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Stabilization of Nonlinear RLC Circuits: Power Shaping and Passivation

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Abstract— **In this paper we prove that for a class of RLC circuits with convex energy function and weak electromagnetic coupling it is possible to "add a differentiation" to the port terminals preserving passivity—with a new storage function that is directly related to the circuit power. The result is of interest in circuits theory, but also has applications in control problems as it suggests the paradigm of power shaping stabilization as an alternative to the well-known method of energy shaping. We show in the paper that, in contrast with energy shaping designs, power shaping is not restricted to systems without pervasive dissipation and naturally allows to add "derivative" actions in the control. These important features, that stymie the applicability of energy shaping control, make power shaping very practically appealing, as illustrated with examples in the paper. To establish our results we exploit the geometric property that voltages and currents in RLC circuits live in orthogonal spaces, i.e., Tellegen's theorem, and heavily rely on the seminal paper of Brayton and Moser in 1964.**

Note: This paper is an abridged version of (Ortega *et al.*, 2003).

I. Introduction

In this paper we are interested in (possibly nonlinear) RLC circuits consisting of arbitrary interconnections of resistors, inductors, capacitors and voltage and current sources. It is well-known that, if the resistors, inductors and capacitors are passive, i.e., if their energy functions are positive, then the overall interconnected circuit is also passive with port variables the external sources voltages and currents, and storage function the total stored energy (Desoer and Kuh, 1969). This property was exploited by Youla in 1959 (Youla *et al.*, 1959) who proved that terminating the port variables of a passive RLC circuit with a passive resistor would ensure that "finite energy inputs will be mapped into finite energy outputs," what in modern parlance says that adding damping injection to a passive system ensures $\mathcal{L}_2$–stability. Passivity can also be used to stabilize a non–zero equilibrium point, but in this case we must modify the storage function to assign a minimum at this point. If the storage function is the total energy we refer to this step as energy shaping, which combined with damping injection constitute the two main stages of passivity–based control (PBC) (Ortega and Spong, 1989). As explained in (Ortega *et al.*, 1998) there are several ways to achieve energy shaping, the most physically appealing being the so–called energy balancing PBC (or control by interconnection) method. With this procedure the storage function assigned to the closed–loop passive map is the difference between the total energy of the system and the energy supplied by the controller, hence the name energy balancing. Unfortunately, energy balancing PBC is stymied by the presence of pervasive dissipation, that is, the existence of resistive elements whose power does not vanish at the desired equilibrium point. Another practical drawback of energy–shaping control is the limited ability to "speed up" the transient response (preserving, of course, a provable stable behavior.) Indeed, as tuning in this kind of controllers is essentially restricted to the damping injection gain, the transients may turn out to be somehow sluggish, and the overall performance level below par.

Our main contribution in this paper is the establishment of a new passivity property for a class of RLC circuits that provides the basis for a novel PBC design methodology that does not suffer from the two aforementioned drawbacks. To define the class, we assume that the energy of the inductors and capacitors are not just positive but actually *convex* functions, and assume that the electromagnetic coupling between the dynamic elements is weak. Indeed, for the case of RC or RL circuits this condition is conspicuous by its absence—as already reported in (Ortega and Shi, 2002).

The new passivity property, which is by itself of interest in circuits theory, has two key features that makes it attractive for control design as well. First, that the storage function is not the total energy, but a function directly related with the *power* in the circuit. Second, that the port variables of the new passive system include *derivatives* of the sources voltages and/or currents. The utilization of power (instead of energy) storage functions immediately suggests the paradigm of power shaping stabilization as an alternative to the well–known method of energy shaping. We show in the paper that, in contrast with energy shaping designs, power shaping is applicable also to systems with pervasive dissipation, the only restriction for stabilization being the degree of underactuation of the circuit. Further, establishing passivity with respect to "differentiated" port variables allows the direct incorporation of (approximate) derivative actions, whose predictive nature can speed-up the transient response.

The remaining of the paper is organized as follows. In Section II we briefly review the method of energy balancing passivity-based control (EB-PBC). Next, in Section III, a simple RL-circuit example is presented to motivate the concept of stabilization via power shaping. To generalize the ideas to a broad class of RLC we need some prelimi-

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nary material from the ground breaking paper (Brayton and Moser, 1964), that is introduced in Section IV. Finally, we present the main result in Section V.

II. Energy balancing passivity–based control

In (Ortega *et al.*, 2001) a new method is presented to stabilize the following class of nonlinear systems—that includes passive systems.

