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Hilbert Spaces Contractively Included in the Hardy Space of the Bidisk

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Abstract. We study the reproducing kernel Hilbert spaces $\mathfrak{H}(\mathbb{D}^2, S)$ with kernels of the form

$$\frac{I - S(z_1, z_2 >)S(w_1, w_2)^*}{(1 - z_1 w_1^*)(1 - z_2 w_2^*)}$$

where $S(z_1, z_2)$ is a Schur function of two variables $z_1, z_2 \in \mathbb{D}$. They are analogs of the spaces $\mathfrak{H}(\mathbb{D}, S)$ with reproducing kernel $(1 - S(z)S(w)^*)/(1 - zw^*)$ introduced by de Branges and Rovnyak in L. de Branges and J. Rovnyak, *Square Summable Power Series* Holt, Rinehart and Winston, New York, 1966. We discuss the characterization of $\mathfrak{H}(\mathbb{D}^2, S)$ as a subspace of the Hardy space on the bidisk. The spaces $\mathfrak{H}(\mathbb{D}^2, S)$ form a proper subset of the class of the so–called sub–Hardy Hilbert spaces of the bidisk.

1. Introduction

Let $\mathfrak{F}$ and $\mathfrak{G}$ be Hilbert spaces, $\mathfrak{L}(\mathfrak{F}, \mathfrak{G})$ ($\mathfrak{L}(\mathfrak{F})$ if $\mathfrak{G} = \mathfrak{F}$) the set of bounded operators from $\mathfrak{F}$ to $\mathfrak{G}$ and let $\mathbb{D}$ be the open unit disk in the set of complex numbers $\mathbb{C}$. The function $S : \mathbb{D} \to \mathfrak{L}(\mathfrak{F}, \mathfrak{G})$ is called a Schur function if it is holomorphic on $\mathbb{D}$ and $\|S(z)\| \leqslant 1$ for all $z \in \mathbb{D}$. The set of such S will be denoted by $S(\mathbb{D}; \mathfrak{F}, \mathfrak{G})$ ($S(\mathbb{D}; \mathfrak{F})$ if $\mathfrak{G} = \mathfrak{F}$). For $S \in S(\mathbb{D}; \mathfrak{F}, \mathfrak{G})$, the $\mathfrak{L}(\mathfrak{G})$–valued kernel

$$K_S(w, z) = \frac{I_{\mathfrak{G}} - S(z)S(w)^*}{1 - zw^*} \tag{1.1}$$

is nonnegative. The corresponding reproducing kernel Hilbert space $\mathfrak{H}(\mathbb{D}, S)$ plays an important role in various questions of operator theory, system theory and interpolation; see [2], [5], [12], [15], [18].

Recall that an $\mathfrak{L}(\mathfrak{G})$–valued kernel $K(w, z)$ on a set Ω (such as $K_S(w, z)$ on $\mathbb{D}$) is a function $K(\cdot, \cdot) : \Omega \times \Omega \to \mathfrak{L}(\mathfrak{G})$; it is called Hermitian if $K(w, z)^* = K(z, w)$ and it is called nonnegative on Ω if it is Hermitian and for every natural number r, all points $w_1, \ldots, w_r \in \Omega$ and all vectors $u_1, \ldots, u_r \in \mathfrak{G}$, the block matrix with ij-th entry $\langle K(w_j, w_i)u_i, u_j\rangle_{\mathfrak{G}}$ is nonnegative. A Hilbert space $\mathfrak{M}$ of functions from Ω into $\mathfrak{G}$ is called a reproducing kernel Hilbert space if there is a nonnegative $\mathfrak{L}(\mathfrak{G})$–valued kernel $K(w, z)$ on Ω such that

(1) The function $z \mapsto K(w, z)g$ belongs to $\mathfrak{M}$ for every choice of $w \in \Omega$ and $g \in \mathfrak{G}$.

(2) For every $f \in \mathfrak{M}$, $\langle f, K(w, \cdot)g\rangle_{\mathfrak{M}} = \langle f(w), g\rangle_{\mathfrak{G}}$.

The kernel, on account of (2), is called the reproducing kernel of $\mathfrak{M}$, it is uniquely determined, and the functions in (1) are dense in $\mathfrak{M}$. The space $\mathfrak{M}$ is denoted by $\mathfrak{H}(K)$. If $\Omega \subset \mathbb{C}$ is open, the kernel $K(w, z)$ is called holomorphic if it is holomorphic in z and w^*, and then the elements in $\mathfrak{H}(K)$ are holomorphic $\mathfrak{G}$–valued functions on Ω. Thus the functions in $\mathfrak{H}(\mathbb{D}, S) := \mathfrak{H}(K_S)$ are holomorphic on $\mathbb{D}$.

In the particular case that $S \equiv 0$, the space $\mathfrak{H}(\mathbb{D}, 0)$ coincides with the Hardy space $\mathbf{H}_2(\mathbb{D}, \mathfrak{G})$ of holomorpic $\mathfrak{G}$–valued functions on $\mathbb{D}$:

$$\mathbf{H}_2(\mathbb{D}, \mathfrak{G}) = \{g(z) = \sum_0^\infty g_n z^n | z \in \mathbb{D}, g_n \in \mathfrak{G}, \sum_{n=0}^\infty \|g_n\|^2_{\mathfrak{G}} < \infty\}$$

with Hilbert inner product

$$\langle h(z), g(z)\rangle_{\mathbf{H}_2(\mathbb{D},\mathfrak{G})} = \sum_{n=0}^\infty \langle h_n, g_n\rangle_{\mathfrak{G}}.$$

Recall also that a Hilbert space is contractively (isometrically) included in a Hilbert space $\mathfrak{H}$ if it is a linear subset of $\mathfrak{H}$ and the inclusion map is a contraction (isometry).

THEOREM 1.1 (i) *For* $S \in S(\mathbb{D}; \mathfrak{F}, \mathfrak{G})$, *the space* $\mathfrak{M} = \mathfrak{H}(\mathbb{D}, S)$ (a) *is contractively included in* $\mathbf{H}_2(\mathbb{D}, \mathfrak{G})$, (b) *is invariant under the backward shift operator*

$$R_0 f(z) = \frac{f(z) - f(0)}{z}$$

and (c) *satisfies the inequality*

$$\|R_0 f\|^2_{\mathfrak{M}} \leqslant \|f\|^2_{\mathfrak{M}} - \|f(0)\|^2_{\mathfrak{G}}, \quad f \in \mathfrak{M}. \tag{1.2}$$

(ii) *Conversely, if* $\mathfrak{M}$ *is a Hilbert space of holomorphic* $\mathfrak{G}$*-valued functions on* $\mathbb{D}$ *for which* (a)-(c) *hold, then* $\mathfrak{M}$ *is a reproducing kernel Hilbert space with reproducing kernel of the form* (1.1) : *There exist a Hilbert space* $\mathfrak{F}$ *and a function* $S \in \mathfrak{L}(\mathbb{D}; \mathfrak{F}, \mathfrak{G})$ *such that* $\mathfrak{M} = \mathfrak{H}(\mathbb{D}, S)$. *The space* $\mathfrak{F}$ *and the function* S *can be chosen such that*

$$x \in \mathfrak{F}, \; S(z)x \equiv 0 \Longrightarrow x = 0.$$

This condition determines S *uniquely up to multiplication from the right by a unitary operator.*

This theorem is a special case of Theorems 3.1.2 and 3.1.3 of [5], which are formulated in the setting of Pontryagin spaces. If in part (ii) the space $\mathfrak{M}$ is finite-dimensional, the function S can be chosen rational. Note that the condition (a) is already implied by (c). Indeed, the inequality (1.2) implies that for all k

$$\|R_0^{k+1}f\|^2_{\mathfrak{M}} \leqslant \|R_0^k f\|^2_{\mathfrak{M}} - \|\frac{1}{k!}f^{(k)}(0)\|^2_{\mathfrak{G}},$$

hence for all n

$$\sum_{k=0}^{n} \|\frac{1}{k!}f^{(k)}(0)\|^2_{\mathfrak{G}} \leqslant \|f\|^2_{\mathfrak{M}} - \|R_0^{n+1}f\|^2_{\mathfrak{M}} \leqslant \|f\|^2_{\mathfrak{M}},$$

which implies $\|f\|_{\mathbf{H}_2(\mathbb{D},\mathfrak{G})} \leqslant \|f\|_{\mathfrak{M}}$, that is, $\mathfrak{M}$ is contractively included in $\mathbf{H}_2(\mathbb{D},\mathfrak{G})$. In particular, $\mathfrak{M}$ is a reproducing kernel Hilbert space since $\mathbf{H}_2(\mathbb{D},\mathfrak{G})$ is itself a reproducing kernel Hilbert space and the inclusion map is a contraction from $\mathfrak{M}$ into $\mathbf{H}_2(\mathbb{D},\mathfrak{G})$. For the basic notions of the one variable case we refer to [9],[20].

In this paper we study the analog of Theorem 1.1 in the case of two variables, when the disk $\mathbb{D}$ is replaced by the bidisk $\mathbb{D}^2 = \mathbb{D}\times\mathbb{D}$, where the situation is quite different. The Hardy space of the bidisk is defined as

$$\mathbf{H}_2(\mathbb{D}^2,\mathfrak{G}) = \left\{ g(z_1,z_2) = \sum_{i,j=0}^{\infty} g_{ij} z_1^i z_2^j \mid z_1,z_2\in\mathbb{D},\ g_{ij}\in\mathfrak{G},\ \sum_{i,j=0}^{\infty}\|g_{ij}\|^2_{\mathfrak{G}} < \infty \right\},$$

with Hilbert inner product

$$\langle h(z_1,z_2), g(z_1,z_2)\rangle_{\mathbf{H}_2(\mathbb{D}^2,\mathfrak{G})} = \sum_{i,j=0}^{\infty} \langle h_{ij}, g_{ij}\rangle_{\mathfrak{G}}.$$

It is a reproducing kernel Hilbert space with reproducing kernel

$$K_0(w_1,w_2;z_1,z_2) = \frac{1}{(1-z_1w_1^*)(1-z_2w_2^*)}.$$

This kernel was considered by Koranyi and Pukanszky [16] in connection with the representation of Herglotz functions in more than one variable.

