $k = 1$ in the above expression.

Small string coupling, which is defined by the dilaton expression (5.65) after the constraint (5.64), can be translated into the constraint

$$e^{\Phi} = \frac{1}{N}\left[g_{\text{eff}}^2 \left(\frac{\beta^x d_p^{1/\tilde{d}}}{c_p}\right)\right]^{\frac{-(D-2)a}{8}\left(\frac{\beta+1}{\beta x}\right)} \ll 1\,. \tag{5.68}$$

This last constraint can always be satisfied for generic u (except for the special points $u \to 0$ or $u \to \infty$) by taking N very large. This is a general feature of DW/QFT dualities, a supergravity approximation at least requires a large N limit. We note that when we take $k = 1$ in (5.67) the overall N dependence will drop out. This means that generically we can only expect supergravity to be a good approximation within a finite region of the complete background [108, 109]. Quantum–gravitational corrections can be included by taking into account string loop diagrams (which as we already mentioned can be identified with $1/N$ effects) and this string loop expansion will be restricted to the same region on the domain–wall background.

As we will see, in most cases the region where supergravity is a valid description is not overlapping with the region in which the perturbative field theory is a good description. This means that at a particular scale u and coupling constant g_f there only exists one perturbatively well–defined theory which, through the conjecture, can be mapped to a non–perturbative regime in the other theory. So a strongly coupled theory can be mapped to a weakly coupled (different) theory and it is precisely in this sense that the theories are said to be dual to each other. In fact we could have expected that, because sure enough we know that perturbative quantum field theory is very different from any (super)gravity theory.

This new kind of duality between large N SYM field theories and closed superstring theories (including quantum gravity) can be used to study non–perturbative physics on either side. First of all, although we did not show this here, in the procedure leading to the conjecture we can also introduce a small non–extremality parameter (giving rise to some excitation energy) in the background solution breaking the supersymmetry. This will correspond to a finite temperature configuration on the dual field theory side. Such a background solution has a true horizon and will Hawking radiate (a quantum process) towards thermal equilibrium[11]. Using the conjectured duality we can map this semiclassical gravity result to a process in a finite temperature large N SYM field theory, which means that Hawking radiation can be described by a unitary process associated with $1/N$ effects in the large N quantum field theory.

On the other hand we could also use the duality map to learn about large N quantum field theory, using (semiclassical) supergravity. Let us also remark that this duality is an explicit manifestation of the holographic principle [105, 106]. It was shown in [123] that the number of degrees of freedom in AdS space satisfy the holographic bound of one per surrounding Planck area. The way the two theories are related is by identifying the coordinate u on the domain–wall supergravity side with the scale parameter in the quantum field theory. Moving from $u = 0$ to the boundary at $u = \infty$ in the AdS spacetime should be interpreted as going from the infrared (IR) to the ultraviolet (UV) in the quantum field theory. This also means that large distances (IR) in the quantum field theory correspond to small distances in the domain–wall supergravity (near the center) and vice versa. The special properties of AdS metrics allow for this so–called UV–IR connection. Let us next discuss more specific examples present in our general setup. Along the way we will give some more details of the duality map.

5.2.2 The AdS/CFT examples

In this subsection we will take a better look at the pure AdS examples, which require a constant dilaton. This means that the dilaton coupling parameter $a = 0$. Remember that we do not need the constraint (5.64) now. Also remember that we want $a = 0$ to cover M–branes as well, in which case the string coupling constant is non–existent and we should replace the string length scale by the appropriate M–theory Planck length scale.

The complete background will have unbroken $SO(2,d)$ isometries, which will correspond to invariance under the conformal group in the dual field theory. The

[11] For a black p–brane in Minkowski spacetime this means that the p–brane will evolve into the extremal BPS p–brane. In AdS spacetimes, which act as finite volume boxes, the equilibrium situation will be the one where the temperature of the black p–brane equals the temperature of the surrounding gas of emitted Hawking radiation.

dual field theory therefore has to be a superconformal field theory, which means the parameter x should vanish (otherwise the coupling would classically run). For Δ and β we find in this case

$$\beta = \frac{\tilde{d}}{d} \quad , \quad \Delta = \frac{2d\tilde{d}}{d+\tilde{d}} . \tag{5.69}$$

Using these results we find that the harmonic function in the field theory limit can be written as (5.58)

$$H = 1 + (l_f)^{-d} \left[g_f^2 \, (u\beta)^{\tilde{d}} \left(\frac{d_p}{c_p} \right) \right]^{-1/\beta} , \tag{5.70}$$

where we replaced the $\sqrt{\alpha'}$ by the appropriate fundamental length scale l_f under consideration (which is the Planck length in M–theory and the string length in string theory). Because $d > 0$ always, the field theory limit is guaranteed to take us into the near–horizon region of the background solution. All the branes listed in Table 5.1 are covered by this analysis, which were studied in the paper by Maldacena [110].

When embedded in a string theory we always find $\beta = 1$, which also means $d = \tilde{d}$ and $\Delta = d$ through (5.69). Using (5.3) we can also conclude that $k = 1$, so the branes under consideration have to be D–branes. This also means the constraint (5.64) is satisfied and using (5.63) we find that $u \propto U$, so in this case the two energy scales are essentially equivalent. The fixed conformal field theory coupling constant is proportional to

$$g_f^2 \propto N g_s . \tag{5.71}$$

This coupling constant has to be small if we want to obtain perturbative conformal field theory results. In theories without supersymmetry classical conformal invariance is usually broken by quantum effects, as is represented by the Wilsonian renormalisation group equations. In supersymmetric theories however the conformal invariance can be maintained at the quantum level, which is necessary for the AdS/CFT duality. Looking at the condition for small AdS curvature (5.67) we find