Definition 1: We say that the m–port system

$$\begin{array}{rcl}\dot{\mathbf{x}} &=& \mathbf{f}(\mathbf{x})+\mathbf{g}(\mathbf{x})\mathbf{u}\\ \mathbf{y} &=& \hat{\mathbf{y}}(\mathbf{x}),\end{array} \tag{1}$$

with state $\mathbf{x} = \mathrm{col}(x_1,\ldots,x_n) \in \mathbb{R}^n$, and power port variables $\mathbf{u},\mathbf{y} \in \mathbb{R}^m$, satisfies the energy balance inequality if, along all trajectories compatible with $\mathbf{u}:[0,t]\rightarrow\mathbb{R}^m$, we have

$$\underbrace{\mathcal{E}[\mathbf{x}(t)]-\mathcal{E}[\mathbf{x}(0)]}_{\text{stored energy}} \leq \underbrace{\int_0^t \mathbf{u}^\top(t')\hat{\mathbf{y}}[\mathbf{x}(t')]dt'}_{\text{supplied energy}} \tag{2}$$

where $\mathcal{E}:\mathbb{R}^n\rightarrow\mathbb{R}$ is the stored energy function. If $\mathcal{E}(\mathbf{x})$ is positive semidefinite then we say that the system is *passive* with port variables $(\mathbf{u},\mathbf{y})$.

The proposition below, established in (Ortega *et al.*, 2001), constitutes the basis for energy–balancing PBC. (For simplicity, we present only the case of static state feedback, the case of dynamic controllers may be found in (Ortega *et al.*, 2001).)

Proposition 1: Consider m–port systems that satisfy the energy balance equation (2). If we can find a vector function $\hat{\mathbf{u}}(\mathbf{x}):\mathbb{R}^n\rightarrow\mathbb{R}^m$ such that the partial differential equation[1]

$$\nabla^\top\mathcal{E}_a(\mathbf{x})[\mathbf{f}(\mathbf{x})+\mathbf{g}(\mathbf{x})\hat{\mathbf{u}}(\mathbf{x})] = -\hat{\mathbf{y}}^\top(\mathbf{x})\hat{\mathbf{u}}(\mathbf{x}), \tag{3}$$

can be solved for the scalar function $\mathcal{E}_a:\mathbb{R}^n\rightarrow\mathbb{R}$, and the function $\mathcal{E}_d(\mathbf{x}) := \mathcal{E}(\mathbf{x})+\mathcal{E}_a(\mathbf{x})$ has an isolated minimum at $\mathbf{x}^\star$, then the state–feedback $\mathbf{u}=\hat{\mathbf{u}}(\mathbf{x})$ is an energy balancing PBC, i.e., $\mathbf{x}^\star$ is a stable equilibrium with the difference between the stored and the supplied energies constituting a Lyapunov function.

This result, although quite general, is of limited interest. First of all, these kind of state models do not reveal the role played by the energy function in the system dynamics. Hence it is difficult to incorporate prior information to select a $\hat{\mathbf{u}}(x)$ to solve the PDE (3). In (Ortega *et al.*, 2002) energy balancing PBC is developed for a more suitable class of models, the so–called port–controlled Hamiltonian systems, that explicitly exhibit the existence of dynamic invariants. Second, and perhaps more importantly, it is shown in (Ortega *et al.*, 1998) that, beyond the realm of mechanical systems, the applicability of energy balancing control is severely stymied by the system's natural dissipation. Indeed, it is easy to see that a necessary condition for the *global* solvability of the PDE (3) is that $\hat{\mathbf{y}}^\top(\mathbf{x})\hat{\mathbf{u}}(\mathbf{x})$ vanishes at all the zeros of $\mathbf{f}(\mathbf{x})+\mathbf{g}(\mathbf{x})\hat{\mathbf{u}}(\mathbf{x})$, that is, the implication

$$\mathbf{f}(\bar{\mathbf{x}})+\mathbf{g}(\bar{\mathbf{x}})\hat{\mathbf{u}}(\bar{\mathbf{x}})=0 \Rightarrow \hat{\mathbf{y}}^\top(\bar{\mathbf{x}})\hat{\mathbf{u}}(\bar{\mathbf{x}})=0 \tag{4}$$

should hold. Now, $\mathbf{f}(\mathbf{x})+\mathbf{g}(\mathbf{x})\hat{\mathbf{u}}(\mathbf{x})$ is obviously zero at the equilibrium $\mathbf{x}^\star$, hence the power extracted from the controller should also be zero at the equilibrium. This means that energy balancing PBC is applicable only if the system does not have pervasive damping, i.e., if it can be stabilized extracting a finite amount of energy from the controller. This is the case in regulation of mechanical systems where the extracted power is the product of force and velocity and we want to drive the velocity to zero. Unfortunately, it is no longer the case for most electrical or electromechanical systems where power involves the product of voltages and currents and the latter may be nonzero for nonzero equilibria. For instance, a series RC circuit is energy–balancing stabilizable (because in steady state there is no current drained from the source), but not an RL circuit—see the following section.