Note that if $g(z_1,z_2)\in\mathbf{H}_2(\mathbb{D}^2,\mathfrak{G})$, then $g(z_1,0)$ and $g(0,z_2)$ belong to $\mathbf{H}_2(\mathbb{D},\mathfrak{G})$. The backward shift operator R_0 and the inequality (1.2) have natural analogs: namely, the backward shift operators

$$R_0^{(1)}f(z_1,z_2) = \frac{f(z_1,z_2)-f(0,z_2)}{z_1} \tag{1.3}$$

and

$$R_0^{(2)} f(z_1, z_2) = \frac{f(z_1, z_2) - f(z_1, 0)}{z_2} \tag{1.4}$$

and the inequalities

$$\|R_0^{(1)} f\|_{\mathfrak{M}}^2 \leqslant \|f\|_{\mathfrak{M}}^2 - \|f(0, z_2)\|_{\mathbf{H}_2(\mathbb{D},\mathfrak{G})}^2, \quad f \in \mathfrak{M}, \tag{1.5}$$

and

$$\|R_0^{(2)} f\|_{\mathfrak{M}}^2 \leqslant \|f\|_{\mathfrak{M}}^2 - \|f(z_1, 0)\|_{\mathbf{H}_2(\mathbb{D},\mathfrak{G})}^2, \quad f \in \mathfrak{M}. \tag{1.6}$$

These formulas warrant the next definition (the terminology comes from the title of D. Sarason's book [20]): A Hilbert space $\mathfrak{M}$ of $\mathfrak{G}$-valued functions will be called a *sub-Hardy Hilbert space of the bidisk* if it is a subspace of $\mathbf{H}_2(\mathbb{D}^2, \mathfrak{G})$, which is invariant under both the backward shifts (1.3) and (1.4), and satisfies both the inequalities (1.5) and (1.6). A natural candidate for a sub-Hardy Hilbert space of the bidisk is the reproducing kernel Hilbert space $\mathfrak{H}(\mathbb{D}^2, S)$ with reproducing kernel

$$K_S(w_1, w_2; z_1, z_2) = \frac{I_{\mathfrak{G}} - S(z_1, z_2)S(w_1, w_2)^*}{(1 - z_1 w_1^*)(1 - z_2 w_2^*)}, \tag{1.7}$$

where S is a Schur function of two variables: a holomorphic function $S : \mathbb{D}^2 \to \mathfrak{L}(\mathfrak{F}, \mathfrak{G})$ with $\|S(z_1, z_2)\| \leqslant 1$, $z_1, z_2 \in \mathbb{D}$. We denote the set of such functions by $S(\mathbb{D}^2; \mathfrak{F}, \mathfrak{G})$ ($S(\mathbb{D}^2; \mathfrak{F})$ if $\mathfrak{G} = \mathfrak{F}$). We prove in Section 2 that these spaces are indeed invariant under the backward shifts (1.3), (1.4) and satisfy the inequalities (1.5), (1.6). In Section 3 we show that these are not the only sub-Hardy Hilbert spaces of the bidisk; there we also show that if $\mathfrak{H}(\mathbb{D}^2, S) \neq \{0\}$ then it is infinite dimensional. In Section 5 we give a characterization of sub-Hardy Hilbert spaces of the bidisk (see Theorem 5.1). The main idea is to reduce the two variable case to the one variable case (compare with, for example, [17]) and to invoke the characterization of semi sub-Hardy Hilbert spaces of the bidisk which we derive in Section 4 (see Theorem 4.2): A Hilbert space $\mathfrak{M}$ of $\mathfrak{G}$-valued functions will be called a *semi sub-Hardy Hilbert space of the bidisk with respect to the variable* z_1 (z_2) if it is a subspace of the Hardy space $\mathbf{H}_2(\mathbb{D}^2, \mathfrak{G})$, which is invariant under the backward shift (1.3) ((1.4)), and satisfies the inequality (1.5) ((1.6), respectively). In Section 6 we show that every Schur function $S \in S(\mathbb{D}^2; \mathfrak{F}, \mathfrak{G})$ is the characteristic function of a coisometric colligation in which $\mathfrak{H}(\mathbb{D}^2, S)$ is the state space. Finally, in Section 7 we show that there exist sub-Hardy Hilbert spaces in $\mathbf{H}_2(\mathbb{D}^2, \mathbb{C}^p)$ whose orthogonal complement is shift invariant and nevertheless not of the form $S\mathbf{H}_2(\mathbb{D}^2, \mathbb{C}^q)$ for some $p \times q$ matrix valued Schur function S on the bidisk.

Some of the results presented here were announced in [4]. Related with this paper are those of Ball and Trent [11], Agler [1] and Cotlar and Sadosky [13, 14]. They contain generalizations of the one variable theory to the case of n variables. The kernels studied in [1] coincide for the case $n = 2$ with the kernel (1.7), but we characterize the more general class of sub-Hardy Hilbert spaces of the bidisk. In [13, 14] subspaces of the Hardy space $\mathbf{H}_2(\mathbb{D}^2, \mathfrak{G})$ that are invariant under the multiplication by z_1 and z_2 are studied.

2. Spaces $\mathfrak{H}(\mathbb{D}^2, S)$

For $S \in S(\mathbb{D}^2; \mathfrak{F}, \mathfrak{G})$, the kernel $K_S(w_1, w_2; z_1, z_2)$ defined by (1.7) is nonnegative on $\mathbb{D}^2$ and we denote the corresponding reproducing kernel Hilbert space by $\mathfrak{H}(\mathbb{D}^2, S)$. The nonnegativity of this kernel in $\mathbb{D}^2$ is equivalent to the fact that the operator M_S of multiplication by S is a contraction from the Hardy space $\mathbf{H}_2(\mathbb{D}^2, \mathfrak{F})$ into the Hardy space $\mathbf{H}_2(\mathbb{D}^2, \mathfrak{G})$. The space $\mathfrak{H}(\mathbb{D}^2, S)$ is contractively included in the Hardy space $\mathbf{H}_2(\mathbb{D}^2, \mathfrak{G})$. This is a consequence of the two characterizations for it, obtained in a similar way to those for the one variable space $\mathfrak{H}(\mathbb{D}, S)$; see [9, 20]. The first characterization is as an operator range. We recall that for an operator $T \in \mathfrak{L}(\mathfrak{F}, \mathfrak{G})$, the *range norm* on $\mathfrak{M} = \operatorname{ran} T$ is the norm which makes T a partial isometry from $\mathfrak{H}$ onto $\mathfrak{M}$. Evidently, this norm comes from the Hilbert space inner product

$$\langle Tf, Th\rangle_{\mathfrak{M}} = \langle (I_{\mathfrak{F}} - P)f, h\rangle_{\mathfrak{F}},$$

where P is the orthogonal projection of $\mathfrak{F}$ onto $\ker T$.

THEOREM 2.1 *The space $\mathfrak{H}(\mathbb{D}^2, S)$ is equal to the range of $(I - M_S M_S^*)^{1/2}$ in the range norm.*

Another equivalent characterization, more convenient to our present purpose, is given in the next theorem. We refer to [9] for the proof. For $f \in \mathbf{H}_2(\mathbb{D}^2, \mathfrak{G})$, let

$$m(f) := \sup_{u \in \mathbf{H}_2(\mathbb{D}^2, \mathfrak{F})} \left\{ \|f + Su\|^2_{\mathbf{H}_2(\mathbb{D}^2, \mathfrak{G})} - \|u\|^2_{\mathbf{H}_2(\mathbb{D}^2, \mathfrak{F})} \right\}. \tag{2.1}$$

THEOREM 2.2 *We have*

$$\mathfrak{H}(\mathbb{D}^2, S) = \left\{ f \in \mathbf{H}_2(\mathbb{D}^2, \mathfrak{G}) \mid m(f) < \infty \right\} \text{ and } \|f\|_{\mathfrak{H}(\mathbb{D}^2, S)} = m(f).$$

These theorems imply that the spaces $\mathfrak{H}(\mathbb{D}^2, S)$ are contractively included in the Hardy space $\mathbf{H}_2(\mathbb{D}^2, \mathfrak{G})$. We give an example where the inclusion is isometric and one where it is contractive but not isometric.

EXAMPLE 2.3 Let $\mathfrak{F} = \mathfrak{G} = \mathbb{C}$ and take $S(z_1, z_2) = z_1 z_2$. Then

$$K_S(w_1, w_2; z_1, z_2) = \frac{1}{1 - z_2 w_2^*} + \frac{z_1 w_1^*}{1 - z_1 w_1^*},$$

$\mathfrak{H}(\mathbb{D}^2, S)$ consists of all functions of the form

$$f(z_1, z_2) = g(z_2) + z_1 h(z_1), \quad g, h \in \mathbf{H}_2(\mathbb{D}, \mathbb{C}),$$

and the norm is given by

$$\|f\|^2_{\mathfrak{H}(\mathbb{D}^2,S)} = \|g\|^2_{\mathbf{H}_2(\mathbb{D},\mathbb{C})} + \|h\|^2_{\mathbf{H}_2(\mathbb{D},\mathbb{C})} = \|f\|^2_{\mathbf{H}_2(\mathbb{D}^2,\mathbb{C})}.$$

Thus the inclusion of $\mathfrak{H}(\mathbb{D}^2, S)$ in $\mathbf{H}_2(\mathbb{D}^2, \mathbb{C})$ is isometric. □

EXAMPLE 2.4 Let $\mathfrak{F} = \mathbb{C}^2$, $\mathfrak{G} = \mathbb{C}$, let α and β be nonzero numbers such that $|\alpha|^2 + |\beta|^2 = 1$, and take $S(z_1, z_2) = (\alpha z_1,\ \beta z_2)$. Then the kernel (1.7) is equal to

$$K_S(w_1, w_2; z_1, z_2) = \frac{|\alpha|^2}{1 - z_2 w_2^*} + \frac{|\beta|^2}{1 - z_1 w_1^*}$$

and the space $\mathfrak{H}(\mathbb{D}^2, S)$ consists of all functions of the form

$$f(z_1, z_2) = g(z_2) + h(z_1), \qquad g, h \in \mathbf{H}_2(\mathbb{D}, \mathbb{C}). \tag{2.2}$$

It is easily seen that the norm of $\mathfrak{H}(\mathbb{D}^2, S)$ is not the $\mathbf{H}_2(\mathbb{D}^2, \mathbb{C})$ norm. Indeed, take for instance $w_1 \neq 0$, $w_2 = 0$ and $f(z_1, z_2) = K(w_1, 0; z_1, z_2)$. The square of its $\mathfrak{H}(\mathbb{D}^2, S)$ norm is the value of the reproducing kernel at $(w_1, 0)$, that is,

$$\|f(z_1, z_2)\|^2_{\mathfrak{H}(\mathbb{D}^2,S)} = |\alpha|^2 + \frac{|\beta|^2}{1 - |w_1|^2} = 1 + |\beta|^2 \frac{|w_1|^2}{1 - |w_1|^2}.$$

This number is not equal to

$$\|f(z_1, z_2)\|^2_{\mathbf{H}_2(\mathbb{D}^2,\mathbb{C})} = (|\alpha|^2 + |\beta|^2)^2 + |\beta|^4 \frac{|w_1|^2}{1 - |w_1|^2} = 1 + |\beta|^4 \frac{|w_1|^2}{1 - |w_1|^2}.$$

□

THEOREM 2.5 *Assume $S \in S(\mathbb{D}^2; \mathfrak{F}, \mathfrak{G})$. The space $\mathfrak{H}(\mathbb{D}^2, S)$ is a sub-Hardy Hilbert subspace. Moreover, for $j = 1, 2$ and $f \in \mathfrak{F}$, $R_0^{(j)} S f \in \mathfrak{H}(\mathbb{D}^2, S)$ and*

$$\|R_0^{(1)} S f\|^2_{\mathfrak{H}(\mathbb{D}^2,S)} \leqslant \|f\|^2_{\mathfrak{F}} - \|S(z_1, 0) f\|^2_{\mathbf{H}_2(\mathbb{D},\mathfrak{G})},$$
$$\|R_0^{(2)} S f\|^2_{\mathfrak{H}(\mathbb{D}^2,S)} \leqslant \|f\|^2_{\mathfrak{F}} - \|S(0, z_2) f\|^2_{\mathbf{H}_2(\mathbb{D},\mathfrak{G})}.$$

