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Abstract

In this paper we construct an overlapping generations model for the small open economy incorporating a realistic description of the mortality process. With age-dependent mortality, the typical life-cycle pattern of consumption and saving results from the maximizing behaviour of individual households. Our "Blanchard-Yaari-Modigliani" model is used to analytically study a number of typical shocks affecting the small open economy, namely a balanced-budget public spending shock, a temporary Ricardian tax cut, and an interest rate shock. The demographic details matter a lot—both the impulse-response functions and the welfare profiles (associated with the different shocks) are critically affected by them. These demographic details furthermore do not wash out in the aggregate. The model is flexible and can be applied to a wide variety of theoretical and policy issues.

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1 Introduction

> It is possible that death may be the consequence of two generally co-existing causes; the one, chance, without previous disposition to death or deterioration; the other, a deterioration or an increased inability to withstand destruction. (Gompertz, 1825)

The opening quotation is a verbal introduction to a phenomenon that is now often called Gompertz' Law of mortality. In his path-breaking paper, Benjamin Gompertz[1] (1825) identified two main causes of death, namely one due to pure chance and another depending on the person's age. He pointed out that if only the first cause were relevant, then "the intensity of mortality" would be constant and the surviving fraction of a given cohort would decline in geometric progression. In contrast, if only the second cause would be relevant, and "if mankind be continually gaining seeds of indisposition, or in other words, an increased liability to death" then the force of mortality would increase with age. Gompertz' Law was subsequently generalized by Makeham (1860) who argued that the instantaneous mortality rate depends both on a constant term (first cause) and on a term that is exponential in the person's age (second cause).[2]

The microeconomic implications for consumption behaviour of lifetime uncertainty—resulting from a positive death probability—were first studied in the seminal paper by Yaari (1965). He showed that, faced with a positive mortality rate, individual agents will discount future felicity more heavily due to the uncertainty of survival. Furthermore, with lifetime uncertainty the consumer faces not only the usual solvency condition but also a constraint prohibiting negative net wealth *at any time*—the agent is simply not allowed by capital markets to expire indebted. Yaari assumes that the household can purchase (annuity) or sell (life insurance) actuarial notes at an actuarially fair interest rate. In the absence of a bequest motive, the household will use such notes to fully insure against the adverse effect of lifetime uncertainty.

The Yaari insights were embedded in a general equilibrium growth model by Blanchard (1985). In order to allow for exact aggregation of individual decision rules, Blanchard simplified the Yaari model by assuming a constant death probability, i.e. only the first cause of death is introduced into the model and households enjoy a

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[1] As Hooker (1965) points out, Benjamin Gompertz can be seen as one of the founding fathers of modern demographic and actuarial theory. See also Preston et al. (2001, p. 192). Blanchard (1985, p. 225) and Faruqee (2003, p. 301) incorrectly refer to the non-existing "Gomperty's Law."

[2] The continuous-time version of the Gompertz-Makeham Law of mortality takes the form $m(u) = \mu_0 + (\mu_1/\mu_2)[e^{\mu_2 u} - 1]$, where $m(u)$ is the instantaneous mortality rate of a person with age u and the $\mu_i's$ are non-negative. This form is estimated below using US demographic data.

perpetual youth. Because of its flexibility, the Blanchard-Yaari model has achieved workhorse status in the last two decades.[3] As Blanchard himself points out, his modelling approach has the disadvantage that it cannot capture the life-cycle aspects of consumption and saving behaviour—the age-independent mortality rate ensures that the propensity to consume out of total wealth is the same for all households.[4]

Blanchard's modelling dilemma is clear: exact aggregation is "bought" at the expense of a rather unrealistic description of the demographic process.[5] Of course, in a closed-economy context, the aggregation step is indispensable because equilibrium factor prices are determined in the aggregate factor markets. However, in the context of a small open economy, factor prices are typically determined in world markets so that the aggregation step is not necessary and life-cycle effects can be modelled. The main objective of this paper is to elaborate on exactly this point. As we demonstrate below, it is quite feasible to construct and *analytically* analyze a Blanchard-Yaari type overlapping-generations model incorporating a realistic description of demography. In addition we show that such a model gives rise to drastically different impulse-response functions associated with various macroeconomic shocks—the demographic realism matters.

The remainder of this paper is organized as follows. Section 2 sets out the model. Following Calvo and Obstfeld (1988) and Faruqee (2003), we assume that the mortality rate is age-dependent and solve for the optimal decision rules of the individual households.[6] We establish that the propensity to consume out of total wealth is an increasing function of the individual's age provided the mortality rate is non-decreasing in age. Next, we postulate a constant birth rate and characterize both the population composition and the implied aggregate population growth rate associated with the de-

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[3]For the purpose of this paper, the most important extension is due to Buiter (1988) who allows for non-zero population growth by using the insights of Weil (1989). For a textbook treatment of the Blanchard-Yaari model, see Blanchard and Fischer (1989, ch. 3) or Heijdra and van der Ploeg (2002, ch. 16).

[4]Blanchard shows that a "saving-for-retirement" effect can be mimicked by assuming that labour income declines wih age. Faruqee and Laxton (2000) use this approach in a calibrated simulation model.

[5]Blanchard suggests that a constant mortality rate may be more reasonable if the model is applied to *dynastic families* rather than to individual agents (1985, p. 225, fn.1). Under this interpretation the mortality rate refers to the probability that the dynasty becomes extinct.

[6]The relationship between these papers and ours is as follows. Calvo and Obstfeld (1988) recognize age-dependent mortality but do not solve the decentralized model. Instead, they characterize the dynamically consistent social optimum in the presence of time- and age-dependent lump-sum taxes. Faruqee (2003) models age-specific mortality in a decentralized setting but is ultimately unsuccessful. Indeed, he confuses the cumulative density function with the mortality rate (by requiring the death rate to go to unity in the limit; see (2003, p. 302)). Furthermore, he is unable to solve the transitional dynamics.

mographic process. Still using the general demographic process we characterize the steady-state age-profiles for consumption, human wealth, and asset holdings.