Remark 1: For linear systems it is, of course, possible to overcome the dissipation obstacle by shifting the equilibrium of the systems equation to zero. As the terms dependent on $\mathbf{x}^\star,\mathbf{u}^\star$ cancel in the incremental model, the original (quadratic) storage function—but expressed now in terms of the incremental variables—qualifies as a storage function for the shifted model. Unfortunately, this simple solution is not applicable for the nonlinear case, as there is no systematic procedure to generate, from the knowledge of $\mathcal{E}(\mathbf{x})$, a storage function for the "input–shifted" system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})+\mathbf{g}(\mathbf{x})\mathbf{u}^\star+\mathbf{g}(\mathbf{x})\mathbf{w},\ \ \mathbf{y}=\hat{\mathbf{y}}(\mathbf{x}),$$

with $\mathbf{w} := \mathbf{u}-\mathbf{u}^\star$, and $(\mathbf{w},\mathbf{y})$ the new port variables. As shown in (Maschke *et al.*, 2000) the natural solution of adding to $\mathcal{E}(\mathbf{x})$ a term $-\int_0^t \mathbf{w}^\top(t')\hat{\mathbf{y}}[\mathbf{x}(t')]dt'$ is also restricted to systems without pervasive damping.

III. Towards Power Shaping Control

Let us illustrate with an example how the limitations of energy balancing PBC can be overcome via power balancing. Consider a voltage–controlled nonlinear series RL circuit. The behavior of the inductor is characterized by a function, $p_L = \hat{p}_L(i_L)$, relating the flux linkages p_L and the current i_L, and Faraday's law: $\dot{p}_L = v_L$, where v_L is the inductor voltage. The resistor is a static element described by its characteristic function $v_R = \hat{v}_R(i_R)$, where v_R, i_R are the resistors voltage and current, respectively. The dynamics of the circuit is obtained from Kirchhoff's voltage law as

$$v_L = L(i_L)\frac{di_L}{dt} = -\hat{v}_R(i_L)+v_S, \tag{5}$$

where v_S is the voltage at the port terminal, which is our control action. Furthermore, we have that $i_R = i_L$, and the property $L(i_L) := \nabla_{i_L}\hat{p}_L(i_L)$. Differentiating the inductor's energy $\mathcal{E}_L(p_L)$ we obtain

$$\begin{array}{rcl}\dot{\mathcal{E}}_L(p_L) &=& \nabla_{p_L}\mathcal{E}_L(p_L)\dot{p}_L\ (= i_L v_L)\\ &=& i_S v_S - i_R\hat{v}_R(i_R),\end{array}$$

[1] We use the notation $\nabla_x := \partial/\partial x, \nabla_x^2 := \partial^2/\partial x^2$. When clear from the context the argument will be omitted. Also, all vectors, including the gradient, are column vectors.

where, to obtain the last equation, we used the fact that i_S, the port current, is equal to i_L. If we assume that the resistor is passive, that is, that the energy that it dissipates is nonnegative, i.e., $\int_0^t i_R(t')\hat{v}_R[i_R(t')]dt' \geq 0$, and integrate from 0 to t, we recover the energy balance inequality (2). If we further assume that the inductor is also passive—that is, its stored energy is nonnegative—we verify that the circuit defines a passive system with port variables (v_S, i_S) and storage function $\mathcal{E}_L(p_L)$.

We define as control objective the stabilization of an equilibrium $i_L^\star$ of (5), whose corresponding equilibrium supply voltage is given by $v_S^\star = \hat{v}_R(i_L^\star)$. If we further assume that the function $\hat{v}_R(i_R)$ is zero only at zero, it is clear that, at any equilibrium $i_L^\star \neq 0$, the extracted power $i_L^\star \hat{v}_R(i_L^\star)$ is nonzero, hence the circuit is not energy–balancing stabilizable—not even in the linear case!

To overcome this problem let us define the function

$$F(i_R) := \int_0^{i_R} \hat{v}_R(i_R')di_R', \tag{6}$$

known in the circuits literature (Millar, 1951) as the resistors *content*, which has units of power—in particular, for linear resistors it is half the dissipated power. Furthermore, notice that for passive resistors the function is nonnegative. Summarizing, we have the following result.

Proposition 2: Consider a series RL circuit. If the inductor is passive and has a twice differentiable *convex* energy function, that is,

$$\nabla^2_{p_L}\mathcal{E}_L(p_L) \geq 0,$$

then, along the trajectories of the system, we have the power balance inequality

$$F[i_L(t)] - F[i_L(0)] \leq \int_0^t v_S^\top(t')\frac{di_S}{dt'}(t')dt'. \tag{7}$$

Furthermore, if the resistor is passive, then the circuit defines a passive system with port variables $(v_S, \frac{di_S}{dt})$ and storage function the resistor content.