$$\tau_s \propto (g_s N)^{2/d} = (g_f^2)^{2/d} \gg 1 . \tag{5.72}$$

The new string coupling constant can be written as $g_s = g_f^2/N$, which has to be small in a supergravity approximation. Combining these two requirements we conclude that we need large $N \gg g_f^2 \gg 1$ for the supergravity approximation to be a valid description. String quantum corrections are governed by the string coupling constant and are therefore $1/N$ corrections, as we concluded before. The size of the sphere $S^{\tilde{d}+1}$ and the radius of the AdS_{d+1} embedding are both equal to r_0 and therefore proportional (in $\tilde{\alpha}'$ units) to

$$r_0 \propto (g_s N)^{1/d} = (g_f^2)^{1/d} \,. \tag{5.73}$$

We can use (5.34) to translate this into the value of the cosmological constant

$$\Lambda \propto \frac{d(d-1)}{(g_f^2)^{2/d}} \,. \tag{5.74}$$

We clearly obtain small $\Lambda \ll 1$ or large $r_0 \gg 1$ in the supergravity approximation.

Similar equations hold for M–branes. We loose the constraint of small string coupling, g_s disappears altogether and the fixed conformal field theory coupling constant is just proportional to N. The only requirement for a supergravity approximation is now small curvature and this will involve taking a large N limit. For M–branes the parameter β can take on two values. For M2–branes (or intersections) we find $\beta = 2$ and for M5–branes (or intersections) we find $\beta = \frac{1}{2}$. The energy parameter u (5.15) can be written as

$$u\beta = \frac{r^\beta}{(d_p N)^{(\beta+1)/\tilde{d}} l_{11}^{(\beta+1)}} \,. \tag{5.75}$$

When $\beta = \frac{1}{2}$ this energy scale can be interpreted as the squareroot of the distance scale between M5–branes times the tension of a membrane $T_{\mathrm{M2}} \propto 1/l_{11}^3$. So in the field theory limit we keep the length of stretched membranes fixed. This is of course very similar to Dp–branes in string theory and confirms the interpretation of M5–branes as topological defects on which open supermembranes can end. For $\beta = 2$ we do not find such a nice interpretation.

Because of exact conformal invariance these dualities are not restricted to particular regimes in the AdS background. The coupling constants are independent of the scale parameter u, and so the supergravity approximation constraints can be satisfied on the complete $AdS_{d+1} \times S^{\tilde{d}+1}$ background by just considering a large N limit. Conformal invariance also makes it easier to perform tests of these dualities, basically because some observables (like 2– and 3–point functions) satisfy very stringent constraints because of conformal invariance. This ensures that some perturbative properties on the conformal field theory side can be extended to strong 't Hooft coupling. These results can then be compared with (semiclassical) AdS

supergravity results to test the conjecture. Of course we need a specific relation between the fields on both sides of the conjecture to be able to compare results. The general AdS/CFT duality map was first constructed by Witten in [124] and was suggested earlier for D3–branes in [125]. These maps position the conformal field theory on the boundary of AdS_{d+1} and couple boundary values of AdS supergravity fields to conformal operators living on the boundary. The choice for the boundary of AdS is a natural one because the AdS isometries indeed act on the boundary as ordinary conformal field theory transformations, as we discussed in 4.1.2.

A first requirement for the duality to be possible is that the symmetries match on both sides. These can be checked case by case and should include the isometries from the sphere $S^{\tilde{d}+1}$, giving a $SO(\tilde{d}+2)$ global symmetry group on the supergravity side. These global symmetries are indeed also found as the so–called $\mathscr{R}$–symmetry group on the superconformal field theory side. Conformal symmetry and the number of supersymmetries also match up in all cases. Using the duality maps many checks were made of the AdS/CFT conjecture, mainly for the D3–brane and the self–dual string (a D1–D5 intersection) in $D=6$. Especially the D3–brane case, where the dual field theory is a $D=4$ superconformal Yang–Mills theory, attracted a lot of attention. All these checks so far confirmed the AdS/CFT conjecture and for more details we refer to the review paper [112] and references therein.

The examples which are not yet understood are the five– and four–dimensional Reissner–Nordström extremal black holes. Those cases are conjectured to give rise to an $AdS_2\times S^2$ and $AdS_2\times S^3/CFT_1$ duality respectively. The field theories in these cases should reduce to supersymmetric conformal quantum mechanics models [126, 127], which are difficult to construct. An immediate problem that arises is that, as opposed to higher dimensional field theories, a (conformal) quantum mechanics model does not describe any internal dynamics if we have to assume that all the BPS particles lie on top of each other. The same problem occurs in the conjectured dual AdS_2 supergravity description which displays a mass gap, telling us that small excitations of the AdS_2 supergravity fields can not exist. The duality conjecture therefore reduces to a map between two non–dynamical theories, which does not seem very interesting. Another basic problem refers to the fact that AdS_2 has two boundaries and the question then arises on which boundary the dual conformal quantum mechanics model should live [128]. Another problem was discussed in [129], where it was observed that the AdS_2 spacetime is not a stable background vacuum solution, but instead can fragment into multiple AdS_2

spacetimes.

Although potentially these AdS_2/CFT_1 dualities could teach us a lot about (extremal) black hole dynamics, as advocated in in [130, 131], the situation at this moment is not well understood. What has become clear is that the AdS_2/CFT_1 case requires an understanding of what the duality conjecture means which is different from all the higher–dimensional cases. More recent progress in constructing supersymmetric conformal quantum mechanics models and the interpretation of the duality can be found in [132, 133, 134, 135] and in the review paper [136].