In Section 3 we employ (projected) US demographic data to estimate a number of parametric mortality models. In addition to the Blanchard model, we also estimate three additional models that allow for age-dependent mortality. Not surprisingly, the Gompertz-Makeham model provides by far the best fit with the data. Interestingly, however, the key aspects of the Gompertz-Makeham Law are also captured quite well by our so-called piece-wise linear model which distinguishes two "phases" of life, namely youth and old-age. During youth, the mortality rate is constant and quite low, but during old-age it rises linearly with age. In our view, the piece-wise linear model is interesting in itself for two reasons. First, it presents a continuous-time generalization of the Diamond (1965) model, allowing for individuals to differ even within each "phase" of life. Second, it gives rise to relatively simple analytical expressions for the propensity to consume and the steady-state age profiles for consumption, human wealth, and financial assets. In the remainder of the section we show that the piece-wise linear and Gompertz-Makeham models both give rise to bell-shaped age profiles of financial assets (Modigliani's life-cycle pattern).

In Section 4 we compute and visualize the effects on the key variables of three typical macroeconomic shocks affecting the small open economy, namely a balanced-budget spending shock, a temporary tax cut (Ricardian equivalence experiment), and an interest rate shock. We compare and contrast the results obtained for the Blanchard and piece-wise linear models. In the second part of Section 4 we also present the welfare effects associated with the shocks and demonstrate that the piece-wise linear model may give rise to non-monotonic welfare effects on existing generations, something which is impossible in the Blanchard case. We conclude Section 4 by showing that the two models also give rise to significantly different impulse-response functions for the *aggregate* variables (especially for asset holdings)—the heterogeneity does not "wash out" in the aggregate.

Finally, in Section 5 we mention a number of possible applications of and extensions to the model and draw some conclusions. The paper is concluded with a brief Appendix containing the main derivations and proofs.

4

2 The model

2.1 Households

2.1.1 Individual consumption

From the perspective of birth, the expected lifetime utility of a household is given by:

$$\Lambda(v,v) \equiv \int_v^\infty [1 - \Phi(\tau - v)] \ln \bar{c}(v,\tau) e^{\theta(v-\tau)} d\tau, \tag{2.1}$$

where v is the birth date, $\bar{c}(v,\tau)$ is consumption of a vintage-v agent at time τ ($\geq v$), and θ is the constant pure rate of time preference ($\theta > 0$). Intuitively, $1 - \Phi(\tau - v)$ is the probability that an agent born at time v is still alive at time τ (at which time the agent's age is $\tau - v$). The instantaneous *mortality rate* (or death probability) of a household of age s is given by the hazard rate of the stochastic distribution of the date of death:

$$m(s) \equiv \frac{\phi(s)}{1 - \Phi(s)}, \tag{2.2}$$

where $\phi(s)$ and $\Phi(s)$ denote, respectively, the density and distribution (or cumulative density) functions. These functions exhibit the usual properties, i.e. $\phi(s) \geq 0$ and $0 < \Phi(s) < 1$ for $s \geq 0$. Since, by definition, $\Phi'(s) = \phi(s)$ and $\Phi(0) = 0$, it follows that the first term on the right-hand side of (2.1) can be simplified to:[7]

$$1 - \Phi(\tau - v) = e^{-M(\tau - v)}, \tag{2.3}$$

where $M(\tau - v)$ is related to the mortality rate according to:[8]

$$M(\tau - v) \equiv \int_0^{\tau - v} m(s)\, ds. \tag{2.4}$$

By using (2.3) in (2.1) we find that the utility function of a newborn agent can be written as:

$$\Lambda(v,v) \equiv \int_v^\infty \ln \bar{c}(v,\tau) e^{-[\theta(\tau-v)+M(\tau-v)]} d\tau. \tag{2.5}$$

As was pointed out by Yaari (1965), future felicity is discounted both because of pure time preference (as $\theta > 0$) and because of life-time uncertainty (as $M(\tau - v) > 0$).[9]

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[7]All derivations are documented in a separate Mathematical Appendix (see Heijdra and Romp, 2005). Some key results are derived in a brief Appendix to the paper.

[8]The function $M(s)$ is a *primitive* of $m(s)$ if $M'(s) = m(s)$ for every s in the relevant interval. The indefinite integral is then $\int m(s) = M(s) + C$, where C is some constant which drops out when the integral is evaluated for a particular interval, $s_0 \leq s \leq s_1$.

[9]Yaari (1965, p. 143) attributes the latter insight to Fisher (1930, pp. 216-217).

From the perspective of some later time period t, the utility function of the agent born at time v takes the following form:

$$\Lambda(v,t) \equiv e^{M(t-v)} \int_t^\infty \ln \bar{c}\left(v,\tau\right) e^{-[\theta(\tau-t)+M(\tau-v)]} d\tau, \qquad (2.6)$$

where the discounting factor due to life-time uncertainty ($M\left(\tau-v\right)$) depends on the *age* of the household at time τ.[10] The household budget identity is given by:

$$\dot{\bar{a}}\left(v,\tau\right) = \left[r + m\left(\tau-v\right)\right] \bar{a}\left(v,\tau\right) + \bar{w}\left(\tau\right) - \bar{z}\left(\tau\right) - \bar{c}\left(v,\tau\right), \qquad (2.7)$$

where $\bar{a}\left(v,\tau\right)$ is real financial wealth, r is the exogenously given (constant) world rate of interest, $\bar{w}\left(\tau\right)$ is the wage rate, and $\bar{z}\left(\tau\right)$ is the lump-sum tax (the latter two variables are assumed to be independent of age). Labour supply is exogenous and each household supplies a single unit of labour. As usual, a dot above a variable denotes that variable's time rate of change, e.g. $\dot{\bar{a}}\left(v,\tau\right) \equiv d\bar{a}(v,\tau)/d\tau$. Following Yaari (1965) and Blanchard (1985), we postulate the existence of a perfectly competitive life insurance sector which offers actuarially fair annuity contracts to the households. Since household age is directly observable, the annuity rate of interest faced by a household of age $\tau - v$ is equal to the sum of the world interest rate and the instantaneous mortality rate of that household.