Proof. We show first that the operator $R_0^{(1)}$ defined by (1.3) is a contraction from $\mathfrak{H}(\mathbb{D}^2, S)$ into itself and satisfies the inequality (1.5). That also $R_0^{(2)}$ in (1.4) defines a contraction on $\mathfrak{H}(\mathbb{D}^2, S)$ and that (1.6) holds can be proved similarly and is omitted. We use Theorem 2.2: For $u \in \mathbf{H}_2(\mathbb{D}^2, \mathfrak{F})$ we have

$$\begin{aligned}
&\| \frac{f(z_1, z_2) - f(0, z_2)}{z_1} + S(z_1, z_2)u(z_1, z_2)\|^2_{\mathbf{H}_2(\mathbb{D}^2,\mathfrak{G})} - \|u\|^2_{\mathbf{H}_2(\mathbb{D}^2,\mathfrak{F})} \\
=&\ \|f(z_1, z_2) + z_1 S(z_1, z_2)u(z_1, z_2) - f(0, z_2)\|^2_{\mathbf{H}_2(\mathbb{D}^2,\mathfrak{G})} - \|u\|^2_{\mathbf{H}_2(\mathbb{D}^2,\mathfrak{F})} \\
=&\ \|f(z_1, z_2) + z_1 S(z_1, z_2)u(z_1, z_2)\|^2_{\mathbf{H}_2(\mathbb{D}^2,\mathfrak{G})} \\
&-2\mathrm{Re}\, \langle f(z_1, z_2) + z_1 S(z_1, z_2)u(z_1, z_2),\, f(0, z_2)\rangle_{\mathbf{H}_2(\mathbb{D}^2,\mathfrak{G})} \\
&+\|f(0, z_2)\|^2_{\mathbf{H}_2(\mathbb{D}^2,\mathfrak{G})} - \|z_1 u\|^2_{\mathbf{H}_2(\mathbb{D}^2,\mathfrak{F})} \\
=&\ \|f(z_1, z_2) + z_1 S(z_1, z_2)u(z_1, z_2)\|^2_{\mathbf{H}_2(\mathbb{D}^2,\mathfrak{G})} \\
&-2\mathrm{Re}\, \langle f(z_1, z_2),\, f(0, z_2)\rangle_{\mathbf{H}_2(\mathbb{D}^2,\mathfrak{G})} + \|f(0, z_2)\|^2_{\mathbf{H}_2(\mathbb{D}^2,\mathfrak{G})} - \|z_1 u\|^2_{\mathbf{H}_2(\mathbb{D}^2,\mathfrak{F})} \\
=&\ \|f(z_1, z_2) + z_1 S(z_1, z_2)u(z_1, z_2)\|^2_{\mathbf{H}_2(\mathbb{D}^2,\mathfrak{G})} \\
&-2\mathrm{Re}\, \langle f(0, z_2),\, f(0, z_2)\rangle_{\mathbf{H}_2(\mathbb{D}^2,\mathfrak{G})} + \|f(0, z_2)\|^2_{\mathbf{H}_2(\mathbb{D}^2,\mathfrak{G})} - \|z_1 u\|^2_{\mathbf{H}_2(\mathbb{D}^2,\mathfrak{F})} \\
=&\ \Big(\|f(z_1, z_2) + z_1 S(z_1, z_2)u(z_1, z_2)\|^2_{\mathbf{H}_2(\mathbb{D}^2,\mathfrak{G})} - \|z_1 u\|^2_{\mathbf{H}_2(\mathbb{D}^2,\mathfrak{F})}\Big) \\
&-\|f(0, z_2)\|^2_{\mathbf{H}_2(\mathbb{D}^2,\mathfrak{G})} \leqslant m(f) - \|f(0, z_2)\|^2_{\mathbf{H}_2(\mathbb{D}^2,\mathfrak{G})}.
\end{aligned}$$

Since $m(f) < \infty$, this implies (1.5).
We come the second part of the theorem and compute an upper bound for $m(R_0^{(2)} Sf)$.
For $u \in \mathbf{H}_2(\mathbb{D}^2, \mathfrak{F})$ we have

$$\begin{aligned}
&\left\| \frac{S(z_1, z_2) - S(z_1, 0)}{z_2} f + S(z_1, z_2)u(z_1, z_2)\right\|^2_{\mathbf{H}_2(\mathbb{D}^2,\mathfrak{G})} - \|u\|^2_{\mathbf{H}_2(\mathbb{D}^2,\mathfrak{F})} \\
=&\ \|(S(z_1, z_2) - S(z_1, 0))f + z_2 S(z_1, z_2)u(z_1, z_2)\|^2_{\mathbf{H}_2(\mathbb{D}^2,\mathfrak{G})} - \|u\|^2_{\mathbf{H}_2(\mathbb{D}^2,\mathfrak{F})} \\
=&\ \|S(z_1, z_2)(f + z_2 u(z_1, z_2)) - S(z_1, 0)f\|^2_{\mathbf{H}_2(\mathbb{D}^2,\mathfrak{G})} - \|u\|^2_{\mathbf{H}_2(\mathbb{D}^2,\mathfrak{F})} \\
=&\ \|S(z_1, z_2)(f + z_2 u(z_1, z_2))\|^2_{\mathbf{H}_2(\mathbb{D}^2,\mathfrak{G})} \\
&-2\mathrm{Re}\, \langle S(z_1, z_2)(f + z_2 u(z_1, z_2)),\ S(z_1, 0)f\rangle^2_{\mathbf{H}_2(\mathbb{D}^2,\mathfrak{G})} \\
&+\|S(z_1, 0)f\|^2_{\mathbf{H}_2(\mathbb{D}^2,\mathfrak{G})} - \|u\|^2_{\mathbf{H}_2(\mathbb{D}^2,\mathfrak{F})} \\
=&\ \|S(z_1, z_2)(f + z_2 u(z_1, z_2))\|^2_{\mathbf{H}_2(\mathbb{D}^2,\mathfrak{G})} - 2\mathrm{Re}\, \langle S(z_1, 0)f,\ S(z_1, 0)f\rangle^2_{\mathbf{H}_2(\mathbb{D}^2,\mathfrak{G})} \\
&+\|S(z_1, 0)f\|^2_{\mathbf{H}_2(\mathbb{D}^2,\mathfrak{G})} - \|u\|^2_{\mathbf{H}_2(\mathbb{D}^2,\mathfrak{F})} \\
\leqslant&\ \|f + z_2 u(z_1, z_2)\|^2_{\mathbf{H}_2(\mathbb{D}^2,\mathfrak{G})} - 2\mathrm{Re}\, \langle S(z_1, 0)f,\ S(z_1, 0)f\rangle^2_{\mathbf{H}_2(\mathbb{D}^2,\mathfrak{G})} \\
&+\|S(z_1, 0)f\|^2_{\mathbf{H}_2(\mathbb{D}^2,\mathfrak{G})} - \|u\|^2_{\mathbf{H}_2(\mathbb{D}^2,\mathfrak{F})} \\
\leqslant&\ \|f\|^2_{\mathfrak{F}} - \|S(z_1, 0)f\|^2_{\mathbf{H}_2(\mathbb{D}^2,\mathfrak{G})}.
\end{aligned}$$

Hence

$$m(R_0^{(2)} Sf) \leqslant \|f\|^2_{\mathfrak{F}} - \|S(z_1, 0)f\|^2_{\mathbf{H}_2(\mathbb{D}^2,\mathfrak{G})}.$$

Theorem 2.2 implies that $R_0^{(2)}Sf \in \mathfrak{H}(\mathbb{D}^2, S)$ and that the last inequality in the theorem holds. The results for $R_0^{(1)}Sf$ can be proved similarly. □

We note that (1.5) and (1.6) are satisfied with equality if $\mathfrak{M}$ coincides with the Hardy space of the bidisk. As in the one variable case, it would be of interest to characterize all sub-Hardy Hilbert spaces $\mathfrak{M}$ for which the equalities hold. The operators $R_0^{(1)}$ and $R_0^{(2)}$ are just special cases of the operators defined by

$$R_\alpha^{(1)} f(z_1, z_2) = \frac{f(z_1, z_2) - f(\alpha, z_2)}{z_1 - \alpha}, \quad R_\alpha^{(2)} f(z_1, z_2) = \frac{f(z_1, z_2) - f(z_1, \alpha)}{z_2 - \alpha},$$

for $\alpha \in \mathbb{D}$. These operators commute. Indeed, for $\alpha,\ \beta \in \mathbb{D}$ and $f \in \mathfrak{H}(\mathbb{D}^2, S)$,

$$\begin{aligned} R_\alpha^{(1)} R_\beta^{(2)} f(z_1, z_2) &= \frac{f(z_1, z_2) - f(z_1, \beta) - f(\alpha, z_2) + f(\alpha, \beta)}{(z_1 - \alpha)(z_2 - \beta)} \\ &= R_\beta^{(2)} R_\alpha^{(1)} f(z_1, z_2). \end{aligned}$$

They also satisfy the resolvent identity

$$R_\alpha^{(j)} - R_\beta^{(j)} = (\alpha - \beta) R_\alpha^{(j)} R_\beta^{(j)}, \qquad j = 1, 2,$$

which also holds for the one variable case; see for example [6, Formula (2.16)].

3. The Finite-Dimensional Case

In this section we first give an example of a sub-Hardy Hilbert space that is not an $\mathfrak{H}(\mathbb{D}^2, S)$-space and then we show that nontrivial $\mathfrak{H}(\mathbb{D}^2, S)$-spaces are necessarily infinite dimensional.

We note the following: a vector function f is a common eigenfunction of the operators (1.3) and (1.4) with (possibly different) eigenvalues λ_1 and λ_2 if and only if it is a multiple of the function

$$f(z_1, z_2) = \frac{1}{(1 - z_1\lambda_1)(1 - z_2\lambda_2)}.$$

The sufficiency part is clear. To verify the necessity part, let

$$\frac{f(z_1, z_2) - f(0, z_2)}{z_1} = \lambda_1 f(z_1, z_2) \quad \text{and} \quad \frac{f(z_1, z_2) - f(z_1, 0)}{z_2} = \lambda_2 f(z_1, z_2).$$

Then

$$f(z_1, z_2) = \frac{f(0, z_2)}{1 - \lambda_1 z_1} = \frac{f(z_1, 0)}{1 - \lambda_2 z_2}.$$

Setting $z_1 = 0$ we get

$$f(0, z_2) = \frac{f(0,0)}{1 - \lambda_2 z_2}$$

and therefore

$$f(z_1, z_2) = \frac{f(0,0)}{(1 - z_1\lambda_1)(1 - z_2\lambda_2)}.$$

This type of function appears in the following example.

EXAMPLE 3.1 Let $a_1,\ a_2 \in \mathbb{D}$. The one-dimensional subspace $\mathfrak{M}$ of $\mathbf{H}_2(\mathbb{D}^2, \mathbb{C})$ spanned by

$$f(z_1, z_2) = \frac{1}{(1 - z_1 a_1^*)(1 - z_2 a_2^*)}$$

is a sub-Hardy Hilbert space of the bidisk and equality holds in (1.5) and (1.6). But there exists no Schur function S such that the reproducing kernel of $\mathfrak{M}$ is of the form (1.7).