The properties of Proposition 2 differ from the classical energy balancing and passivity properties in two important respects: the presence of the derivative of i_S and the use of a new power-like storage function. These two properties suggest, similarly to energy balancing PBC, to shape the resistors content. That is, to look for functions $\hat{v}_S(i_L), F_a(i_L)$ such that

$$\dot{F}_a(i_L) \equiv -\hat{v}_S(i_L)\frac{di_L}{dt}. \tag{8}$$

If we furthermore ensure that

$$i_L^\star = \arg\min\{F(i_L) + F_a(i_L)\},$$

then i_{L*} will be a stable equilibrium with Lyapunov function $F_d(i_L) := F(i_L) + F_a(i_L)$, that is, the system is stabilized via power shaping!

Clearly, for any choice of $F_a(i_L)$, (8) is trivially solved with the control $v_S = \hat{v}_S(i_L)$, where

$$\hat{v}_S(i_L) = -\nabla_{i_L}F_a(i_L).$$

If the resistance characteristic is exactly known we can take $F_a(i_L) = -F(i_L) + \frac{R_a}{2}(i_L - i_L^\star)^2$, with $R_a > 0$ some tuning parameter. But clearly, we only need to "dominate" $F(i_L)$ to assign the desired minimum, which (together with the fact that $L(i_L)$ is completely unknown) exhibits the robustness of the design procedure.

Detailed proofs for general RL and RC circuits can be found in (Jeltsema *et al.*, 2003; Ortega and Shi, 2002). An important observation, that will be proved for more general nonlinear RLC circuits in the following section, is that we can express the circuit dynamics (5) in terms of the resistor content as

$$L(i_L)\frac{di_L}{dt} = -\nabla_{i_L}F(i_L) + v_S.$$

The identification of a gradient–like description of RLC circuits is the main contribution of the seminal paper (Brayton and Moser, 1964).

IV. Passivity of Brayton-Moser Circuits

The previous developments show that, using the content (resp. co-content in the RC case (Jeltsema *et al.*, 2003; Ortega and Shi, 2002)) as a storage function, we can identify new passivity properties of RL (resp. RC) circuits. In this section we will establish similar properties for RLC circuits. Towards this end, we strongly rely on some fundamental results reported in (Brayton and Moser, 1964). Furthermore, we assume that the current-controlled resistors are contained in Σ_L and the voltage-controlled resistors are contained in Σ_C. The class of RLC considered here is then composed by an interconnection of Σ_L and Σ_C. For a detailed derivation, see (Ortega *et al.*, 2003).

A. Brayton and Moser's Equations

In the early sixties Brayton and Moser (Brayton and Moser, 1964) have shown that the dynamic behavior of a topologically complete circuit (where we restrict, for simplicity, to circuits having only voltage sources in series with the inductors) is governed by the following differential equation

$$\mathbf{Q}(\mathbf{x})\dot{\mathbf{x}} = \nabla_{\mathbf{x}}P(\mathbf{x}) - \mathbf{B}\mathbf{v}_S \tag{9}$$

where $\mathbf{x} = \text{col}(\mathbf{i}_L, \mathbf{v}_C)$, $\mathbf{B} = \text{col}(\mathbf{B}_S, 0)$ with $\mathbf{B}_S \in \mathbb{R}^{n_L \times n_s}$, $\mathbf{Q}(\mathbf{x}) = \text{diag}(-\mathbf{L}(\mathbf{i}_L), \mathbf{C}(\mathbf{v}_C)) \in \mathbb{R}^{n \times n}$, $n = n_L + n_C$, and $P : \mathbb{R}^n \to \mathbb{R}$ is called the mixed-potential and is given by

$$P(\mathbf{x}) = \mathbf{i}_L^\top \mathbf{\Gamma}\mathbf{v}_C + F(\mathbf{i}_L) - G(\mathbf{v}_C), \tag{10}$$

where $\Gamma \in \mathbb{R}^{n_L \times n_C}$ is a (full rank) matrix that captures the interconnection structure between the inductors and capacitors. The functions $F(\mathbf{i}_L)$ and $G(\mathbf{v}_C)$ are the resistors content $F(i_L)$ (like in (6)) and co-content $G(v_C)$ having the form

$$G(v_R) := \int_0^{v_R} \hat{i}_R(v_R')dv_R', \tag{11}$$

respectively.

B. Generation of New Storage Function Candidates

Let us next see how the Brayton-Moser equations (9) can be used to generate storage functions for RLC circuits. From (9) we have that

$$\dot{P}(\mathbf{x}) = \dot{\mathbf{x}}^\top \mathbf{Q}(\mathbf{x})\dot{\mathbf{x}} + \dot{\mathbf{x}}^\top \mathbf{B}\mathbf{v}_S. \qquad (12)$$