Abstracting from physical capital, financial wealth can be held in the form of domestic government bonds ($\bar{d}\left(v,\tau\right)$) or foreign bonds ($\bar{f}\left(v,\tau\right)$).

$$\bar{a}\left(v,\tau\right) \equiv \bar{d}\left(v,\tau\right) + \bar{f}\left(v,\tau\right). \qquad (2.8)$$

The two assets are perfect substitutes in the households' portfolios and thus attract the same rate of return.

In the planning period t, the household chooses paths for consumption and financial assets in order to maximize lifetime utility (2.6) subject to the flow budget identity (2.7) and a solvency condition, taking as given its initial level of financial assets $\bar{a}(v,t)$.

[10]The appearance of the term $e^{M(t-v)}$ in front of the integral is a consequence of the fact that the distribution of expected remaining lifetimes is not memoryless in general. Blanchard (1985) uses the memoryless exponential distribution for which $M\left(s\right) = \mu_0 s$ (where μ_0 is a constant) and thus $M\left(t-v\right) - M\left(\tau-v\right) = -M\left(\tau-t\right)$. Equation (2.6) can then be written in a more familiar format as:

$$\Lambda(v,t) \equiv \int_t^\infty \ln \bar{c}\left(v,\tau\right) e^{-(\theta+\mu_0)(\tau-t)} d\tau.$$

The household optimum is fully characterized by:

$$\frac{\dot{\bar{c}}(v,\tau)}{\bar{c}(v,\tau)} = r - \theta, \tag{2.9}$$

$$\Delta(u,\theta)\,\bar{c}(v,t) = \bar{a}(v,t) + \bar{h}(v,t), \tag{2.10}$$

$$\bar{h}(v,t) \equiv e^{ru+M(u)} \int_u^\infty [\bar{w}(s+v) - \bar{z}(s+v)]\, e^{-[rs+M(s)]} ds \tag{2.11}$$

where $u \equiv t - v$ is the *age* of the household in the planning period and $\Delta(u,\lambda)$ is defined in general terms as:

$$\Delta(u,\lambda) \equiv e^{\lambda u + M(u)} \int_u^\infty e^{-[\lambda s + M(s)]} ds, \qquad \text{(for } u \geq 0,\ \lambda > 0). \tag{2.12}$$

Equation (2.9) is the consumption Euler equation, relating the optimal time profile of consumption to the difference between the interest rate and the pure rate of time preference. The instantaneous mortality rate does not feature in this expression because households fully insure against the adverse effects of lifetime uncertainty (Yaari, 1965). In order to avoid having to deal with a taxonomy of different cases, we restrict attention in the remainder of this paper to the case of a nation populated by patient agents, i.e. $r > \theta$.[11] Equation (2.10) shows that consumption in the planning period is proportional to total wealth, consisting of financial wealth ($\bar{a}(v,t)$) and human wealth ($\bar{h}(v,t)$). The proportionality factor is obtained by evaluating (2.12) for $\lambda = \theta$.[12] Clearly, $\Delta(u,\theta)$ depends only on the household's age in the planning period and *not* on time itself. For future reference, Lemma 1 establishes some important properties of the $\Delta(u,\lambda)$ function. Finally, human wealth is defined in (2.11) and represents the market value of the unit time endowment, i.e. the present value of after-tax wage income, using the annuity rate of interest for discounting purposes. Unless after-tax wage income is time-invariant, human wealth depends on both time and on the household's age in the planning period.

Lemma 1 *Let $\Delta(u,\lambda)$ be defined as in (2.12) and assume that the mortality rate is non-decreasing, i.e. $m'(s) \geq 0$ for all $s \geq 0$. Then the following properties can be established for $\Delta(u,\lambda)$: (i) decreasing in λ, $\partial\Delta(u,\lambda)/\partial\lambda < 0$; (ii) non-increasing in household age, $\partial\Delta(u,\lambda)/\partial u \leq 0$; (iii) upper bound, $\Delta(u,\lambda) \leq 1/[\lambda + m(u)]$; (iv) $\Delta(u,\lambda) > 0$ for $u < \infty$; (v) for $\lambda \to \infty$, $\Delta(u,\lambda) \to 0$.*

Proof: see Appendix.

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[11]The results for the other cases (with $r < \theta$ or $r = \theta$) are easily deduced from our mathematical expressions.

[12]As we demonstrate below, $\Delta(u,\lambda)$ plays a very important role in the model. Evaluated for $\lambda = \theta$, $1/\Delta(u,\theta)$ represents the marginal (and average) propensity to consume out of total wealth.