Discussion. For $i = 1, 2$, let

$$f_i(z_i) = \frac{1}{1 - z_i a_i^*}, \quad p_i = \frac{1}{1 - |a_i|^2}, \quad b_i(z_i) = \frac{z_i - a_i}{1 - z_i a_i^*}.$$

Then

$$\frac{f_i(z_i) f_i(w_i)^*}{p_i} = \frac{1 - b_i(z_i) b_i(w_i)^*}{1 - z_i w_i^*}, \quad i = 1, 2,$$

and since $f(z_1, z_2) = f_1(z_1) f_2(z_2)$,

$$\frac{f(z_1, z_2) f(w_1, w_2)^*}{p_1 p_2} = \frac{1 - b_1(z_1) b_1(w_1)^*}{1 - z_1 w_1^*} \cdot \frac{1 - b_2(z_2) b_2(w_2)^*}{1 - z_2 w_2^*}. \tag{3.1}$$

The square of the $\mathbf{H}_2(\mathbb{D}^2, \mathbb{C})$ norm of f is equal to $p_1 p_2$ and thus the left side of (3.1) is the reproducing kernel of $\mathfrak{M}$; see [15, formula (2.4)]. Assume there are a Hilbert space $\mathfrak{F}$ and an $S \in S(\mathbb{D}^2, \mathfrak{F}, \mathbb{C})$ such that this kernel is of the form (1.7). Then

$$\frac{1 - b_1(z_1) b_1(w_1)^*}{1 - z_1 w_1^*} \cdot \frac{1 - b_2(z_2) b_2(w_2)^*}{1 - z_2 w_2^*} = \frac{1 - S(z_1, z_2) S(w_1, w_2)^*}{(1 - z_1 w_1^*)(1 - z_2 w_2^*)}$$

and hence

$$\begin{aligned} &b_1(z_1) b_1(w_1)^* + b_2(z_2) b_2(w_2)^* - b_1(z_1) b_1(w_1)^* b_2(z_2) b_2(w_2)^* \\ &\quad = S(z_1, z_2) S(w_1, w_2)^*. \end{aligned}$$

The right side is a nonnegative function on $\mathbb{D}^2$ while the left side has one negative square, and hence there is a contradiction. □

Example 3.1 shows that the tensor product of $\mathfrak{H}(\mathbb{D}, b_1)$ and $\mathfrak{H}(\mathbb{D}, b_2)$ is not an $\mathfrak{H}(\mathbb{D}^2, S)$ space. More is true than is shown in this example: There is no norm on $\mathfrak{M}$ defined in the above example for which the reproducing kernel is of the form (1.7). This follows from the main result of this section:

THEOREM 3.2 *Every space $\mathfrak{H}(\mathbb{D}^2, S)$, $S \in S(\mathbb{D}^2; \mathfrak{F}, \mathfrak{G})$, is either trivial or infinite dimensional.*

It is well known (see [10, 19]) that the product of two nonnegative scalar kernels is still nonnegative. The following result complements this and will be used in the proof of the theorem.

LEMMA 3.3 *Let $M(w, z)$ be a nonnegative $\mathfrak{L}(\mathfrak{G})$–valued kernel on $\mathbb{D}$ and let $s(z)$ be a nonconstant scalar Schur function on $\mathbb{D}$. Then the kernel $K(w, z) = 1/1 - s(z)s(w)^* M(z, w)$ is nonnegative and if $\mathfrak{H}(M) \neq \{0\}$ then the reproducing kernel Hilbert space $\mathfrak{H}(K)$ is infinite dimensional.*

Proof. Since $M(w, z)$ is nonnegative, the reproducing kernel space $\mathfrak{H}(M)$ is well defined and the kernel $M(w, z)$ can be factorized as $M(z, w) = F(z)F(w)^*$, where $F : \mathbb{D} \to \mathfrak{L}(\mathfrak{H}(M), \mathfrak{G})$ is the evaluation mapping defined by $F(w)f = f(w)$, $f \in \mathfrak{H}(M)$; see for example [5, Theorem 1.1.2]. It follows that $K(w, z) = \sum_0^\infty s(z)^n s(w)^{*n} F(z)F(\omega)^*$ is nonnegative, the space $\mathfrak{H}(K)$ is well defined, and for every $n \in \mathbb{N}$, $s(z)^n \mathfrak{H}(M) \subset \mathfrak{H}(K)$; see [5, Theorems 1.5.5 and 1.5.7] in the definite setting. Thus if f is a nonzero element of $\mathfrak{H}(M)$, the linearly independent functions $f(z), s(z)f(z), s(z)^2 f(z), \ldots$ belong to $\mathfrak{H}(K)$. □

Proof of Theorem 3.2: Assume $\mathfrak{H}(\mathbb{D}^2, S)$ is finite dimensional. Let $b(z)$ be a nonconstant scalar Schur function and replace z_2 in (1.7) by $b(z_1)$ and w_2 by $b(w_1)$. Then the kernel

$$K(w_1, z_1) := \frac{1 - S(z_1, b(z_1))S(w_1, b(w_1))^*}{(1 - z_1 w_1^*)(1 - b(z_1)b(w_1)^*)}$$

is nonnegative and since $\mathfrak{H}(\mathbb{D}^2, S)$ is finite dimensional, the reproducing kernel space $\mathfrak{H}(K)$ is finite dimensional also. On the other hand $K(w_1, z_1)$ is the product of the two nonnegative kernels

$$L(w_1, z_1) := \frac{1}{1 - b(z_1)b(w_1)^*}, \quad M_b(w_1, z_1) := \frac{1 - S(z_1, b(z_1))S(w_1, b(w_1))^*}{1 - z_1 w_1^*}$$

By Lemma 3.3, $\mathfrak{H}(M_b) = \{0\}$. Hence $M_b(w_1, z_1)$ is identically zero for every nonzero scalar Schur function $b(z)$, and so

$$S(z_1, b(z_1))S(w_1, b(w_1))^* \equiv 1. \tag{3.2}$$

Fix z_1 and z_2 in $\mathbb{D}$, choose $w_1 = z_1$ and take for b the Blaschke factor

$$b(z) = \frac{z - \alpha}{1 - z\bar{\alpha}}, \quad \text{where} \quad \alpha = \frac{z_1 - z_2}{1 - z_1\bar{z}_2}.$$

Then $b(z_1) = z_2$ and by (3.2), $S(z_1, z_2)S(z_1, z_2)^* \equiv 1$, $z_1, z_2 \in \mathbb{D}$. The positivity of the kernel $K_S(w_1, w_2; z_1, z_2)$ implies that more generally $S(z_1, z_2)S(w_1, w_2)^* \equiv 1$, $z_1, z_2, w_1, w_2 \in \mathbb{D}$, and so $\mathfrak{H}(\mathbb{D}^2, S) = \{0\}$. □

In going from one variable to two variables, we replaced the denominator $1 - zw^*$ in (1.1) by $(1 - z_1 w_1^*)(1 - z_2 w_2^*)$; see (1.7). This corresponds to case of the bidisk $\mathbb{D}^2 = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1| < 1, |z_2| < 1\}$. But another possibility would be to replace $1 - zw^*$ by $1 - z_1 w_1^* - z_2 w_2^*$ and to consider the kernel

$$L_S(w_1, w_2; z_1, z_2) := \frac{I - S(z_1, z_2)S(w_1, w_2)^*}{1 - z_1 w_1^* - z_2 w_2^*}.$$

This case corresponds to the unit ball $\mathbb{B}_2 := \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 < 1\}$ and differs from the bidisk case as can be seen from the following example.

EXAMPLE 3.4 There exist finite–dimensional reproducing kernel Hilbert spaces of the form $\mathfrak{H}(L_S)$.

Discussion. For $n \in \mathbb{N}$,

$$(z_1 w_1^* + z_2 w_2^*)^n = \sum_{k=0}^{n} C(n, k)\, (z_1 w_1^*)^k (z_2 w_2^*)^{n-k}, \quad C(n, k) = \frac{n!}{(n-k)!\, k!},$$

hence if $S(z_1, z_2)$ is the $1 \times (n+1)$ matrix valued function

$$S(z_1, z_2) = (z_1^n \;\; \sqrt{C(n,1)} z_1^{n-1} z_2 \;\; \sqrt{C(n,2)} z_1^{n-2} z_2^2 \;\; \cdots \;\; z_2^n),$$

then $S(z_1, z_2)S(w_1, w_2)^* = (z_1 w_1^* + z_2 w_2^*)^n$ and

$$L_S(w_1, w_2; z_1, z_2) = 1 + (z_1 w_1^* + z_2 w_2^*) + (z_1 w_1^* + z_2 w_2^*)^2 + \cdots + (z_1 w_1^* + z_2 w_2^*)^{n-1}.$$

It follows that the kernel $L_S(w_1, w_2; z_1, z_2)$ is nonnegative and that the space $\mathfrak{H}(L_S)$ is spanned by the linearly independent functions $1, z_1, z_2, z_1^2, z_1 z_2, z_2^2, z_1^3, \ldots, z_2^{n-1}$. □

In Section 5 we characterize the finite-dimensional sub-Hardy Hilbert spaces in the bidisk; see Theorem 5.2.

4. Semi Sub-Hardy Hilbert Spaces of the Bidisk

The following simple lemma shows how analytic vector valued functions of two variables can be reduced to analytic vector valued functions of one variable. The idea is not new, see for example [17]. The lemma is the key to the paper [3] in which interpolation problems are studied in the Hardy space of the bidisk.

LEMMA 4.1 *Let $g(z_1, z_2)$ be a function from $\mathbf{H}_2(\mathbb{D}^2, \mathfrak{G})$ and consider its power series expansion*

$$g(z_1, z_2) = \sum_{i,j=0}^{\infty} g_{ij} z_1^i z_2^j, \quad g_{ij} \in \mathfrak{G}. \tag{4.1}$$

If

$$f_j(z_1) := \sum_{i=0}^{\infty} g_{ij} z_1^i \quad \text{and} \quad h_i(z_2) := \sum_{j=0}^{\infty} g_{ij} z_2^j, \qquad i, j = 0, 1, \ldots,$$

then the functions

$$f(z_1) = \begin{pmatrix} f_0(z_1) \\ f_1(z_1) \\ \vdots \end{pmatrix} \quad \text{and} \quad h(z_2) = \begin{pmatrix} h_0(z_2) \\ h_1(z_2) \\ \vdots \end{pmatrix} \tag{4.2}$$

belong to $\mathbf{H}_2(\mathbb{D}, \ell_2(\mathfrak{G}))$ and satisfy

$$g(z_1, z_2) = E_{\mathfrak{G}}(z_2) f(z_1) = E_{\mathfrak{G}}(z_1) h(z_2), \tag{4.3}$$

where

$$E_{\mathfrak{G}}(z) = \left(I_{\mathfrak{G}} \; zI_{\mathfrak{G}} \; z^2 I_{\mathfrak{G}}, \; \cdots \right). \tag{4.4}$$

Moreover,

$$\|g\|_{\mathbf{H}_2(\mathbb{D}^2, \mathfrak{G})} = \|f\|_{\mathbf{H}_2(\mathbb{D}, \ell_2(\mathfrak{G}))} = \|h\|_{\mathbf{H}_2(\mathbb{D}, \ell_2(\mathfrak{G}))}. \tag{4.5}$$

The proof of this lemma is straightforward and therefore omitted. We usually write E for $E_{\mathfrak{G}}$; it should be clear from the context to which space it is related.

We now come to the characterization of a semi sub-Hardy Hilbert space of $\mathbb{D}^2$; for the definition we refer to the Introduction.

THEOREM 4.2 *Let $\mathfrak{G}$ be a Hilbert space and $\mathfrak{M}$ a Hilbert space of $\mathfrak{G}$-valued functions on $\mathbb{D}^2$.*

(1) *$\mathfrak{M}$ is a semi sub-Hardy Hilbert space of $\mathbb{D}^2$ with respect to z_1 if and only if it is a reproducing kernel Hilbert space with a reproducing kernel of the form*

$$K_{\mathfrak{M}}(w_1, w_2; z_1, z_2) = E(z_2) \frac{I_{\mathfrak{G}} - S(z_1) S(w_1)^*}{1 - z_1 w_1^*} E(w_2)^* \tag{4.6}$$

for some Schur function $S(z) \in S(\mathbb{D}; \mathfrak{F}, \ell_2(\mathfrak{G}))$ and some Hilbert space $\mathfrak{F}$. In this case, $g \in \mathfrak{M}$ if and only if it can be written as

$$g(z_1, z_2) = E(z_2) f(z_1), \qquad f \in \mathfrak{H}(\mathbb{D}, S), \tag{4.7}$$

and then $\|g\|_{\mathfrak{M}} = \|f\|_{\mathfrak{H}(\mathfrak{D},S)}$.