5.2.3 Non–trivial dilaton Dp–branes in $D = 10$

Let us first discuss all ten–dimensional Dp–branes, except for the already covered D3–brane. These were first discussed in [108, 109], except for the special case of the D8–brane. The only difference with the pure *AdS* cases is the appearance of a non–trivial dilaton, forcing us to use the constraint (5.64) and signalling the breaking of the conformal isometry group of the complete background solution. The non–trivial dilaton, turning the background solution into a generic domain–wall, is also responsible for the breaking of supersymmetry. The number of broken supersymmetries is obviously the same as the number of broken supersymmetries in the original Dp–brane solution, which is $\frac{1}{2}$ of the maximum number 32. Remember that the pure *AdS* vacuum solution preserved all of the supersymmetries and with respect to the original p–brane soliton solution this represented supersymmetry enhancement in the near–horizon geometry.

The string coupling constant will depend on the (radial) *AdS* energy scale u and therefore naturally represents a (classically) running coupling constant in the dual field theory. This also means that when we want to use a supergravity approximation we will be forced into a region (an energy scale u interval) of the complete domain–wall supergravity background where the string coupling is small. Similarly the spacetime curvature of the background, when probed with strings, is no longer constant because the dilaton scalar will now contribute as well. Therefore it is non–trivial to find energy regions where supergravity will be a good approximation.

The constraint (5.64) for $D = 10$ and $\Delta = 4$ gives $k = 1$ and more importantly $x = p - 3$. The fixed 't Hooft coupling constant (5.54) therefore equals

$$g_f^2 = c_p N g_s \sqrt{\alpha'}^{(p-3)} = N g_{YM}^2 \,, \tag{5.76}$$

which is the appropriate scaling for a coupling constant of a $p+1$–dimensional Yang–Mills gauge theory. We will keep this quantity fixed, which for $p > 3$ means we have to take $N \to \infty$ if we want to be sure that the bulk supergravity decouples. From the Yang–Mills theory point of view this is the 't Hooft limit of an infinite number of colors. The supersymmetric Yang–Mills theory under consideration will of course be the one obtained as a low–energy limit of Dp–branes, which as we discussed in section 2.3.2 can all be obtained from a toroidal reduction of the $D = 10$ SYM theory. These include scalars which for $p \geq 1$ will be frozen into their vacuum expectation values[12] and will play no role in the actual dynamics. Remember that even p D–branes are string solitons of IIA superstrings, whereas odd p D–branes are part of IIB superstring theory.

Other $D = 10$ Dp–brane parameters we will need are

$$\beta = \tfrac{1}{2}(5-p) \quad , \quad a = \tfrac{1}{2}(3-p)\,. \tag{5.77}$$

For $p = 5$ we see that $\beta = 0$ and therefore the near–horizon geometry becomes $d+1$–dimensional flat space in the dual frame metric. This excludes the D5–branes from our discussion. Also remember that the IIB D7–brane is excluded because $\tilde{d} = 0$. The relation between the Dp–brane energy scale U and the holographic energy scale u now becomes

$$U^{5-p} = \tfrac{1}{4}(5-p)^2 \left(\frac{d_p}{c_p}\right) g_f^2 u^2\,. \tag{5.78}$$

Plugging in the different $D = 10$ Dp–brane parameters the dilaton background (5.65) becomes

$$e^{\Phi} = \frac{1}{N}\left[\tfrac{1}{2}|5-p|(g_f^2)^{\frac{1}{p-3}}\, u \left(\frac{d_p^{1/(7-p)}}{c_p^{1/(p-3)}}\right)\right]^{\frac{(p-3)(7-p)}{2(5-p)}}\,. \tag{5.79}$$

Let us analyze for each Dp–brane where we can trust a supergravity approximation on the one hand and a perturbative supersymmetric Yang–Mills gauge theory approximation on the other hand.

The effective dimensionless coupling constant (5.66) in the perturbative Yang–Mills theory equals

$$g_{eff}^2 = g_f^2\, u^{(p-3)}\,. \tag{5.80}$$

This represents the classical scaling of the coupling constant which because of supersymmetry should not be affected by quantum effects in the field theory. Necessarily a perturbative field theory analysis can only be trusted in the following

[12] Put differently, for $p \geq 1$ the positions of the N Dp–branes are fixed and determine the gauge theory vacuum.

energy regimes

$$\begin{aligned} u &\gg (g_f^2)^{\frac{-1}{(p-3)}} \quad \forall\, p<3 \\ u &\ll (g_f^2)^{\frac{-1}{(p-3)}} \quad \forall\, p>3\,. \end{aligned} \tag{5.81}$$

In a supergravity approximation we need small curvature (5.67) and at the same time small string coupling e^{Φ} (5.79). Small curvature means[13]

$$\tau_s \propto \left(g_{eff}^2\right)^{\frac{1}{(5-p)}} \gg 1\,. \tag{5.82}$$

Similarly small string coupling (5.79), when rewritten using g_{eff}, gives the condition

$$e^{\Phi} \propto \frac{1}{N}\left(g_{eff}^2\right)^{\frac{(7-p)}{2(5-p)}} \ll 1\,. \tag{5.83}$$

Translating the conditions (5.82) and (5.83) into conditions on the energy scale u we find

$$\begin{aligned} u &\ll (g_f^2)^{\frac{-1}{(p-3)}} \quad \forall\, p<3 \\ u &\gg (g_f^2)^{\frac{-1}{(p-3)}} \quad \forall\, 3<p<5 \\ u &\ll (g_f^2)^{\frac{-1}{(p-3)}} \quad \forall\, p>5\,, \end{aligned} \tag{5.84}$$

for small curvature. Notice that these regimes are opposite to the perturbative field theory regimes (5.81) for $p<5$. For $p>5$ these regimes overlap. Small string coupling (5.83) translates into

$$\begin{aligned} u &\gg (g_f^2)^{\frac{-1}{(p-3)}} N^{\frac{(p-3)(7-p)}{2(5-p)}} \quad \forall\, p<3 \\ u &\ll (g_f^2)^{\frac{-1}{(p-3)}} N^{\frac{(p-3)(7-p)}{2(5-p)}} \quad \forall\, 3<p<5 \\ u &\gg (g_f^2)^{\frac{-1}{(p-3)}} N^{\frac{(p-3)(7-p)}{2(5-p)}} \quad \forall\, 5<p<7 \\ u &\ll (g_f^2)^{\frac{-1}{(p-3)}} N^{\frac{(p-3)(7-p)}{2(5-p)}} \quad \forall\, p>7\,. \end{aligned} \tag{5.85}$$