2.1.2 Demography

In order to allow for non-zero population growth, we employ the analytical framework developed by Buiter (1988) which distinguishes the instantaneous mortality rate $m(s)$ and the birth rate $b\ (> 0)$ and thus allows for net population growth or decline. The population size at time t is denoted by $L(t)$ and the size of a newborn generation is assumed to be proportional to the current population:

$$L(v, v) = bL(v). \tag{2.13}$$

The size of cohort v at some later time τ is:

$$L(v, \tau) = L(v, v)[1 - \Phi(\tau - v)] = bL(v)e^{-M(\tau - v)}, \tag{2.14}$$

where we have used (2.3) and (2.13). The aggregate mortality rate, $\bar{m}$, is defined as:

$$\bar{m}L(t) = \int_{-\infty}^{t} m(t - v)L(v, t)\, dv, \tag{2.15}$$

and it is assumed that $\bar{m}$ is constant (see also below). Despite the fact that the expected remaining lifetime of each individual is stochastic, there is no aggregate uncertainty in the economy. In the absence of international migration, the growth rate of the aggregate population, n, is equal to the difference between the birth rate and the aggregate mortality rate, i.e. $n \equiv b - \bar{m}$. It follows that $L(v) = A_0 e^{nv}$, $L(t) = A_0 e^{nt}$ and thus $L(v) = L(t)e^{-n(t-v)}$. Using this result in (2.14) we obtain the generational population weights:

$$l(v, t) \equiv \frac{L(v, t)}{L(t)} = be^{-[n(t-v) + M(t-v)]}, \qquad t \geq v. \tag{2.16}$$

The key thing to note about (2.16) is that the population proportion of generation v at time t only depends on the age of that generation and not on time itself.

2.1.3 Per capita household sector

Per capita variables are calculated as the integral of the generation-specific values weighted by the corresponding generation weights. For example, per capita consumption, $c(t)$, is defined as:

$$c(t) \equiv \int_{-\infty}^{t} l(v, t)\bar{c}(v, t)\, dv, \tag{2.17}$$

where $l(v, t)$ and $\bar{c}(v, t)$ are defined in, respectively, (2.16) and (2.10) above. Exact aggregation of (2.10) is impossible because both $\Delta(u, \theta)$ and the wealth components,

$\bar{a}(v, t)$ and $\bar{h}(v, t)$, depend on the generations index v. The "Euler equation" for per capita consumption can nevertheless be obtained by differentiating (2.17) with respect to time and noting (2.9) and (2.16):

$$\dot{c}(t) = b\bar{c}(t, t) + (r - \theta) c(t) - \int_{-\infty}^{t} [n + m(t - v)] l(v, t) \bar{c}(v, t) dv.$$
(2.18)

Per capita consumption growth is boosted by the arrival of new generations who start to consume out of human wealth (first term on the right-hand side) and by individual consumption growth (second term). The third term on the right-hand side of (2.18) corrects for population growth and (age-dependent) mortality.[13]

Per capita financial wealth is defined as $a(t) \equiv \int_{-\infty}^{t} l(v, t)\bar{a}(v, t)dv$. By differentiating this expression with respect to t we obtain:

$$\dot{a}(t) = (r - n) a(t) + w(t) - z(t) - c(t),$$
(2.19)

where $w(t) = \bar{w}(t)$, $z(t) = \bar{z}(t)$, and we have used equation (2.7) and noted the fact that newborns are born without financial assets ($\bar{a}(t, t) = 0$). The interest rate net of population growth is assumed to be positive, i.e. $r > n$. As in the standard Blanchard model, annuity payments drop out of the expression for per capita asset accumulation because they constitute transfers (via the life insurance companies) from those who die to agents who stay alive.

Finally, per capita human wealth is defined as $h(t) \equiv \int_{-\infty}^{t} l(v, t)\bar{h}(v, t)dv$ so that $\dot{h}(t)$ can be written as:

$$\dot{h}(t) = (r - n) h(t) + b\bar{h}(t, t) - w(t) + z(t).$$
(2.20)

In the standard Buiter model per capita human wealth is the same for all generations and accumulates at the constant annuity rate of interest $(r + m)$. In contrast, in the present model the effects of the net interest rate $(r - n)$ and the birth rate (b) are separate, with the former applying to per capita human wealth and the latter applying to the human wealth of newborn generations.

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[13]If the mortality rate were constant, as in Blanchard (1985) and Buiter (1988), then $n \equiv b - m$ and equation (2.18) would simplify to:

$$\dot{c}(t) = (r - \theta) c(t) - b[c(t) - c(t, t)].$$

2.2 Firms, government, and foreign sector

Following Buiter (1988) we keep the production side of the model as simple as possible by abstracting from physical capital altogether.[14] Competitive firms face the technology $Y(t) = k(t) L(t)$ where $k(t)$ is an exogenous productivity index and $L(t)$ is the aggregate supply of labour. The real wage rate is then given by $w(t) = k(t)$.

The government budget identity is given by:

$$\dot{d}(t) = (r - n) d(t) + g(t) - z(t), \tag{2.21}$$

where $d(t) \equiv \int_{-\infty}^{t} l(v,t) \bar{d}(v,t) dv$ is the per capita stock of domestic bonds, and $g(t)$ is per capita government goods consumption. The government solvency condition is $\lim_{\tau \to \infty} d(\tau) e^{(r-n)(t-\tau)} = 0$, so that the intertemporal budget constraint of the government can be written as:

$$d(t) = \int_{t}^{\infty} [z(\tau) - g(\tau)] e^{(r-n)(t-\tau)} d\tau. \tag{2.22}$$

To the extent that there is outstanding debt (positive left-hand side), it must be exactly matched by the present value of current and future primary surpluses (positive right-hand side), using the net interest rate $(r - n)$ for discounting purposes.

Finally, the evolution of the per capita stock of net foreign assets is explained by the current account:

$$\dot{f}(t) = (r - n) f(t) + w(t) - c(t) - g(t), \tag{2.23}$$

where we have used $y(t) \equiv Y(t)/L(t) = w(t)$ and where $f(t) \equiv \int_{-\infty}^{t} l(v,t) f(v,t) dv$ denotes the per capita stock of foreign bonds in the hands of domestic households.