(2) $\mathfrak{M}$ *is a semi sub–Hardy Hilbert space of* $\mathbb{D}^2$ *with respect to* z_2 *if and only if it is a reproducing kernel Hilbert space with a reproducing kernel of the form*

$$K_{\mathfrak{M}}(w_1, w_2; z_1, z_2) = E(z_1)\frac{I_{\mathfrak{G}} - S(z_2)S(w_2)^*}{1 - z_2 w_2^*}E(w_1)^* \tag{4.8}$$

for some Schur function $S(z) \in S(\mathbb{D}; \mathfrak{F}, \ell_2(\mathfrak{G}))$ *and some Hilbert space* $\mathfrak{F}$. *In this case,* $g \in \mathfrak{M}$ *if and only if it can be written as*

$$g(z_1, z_2) = E(z_1)f(z_2), \qquad f \in \mathfrak{H}(\mathbb{D}, S), \tag{4.9}$$

and then $\|g\|_{\mathfrak{M}} = \|f\|_{\mathfrak{H}(\mathfrak{D},S)}$.

Proof. Let $\mathfrak{M}$ be a semi sub-Hardy Hilbert space of $\mathbb{D}^2$ with respect to z_1 and consider the subspace

$$\mathfrak{M}_1 = \{f \in \mathbf{H}_2(\mathbb{D}, \ell_2(\mathfrak{G})) : \quad E(z_2)f(z_1) \in \mathfrak{M}\}$$

with the induced norm

$$\|f\|_{\mathfrak{M}_1} = \|g\|_{\mathfrak{M}}, \quad g(z_1, z_2) = E(z_2)f(z_1).$$

Then $\mathfrak{M}_1$ is contractively included in $\mathbf{H}_2(\mathbb{D}, \ell_2(\mathfrak{G}))$. Furthermore, since

$$\begin{aligned} R_0^{(1)}g(z_1, z_2) &= \frac{E(z_2)f(z_1) - E(z_2)f(0)}{z_1} \\ &= E(z_2)\frac{f(z_1) - f(0)}{z_1} = E(z_2)R_0 f(z_1), \end{aligned} \tag{4.10}$$

the space $\mathfrak{M}_1$ is backward shift invariant, and by (1.5) applied to elements in $\mathfrak{M}$,

$$\|R_0 f\|^2_{\mathfrak{M}_1} = \|R_0^{(1)}g\|^2_{\mathfrak{M}} \leqslant \|g\|^2_{\mathfrak{M}} - \|g(0, z_2)\|^2_{\mathbf{H}_2(\mathbb{D},\mathfrak{G})} = \|f\|^2_{\mathfrak{M}_1} - \|f(0)\|^2_{\ell_2(\mathfrak{G})}.$$

Thus Theorem 1.1 can be applied: The reproducing kernel of the space $\mathfrak{M}_1$ is of the form $\dfrac{I - S(z_1)S(w_1)^*}{1 - z_1 w_1^*}$ for some $S(z) \in S(\mathbb{D}; \mathfrak{F}, \ell_2(\mathfrak{G}))$ and some Hilbert space $\mathfrak{F}$. Hence every $g \in \mathfrak{M}$ can be written as in (4.7) and $\|g\|_{\mathfrak{M}} = \|f\|_{\mathfrak{H}(\mathfrak{D},S)}$. To show that (4.6) is the reproducing kernel of $\mathfrak{M}$, let $k \in \mathfrak{G}$ and $(w_1, w_2) \in \mathbb{D}^2$. Then the vector $E(w_2)^*k \in \ell_2(\mathfrak{G})$ and so the function

$$K_{\mathfrak{M}}(w_1, w_2; z_1, z_2)k = E(z_2)\frac{I - S(z_1)S(w_1)^*}{1 - z_1 w_1^*}E(w_2)^*k$$

belongs to $\mathfrak{M}$. Finally, for any function g of the form (4.7) we have

$$\begin{aligned} \langle g, K_{\mathfrak{M}}(w_1, w_2; z_1, z_2)k\rangle_{\mathfrak{M}} &= \langle f, \frac{I_{\ell_2(\mathfrak{G})} - S(z_1)S(w_1)^*}{1 - z_1 w_1^*}E(w_2)^*k\rangle_{\mathfrak{H}(\mathbb{D},S)} = \\ &= \langle f(w_1), E(w_2)^*k\rangle_{\ell_2(\mathfrak{G})} = \langle E(w_2)f(w_1), k\rangle_{\mathfrak{G}} = \langle g(w_1, w_2), k\rangle_{\mathfrak{G}}. \end{aligned}$$

Conversely, assume $\mathfrak{M}$ is the reproducing kernel Hilbert space with the reproducing kernel $K_{\mathfrak{M}}$ of the form (4.6). Then every element $g \in \mathfrak{M}$ admits a representation (4.7). Since $R_0 f \in \mathfrak{H}(\mathbb{D}, S)$ and by (4.10), $R_0^{(1)} g \in \mathfrak{M}$. Furthermore,

$$\begin{aligned} \|R_0^{(1)} g\|_{\mathfrak{M}}^2 &= \|R_0 f\|_{\mathfrak{H}(\mathbb{D},S)}^2 \leqslant \|f\|_{\mathfrak{H}(\mathbb{D},S)}^2 - \|f(0)\|_{\ell_2(\mathfrak{G})}^2 \\ &= \|g\|_{\mathfrak{M}}^2 - \|g(0, z_2)\|_{\mathbf{H}_2(\mathbb{D}^2,\mathfrak{G})}^2, \end{aligned}$$

which proves the inequality (1.5) and completes the proof of part (1) of the theorem. The assertions in part (2) can be proved in much the same way. □

EXAMPLE 4.3 The kernel (4.8) is not necessarily of the form (1.7). For instance the choice $S(z_2) = \text{diag}\ (1, z_2, 1, 1, \cdots)$ leads to $K_{\mathfrak{M}}(w_1, w_2; z_1, z_2) = z_1 w_1^*$, and the corresponding space $\mathfrak{H}(K_{\mathfrak{M}})$ coincides with $z_1\mathbb{C}$. This space is not $R_0^{(1)}$-invariant and thus the kernel (4.8) is not of the form (1.7). On the other hand, when the function S is scalar, one obtains a kernel of the form (1.7). □

If $\mathfrak{F}$ in Theorem 4.2 can be identified with an ℓ_2-space, then the semi sub-Hardy Hilbert space $\mathfrak{M}$ admits characterizations analogous to Theorems 2.1 and 2.2. To show this we use the following simple observation.

LEMMA 4.4 *Let $S(z) \in S(\mathbb{D}; \ell_2(\mathfrak{F}), \ell_2(\mathfrak{G}))$. Then in the notation of Lemma* 4.1 *the formulas*

$$(\mathbf{S_1} g)(z_1, z_2) = E(z_2) S(z_1) f(z_1), \quad (\mathbf{S_2} g)(z_1, z_2) = E(z_1) S(z_2) h(z_2), \tag{4.11}$$

where $g(z_1, z_2) = E(z_2) f(z_1) = E(z_1) h(z_2)$, define two contractions $\mathbf{S_1}$ and $\mathbf{S_2}$ from $\mathbf{H}_2(\mathbb{D}^2, \mathfrak{F})$ to $\mathbf{H}_2(\mathbb{D}^2, \mathfrak{G})$, whose adjoints are given by

$$(\mathbf{S_1}^* g)(z_1, z_2) = E(z_2)(\mathbf{p}\left(S^* f\right))(z_1), \quad (\mathbf{S_2}^* g)(z_1, z_2) = E(z_1)(\mathbf{p}\left(S^* h\right))(z_2), \tag{4.12}$$

where the symbol $\mathbf{p}$ denotes the orthogonal projection of the Lebesgue space $\mathbf{L}_2(\partial\mathbb{D}, \ell_2(\mathfrak{F}))$ onto $\mathbf{H}_2(\mathbb{D}, \ell_2(\mathfrak{F}))$. In particular, if $h \in \mathfrak{G}$ and $g(z_1, z_2) = \frac{h}{(1-z_1 w_1^*)(1-z_2 w_2^*)}$, then

$$\begin{aligned} (\mathbf{S_1}^* g)(z_1, z_2) &= E(z_2) \frac{S(w_1)^* E(w_1)^* h}{1 - z_1 w_1^*}, \\ (\mathbf{S_2}^* g)(z_1, z_2) &= E(z_1) \frac{S(w_2)^* E(w_2)^* h}{1 - z_2 w_2^*}. \end{aligned} \tag{4.13}$$

Proof. These results follow from the one variable case. Since M_S is a contraction, for $g(z_1, z_2) = E(z_2) f(z_1) \in \mathbf{H}_2(\mathbb{D}^2, \mathfrak{F})$ we have

$$\|\mathbf{S_1} g\|_{\mathbf{H}_2(\mathbb{D}^2,\mathfrak{G})} = \|M_S f\|_{\mathbf{H}_2(\mathbb{D},\ell_2(\mathfrak{G}))} \leqslant \|f\|_{\mathbf{H}_2(\mathbb{D},\ell_2(\mathfrak{F}))} = \|g\|_{\mathbf{H}_2(\mathbb{D}^2,\mathfrak{F})},$$

which shows that $\mathbf{S_1}$ is a contraction. For the computation of the adjoint of $\mathbf{S_1}$, let $g(z_1, z_2) = E(z_2)f(z_1) \in \mathbf{H}_2(\mathbb{D}^2, \mathfrak{F})$ and $k(z_1, z_2) = E(z_2)h(z_1) \in \mathbf{H}_2(\mathbb{D}^2, \mathfrak{G})$, then

$$\begin{aligned}\langle \mathbf{S_1}^* k, g\rangle_{\mathbf{H}_2(\mathbb{D}^2,\mathfrak{F})} &= \langle k, \mathbf{S_1} g\rangle_{\mathbf{H}_2(\mathbb{D}^2,\mathfrak{G})} = \langle h, M_S f\rangle_{\mathbf{H}_2(\mathbb{D},\ell_2(\mathfrak{G}))} = \\ &= \langle \mathbf{p}\left(S^* h\right), f\rangle_{\mathbf{H}_2(\mathbb{D},\ell_2(\mathfrak{F}))} = \langle E(z_2)(\mathbf{p}(S^* f))(z_1), k\rangle_{\mathbf{H}_2(\mathbb{D}^2,\mathfrak{F})},\end{aligned}$$

which proves the first equality in (4.12). To obtain first equality in (4.13), use the first equality in (4.12) for $g(z_1, z_2) = E(z_2)E(w_2)^* h/1 - z_1 w_1^*$ and recall (see [15, Lemma 2.1] for matrix valued Schur functions) that with $h' = E(w_2)^* h \in \ell_2(\mathfrak{G})$

$$\mathbf{p}\left(\frac{S(z_1)^* h'}{1 - z_1 w_1^*}\right) = \frac{S(w_1)^* h'}{1 - z_1 w_1^*}.$$

The assertions concerning the operator $\mathbf{S_2}$ are proved in much the same way. □

The lemma readily implies the following analogs of Theorems 2.1 and 2.2.