Combining the conditions (5.84) and (5.85) to find the regions where supergravity is a good approximation, will necessarily give a condition on N to be able to satisfy both conditions at the same time (except for the D8–brane). We find

$$N^{\left|\frac{(p-3)(7-p)}{2(5-p)}\right|} \gg 1 \quad \forall\, p<7\,. \tag{5.86}$$

[13] In our analysis of small curvature and string coupling we suppress all constants of order 1, e.g. p, c_p and d_p.

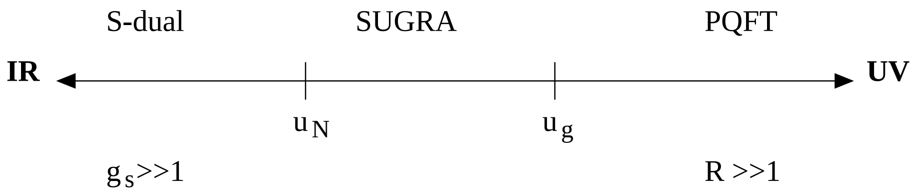

Figure 5.2: Different regimes in the energy plot for Dp–branes with $0 \leq p < 3$ and $N \gg 1$. The terminology should be self–explanatory and was discussed in the main text.

A supergravity approximation therefore always involves a large N limit. The only exception being the D8–brane, but remember that in that case we had to take $N \to \infty$ anyhow to keep g_f^2 fixed and to decouple gravity. The D8–brane was first discussed in [107].

The $D = 10$ Dp–branes can be divided into four types of qualitatively different behavior [108, 109, 107].

- Dp–branes with $0 \leq p < 3$. Summarizing the previous discussion we find that the supergravity approximation is valid in the IR when

$$u_N = (g_f^2)^{\frac{-1}{(p-3)}} N^{\frac{(p-3)(7-p)}{2(5-p)}} \ll u \ll u_g = (g_f^2)^{\frac{-1}{(p-3)}} , \tag{5.87}$$

which implies large N. The perturbative quantum field theory description applies in the UV, when

$$u \gg u_g . \tag{5.88}$$

We end up in a strongly coupled string theory regime when

$$u \ll u_N . \tag{5.89}$$

In the inequality (5.87) we defined the two critical points u_N and u_g, where the subscript refers to their (different) dependence on the fixed quantities. The critical point u_g describes the crossover from small to large curvature ($R \gg 1$) in the string theory description and the transition from perturbative (PQFT) to non–perturbative in the field theory description. The critical point u_N describes the crossover from weak to strong string coupling. When $N = 1$ the two critical points become equal and the separation between u_g and u_N increases with N, which motivates the subscript N on u_N. We plotted the different regimes in Figure 5.2.

Depending on whether we are in Type IIA ($p = 0$ and $p = 2$) or Type IIB superstring theory ($p = 1$) we can invoke M–theory or S–dual Type IIB theory respectively to try to obtain another weakly coupled description of this system of Dp–branes in the far IR. The Dp–brane background itself should then also be transformed. For the D2–brane the solution transforms into the M–theory M2–brane, which we already discussed in the AdS/CFT subsection. It follows that the $(2+1)$–dimensional field theory flows to a conformal fixed point in the IR, which has a dual description as the $AdS_4 \times S^7$ M2–brane near–horizon supergravity in the large N limit [108].

The S–dual background for the Type IIB D1–brane is the fundamental ($k = 0$) Type IIB F1–brane, which has a curvature singularity at $u = 0$ and therefore a supergravity approximation is not possible in the far IR. However the $(1+1)$–dimensional Yang–Mills gauge theory flows to a conformal fixed point in the IR [75]. This suggests that the curvature singularity in the S–dual F1 background is an artifact and is resolved by a description as a conformal fixed point in the $1+1$–dimensional gauge theory [108].

The S–dual D0–brane background is the M–wave, which just represents $D = 11$ momentum. Wrapping the M–wave on a light–like compact direction with N units of momentum will give the N D0–branes near–horizon solution [137]. The S–dual gravitational theory in the IR is therefore M–theory on a compact light–like direction with N units of momentum. This is the strongly coupled region of the dual N D0–branes quantum mechanics model. This quantum mechanics model is nothing but the Matrix model and we now conclude that only the strongly coupled IR limit of the Matrix model will describe DLCQ M–theory. We can now also understand why we reached a different conclusion in section 3.1.1. When we "deduced" the Matrix model we neglected the gravitational backreaction of the N units of M–theory momentum. The agreement between *perturbative* Matrix model results and DLCQ M–theory (in a supergravity limit) should probably be understood as a consequence of non–renormalisation theorems due to supersymmetry. The Matrix theory conjecture has now been reduced to a special example of the more general DW/QFT correspondence [137, 138]. Although the above sketched scenario seems plausible, a remaining problem is that we do not understand the duality map between quantum mechanics models and DW_2 backgrounds very well. These problems were already explained in section 4.2.2 where we discussed the AdS_2/CFT_1 examples.