2.3 Steady-state equilibrium

It is relatively straightforward to characterize the steady state of the model. The steady-state values for all variables are designated by means of a hat overstrike, e.g. $\hat{c}$ is steady-state per capita consumption. Where no confusion can arise, the time index is also suppressed. For a constant level of technology, $k(t) = \hat{k}$, the steady-state wage rate is time-invariant, i.e. $w(t) = \hat{w} = \hat{k}$. If the government variables are also held constant, so that $z(t) = \hat{z}$, $g(t) = \hat{g}$, and $d(t) = \hat{d} \equiv (\hat{z} - \hat{g})/(r - n)$, then the

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[14]In the context of a small open economy with firms facing convex investment adjustment costs, our approach does not entail much loss of generality because the investment and savings systems decouple in that case. See Matsuyama (1987), Bovenberg (1993, 1994), Heijdra and Meijdam (2002), and Heijdra and van der Ploeg (2002, pp. 571-581).

economy settles into a unique saddle-point stable steady-state equilibrium in which $c(t) = \hat{c}$, $h(t) = \hat{h}$, $a(t) = \hat{a}$, and $f(t) = \hat{f}$.[15]

In the steady-state equilibrium, all individual household variables can be rewritten solely in terms of their age, $u \equiv t - v$ (as is also the case outside the steady state for $\Delta(u, \theta)$—see equation (2.12) above). By substituting $w(t) = \hat{w}$ and $z(t) = \hat{z}$ into (2.11) we find the expression for age-dependent human wealth:

$$\hat{\bar{h}}(u) \equiv \hat{\bar{h}}(v, t) = [\hat{w} - \hat{z}] \Delta(u, r), \tag{2.24}$$

where $\Delta(u, r)$ is obtained from (2.12) by setting $\lambda = r$. Since a newborn has no financial wealth, it follows from (2.10) that $\hat{\bar{c}}(v, v) = \hat{\bar{h}}(0) / \Delta(0, \theta)$. The Euler equation (2.9) shows that $\hat{\bar{c}}(v, t) = \hat{\bar{c}}(v, v) e^{(r-\theta)u}$ so that, by combining the two results, we obtain:

$$\hat{\bar{c}}(u) \equiv \hat{\bar{c}}(v, t) = \frac{\hat{\bar{h}}(0)}{\Delta(0, \theta)} e^{(r-\theta)u}. \tag{2.25}$$

Steady-state asset holdings can be computed by using (2.10):

$$\hat{\bar{a}}(u) = \Delta(u, \theta) \hat{\bar{c}}(u) - \hat{\bar{h}}(u). \tag{2.26}$$

The steady-state per capita variables can be expressed in terms of individual variables. Using equation (2.16), (2.17) and (2.25) we write steady-state per capita consumption as:

$$\hat{c} = \frac{\hat{\bar{h}}(0)}{\Delta(0, \theta)} \int_0^\infty b e^{-[(\theta+n-r)u + M(u)]} du$$
$$= \frac{\hat{\bar{h}}(0)}{\Delta(0, \theta)} b \Delta(0, \theta + n - r). \tag{2.27}$$

From (2.20) we find the expression for steady-state per capita human capital:

$$\hat{h} = \frac{\hat{w} - \hat{z} - b\hat{\bar{h}}(0)}{r - n} = \frac{\hat{w} - \hat{z}}{r - n} [1 - b\Delta(0, r)], \tag{2.28}$$

where we have used equation (2.24) (for $v = 0$) to get to the second expression. Finally, from equation (2.19) and the per capita version of (2.8) we obtain the expressions for steady-state per capita financial assets:

$$\hat{a} \equiv \hat{d} + \hat{f} = \frac{\hat{c} + \hat{z} - \hat{w}}{r - n}. \tag{2.29}$$

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[15]Saddle-point stability follows trivially from the fact that all agents in the economy satisfy their respective solvency conditions. Consumption and human wealth are forward-looking (jumping) variables whilst total financial assets and net foreign assets are predetermined (sticky) variables.

Armed with these expressions it is straightforward to derive the long-run effects of various shocks impacting the economy.[16] A balanced-budget increase in government consumption ($d\hat{z} = d\hat{g} > 0$) leads to a decrease in steady-state human wealth and consumption for all cohorts:

$$\frac{d\hat{\bar{h}}(u)}{d\hat{z}} = -\Delta(u,r) < 0, \tag{2.30}$$

$$\frac{d\hat{\bar{c}}(u)}{d\hat{z}} = \frac{d\hat{\bar{h}}(0)}{d\hat{z}} \frac{e^{(r-\theta)u}}{\Delta(0,\theta)} < 0. \tag{2.31}$$

Obviously, per capita steady-state consumption and human wealth also fall (see equations (2.27) and (2.28)). It follows from (2.29) that per capita steady-state financial assets decline because consumption is crowded out more than one for one:

$$\frac{d\hat{a}}{d\hat{z}} = \frac{1}{r-n}\left[1 + \frac{d\hat{c}}{d\hat{z}}\right] < 0. \tag{2.32}$$

Finally, since government debt is unchanged (by design) it follows from the first equality in (2.29) that $d\hat{f}/d\hat{z} = d\hat{a}/d\hat{z}$. The balanced-budget increase in government consumption thus leads to a long-run reduction in financial assets and a reduction in net imports, just as in the standard open-economy Blanchard (1985, p. 230-231) model with $r > \theta$. (An decrease in steady-state productivity ($d\hat{w} < 0$) has the same effects on $\hat{\bar{h}}(u)$, $\hat{\bar{c}}(u)$, $\hat{c}$, $\hat{a}$, and $\hat{f}$ as a balanced-budget increase in government consumption.)

A long-run tax-financed increase in public debt ($(r-n)d\hat{d} = d\hat{z} > 0$) leads to a decrease in generation-specific and per capita steady-state consumption and human wealth (see (2.30)-(2.31)). It follows from (2.29) that:

$$(r-n)\frac{d\hat{f}}{d\hat{z}} \equiv -(r-n)\frac{d\hat{d}}{d\hat{z}} + \frac{d\hat{c}}{d\hat{z}} + 1 = \frac{d\hat{c}}{d\hat{z}} < -1. \tag{2.33}$$

As in the standard Blanchard model (with $r > \theta$), government debt more than displaces foreign assets in the households' portfolios (1985, p. 242).