Compare the latter with the right-hand side of (7) of Proposition 2 (notice that $\dot{\mathbf{x}}^\top \mathbf{B}\mathbf{v}_S = \mathbf{i}_S^\top \mathbf{v}_S$). Unfortunately, even under the reasonable assumption that the inductor and capacitor have convex energy functions, the presence of the negative sign in the first main diagonal block of $\mathbf{Q}(\mathbf{x})$ makes the quadratic form sign–indefinite, and not negative (semi–)definite as desired. Hence, we cannot establish a power-balance inequality from (12). Moreover, to obtain the passivity property an additional difficulty stems from the fact that $P(\mathbf{x})$ is also not sign-definite. To overcome these difficulties we borrow inspiration from (Brayton and Moser, 1964) and look for other suitable pairs, say $\mathbf{Q}_A(\mathbf{x})$ and $P_A(\mathbf{x})$, which we call *admissible*, that preserve the form of (9). More precisely, we want to find matrix functions $\mathbf{Q}_A(\mathbf{x})$ verifying

$$\mathbf{Q}_A^\top(\mathbf{x}) + \mathbf{Q}_A(\mathbf{x}) \leq 0, \qquad (13)$$

and scalar functions $P_A : \mathbb{R}^n \to \mathbb{R}$ (if possible, positive semi-definite) that describe the same dynamics as (9). If (13) holds, it is clear that $\dot{P}_A(\mathbf{x}) \leq \dot{\mathbf{x}}^\top \mathbf{B}\mathbf{v}_S$, from which we obtain a power balance equation with the desired port variables. Furthermore, if $P_A(\mathbf{x})$ is positive semi-definite we are able to establish the required passivity property.

A complete characterization of the admissible pairs $(\mathbf{Q}_A, P_A)$ has been reported in (Ortega *et al.*, 2002), but it requires the solution of a partial differential equation. A more constructive procedure to generate admissible pairs is given in the following proposition which, for ease of reference, is enunciated in terms of the original RLC circuit data.[2] For ease of notation, we write (9) in the more compact form

$$\mathbf{Q}(\mathbf{x})\dot{\mathbf{x}} = \nabla_{\mathbf{x}}\tilde{P}(\mathbf{x}), \qquad (14)$$

where $\tilde{P}(\mathbf{x}) = P(\mathbf{x}) - \mathbf{x}^\top \mathbf{B}\mathbf{v}_S$.

Proposition 3: Consider a complete RLC circuit with regulated voltage sources in series with the inductors. Assume that the energy functions of the dynamic elements are *strictly* convex, *i.e.*, $\nabla^2_{\mathbf{v}_C}\mathcal{E}_C(\mathbf{v}_C), \nabla^2_{\mathbf{i}_L}\mathcal{E}_L(\mathbf{i}_L) > 0$. Then,

(i) (Sufficiency) For all $\lambda \in \mathbb{R}$, and symmetric matrix functions $\mathbf{M}(\mathbf{i}_L, \mathbf{v}_C)$, with $\mathbf{M} : \mathbb{R}^n \to \mathbb{R}^{n\times n}$, the pair

$$\tilde{P}_A := \lambda\tilde{P}_A + \frac{1}{2}\nabla^\top\tilde{P}_A\mathbf{M}\nabla^\top\tilde{P}_A \qquad (15)$$

$$\mathbf{Q}_A := \left[\frac{1}{2}(\nabla^2 P_A)\mathbf{M} + \frac{1}{2}\nabla(\mathbf{M}\nabla P_A) + \lambda\mathbf{I}\right]\mathbf{Q} \qquad (16)$$

is admissible, *i.e.*, is such that the time integrals of

$$\mathbf{Q}_A(\mathbf{x})\dot{\mathbf{x}} = \nabla_{\mathbf{x}}\tilde{P}_A(\mathbf{x})$$

coincide with the time integrals of (14).

(ii) (Partial converse) Assume the circuit (9) admits only isolated equilibrium points. Then, given any admissible pair $(\mathbf{Q}_A, \tilde{P}_A)$ there exists λ, and $\mathbf{M}$ such that, almost everywhere,[3] $\tilde{P}_A$ takes the form (15).

Proof: See (Ortega *et al.*, 2003). ∎

An important observation regarding Proposition 3 is that, for suitable choices of λ and $\mathbf{M}$, we can now try to generate a matrix $\mathbf{Q}_A(\mathbf{x})$ with the required negativity property (13).

Remark 2: Some simple calculations show that a change of (state) coordinates on the dynamical system (14) acts as a similarity transformation on $\mathbf{Q}$. Therefore, is of no use for our purposes where we want to change the sign of $\mathbf{Q}$ to render the quadratic form sign–definite.

C. Power Balance Inequality

Before we present our main result we first remark that in order to preserve the port variables $(\mathbf{v}_S, \frac{d\mathbf{i}_S}{dt})$, we must ensure that the transformed dynamics can be expressed in the form (9), which is equivalent to requiring that $\tilde{P}_A(\mathbf{x}) = P_A(\mathbf{x}) - \mathbf{x}^\top \mathbf{B}\mathbf{v}_S$. This naturally restricts the freedom in the choices for λ and $\mathbf{M}$ in Proposition 3.

Theorem 1: Consider a (possibly nonlinear) RLC circuit described by (9). Assume:

A.1 The inductors and capacitors are passive and have strictly convex energy functions.