The established relation between the (uncompactified) Matrix model and

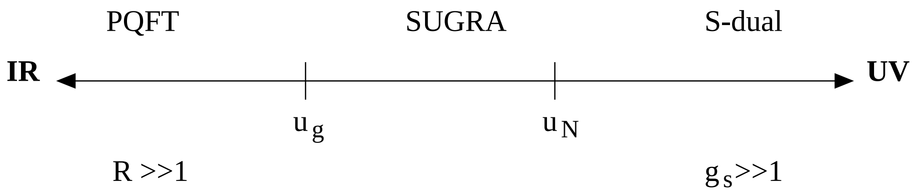

Figure 5.3: Different regimes in the energy plot for the D4–brane with $N \gg 1$.

the more general DW_2/QFT_1 correspondence seems to break down when we consider Matrix model compactifications. According to the Matrix model conjecture $p+1$–dimensional Yang–Mills theory defined on a torus T_p will describe DLCQ M–theory on the T–dual torus. At first sight these Matrix model conjectures can not be obtained as special examples of the DW/QFT correspondence if $p > 0$. Supposedly a relation should exist [138], but a detailed understanding seems to be missing so far.

- The D4–brane. The supergravity regime is defined in the UV by

$$u_g = (g_f^2)^{-1} \ll u \ll u_N = (g_f^2)^{-1} N^{\frac{3}{2}}\,. \tag{5.90}$$

The $D = 5$ perturbative quantum field theory description applies in the IR, when

$$u \ll u_g\,. \tag{5.91}$$

Strong string coupling is encountered in the far UV when

$$u \gg u_N\,. \tag{5.92}$$

We plotted the different regimes in Figure 5.3.

In the strong string coupling regime we can try to go to the S–dual description, which would be the M–theory M5–brane. In the UV regime we therefore obtain the AdS/CFT duality between $AdS_7 \times S^4$ supergravity and a $(5+1)$–dimensional conformal field theory. This also means that the $(4+1)$–dimensional gauge theory of the D4–branes should flow in the UV to a $(5+1)$–dimensional conformal fixed point theory. This was also suggested earlier in studies involving Matrix theory on T^4 and T^5 [61, 62] . To decouple gravity and to fix the 't Hooft coupling constant we mentioned that we had to take $N \to \infty$ and small g_s. However in this case we can

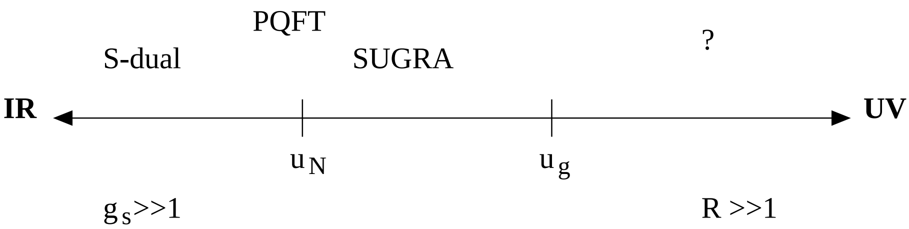

Figure 5.4: Different regimes in the energy plot for the D6–brane with $N \gg 1$. The question mark represents our lack of knowledge for both the QFT and the string theory in the UV. When $N \to \infty$, $u_N \to 0$ and the S–dual string theory region will shrink to zero size.

take $g_s \to \infty$ and finite N to fix the coupling constant and decouple *eleven–dimensional* bulk gravity to leave us with the (conformal) M5–brane worldvolume field theory. Classically this is a self–dual rank 2 gauge theory, whose quantum version is not yet understood.

- The D6–brane. In the IR there exists a valid supergravity regime bounded by

$$u_N = (g_f^2)^{\frac{-1}{3}} N^{\frac{-3}{2}} \ll u \ll u_g = (g_f^2)^{\frac{-1}{3}}, \qquad (5.93)$$

which partially overlaps with the perturbative field theory regime, defined by

$$u \ll u_g . \qquad (5.94)$$

At the same IR end of the energy spectrum, when

$$u \ll u_N , \qquad (5.95)$$

we end up in a region of strong string coupling. We plotted the different regimes in Figure 5.4.

The strong string coupling regime can perhaps be resolved by going to eleven–dimensional M–theory. This will amount to considering Kaluza–Klein monopoles, their near–horizon region and their worldvolume field theory description. This time it is impossible to consider a limit $g_s \to \infty$ and finite N to fix the worldvolume field theory coupling constant and decouple *eleven–dimensional* gravity. So the best we can do is consider the limit $N \to \infty$. Essentially this excludes the appearance of a strong string

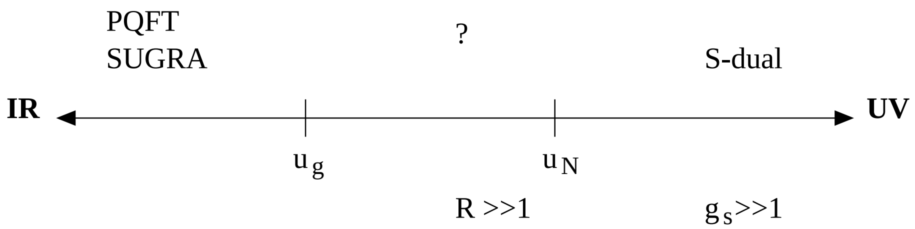

Figure 5.5: Different regimes in the energy plot for the D8–brane with $N \gg 1$. The question mark represents our lack of knowledge for both the QFT and the string theory. When $N \to \infty$, $u_N \to \infty$ and the S–dual string theory region will shrink to zero size.

coupling regime and describes a free string theory limit. The *DW/QFT* duality then suggests that in the IR a perturbative 't Hooft limit ($N \to \infty$) of the $(6+1)$–dimensional (non–renormalizable) gauge theory *equals* a free string theory on the domain–wall background. Different interpretations of the correspondence in this D6–branes example, although investigated in a slightly different context, were pointed out in [108, 121].