An increase in the world interest rate leads to higher discounting of after-tax wages and a reduction in both individual and aggregate human wealth:

$$\frac{d\hat{\bar{h}}(u)}{dr} = [\hat{w} - \hat{z}]\frac{\partial\Delta(u,r)}{\partial r} < 0, \tag{2.34}$$

$$\frac{d\hat{h}}{dr} = \int_0^\infty l(u)\frac{d\hat{\bar{h}}(u)}{dr}du < 0, \tag{2.35}$$

where we have used Lemma 1(i) to establish the sign in (2.34). By using (2.25) we find for the interest elasticity of individual consumption:

$$\frac{r}{\hat{\bar{c}}(u)}\frac{d\hat{\bar{c}}(u)}{dr} = ru + \frac{d\hat{\bar{h}}(0)}{dr}\frac{r}{\hat{\bar{h}}(0)} = ru + \frac{d\Delta(0,r)}{dr}\frac{r}{\Delta(0,r)}, \tag{2.36}$$

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[16]The impact and transitional effects of these shocks are studied in Section 4 of the paper.

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where we have used (2.24) to get to the second expression. The effect on consumption depends on the age of the household. Clearly, for newborns ($u = 0$) consumption falls because of the drop in the level of human wealth. Since the interest elasticity of $\Delta(0, r)$ is finite, however, it follows from (2.36) that for sufficiently old households consumption will rise. The negative level effect on consumption (operating via human wealth) is dominated by the positive growth effect (operating via the Euler equation (2.9)).

The effect on aggregate consumption is thus also ambiguous in general. If the hazard rate is very high around and after the point where the effect on individual consumption becomes positive, there will be very few people for whom consumption actually rises. The effect on aggregate consumption is negative for such demographies. In contrast, if a lot of people are still alive after the positive growth effect dominates the initial negative wealth effect, then the weight of this positive effect dominates and the aggregate effect is positive.

The effect on individual financial asset holdings can be deduced from (2.26):

$$\frac{d\hat{\bar{a}}(u)}{dr} = \Delta(u, \theta) \frac{d\hat{\bar{c}}(u)}{dr} - \frac{d\hat{\bar{h}}(u)}{dr} > 0, \qquad \text{(for } u > 0\text{)}, \tag{2.37}$$

and $d\hat{\bar{a}}(0)/dr = 0$ (newborns possess no assets). Despite the ambiguity of the sign of $d\hat{\bar{c}}(u)/dr$, individual assets must increase for all generations.[17] As a result, per capita financial assets also increase unambiguously. In the absence of pre-existing government debt ($\hat{z} = \hat{g}$ and $\hat{d} = 0$), per capita net foreign assets increases by the same amount as total financial assets, i.e. $d\hat{a}/dr = d\hat{f}/dr > 0$).

3 Demography

As was stressed by Blanchard (1985, p. 223), exact aggregation of the consumption function is generally impossible because both the propensity to consume (our $1/\Delta(u, \theta)$) and the wealth components (our $\bar{a}(v, t)$ and $\bar{h}(v, t)$) are age dependent. Blanchard cuts this Gordian knot by assuming the mortality rate to be constant, i.e. $m(s) = \mu_0 > 0$ and $M(u) = \mu_0 u$. The advantages of his approach are its simplicity and its undoubted flexibility—the expected remaining planning horizon is $1/\mu_0$ so, by letting $\mu_0 \to 0$, the infinite-horizon Ramsey model is obtained as a special case. The main disadvantage of the Blanchard approach is that it cannot capture the life-cycle

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[17]This result follows from the fact that $d\hat{\bar{c}}(u)/dr$ is smallest for $u = 0$ at which point $d\hat{\bar{a}}(u)/dr = 0$. As u rises, $d\hat{\bar{c}}(u)/dr$ increases. Since $d\hat{\bar{h}}(u)/dr$ is negative for all u, the inequality in (2.37) follows readily.

aspect of consumption behaviour. In addition, the perpetual youth assumption is of course easily refuted empirically as it runs foul of the Gompertz-Makeham Law of mortality (see Preston et al. (2001) and below).

In the context of a small open economy, however, it is quite feasible to incorporate a realistic demographic structure because the aggregation step is not necessary. The interest rate is determined in world capital markets and is exogenous to the small open economy. Conditional on the world interest rate, the factor price frontier pins down the real wage rate (which may also depend on an exogenous productivity index). With factor prices determined, the macroeconomic equilibrium can be studied directly at the level of individual households.

3.1 Estimates

In this paper we estimate the survival function $(1 - \Phi(\tau - v))$ by using actual US projections on expected survival rates for people born in 2001 (Arias et al., 2003, p. 26, Table 6, Column 3). Surviving fractions are reported for 5-year intervals and at birth. Denoting the actual expected surviving fraction up until age u_i of the people born in 2001 by $S(u_i)$, we can estimate the parameters of a given parametric distribution function by means of non-linear least squares. Denoting the parameter vector by μ, the model to be estimated is:

$$S(u_i) = 1 - \Phi(u_i, \mu) + \varepsilon_i = e^{-M(u_i, \mu)} + \varepsilon_i, \tag{3.1}$$

where $M(u_i) = \int_0^{u_i} m(s, \mu)ds$ and ε_i is the stochastic error term. The estimates are reported in Table 1 for various specifications of the mortality process. In that table, $\hat{\sigma}$ is the estimated standard error of the regression, the t-statistics are given in round brackets below the estimates, and $1 - \widehat{\Phi}(100)$ represents the estimated proportion of centenarians. Finally, $\hat{n}(b)$ is the estimated population growth rate (in percent per annum), conditional on a given birth rate b (which is held constant at 1.5% per annum). The growth rate of the population depends on the form of the mortality process and is computed by combining (2.15) and (2.16) and simplifying:

$$b = \frac{1}{\Delta(0, n)}. \tag{3.2}$$

For a given birth rate b, equation (3.2) implicitly defines the coherent solution for n and thus for the aggregate mortality rate, $\bar{m} \equiv b - n$.[18]

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[18]For a constant mortality rate m, we have $1/\Delta(0, n) = n + m$ so that (3.2) implies $n = b - m$. Blanchard (1985) sets $b = m$ so that $n = 0$ (constant population).