THEOREM 4.5 *For $S(z) \in S(\mathbb{D}; \ell_2(\mathfrak{F}), \ell_2(\mathfrak{G}))$, the space $\mathfrak{M}$ with reproducing kernel*

$$K_{\mathfrak{M}}(w_1, w_2; z_1, z_2) = E(z_2)\frac{I_{\mathfrak{G}} - S(z_1)S(w_1)^*}{1 - z_1 w_1^*}E(w_2)^*$$

$$\left(K_{\mathfrak{M}}(w_1, w_2; z_1, z_2) = E(z_1)\frac{I_{\mathfrak{G}} - S(z_2)S(w_2)^*}{1 - z_2 w_2^*}E(w_1)^*\right)$$

coincides (a) *with the operator range* ran $(I - \mathbf{S_1}\mathbf{S_1}^*)^{\frac{1}{2}}$ (ran $(I - \mathbf{S_2}\mathbf{S_2}^*)^{\frac{1}{2}}$, *resp.*) *in the range norm, and* (b) *with the space*

$$\mathfrak{M}' = \left\{k \in \mathbf{H}_2(\mathbb{D}^2, \mathfrak{G}) \mid \sup_{g \in \mathbf{H}_2(\mathbb{D}^2,\mathfrak{F})} \left(\|k + \mathbf{S_1} g\|^2_{\mathbf{H}_2(\mathbb{D}^2,\mathfrak{G})} - \|g\|^2_{\mathbf{H}_2(\mathbb{D}^2,\mathfrak{F})}\right) < \infty\right\},$$

$$\left(\mathfrak{M}' = \left\{k \in \mathbf{H}_2(\mathbb{D}^2, \mathfrak{G}) \mid \sup_{g \in \mathbf{H}_2(\mathbb{D}^2,\mathfrak{F})} \left(\|k + \mathbf{S_2} g\|^2_{\mathbf{H}_2(\mathbb{D}^2,\mathfrak{G})} - \|g\|^2_{\mathbf{H}_2(\mathbb{D}^2,\mathfrak{F})}\right) < \infty\right\}, \quad resp.\right)$$

and the norm of $k \in \mathfrak{M}$ is exactly the supremum in the formula for $\mathfrak{M}'$.

The sub-Hardy Hilbert space $\mathfrak{M}$ in Example 3.1 is an operator range:

$$\mathfrak{M} = \text{ran}\,(I - \mathbf{V}\mathbf{V}^*)^{\frac{1}{2}} = \text{ran}\,(I - \mathbf{V}\mathbf{V}^*),$$

where $\mathbf{V}$ is an isometric operator from $\mathbf{H}_2(\mathbb{D}^2, \mathbb{C})$ into itself: We can take $\mathbf{V} = \mathbf{S}_2$ defined by the second equality in (4.11) with $\mathfrak{F} = \mathfrak{G} = \mathbb{C}$ and for an arbitrary μ on the unit circle

$$S(z_2) = I_{\ell_2} + (z_2 - \mu)\frac{E(a_1)^*}{\sqrt{p_1}}\frac{1}{1 - z_2 a_2^*}\frac{1}{p_2}\frac{1}{\mu - a_2}\frac{E(a_1)}{\sqrt{p_1}}.$$

Indeed, this follows from the identity (see [3, Lemmas 3.1 and 3.4])

$$\begin{aligned}\frac{f(z_1,z_2)f(w_1,w_2)^*}{p_1p_2} &= E(z_1)\,\frac{E(a_1)^*}{\sqrt{p_1}}\,\frac{1-b_2(z_2)b_2(w_2)^*}{1-z_2w_2^*}\,\frac{E(a_1)}{\sqrt{p_1}}\,E(w_1)^*\\ &= E(z_1)\,\frac{I_{\ell_2}-S(z_2)S(w_2)^*}{1-z_2w_2^*}\,E(w_1)^*.\end{aligned}$$

Similarly we could have chosen $\mathbf{V}=\mathbf{S_1}$. Now we turn to finite-dimensional semi sub-Hardy Hilbert spaces of $\mathbb{D}^2$. We first recall the following well known result

LEMMA 4.6 *Let $\mathfrak{G}$ be a Hilbert space, Ω some set, and let $\mathfrak{M}$ be a finite-dimensional Hilbert space of $\mathfrak{G}$-valued functions on Ω with basis $\{g_1, g_2, \ldots, g_n\}$. Let $\mathbb{P}$ be the strictly positive $n\times n$ matrix with ℓj-th entry $p_{\ell j}=\langle g_j, g_\ell\rangle_{\mathfrak{M}}$ and let $G(z)$ be the $n\times 1$-vector function $G(z)=(g_1(z)\ g_2(z)\ \cdots\ g_n(z))$. Then $\mathfrak{M}$ is a reproducing kernel Hilbert space with reproducing kernel given by the formula $K(w,z)=G(z)\mathbb{P}^{-1}G(w)^*$.*

For a proof see for example [7, Theorem 4.1].

THEOREM 4.7 *Let $\mathfrak{G}$ be a Hilbert space and $\mathfrak{M}$ an n-dimensional subspace of $\mathbf{H}_2(\mathbb{D}^2,\ \mathfrak{G})$. Then $\mathfrak{M}$ is a semi sub-Hardy Hilbert space of $\mathbb{D}^2$ with respect to z_1 (z_2) if and only if it is a reproducing kernel Hilbert space with reproducing kernel*

$$K(w_1,w_2;z_1,z_2)=E(z_2)C\,(I_n-z_1A)^{-1}\,\mathbb{P}^{-1}\left(I_n-w_1^*A^*\right)^{-1}C^*E(w_2)^* \tag{4.14}$$

$$\Big(K(w_1,w_2;z_1,z_2)=E(z_1)C\,(I_n-z_2A)^{-1}\,\mathbb{P}^{-1}\left(I_n-w_2^*A^*\right)^{-1}C^*E(w_1)^*,\quad \textit{respectively}\Big),$$

where $C\in\mathfrak{L}(\mathbb{C}^n,\ \ell_2(\mathfrak{G}))$, $A\in\mathbb{C}^{n\times n}$, and $\mathbb{P}\in\mathbb{C}^{n\times n}$ is a strictly positive matrix such that

$$\mathbb{P}-A^*\mathbb{P}A\geqslant C^*C. \tag{4.15}$$

Proof. By Lemma 4.6, $\mathfrak{M}$ is the reproducing kernel Hilbert space with reproducing kernel $K(w,z)=G(z)\mathbb{P}^{-1}G(w)^*$, where

$$G(z_1,z_2)=(g_1(z_1,z_2)\ g_2(z_1,z_2)\ \ldots\ g_n(z_1,z_2))\,, \tag{4.16}$$

the entries $g_j(z_1,z_2)$ form a basis of $\mathfrak{M}$, and $\mathbb{P}=(\langle g_j, g_\ell\rangle_{\mathfrak{M}})_{j,\ell=1}^n$ is strictly positive.
Necessity: Since $\mathfrak{M}$ is $R_0^{(1)}$–invariant,

$$R_0^{(1)}G(z_1,z_2)=\frac{G(z_1,z_2)-G(0,z_2)}{z_1}=G(z_1,z_2)A \tag{4.17}$$

for some matrix $A \in \mathbb{C}^{n\times n}$ and therefore

$$G(z_1, z_2) = G(0, z_2)\,(I_n - z_1 A)^{-1}\,. \tag{4.18}$$

Since the $g_j \in \mathbf{H}_2(\mathbb{D}^2, \mathfrak{G})$ admit an expansion of the form (4.1), there is an operator

$$C = \begin{pmatrix} G_{00} & G_{01} & \dots \end{pmatrix}^T \in \mathfrak{L}(\mathbb{C}^n, \ell_2(\mathfrak{G}))$$

such that

$$G(0, z_2) = E(z_2)C \tag{4.19}$$

and therefore

$$G(z_1, z_2) = E(z_2)C\,(I_n - z_1 A)^{-1}\,. \tag{4.20}$$

Now the formula (4.14) for the kernel follows. For every vector $x \in \mathbb{C}^n$,

$$\|Gx\|^2_{\mathfrak{M}} = x^*\mathbb{P}x. \tag{4.21}$$

Furthermore, on account of (4.17),

$$\|R_0^{(1)}Gx\|^2_{\mathfrak{M}} = \|GAx\|^2_{\mathfrak{M}} = x^*A^*\mathbb{P}Ax, \tag{4.22}$$

whereas, by Lemma 4.1, the equality (4.19) implies

$$\|G(0, z_2)x\|^2_{\mathbf{H}_2(\mathbb{D},\ell_2(\mathfrak{G}))} = \|Cx\|^2_{\ell_2(\mathfrak{G})} = x^*C^*Cx. \tag{4.23}$$

Substituting the latter three equalities into (1.5) we get

$$x^*A^*\mathbb{P}Ax \leqslant x^*\mathbb{P}x - x^*C^*Cx.$$

Since x is arbitrary, the latter inequality is equivalent to (4.15).
Sufficiency: If $\mathfrak{M}$ is a reproducing kernel Hilbert space with reproducing kernel K of the form (4.14), it is spanned by the columns of the $\mathfrak{L}(\mathbb{C}^n; \mathfrak{G})$-valued function $G(z_1, z_2)$ given by (4.20), that is, $\mathfrak{M}$ consists of all the functions g of the form

$$g(z_1, z_2) = G(z_1, z_2)x, \quad x \in \mathbb{C}^n, \tag{4.24}$$

with the norm given by (4.21). The $R_0^{(1)}$-invariance of $\mathfrak{M}$ follows from (4.17) whereas, on account of (4.15), the equalities (4.21)–(4.23) imply that the inequality (1.5) holds for every function g of the form (4.24). The statements in brackets in the theorem can be proved quite similarly. □

5. Sub-Hardy Hilbert Spaces of the Bidisk

In the preceding section we considered the invariance under the backward shifts $R_0^{(1)}$ and $R_0^{(1)}$ separately. The spaces we are interested in are invariant under both of them.

THEOREM 5.1 *Let $\mathfrak{G}$ be a Hilbert space, let $\mathfrak{M}$ be a Hilbert space of $\mathfrak{G}$-valued functions and let T be the shift of $\ell_2(\mathfrak{G})$ defined by the matrix*

$$T = \begin{pmatrix} 0 & I_{\mathfrak{G}} & 0 & 0 & \cdots \\ 0 & 0 & I_{\mathfrak{G}} & 0 & \cdots \\ 0 & 0 & & & \end{pmatrix}. \tag{5.1}$$

The following statements are equivalent:
(1) $\mathfrak{M}$ is a sub–Hardy Hilbert space of $\mathbb{D}^2$.
(2) $\mathfrak{M}$ is a reproducing kernel Hilbert space with the reproducing kernel of the form (4.6) *with a Schur function S such that the space $\mathfrak{H}(\mathbb{D}, S)$ is T-invariant.*
(3) $\mathfrak{M}$ is a reproducing kernel Hilbert space with the reproducing kernel of the form (4.8) *with a Schur function S such that the space $\mathfrak{H}(\mathbb{D}, S)$ is T-invariant.*

Proof: Assume that $\mathfrak{M}$ is a sub-Hardy Hilbert space of $\mathbb{D}^2$. Then it is contractively included in $\mathbf{H}_2(\mathbb{D}^2, \mathfrak{G})$ and invariant under $R_0^{(1)}$. By Theorem 4.2, it is the reproducing kernel Hilbert space with the reproducing kernel $K_{\mathfrak{M}}$ of the form (4.6). This means that every $g \in \mathfrak{M}$ admits a representation (4.7). By (4.7) and (5.1),

$$R_0^{(2)} g(z_1, z_2) = \frac{g(z_1, z_2) - g(z_1, 0)}{z_2} = \frac{E(z_2) - E(0)}{z_2} f(z_1) = E(z_2) T f(z_1),$$

and, since $\mathfrak{M}$ is also $R_0^{(2)}$-invariant, $E(z_2) T f(z_1) \in \mathfrak{M}$, therefore

$$T f(z_1) \in \mathfrak{H}(\mathbb{D}, S).$$

Since $f(z_1) \in \mathfrak{H}(\mathbb{D}, S)$ is arbitrary, the latter means that $\mathfrak{H}(\mathbb{D}, S)$ is T-invariant.