- The D8–brane. This case is rather special, although somewhat similar to the D6–brane discussion. As we already pointed out the supergravity approximation does not need to involve a large N limit and is valid in the energy range

$$u \ll u_g = (g_f^2)^{\frac{-1}{5}}\,, \tag{5.96}$$

which becomes larger in the limit of large N. The perturbative gauge theory regime is bounded by

$$u \ll u_g\,, \tag{5.97}$$

which is the same energy range as in the supergravity approximation. Large string coupling is encountered in the UV when

$$u \gg u_N = (g_f^2)^{\frac{-1}{5}} N^{\frac{5}{6}}\,, \tag{5.98}$$

which gets larger when N is increasing. We plotted the different regimes in Figure 5.5.

Again we would like to go to M–theory to resolve this regime. We need M9–branes for that but these solutions will still have large curvature in the UV, so a supergravity approximation does not exist in that regime. Also the worldvolume theory of M9–branes is not yet well understood [20, 23]. As in the case of the D6–brane it is again impossible to take the limit $g_s \to \infty$ and finite N to fix the worldvolume coupling constant and decouple *eleven–dimensional* gravity. So we have to consider the limit $N \to \infty$, which is a free string theory limit. The analysis above suggests that in the IR a 't Hooft limit of the $(8+1)$–dimensional (non–renormalizable) gauge theory can also be described by a free string theory on the domain–wall background.

This ends our discussion on $D = 10$ Dp–branes. We will now move on to their six–dimensional analogues.

5.2.4 Non–trivial dilaton dp–branes in $D = 6$

The examples in $D = 6$ were first discussed in [107]. The p–brane solutions in $D = 6$ we will treat all have $\Delta = n = 2$, where n is the integer counting the (higher–dimensional) constituents which make up the $D = 6$ p–brane solution. This can be understood as follows. In ten dimensions one can consider intersecting or overlapping Dp–branes, let us say we consider a Dp–brane intersecting with a Dq–brane with p' common worldvolume directions. These solutions are stable and BPS, breaking $1/4$ of the 32 supersymmetries, only when the number of relative transverse directions[14] equals 4 [139, 140]. Reducing all 4 relative transverse directions on a torus T^4 we end up in $D = 6$ with a p'–brane solution. Reducing on the T^4 relative transverse space also means the constituent branes are *delocalized*, the solution can not depend on the relative transverse directions anymore. All information of the $D = 10$ intersection is hidden in the 4 small compact directions, from the $D = 6$ point of view this can not be distinguished from an ordinary p'–brane solution. To obtain a solution with just one harmonic functions we will decide to identify the $D = 10$ Dp– and Dq–brane charges. This means we identify the number N_1 of Dp–branes and the number N_2 of Dq–branes to equal the number N of p'–branes. The resulting $D = 6$ p'–brane can be found in our general solution (5.4) if we take $D = 6$ and $\Delta = 2$ and inherits many properties of the $D = 10$ Dp–brane parents, so we decide to call them dp'–branes. For example their tension again scales as $1/g_s$, which is customary for D–branes.

It is also possible to give these string solitons an interpretation without referring to $D = 10$ Dp–brane intersections. Instead we can relate them directly to $D = 10$ Dp–branes by considering $K3$ compactifications. A $K3$–manifold is

[14] These are transverse directions of one of the constituent branes and worldvolume directions of the other brane.

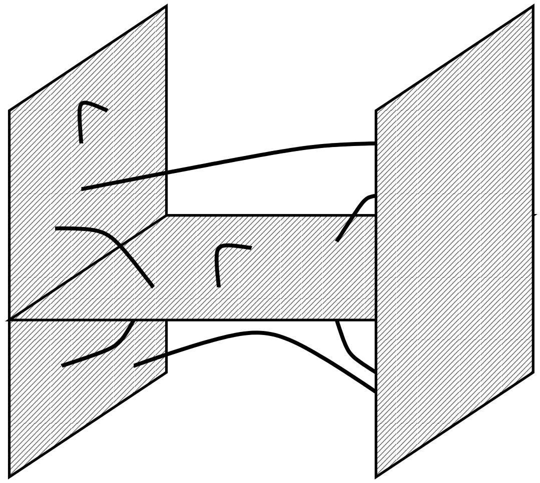

Figure 5.6: Intersecting D–branes and some different open string excitations.

a 4–dimensional Calabi–Yau manifold breaking $1/2$ of the $D = 10$ supersymmetries. Compactification of Type IIA or Type IIB superstrings on a $K3$–manifold gives a corresponding $D = 6$ superstring theory with 16 supersymmetries. String solitons in this $D = 6$ theory will break another half and therefore correspond to supergravity soliton solutions preserving 8 supersymmetries. When we consider Dp–branes in Type IIA or Type IIB superstring theory and compactify the theory on a $K3$–manifold, the $D = 6$ dp'–branes with $\Delta = 2$ will arise naturally.

The worldvolume theory of these dp'–branes can not be the same as their Dp–brane parents. For one thing they should preserve a smaller number of supersymmetries. To construct them the easiest approach is to use their interpretation as intersections of Dp–branes in $D = 10$. Open strings, which will determine the worldvolume field theory fluctuations, now have the possibility to stretch from the Dp–branes to the Dq–branes, see Figure 5.6. These will give rise to extra states in the worldvolume field theory. Extra fermions and scalars denoting the relative position in the intersection space (which is the space of relative transverse directions), called a supersymmetry hypermultiplet, will appear on both the Dp–brane and Dq–brane worldvolume theory. These scalars and fermions will transform in the fundamental representation of the (different) gauge groups. The appearance of the hypermultiplet will break the supersymmetry of the system to one preserving only $1/4$ of the maximum of 32 supersymmetries. Most important for our discussion will be that the $D = 6$ dp'–brane worldvolume field theory will have extra hypermultiplet scalars which after identification of the number of branes transform in a single fundamental representation of the $U(2N)$ gauge group.