A.2 The voltage-controlled resistors in Σ_C are passive, linear and time-invariant. Also, $\det(\mathbf{R}_C) \neq 0$, and thus $G(\mathbf{v}_C) = \frac{1}{2}\mathbf{v}_C^\top\mathbf{R}_C^{-1}\mathbf{v}_C \geq 0$.

A.3 Uniformly in $\mathbf{x}$ we have

$$\|\mathbf{C}^{\frac{1}{2}}(\mathbf{v}_C)\mathbf{R}_C\mathbf{\Gamma}^\top\mathbf{L}^{-\frac{1}{2}}(\mathbf{i}_L)\| < 1,$$

where $\|\cdot\|$ denotes the spectral norm of a matrix.

Under these conditions, we have the following power balance inequality

$$P_A[\mathbf{x}(t)] - P_A[\mathbf{x}(0)] \leq \int_0^t \mathbf{v}_S^\top(t')\frac{d\mathbf{i}_S}{dt'}(t')dt', \qquad (17)$$

where the transformed mixed-potential function is defined as

$$P_A(\mathbf{x}) = F(\mathbf{i}_L) + \tfrac{1}{2}\mathbf{i}_L\mathbf{\Gamma}\mathbf{R}_C\mathbf{\Gamma}^\top\mathbf{i}_L + \tfrac{1}{2}(\mathbf{\Gamma}^\top\mathbf{i}_L - \mathbf{R}_C^{-1}\mathbf{v}_C)^\top\mathbf{R}_C(\mathbf{\Gamma}^\top\mathbf{i}_L - R_C^{-1}\mathbf{v}_C).$$

If, furthermore

A.4 The current-controlled resistors are passive,

then, the circuit defines a passive system with port variables $(\mathbf{v}_S, \frac{d\mathbf{i}_S}{dt})$ and storage function the transformed mixed-potential $P_A(\mathbf{x})$.

Proof: The proof consists in first defining the parameters λ and $\mathbf{M}$ of Proposition 3 so that, under the conditions A.1–A.4 of the theorem, the resulting $\mathbf{Q}_A(\mathbf{x})$ satisfies (13) and $P_A(\mathbf{x})$ is a positive semi-definite function. First, notice

[2] To simplify the notation, in the sequel we omit the arguments of the functions, writing them explicitly only when the function is first defined.

[3] As shown in the proof, the qualifier (a.e.) stands for the existence of possible singular points. These points can be avoided with standard regularization procedures, but is omitted here for brevity.

that under assumption A.2 the co-content is linear and quadratic. To ensure that $P(\mathbf{x})$ is linear in v_S, as is required to preserve the desired port variables, we may select $\lambda = 1$ and $\mathbf{M} = \text{diag}(0, 2\mathbf{R}_C)$. Now, using (16) we obtain after some straight forward calculations

$$\mathbf{Q}_A(\mathbf{x}) = \begin{bmatrix} -\mathbf{L}(\mathbf{i}_L) & 2\mathbf{\Gamma}\mathbf{R}_C\mathbf{C}(\mathbf{v}_C) \\ 0 & -\mathbf{C}(\mathbf{v}_C) \end{bmatrix}.$$

Assumption A.1 ensures that $\mathbf{L}(\mathbf{i}_L)$ and $\mathbf{C}(\mathbf{v}_C)$ are positive definite. Hence, a Schur complement analysis proves that, under Assumption A.3, (17) holds. This proves the power balance inequality. Passivity follows from the fact that, under Assumption A.2 and A.4, the mixed-potential function $P_A(\mathbf{x})$ is positive semi-definite for all $\mathbf{x}$. ■

Remark 3: Assumption A.3 is satisfied if the voltage-controlled resistances in $\mathbf{R}_C$ are 'small'. Recalling that these resistors are contained in Σ_C, this means that the coupling between Σ_L and Σ_C, that is, the coupling between the inductors and capacitors, is weak.

V. Stabilization via Power Shaping

The theorem below proves that complete RLC circuits with strictly convex energy function and linear voltage controlled resistors are stabilizable via power-shaping—*without* requiring Assumptions A.3 or A.4—but only provided that the number of control signals is "sufficiently large" to shape the mixed potential function and add the damping.

*Theorem 2 (*Stabilization via power shaping): Consider a complete RLC circuit satisfying Assumptions A.1 and A.2 of Theorem 1, and a desired (admissible) equilibrium $(\mathbf{i}_L^*, \mathbf{v}_C^*) \in \mathbb{R}^n$. Assume there exists a function $P_a : \mathbb{R}^{n_L} \to \mathbb{R}$ verifying:

A.5 *(Realizability)* $\mathbf{B}_S^{\perp}\nabla P_a = 0$, where $\mathbf{B}_S^{\perp}\mathbf{B}_S = 0$.