Conversely, let $\mathfrak{M}$ be the reproducing kernel Hilbert space with the reproducing kernel of the form (4.6) and assume that $\mathfrak{H}(\mathbb{D}, S)$ is T-invariant. Let $f(z_1) \in \mathfrak{H}(\mathbb{D}, S)$; then $T f(z_1) \in \mathfrak{H}(\mathbb{D}, S)$ and by Theorem 4.2,

$$E(z_2) T f(z_1) = R_0^{(2)}(E(z_2) f(z_1)) \in \mathfrak{M}.$$

Since the representation formula (4.7) holds for all functions in $\mathfrak{M}$, the latter means that $\mathfrak{M}$ is $R_0^{(2)}$-invariant. According to Theorem 4.2, it is also $R_0^{(1)}$-invariant and contractively included in $\mathbf{H}_2(\mathbb{D}^2, \mathfrak{G})$. This proves the equivalence of the two first statements. The equivalence between (1) and (3) can be proved in a similar way. □

In Example 4.3 with $S(z_2) = \text{diag}\,(1, z_2, 1, 1, \cdots)$, the space $\mathfrak{H}(\mathbb{D}, S)$ consists of vectors of the form $(\,0 \;\; c \;\; 0 \;\; 0 \;\; \ldots\,)^T$ with $c \in \mathbb{C}$ and is therefore not invariant under the shift T.

THEOREM 5.2 *Let $\mathfrak{G}$ be a Hilbert space and let $\mathfrak{M}$ be an n-dimensional subspace of $\mathbf{H}_2(\mathbb{D}^2, \mathfrak{G})$. Then $\mathfrak{M}$ is a sub–Hardy Hilbert space of $\mathbb{D}^2$ if and only if it is a* *reproducing kernel Hilbert space with reproducing kernel*

$$K(w_1, w_2; z_1, z_2) = \\ = C\,(I_n - z_1 B_1)^{-1}\,(I_n - z_2 B_2)^{-1}\,\mathbb{P}^{-1}\left(I_n - w_2^* B_2^*\right)^{-1}\left(I_n - w_1^* B_1^*\right)^{-1} C^*, \tag{5.2}$$

where $C \in \mathfrak{L}(\mathbb{C}^n, \mathfrak{G})$, B_1, $B_2 \in \mathbb{C}^{n\times n}$, and $\mathbb{P} \in \mathbb{C}^{n\times n}$ is a strictly positive matrix such that

$$CB_1^{\ell} B_2^{j} = CB_2^{j} B_1^{\ell}, \quad \ell, j = 0, 1, \dots, \tag{5.3}$$

$$\mathbb{P} - B_1^*\mathbb{P}B_1 \geqslant \sum_{j=0}^{\infty} B_2^{j*} C^* C B_2^{j}, \quad \mathbb{P} - B_2^*\mathbb{P}B_2 \geqslant \sum_{j=0}^{\infty} B_1^{j*} C^* C B_1^{j}. \tag{5.4}$$

Proof. As in the proof of Theorem 4.7, we apply Lemma 4.6: $\mathfrak{M}$ is the reproducing kernel Hilbert space with reproducing kernel $K(w, z) = G(z)\mathbb{P}^{-1}G(w)^*$, where

$$G(z_1, z_2) = \left(g_1(z_1, z_2)\ g_2(z_1, z_2)\ \dots\ g_n(z_1, z_2)\right),$$

the entries $g_j(z_1, z_2)$ form a basis of $\mathfrak{M}$, and $\mathbb{P} = (\langle g_j, g_\ell\rangle_{\mathfrak{M}})_{j,\ell=1}^n$ is strictly positive. We derive a formula for G: Since $\mathfrak{M}$ is $R_0^{(1)}$-invariant, $G(z_1, z_2)$ admits a representation (4.18); we write A_1 for A. Since $\mathfrak{M}$ is $R_0^{(2)}$-invariant,

$$\begin{aligned} R_0^{(2)} G(z_1, z_2) &= \frac{G(0, z_2) - G(0, 0)}{z_2}\,(I_n - z_1 A_1)^{-1} \\ &= G(z_1, z_2) B_2 = G(0, z_2)\,(I_n - z_1 A_1)^{-1} B_2 \end{aligned}$$

for some matrix $B_2 \in \mathbb{C}^{n\times n}$ and therefore,

$$G(0, z_2) = G(0, 0)\,(I_n - z_1 A_1)^{-1}\,(I_n - z_2 B_2)^{-1}\,(I_n - z_1 A_1).$$

Substituting this $G(0, z_2)$ in (4.18) and setting $C = G(0, 0)$, we get

$$G(z_1, z_2) = C\,(I_n - z_1 A_1)^{-1}\,(I_n - z_2 B_2)^{-1}. \tag{5.5}$$

Similarly one can derive the representation

$$G(z_1, z_2) = C\,(I_n - z_2 A_2)^{-1}\,(I_n - z_1 B_1)^{-1}, \tag{5.6}$$

for some matrices B_1, $A_2 \in \mathbb{C}^{n\times n}$. If we compare the last two equalities for $z_1 = 0$ and $z_2 = 0$ we obtain the equalities

$$C\,(I_n - z_2 A_2)^{-1} \equiv C\,(I_n - z_2 B_2)^{-1} \quad \text{and} \quad C\,(I_n - z_1 A_1)^{-1} \equiv C\,(I_n - z_1 B_1)^{-1}.$$

Hence (5.6) and (5.5) can be rewritten as

$$G(z_1, z_2) = C\,(I_n - z_2 B_2)^{-1}\,(I_n - z_1 B_1)^{-1} = C\,(I_n - z_1 B_1)^{-1}\,(I_n - z_2 B_2)^{-1}. \tag{5.7}$$

This readily implies (5.3) as well as the formula 5.2 for the kernel of $\mathfrak{M}$. The space $\mathfrak{M}$ consists of all functions of the form $f(z_1, z_2) = G(z_1, z_2)x$, $x \in \mathbb{C}^n$, with the norm

$$\|f\|^2_{\mathfrak{M}} = x^*\mathbb{P}x.$$

Furthermore, for f of this form we have, on account of (5.7),

$$R_0^{(j)} f(z_1, z_2) = G(z_1, z_2)B_j x,\ j = 1, 2,$$

$$f(z_1, 0) = C\,(I_n - z_1B_1)^{-1}\,x,\ \text{ and } f(0, z_2) = C\,(I_n - z_2B_2)^{-1}\,x.$$

Therefore

$$\|R_0^{(j)} f\|^2_{\mathfrak{M}} = x^*A_j^*\mathbb{P}A_jx, \qquad j = 1, 2,$$

and

$$\|f(z_1, 0)x\|^2_{\mathbf{H}_2(\mathbb{D},\mathfrak{G})} = \sum_{j=0}^{\infty} x^*B_1^{j*}C^*CB_1^jx,$$

$$\|f(0, z_2)x\|^2_{\mathbf{H}_2(\mathbb{D},\mathfrak{G})} = \sum_{j=0}^{\infty} x^*B_2^{j*}C^*CB_2^jx,$$

where the two series converge in the matrix norm, since $f(z_1, 0)$ and $f(0, z_2)$ belong to $\mathbf{H}_2(\mathbb{D}, \mathfrak{G})$. Substituting the right sides of the last four norm equalities into (1.5) and (1.6) and taking into account that x is an arbitrary vector from $\mathbb{C}^n$, we get (5.4).

The converse can be proved in much the same way as in Theorem 4.7. □

REMARKS 5.3 (1) Note that each one of inequalities in (5.4) implies

$$\sum_{\ell,j=0}^{\infty} B_2^{j*}B_1^{\ell*}C^*CB_1^{\ell}B_2^j \leqslant \mathbb{P}. \tag{5.8}$$

This leads to the following so far unsolved problem: Characterize in terms of invariance properties an n-dimensional subspace $\mathfrak{M}$ of $\mathbf{H}_2(\mathbb{D}^2,\ \mathfrak{G})$ with the inequality (5.8) instead of the pair of inequalities in (5.4).

(2) Note also that the corresponding equalities are mutually equivalent.

6. Realizations

In this section we derive a special realization for Schur functions on the bidisk.

THEOREM 6.1 *Let $\widehat{S}(z_1, z_2) \in S(\mathbb{D}^2; \mathfrak{F}, \mathfrak{G})$, $\mathfrak{F}$ and $\mathfrak{G}$ Hilbert spaces. Then there is an $S(z) \in S(\mathbb{D}; \ell_2(\mathfrak{F}), \ell_2(\mathfrak{G}))$ such that $\mathfrak{H}(\mathbb{D}, S)$ is invariant under the backward shift T given by* (5.1),

$$\mathfrak{H}(\mathbb{D}^2, \widehat{S}) = \{g \mid g(z_1, z_2) = E(z_2)f(z_1),\ f \in \mathfrak{H}(\mathbb{D}, S)\}$$

and $\|g\|_{\mathfrak{H}(\mathbb{D}^2,\widehat{S})} = \|f\|_{\mathfrak{H}(\mathbb{D},S)}$ if $g(z_1, z_2) = E(z_2)f(z_1)$, $f \in \mathfrak{H}(\mathbb{D}, S)$.

Proof. By Theorem 2.5, $\mathfrak{H}(\mathbb{D}^2, \widehat{S})$ is a sub-Hardy space of $\mathbb{D}^2$, and by Theorems 5.1 and 4.2, there exist a Hilbert space $\mathfrak{F}_1$ and a Schur function $S_1(z) \in S(\mathbb{D}; \mathfrak{F}_1, \ell_2(\mathfrak{G}))$ such that

$$\frac{I_{\mathfrak{G}} - \widehat{S}(z_1, z_2)\widehat{S}(w_1, w_2)^*}{(1 - z_1 w_1^*)(1 - z_2 w_2^*)} = E(z_2)\frac{I_{\ell_2(\mathfrak{G})} - S_1(z_1)S_1(w_1)^*}{1 - z_1 w_1^*}E(w_2)^*. \tag{6.1}$$

Now write $\widehat{S}(z_1, z_2) = E(z_2)R(z_1)$; then $R(z)$ is a bounded operator from $\mathfrak{F}$ to $\ell_2(\mathfrak{G})$ and it is analytic in $z \in \mathbb{D}$. Define

$$S(z) = \left(R(z)\ T^* R(z)\ T^{*2} R(z),\ \cdots\right), \tag{6.2}$$

T^* being the forward shift on $\ell_2(\mathfrak{G})$. To prove the theorem, we only need to show that

(i) $S(z) \in S(\mathbb{D}; \ell_2(\mathfrak{F}), \ell_2(\mathfrak{G}))$, and
(ii) in the formula (6.1) we may replace $\mathfrak{F}_1$ by $\ell_2(\mathfrak{F})$ and $S_1(z)$ by $S(z)$.