We already encountered one example of a $D=6$ dp–brane in our $AdS/\,CFT$ discussion, where we described the $D=6$ self–dual string ($p=1$) as the pure $AdS_3\times S^3$ near–horizon geometry example which is dual to a 2–dimensional conformal field theory. The d2–brane is the special flat Minkowski spacetime near–horizon geometry example, the d3–brane has $\tilde{d}=0$, so we will discuss the d0–brane and the (domain–wall) d4–brane as new examples of DW/QFT dualities in $D=6$.

Interestingly enough the constraint (5.64) tells us that in this case

$$x=p-1\,, \tag{5.99}$$

which means that the dimension of the fixed coupling constant (5.54) is that of a scalar field theory[15]. This is a very important difference as compared with the $D=10$ Dp–branes. It means that the Yang–Mills coupling constant $g_{YM}\to\infty$ in the field theory limit. Generically the dp–brane worldvolume field theory will be in a vacuum with non–zero vacuum expectation values for the hypermultiplet scalars, which is called the Higgs branch[16]. This will give rise to a Higgs effect giving mass to all the vector bosons on the worldvolume proportional to the Yang–Mills coupling constant. Because the Yang–Mills coupling constant diverges in the field theory limit, all vector bosons will become infinitely massive and decouple. The effective field theory which is left–over is a worldvolume scalar field theory (with the necessary fermions of course to make it supersymmetric). These kinds of limits were discussed earlier in the context of Matrix theories [141, 142] and in the context of the $D=6$ self–dual string AdS/CFT duality in [110].

In principle the dynamics of the positions of the dp–branes, represented by the vector multiplet scalars (as opposed to the hypermultiplet scalars) is now included as well, which is called the Coulomb branch when the hypermultiplet scalars have vanishing expectation values. In [141, 142] it is argued that the Coulomb branch decouples and one should only consider the Higgs branch scalar field theory. We will assume this conclusion to be correct and will only consider, when needed, the Higgs branch of the corresponding dp–brane worldvolume field theory.

Let us now analyze the $D=6$ supergravity near–horizon backgrounds. We are mainly interested in the dilaton–expression which governs the analysis of where

[15] This is of course consistent with the fact that the $p=1$ case should be a conformal field theory.

[16] Strictly speaking, this is only called the Higgs branch if the scalars in the vector multiplet have vanishing expectation values, which means that all the dp–branes are on top of each other.

we will be able to use a supergravity approximation. We are going to repeat the analysis done for the $D = 10$ Dp–branes, just replacing the $D = 10$ parameters by the appropriate six–dimensional ones. Using that $\Delta = 2$ and $D = 6$ we find

$$\beta = 2 - p \quad , \quad a = 1 - p\,. \tag{5.100}$$

The D–brane constraint (5.64) gives the following relation between the D–brane energy scale U and the holographic energy scale u

$$U^{2-p} = |2-p| \left(\frac{d_p}{c_p}\right) g_f^2 u\,. \tag{5.101}$$

The singular case $p = 2$ is the one with the flat Minkowski near–horizon region, as the $p = 5$ case in $D = 10$. The effective dimensionless coupling constant equals

$$g_{eff}^2 = g_f^2 u^{p-1}\,. \tag{5.102}$$

In terms of this effective coupling constant the dilaton background (5.65) can be expressed as

$$e^{\Phi} = \frac{1}{d_p N} |2-p|^{\frac{(p-1)(3-p)}{2(2-p)}} \left(\frac{d_p g_{eff}^2}{c_p}\right)^{\frac{(3-p)}{2(2-p)}}\,. \tag{5.103}$$

This is of course very similar to the expression in (5.83).

From (5.102) it follows that a perturbative field theory analysis is valid when

$$\begin{aligned} u &\gg (g_f^2)^{\frac{1}{(1-p)}} \quad \forall\, p < 1 \\ u &\ll (g_f^2)^{\frac{1}{(1-p)}} \quad \forall\, p > 1\,. \end{aligned} \tag{5.104}$$

The supergravity approximation requires us to satisfy the following two conditions simultaneously (we neglect constants of order 1, e.g. p, c_p and d_p)

$$\begin{aligned} \tau_s &\propto \left(g_{eff}^2\right)^{\frac{1}{(2-p)}} \gg 1 \\ e^{\Phi} &\propto \frac{1}{N} \left(g_{eff}^2\right)^{\frac{(3-p)}{2(2-p)}} \ll 1\,. \end{aligned} \tag{5.105}$$

It should be clear that the first condition represents small curvature and the second condition represents small string coupling. Translating these conditions into conditions on the energy u will give us the energy regimes where supergravity

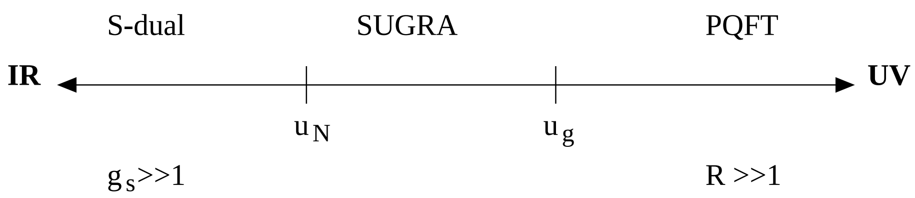

Figure 5.7: Different regimes in the energy plot for d0–branes with $N \gg 1$. The terminology should be self–explanatory and was discussed in section 4.2.3.

is a good description. For $p = 0$ these conditions can again only be satisfied simultaneously by taking a large N limit, whereas for $p = 4$ a large N limit is not implied by (5.105). However, we need to take $N \to \infty$ in the d4–brane case to fix the coupling constant (5.54) and to decouple the worldvolume field theory from gravity.

Below we discuss the two $D = 6$ DW/QFT examples in more detail.