A.6 *(Equilibrium assignment)* $\nabla P_a(\mathbf{i}_L^*) + \nabla_{\mathbf{i}_L}F(\mathbf{i}_L^*) + \mathbf{\Gamma}\tilde{\mathbf{R}}_C\mathbf{\Gamma}^\top\mathbf{i}_L^* = 0$.

A.7 *(Damping injection)* Uniformly in $\mathbf{i}_L$, $\nabla^2 P_a + \nabla^2_{\mathbf{i}_L}F \geq R_a\mathbf{I}$, for some sufficiently large $R_a > 0$.

Under these conditions, the circuit is stabilizable via *power-shaping*. More precisely, the control law

$$\mathbf{v}_S = -\left(\mathbf{B}_S^\top\mathbf{B}_S\right)^{-1}\mathbf{B}_S^\top\nabla P_a \tag{18}$$

ensures that all *bounded* trajectories satisfy

$$\lim_{t\to\infty}(\mathbf{i}_L(t), \mathbf{v}_C(t)) = (\mathbf{i}_L^*, \mathbf{v}_C^*).$$

Furthermore, if the characteristic functions of the dynamic elements are such that $(\mathbf{p}_L, \mathbf{q}_C) = (\hat{\mathbf{p}}_L(\mathbf{i}_L), \hat{\mathbf{q}}_C(\mathbf{v}_C))$ is a global diffeomorphism then all trajectories are bounded and the equilibrium is *globally attractive*.

Proof: The circuit dynamics are described by (9) and (10). Now, under Assumption A.5, the control law (18) satisfies $\mathbf{B}_S\mathbf{v}_S = -\nabla P_a$. This leads to the closed-loop dynamics

$$\mathbf{Q}\frac{d}{dt}\begin{bmatrix} \mathbf{i}_L \\ \mathbf{v}_C \end{bmatrix} = \nabla P_d,$$

where $P_d(\mathbf{i}_L, \mathbf{v}_C) := P + P_a$. From Assumption A.1 we have that $\mathbf{Q}$ is full rank and consequently the equilibria are the extrema of P_d. Furthermore, from (10) and Assumption A.2 we have that

$$\nabla P_d = \begin{bmatrix} \mathbf{\Gamma}\mathbf{v}_C + \nabla_{\mathbf{i}_L}F + \nabla P_a \\ \mathbf{\Gamma}^\top\mathbf{i}_L - \tilde{\mathbf{R}}_C^{-1}\mathbf{v}_C \end{bmatrix}.$$

Since all admissible equilibria satisfy $\mathbf{v}_C^* = \tilde{\mathbf{R}}_C\mathbf{\Gamma}^\top\mathbf{i}_L^*$, we clearly have that $\nabla_{\mathbf{v}_C}P_d(\mathbf{i}_L^*, \mathbf{v}_C^*) = 0$. On the other hand, Assumption A.2 and A.7 ensure that the function $P_a(\mathbf{i}_L) + F(\mathbf{i}_L) + \frac{1}{2}\mathbf{i}_L^\top\mathbf{\Gamma}\tilde{\mathbf{R}}_C\mathbf{\Gamma}^\top\mathbf{i}_L$ is *strongly* convex, and consequently that it has a unique global minimum at the point where its gradient is zero. This, together with Assumption A.6, ensures $(\mathbf{i}_L^*, \mathbf{v}_C^*)$ is the unique equilibrium of the closed-loop system.

Once we have achieved the power shaping we will now apply Proposition 3 to generate another admissible pair $(\mathbf{Q}_A, \tilde{P}_d)$ with $\mathbf{Q}_A + \mathbf{Q}_A^\top < 0$—notice the strict inequality. We make at this point the important observation that, since $\nabla\tilde{P}_d = \mathbf{Q}_A\mathbf{Q}^{-1}\nabla P_d$, the extrema of all new mixed potentials $\tilde{P}_d$ will coincide with the extrema of P_d. We apply the transformations of Proposition 3 to the closed-loop system above with the parameters $\lambda = -1$, $\mathbf{M} = \text{diag}(2\mathbf{I}/R_a, 0)$ that yields

$$\mathbf{Q}_A = \begin{bmatrix} -[\frac{2}{R_a}(\nabla^2 P_a + \nabla^2_{\mathbf{i}_L}F) + \mathbf{I}]\mathbf{L} & 0 \\ -\frac{2}{R_a}\mathbf{\Gamma}^\top\mathbf{L} & -\mathbf{C} \end{bmatrix},$$

whose symmetric part is negative definite for sufficiently large R_a. Consequently, along the closed-loop dynamics, which can also be described by

$$\mathbf{Q}_A\frac{d}{dt}\begin{bmatrix} \mathbf{i}_L \\ \mathbf{v}_C \end{bmatrix} = \nabla\tilde{P}_d,$$

we have

$$\dot{\tilde{P}}_d = \frac{1}{2}\nabla\tilde{P}_d^\top\mathbf{Q}_A^{-\top}(\mathbf{Q}_A + \mathbf{Q}_A^\top)\mathbf{Q}_A^{-1}\nabla\tilde{P}_d \leq -\alpha|\nabla\tilde{P}_d|^2$$

for some $\alpha > 0$, where $|\cdot|$ is the euclidian norm. Convergence of all bounded trajectories follows immediately from LaSalle's invariance principle and the fact that $|\nabla\tilde{P}_d| = 0$ only at the desired equilibrium.[4]