We consider four different functional forms for the instantaneous mortality rate and the associated $M(u_i)$ functions. The Blanchard model based on a *constant* mortality rate (model 1) yields an estimated mortality rate of 0.7% per annum and displays the worst fit of all cases considered–the estimated standard error is 0.23 which far exceeds the standard errors for the other models. Model 2 is based on the notion that the mortality rate increases with age. This *linear-in-age* model fits a little better than the constant model but it predicts a *negative* mortality rate for newborns. Constraining the constant to zero, the fit deteriorates somewhat though it is still better than that of the constant model. Models 1 and 2 both spectacularly overestimate the proportion of centenarians (almost 50% and 34% for models 1 and 2 respectively).

Model 3 postulates that the mortality rate is constant up to a certain age $\bar{u}$, after which it increases linearly with age. The so-called *piece-wise linear* (PWL hereafter) model fits much better than the first two models. The estimated standard error is 0.03 and the parameters are highly significant. Interestingly, the model predicts quite realistically that mortality starts to increase with age only after households reach the critical age of about 61 years. Finally, for model 4 the mortality rate follows the Gompertz-Makeham (GM hereafter) process. The GM model clearly displays the best fit of all cases considered–the estimated standard error is only one-sixteenth that of the next-best (PWL) model and all coefficients are highly significant.[19] Both models 3 and 4 yield reasonable predictions for the proportion of centenarians.

In the top panel of Figure 1 we illustrate the data points (stars) as well as the estimated survival functions for the different models. The poor fit of models 1 and 2 is confirmed–the surviving fraction is underestimated up to about age 80 and overestimated thereafter. Models 3 and 4 both track the data quite well. The key difference between these models lies in their predicted mortality rates and expected remaining lifetimes that are plotted in, respectively, the middle and bottom top panels of Figure 1. After about age 88, the mortality rate is steepest for the GM model. It is this non-linear feature of the mortality process that the PWL model fails to capture adequately. The expected remaining lifetimes for the GM and PWL models are, however, quite similar.

3.2 Steady-state profiles

In Figure 2 we visualize (for all estimated models) the steady-state age profiles for the propensity to consume ($1/\Delta(u, \theta)$), human wealth ($\hat{\bar{h}}(u)$), consumption ($\hat{\bar{c}}(u)$),

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[19]This good fit may be a consequence of the fact that demographers often use the GM model to generate demographic predictions especially at high ages. See Preston et al. (2001, p. 192) on this point.

and financial assets ($\hat{\hat{a}}(u)$). The analytical expressions for these variables are given in, respectively, equations (2.12), (2.24), (2.25), and (2.26). Especially the $\Delta(u, \lambda)$ function (defined in (2.12)) plays a key role in the model. For models 1-3, closed-form solutions for $\Delta(u, \theta)$ can be derived. Indeed, for model 1 (the Blanchard case) it reduces to $\Delta(u, \theta) = 1/(\theta + \mu_0)$ and is thus independent of the age of the household. For model 2, the solution is:

$$\Delta(u, \theta) \equiv \frac{\sqrt{\pi}}{2\mu_1} \operatorname{erfcx}\left(\mu_1 u + \frac{\theta + \mu_0}{2\mu_1}\right), \tag{3.3}$$

where $\operatorname{erfcx}(x)$ is the so-called scaled complementary error function (Kreyszig, 1988, p. A 78). The properties of this function and its close relatives are covered in Lemma 2. Since $\operatorname{erfcx}(u)$ is a downward sloping function of the household's age, it follows from (3.3) that the marginal propensity to consume, $1/\Delta(u, \theta)$, increases with age. This is confirmed in the top left-hand panel of Figure 2.

For the PWL model the expression for $\Delta(u, \theta)$ features two branches, depending on whether the household is still "young" ($0 < u < \bar{u}$) or has entered "old age" ($u > \bar{u}$):

$$\Delta(u, \theta) = \begin{cases} \dfrac{1 - e^{-(\theta + \mu_0)(\bar{u} - u)}}{\theta + \mu_0} & \\ \quad + e^{-(\theta + \mu_0)(\bar{u} - u)} \dfrac{\sqrt{\pi}}{2\mu_1} \operatorname{erfcx}\left(\dfrac{\theta + \mu_0}{2\mu_1}\right) & \text{for } 0 < u < \bar{u} \\ \dfrac{\sqrt{\pi}}{2\mu_1} \operatorname{erfcx}\left(\mu_1(u - \bar{u}) + \dfrac{\theta + \mu_0}{2\mu_1}\right) & \text{for } u \geq \bar{u} \end{cases} \tag{3.4}$$

Young households are still on the flat part of the mortality curve and for them $\Delta(u, \theta)$ can be written as a weighed average of $1/(\theta + \mu_0)$ and $\Delta(\bar{u}, \theta)$, with respective exponential weights $1 - e^{-(\theta + \mu_0)(\bar{u} - u)}$ and $e^{-(\theta + \mu_0)(\bar{u} - u)}$. Intuitively, $\bar{u} - u$ measures *how young* such households are, i.e. how far away they are from entering old age.[20] For old households, whose u exceeds $\bar{u}$, the lower branch of (3.4) is relevant. For such households, it matters *how old* they are, i.e. how far along in old age they are as measured by $u - \bar{u}$. It follows readily from (3.4) that $\Delta(u, \theta)$ declines with age, i.e. the marginal propensity to consume increases with age. This pattern is confirmed in the top left-hand panel of Figure 2.