The key to the proof of (i) is the simple observation that

$$z_2^n E(z_2)R(z_1) = E(z_2)(T^{*n} R(z_1)).$$

Indeed, if for $f = (f_0\ f_1 \ldots f_n \ldots)^T \in \ell_2(\mathfrak{F})$ and $\ell \geqslant k$ we set

$$f_{k,\ell} = (0 \ldots 0, f_k, \ldots, f_\ell, 0 \ldots)^T \in \ell_2(\mathfrak{F}),$$

then for $z_1, z_2 \in \mathbb{D}$ we have

$$\begin{aligned} E(z_2)S(z_1)f_{k,\ell} &= E(z_2)\sum_{n=k}^{\ell} T^{*n} R(z_1) f_n \\ &= E(z_2)R(z_1)\sum_{n=k}^{\ell} z_2^n f_n = \widehat{S}(z_1, z_2)\sum_{n=k}^{\ell} z_2^n f_n. \end{aligned}$$

Hence, since $\widehat{S}(z_1, z_2)$ is a contraction in $\mathfrak{L}(\mathfrak{F}, \mathfrak{G})$,

$$\left\| E(z_2)S(z_1)f_{k,\ell} \right\|_{\mathfrak{G}}^2 \leqslant \left\| \sum_{n=k}^{\ell} z_2^n f_n \right\|_{\mathfrak{F}}^2.$$

If we write $z_2 = re^{i\varphi}$, integrate both sides of this inequality over φ from 0 to 2π and then take the limit for $r \uparrow 1$, we obtain

$$\left\| S(z_1)f_{k,\ell} \right\|_{\ell_2(\mathfrak{G})}^2 \leqslant \left\| f_{k,\ell} \right\|_{\ell_2(\mathfrak{F})}^2.$$

It follows that the series in the equality

$$S(z_1)f = \sum_{n=0}^{\infty} T^{*n} R(z_1) f_n, \quad f = (f_0\ f_1 \ldots f_n \ldots)^T \in \ell_2(\mathfrak{F}),$$

converges in $\ell_2(\mathfrak{G})$ and for all $z \in \mathbb{D}$, $\|S(z)f\|_{\ell_2(\mathfrak{G})} \leqslant \|f\|_{\ell_2(\mathfrak{F})}$. This proves (i). To see (ii), we observe that

$$\begin{aligned}\frac{\widehat{S}(z_1, z_2)\widehat{S}(w_1, w_2)^*}{1 - z_2 w_2^*} &= E(z_2)\frac{R(z_1)R(w_1)^*}{1 - z_2 w_2^*}E(w_2)^* \\ &= E(z_2)\left(\sum_{n=0}^{\infty} z_2^n R(z_1)R(w_1)^* w_2^{*n}\right)E(w_2)^* \\ &= E(z_2)\left(\sum_{n=0}^{\infty} \left(T^{*n}R(z_1)\right)\left(T^{*n}R(w_1)\right)^*\right)E(w_2)^* \\ &= E(z_2)S(z_1)S(w_1)^*E(w_2)^*.\end{aligned}$$

On the other hand, on account of (6.1), we have

$$\frac{\widehat{S}(z_1, z_2)\widehat{S}(w_1, w_2)^*}{1 - z_2 w_2^*} = E(z_2)S_1(z_1)S_1(w_1)^*E(w_2)^*.$$

Thus

$$S(z_1)S(w_1)^* = S_1(z_1)S_1(w_1)^*,$$

and this implies (ii). □

As a Schur function of one variable, the function S in Theorem 6.1 admits a coisometric realization with $\mathfrak{H}(\mathbb{D}, S)$ in the role of the state space:

$$S(z) = D + zC\left(I_{\mathfrak{H}(\mathbb{D},S)} - zA\right)^{-1}B, \tag{6.3}$$

where the operators defined by the rules: for $g \in \mathfrak{H}(\mathbb{D}, S)$ and $f \in \ell_2(\mathfrak{F})$,

$$\begin{aligned}(Ag)(z) &= \frac{g(z) - g(0)}{z}, \quad Cg = g(0), \\ (Bf)(z) &= \frac{S(z) - S(0)}{z}f, \quad Df = S(0)f,\end{aligned} \tag{6.4}$$

are bounded operators such that the operator matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} : \begin{pmatrix} \mathfrak{H}(\mathbb{D}, S) \\ \ell_2(\mathfrak{F}) \end{pmatrix} \to \begin{pmatrix} \mathfrak{H}(\mathbb{D}, S) \\ \ell_2(\mathfrak{G}) \end{pmatrix} \tag{6.5}$$

is coisometric and closely outer connected, which means that

$$\mathfrak{H}(\mathbb{D}, S) = \overline{\operatorname{span}}\,\{\operatorname{ran}(1 - zA^*)^{-1}C^* | z \in \mathbb{D}\}.$$

THEOREM 6.2 *Let* $\widehat{S} \in S(\mathbb{D}^2; \mathfrak{F}, \mathfrak{G})$. *Let* $S \in S(\mathbb{D}, \ell_2(\mathfrak{F}), \ell_2(\mathfrak{G}))$ *be as in Theorem* 6.1 *and assume it has the closely outer connected coisometric realiza-*

tion (6.3). *Define the operators* $\mathbf{A}, \mathbf{B}, \mathbf{C}$ *and* $\mathbf{D}$ *by the rules: for* $g \in \mathfrak{H}(\mathbb{D}, S)$ *and* $f \in \ell_2(\mathfrak{F})$,

$$\begin{aligned} \mathbf{A}E(z_2)g(z_1) &= E(z_2)(Ag)(z_1), & (6.6)\\ \mathbf{B}E(z_2)f &= E(z_2)(Bf)(z_1), & (6.7)\\ \mathbf{C}E(z_2)g(z_1) &= E(z_2)(Cg), & (6.8)\\ \mathbf{D}E(z_2)f &= E(z_2)Df. & (6.9) \end{aligned}$$

Then the operator

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} : \begin{pmatrix} \mathfrak{H}(\mathbb{D}^2, \widehat{S}) \\ \mathbf{H}_2(\mathbb{D}, \mathfrak{F}) \end{pmatrix} \to \begin{pmatrix} \mathfrak{H}(\mathbb{D}^2, \widehat{S}) \\ \mathbf{H}_2(\mathbb{D}, \mathfrak{G}) \end{pmatrix}$$

is coisometric and represents the operator $\mathbf{S}_1$ *defined by* (4.11) *as*

$$(\mathbf{S_1}E(z_2)f)\,(z_1) = \big(\mathbf{D} + z_1\mathbf{C}(I - z_1\mathbf{A})^{-1}\mathbf{B}\big)\,(E(z_2)f), \quad f \in \ell_2(\mathfrak{F}). \quad (6.10)$$

In particular, for $h \in \mathfrak{F}$,

$$\widehat{S}(z_1, z_2)h = \big(\mathbf{D} + z_1\mathbf{C}(I - z_1\mathbf{A})^{-1}\mathbf{B}\big)\,h.$$

Proof. The representation (6.10) and that it comes from a coisometric operator matrix follow easily from the coisometric representation for $S(z)$, and is omitted. The last equality follows from (6.10) if we take $f = (h\ 0\ \underline{0} \dots)^T \in \ell_2(\mathfrak{F})$ and observe that $\mathbf{S_1}(E(z_2)f) = E(z_2)S(z_1)f = E(z_2)R(z_1)h = \widehat{S}(z_1, z_2)h$. □

REMARKS 6.3 (1) In some sense, the constants in the above realization are the elements of $\mathbf{H}_2(\mathbb{D})$. Similar realizations in spirit (but the parallel is deeper) appear in the setting of upper triangular operators in [8].
(2) Theorem 6.2 in particular applies to functions such that the kernel (1.7) is positive in the bidisk. It is of interest to connect the present colligation to the ones associated by Agler in the setting of this class of functions.

7. Shift–Invariant Subspaces

In this section we prove the following corollary of Theorem 3.2 about shift invariant subspaces in the Hardy space of the bidisk. We write $\mathbf{H}_2^p(\mathbb{D}^2)$ for $\mathbf{H}_2(\mathbb{D}^2, \mathbb{C}^p)$, $p \in \mathbb{N}$, and $\mathbb{T}$ for the unit circle $\partial\mathbb{D}$.

THEOREM 7.1 *For* $n \in \mathbb{N}$, $(a_1, b_1), (a_2, b_2), \dots, (a_n, b_n) \in \mathbb{D}^2$, *and* $x_1, x_2, \dots, x_n \in \mathbb{C}^p$, *set*

$$\mathfrak{M} = \{f \in \mathbf{H}_2^p(\mathbb{D}^2) \mid x_i^* f(a_i, b_i) = 0, i = 1, 2, \dots, n\}.$$

Then there exists no $p \times q$ *matrix valued Schur function* $S(z_1, z_2)$ *which takes isometric values on* $\mathbb{T}^2$ *such that*

$$\mathfrak{M} = S\mathbf{H}_2^q(\mathbb{D}^2). \tag{7.1}$$

Proof. Evidently, $\mathfrak{M}^\perp$ is spanned by the linearly independent functions

$$f_j(z_1, z_2) = \frac{x_j}{(1 - z_1 a_j^*)(1 - z_2 b_j^*)}, \quad j = 1, 2, \ldots n,$$

and hence $\dim \mathfrak{M}^\perp = n$. Assume there exists a $p \times q$ matrix valued Schur function $S(z_1, z_2)$ which takes isometric values on $\mathbb{T}^2$ such that (7.1) holds, that is, such that

$$\mathfrak{M}^\perp = \mathbf{H}_2^p(\mathbb{D}^2) \ominus S\mathbf{H}_2^q(\mathbb{D}^2).$$

We claim that then $\mathfrak{M}^\perp = \mathfrak{H}(\mathbb{D}^2, S)$, and so $\dim \mathfrak{H}(\mathbb{D}^2, S) = n$, which is impossible since by Theorem 3.2, such spaces are either trivial or infinite dimensional. This contradiction proves the theorem. It remains to show the claim, or equivalently, that the space $\mathfrak{M}^\perp$ has reproducing kernel K_S defined by (1.7):

$$K_S(w_1, w_2; z_1, z_2) = \frac{I - S(z_1, z_2)S(w_1, w_2)^*}{(1 - z_1 w_1^*)(1 - z_2 w_2^*)}.$$

The proof is similar to the one variable case, and is recalled for completeness. We first show that the function $(z_1, z_2) \mapsto K_S(w_1, w_2; z_1, z_2)x$ belongs to $\mathbf{H}_2^p(\mathbb{D}^2) \ominus S\mathbf{H}_2^q(\mathbb{D}^2)$ for every $x \in \mathbb{C}^p$ and $(w_1, w_2) \in \mathbb{D}^2$. For $f = Su$ with $u \in \mathbf{H}_2^p(\mathbb{D}^2)$ we have

$$\begin{aligned}
&\langle Su, \frac{I - S(z_1, z_2)S(w_1, w_2)^*}{(1 - z_1 w_1^*)(1 - z_2 w_2^*)} x\rangle_{\mathbf{H}_2^p(\mathbb{D}^2)} \\
&= \langle Su, \frac{x}{(1 - z_1 w_1^*)(1 - z_2 w_2^*)}\rangle_{\mathbf{H}_2^p(\mathbb{D}^2)} \\
&-\langle Su, \frac{S(z_1, z_2)S(w_1, w_2)^*}{(1 - z_1 w_1^*)(1 - z_2 w_2^*)} x\rangle_{\mathbf{H}_2^p(\mathbb{D}^2)} \\
&= x^* S(w_1, w_2)u(w_1, w_2) - \langle u, \frac{S(w_1, w_2)^*}{(1 - z_1 w_1^*)(1 - z_2 w_2^*)} x\rangle_{\mathbf{H}_2^p(\mathbb{D}^2)} \\
&= 0.
\end{aligned}$$

The second equality is true because S is assumed to be isometric on $\mathbb{T}^2$. This proves $K_S(w_1, w_2; \cdot, \cdot)x \in \mathfrak{M}^\perp$. Next we show the reproducing kernel property: We have for $f \in \mathbf{H}_2^p(\mathbb{D}^2) \ominus S\mathbf{H}_2^q(\mathbb{D}^2)$,

$$\begin{aligned}
\langle f, K_S(w_1, w_2; z_1, z_2)x\rangle_{\mathbf{H}_2^p(\mathbb{D}^2)} &= \langle f, \frac{x}{(1 - z_1 w_1^*)(1 - z_2 w_2^*)}\rangle_{\mathbf{H}_2^p(\mathbb{D}^2)} \\
&= x^* f(w_1, w_2).
\end{aligned}$$

Thus $\mathfrak{M}^\perp = \mathfrak{H}(\mathbb{D}^2, S)$, and the claim is proved. □

Observe that $\mathfrak{M}^{\perp}$ is an example of a sub-Hardy Hilbert space for which (1.5) and (1.6) are satisfied as equalities.

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