To prove boundedness of trajectories we apply the change of coordinates $(\mathbf{p}_L, \mathbf{q}_C) = (\hat{\mathbf{p}}_L(\mathbf{i}_L), \hat{\mathbf{q}}_C(\mathbf{v}_C))$ to the closed-loop system to obtain

$$\begin{bmatrix} \dot{\mathbf{p}}_L \\ \dot{\mathbf{q}}_C \end{bmatrix} = \begin{bmatrix} 0 & -\mathbf{\Gamma} \\ \mathbf{\Gamma}^\top & -\tilde{\mathbf{R}}_C^{-1} \end{bmatrix}\begin{bmatrix} \nabla\mathcal{E}_L(\mathbf{p}_L) \\ \nabla\mathcal{E}_C(\mathbf{q}_C) \end{bmatrix} - \begin{bmatrix} \nabla_{\mathbf{i}_L}F(\hat{\mathbf{i}}_L(\mathbf{p}_L)) + \nabla P_a(\hat{\mathbf{i}}_L(\mathbf{p}_L)) \\ 0 \end{bmatrix},$$

where we have denoted the inverse function of $\mathbf{p}_L = \hat{\mathbf{p}}_L(\mathbf{i}_L)$ by $\mathbf{i}_L := \hat{\mathbf{i}}_L(\mathbf{p}_L)$ and $\mathcal{E}_L : \mathbb{R}^{n_L} \to \mathbb{R}$, $\mathcal{E}_C : \mathbb{R}^{n_C} \to \mathbb{R}$ denote the total energy stored in the inductors and capacitors, respectively. (Notice that $\mathbf{i}_L = \nabla\mathcal{E}_L$ and $\mathbf{v}_C = \nabla\mathcal{E}_C$.) From

[4]The explicit expression of $\tilde{P}_d$ is of no interest for our derivations, as LaSalle's invariance principle imposes no particular positivity constraint on this function.

Assumption A.1 we have that the total energy, $\mathcal{E} = \mathcal{E}_C + \mathcal{E}_L$, is a *positive radially unbounded* function. Evaluating its time derivative we get

$$\dot{\mathcal{E}} = -\nabla^\top \mathcal{E}_C \tilde{\mathbf{R}}_C^{-1} \nabla \mathcal{E}_C - \nabla \mathcal{E}_L^\top [\nabla_{\mathrm{i}_L} F(\hat{\mathrm{i}}_L(\mathrm{p}_L)) + \nabla P_a(\hat{\mathrm{i}}_L(\mathrm{p}_L))] \quad (19)$$

Assumption A.7 states that the function $F(\mathrm{i}_L) + P_a(\mathrm{i}_L)$ is *strongly convex*. The latter ensures that the lower term in (19) is positive outside some ball $|\mathrm{i}_L| = b$, and consequently $\dot{\mathcal{E}}$ is negative outside a compact set. This proves global boundedness of the solutions and completes the proof. ■

Remark 4: Clearly, all assumptions of Theorem 2 are constraints related with the "degree of under–actuation" of the circuit. All conditions are obviated in the extreme case where $\mathbf{B}_S = \mathbf{I}$ when we can add an arbitrary power function P_a. Also, the rather restrictive Assumption A.3 of Theorem 1 is conspicuous by its absence—this means that we *do not* assume that the circuit to be controlled is already passive.

VI. Conclusion

Our main motivation in this paper was to propose an alternative to the well-known method of energy shaping stabilization of physical systems—which as pointed out in (Ortega *et al.*, 2002; Ortega *et al.*, 2001; Schaft, 2000) is severely stymied by the existence of pervasive damping. In this paper we have, for nonlinear RLC circuits, put forth the paradigm of power shaping and shown that it is not restricted to systems without pervasive dissipation. The starting point for the formulation of the power shaping idea are some new power balancing and passivity properties established for a class of nonlinear RLC circuits with convex energy function and weak electromagnetic coupling. To enlarge the class of circuits that enjoy these properties we have made extensive use of Proposition 3 which provides a procedure to generate alternative circuit topologies that reveal, through the new admissible pairs $(\mathbf{Q}_A, P_A)$, properties of the original circuit that we can exploit in our controller design. Future research includes the extension of our results beyond the realm of RLC circuits, e.g., to mechanical or electromechanical systems. A related question is whether we can find Brayton–Moser like models for this class of systems.

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