- The $D = 6$ d0–brane. A supergravity approximation can be used in the following IR energy regime

$$u_N = g_f^2 N^{\frac{-4}{3}} \ll u \ll u_g = g_f^2 \,, \tag{5.106}$$

which can only be satisfied for large N. In the UV we can use perturbative field theory

$$u \gg u_g \,, \tag{5.107}$$

which in this case reduces to a quantum mechanics model. We therefore find the typical DW/QFT behavior that the supergravity regime and the perturbative field theory regime do not overlap, avoiding inconsistencies. In the far IR, when

$$u \ll u_N \,, \tag{5.108}$$

the string coupling becomes large and we could try to use an S–dual description. The different regimes are plotted in Figure 5.7.

The d0–brane is related to the D4–brane by considering a IIA compactification on a $K3$–manifold. It is conjectured (and by now well established) that Type IIA superstring theory on a $K3$–manifold is S–dual to Heterotic superstring theory on a T^4 [26]. On the Heterotic side the S–dual soliton solution would be a fundamental state ($k = 0$) and has a curvature singularity

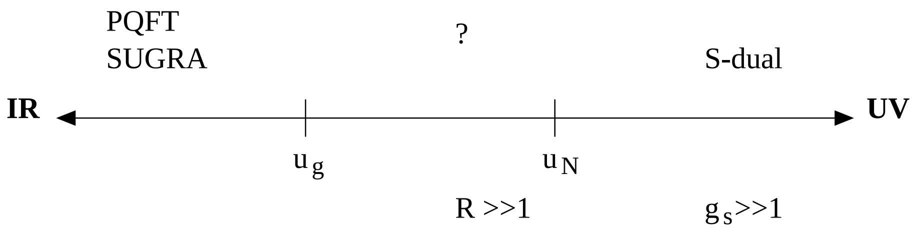

Figure 5.8: Different regimes in the energy plot for the d4–brane with $N \gg 1$. The question mark represents our lack of knowledge for both the QFT and the string theory. When $N \to \infty$, $u_N \to \infty$ and the S–dual string theory region will shrink to zero size.

at $u = 0$, so a supergravity approximation will not make sense. The situation resembles the D1–brane case in $D = 10$ Type IIB theory. There the curvature singularity in the S–dual F1–brane solution was resolved by the strong coupling conformal fixed point of the $(1+1)$–dimensional gauge theory. It is suggestive to propose the occurrence of a similar phenomenon in this case. It would therefore be interesting to determine the strongly coupled IR limit of the corresponding quantum mechanics model. As mentioned in section 4.2.2 and 4.2.3, we will have to deal with the problems involving the interpretation of the DW_2/QFT_1 correspondence if we want to understand this d0–branes example in all its detail.

- The d4–brane. Just like the $D = 10$ D8–brane this is a special case, because it is a $D = 6$ domain–wall solution. The supergravity regime is bounded from above by

$$u \ll u_g = (g_f^2)^{\frac{-1}{3}} \tag{5.109}$$

and we do not need large N. The perturbative field theory is valid in

$$u \ll u_g\,, \tag{5.110}$$

so supergravity and perturbative field theory are valid in the same regime, which seems implausible. Large string coupling is encountered in the UV when

$$u \gg u_N = (g_f^2)^{\frac{-1}{3}} N^{\frac{4}{3}}\,. \tag{5.111}$$

We plotted the different regimes in Figure 5.8.

As in the D8–brane case however, we should remember that we had to take $N \to \infty$ to fix the coupling constant and to decouple gravity. So the statement should be that the IR 't Hooft limit of the $(4+1)$–dimensional (non–renormalizable) field theory can equivalently be described by a *free* string theory on the corresponding domain–wall background. Again we can not consider a limit in which we take $g_s \to \infty$ and keep N finite to decouple an S–dual gravity theory (like we could for the Type IIA D4–brane). In the UV, after taking the limit $N \to \infty$ which shrinks the S–dual region to zero size, a well–defined description is unknown. This is also suggested by the non–renormalizability of the worldvolume field theory.

At the end of this subsection let us make the following remarks. We did not present a detailed investigation of the dp–brane worldvolume field theories. Our discussion was focussed on the dp–brane geometry in the field theory limit and the search for well–defined supergravity regions. We showed that the near–horizon geometries of the $D = 6$ dp–branes indeed have regions where a supergravity approximation seems valid and the analysis is strikingly similar to that of the $D = 10$ Dp–branes. We did make some general remarks on the nature of the field theory, which is governed by scalar dynamics, presumably in the Higgs branch of the $p+1$–dimensional gauge theory, consisting of supersymmetry vector multiplets and hypermultiplets.

We should point out that other work was done on *localized* Dp–brane intersections and the field theory limit [143, 144, 145]. In these investigations a limit is considered taking one into the near–horizon geometry of the lower–dimensional D–brane in the intersection and the dual field theory should then also be the one living on the lower–dimensional D–brane. Although the field theory limit in that case fixes the Yang–Mills coupling constant, there could be a connection with the results presented here in the sense that both investigations start off with the same intersecting D–brane system.

The status of these DW/QFT dualities, in $D = 10$ as well as in $D = 6$, is not entirely clear at this moment because they are hard to check explicitly. Basically this is because the supergravity approximation and the perturbative field theory are generically valid in opposite energy regimes, making it very hard to perform explicit checks of the duality conjecture. What can be checked of course are the symmetries and it is not very hard to show that these match in all examples presented. However, the very general mechanism leading to these proposed dualities, the explicit checks of the AdS/CFT duality and the string theory interpretation of Dp–branes as discussed in 1.1.5, can all be considered strong circumstantial evidence for the correctness of the DW/QFT duality conjecture.

This ends the chapter on string solitons and the field theory limit. In the next concluding chapter we will summarize our results presented here and in the previous chapter and try to establish a common understanding of these results, apparently teaching us that gravity and gauge field theories, as limits of an underlying string theory, are connected in a very interesting way.