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[20]Obviously, if old age were to set in only after a very long time ($\bar{u} \to \infty$), then one is back in the standard Blanchard case with $\Delta(u, \theta) = 1/(\theta + \mu_0)$ indefinitely.

Lemma 2 *The error function (*$\mathrm{erf}(x)$*), complementary error function (*$\mathrm{erfc}(x)$*), and scaled complementary error function (*$\mathrm{erfcx}(x)$*) are defined as follows.*

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt,$$

$$\mathrm{erfc}(x) \equiv \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\, dt = 1 - \mathrm{erf}(x),$$

$$\mathrm{erfcx}(x) \equiv e^{x^2} \mathrm{erfc}(x).$$

For non-negative values of x, *these functions have the following properties:*

(i) $0 < \mathrm{erf}(x), \mathrm{erfc}(x), \mathrm{erfcx} < 1$ *for* $0 < x \ll \infty$.

(ii) $\mathrm{erf}(0) = 1 - \mathrm{erfc}(0) = 1 - \mathrm{erfcx}(0) = 0$.

(iii) $\lim_{x \to \infty} \mathrm{erf}(x) = 1$, $\lim_{x \to \infty} \mathrm{erfc}(x) = \lim_{x \to \infty} \mathrm{erfcx}(x) = 0$.

(iv) $\mathrm{erf}'(x) > 0$, $\mathrm{erfc}'(x) < 0$, $\mathrm{erfcx}'(x) < 0$.

(v) $\mathrm{erfcx}(x) \approx 1/(x\sqrt{\pi})$ *for large* x.

For the GM model no closed-form solutions for $\Delta(u, \theta)$ can be obtained, and numerical integration techniques must be used. As is shown in the top left-hand panel of Figure 2, the marginal propensity to consume for these models closely tracks the solution for the PWL model up to about age $u = 80$. Thereafter the non-linearity of the mortality rate starts to cut in and $1/\Delta(u, \theta)$ increases more rapidly than is implied by the PWL model.

In the top right-hand panel of Figure 2 the age profile for steady-state human wealth ($\hat{\bar{h}}(u)$, defined in (2.24) above) is plotted for the different mortality models.[21] For the standard Blanchard model the annuity rate of interest is age-independent because the mortality rate is constant. As a result, human wealth is age-independent also. For the linear model the annuity rate of interest rises with age so that discounting of after-tax wage income is heavier the older the household is. Human wealth gradually falls with age as a result. Indeed, it follows from (2.24) that $\hat{\bar{h}}(u)$ is proportional to $\Delta(u, r)$ which is downward sloping in u for any demography with a non-decreasing mortality rate (see Lemma 1).

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[21] As parameter values we used $b = 0.015$, $\theta = 0.035$, $r = 0.04$, $w = 5$, and $z = 0$. The implied values for the population growth rate (n) are reported in Table 1. The simulation results are quite robust for different parameter values.

The pattern for human wealth looks rather similar for the remaining models 3 and 4. Exploiting the proportionality between $\hat{\bar{h}}(u)$ and $\Delta(u, r)$, we find that the slope of the human wealth profile is given by:

$$\frac{d\hat{\bar{h}}(u)}{du} = [\hat{w} - \hat{z}]\left[(r + m(u))\Delta(u, r) - 1\right] < 0, \tag{3.5}$$

where the term in square brackets on the right-hand side is equal to $\partial\Delta(u, r)/\partial u$. During the early phase of life, the annuity rate $r + m(u)$ is relatively low, $\Delta(u, r)$ is relatively high, and human wealth falls only slightly as young agents are still on the flat part of the mortality curve. At high ages, $r + m(u)$ is high, $\Delta(u, r)$ is low, and $d\hat{\bar{h}}(u)/du$ is again relatively low. The PWL and GM models both give rise to inverse-S-shaped profiles for human wealth with a point of inflexion located at the approximate age of 60. Only after about age 80 do the paths implied by the two models diverge somewhat, with the GM model showing the sharpest decline.

In the bottom left-hand panel of Figure 2 the age profile of steady-state consumption ($\hat{\bar{c}}(u)$) is visualized. As follows readily from (2.25), the slope of the consumption age profile is the same for all models. Interestingly, the estimated mortality models all predict very similar steady-state consumption paths (in level terms).

Finally, in the bottom right-hand panel of Figure 2 the age profile of steady-state financial assets ($\hat{\bar{a}}(u)$ as defined in (2.26) above) is visualized. For both models 1 and 2, financial assets rise with age. Matters are vastly different for models 3 and 4. For these models financial asset holdings follow the classic life-cycle pattern stressed by Modigliani and co-workers. i.e. households save up until middle age after which dissaving takes place. Again the most pronounced dissaving effect takes place for the GM model. Despite the fact that very old agents have hardly any financial assets left, the annuity rate of interest is so high that a high consumption level can nevertheless be maintained.

The upshot of the discussion so far is as follows. The constant and linear models track the demographic data very poorly and predict unrealistic age patterns for the consumption propensity, human wealth, and financial wealth. In contrast, the PWL and GM models track the data rather well and predict the relevant life-cycle patterns. While the GM model slightly outperforms the PWL model, it carries a (minor) disadvantage in that it can only be analyzed numerically, whereas the PWL model can be solved analytically in terms of well-known functions. Indeed, the salient features of the Gompertz-Makeham Law seem to be approximated rather well by means of a piece-wise linear mortality rate. A further theoretical advantage of the PWL model is that it enables a conceptual distinction between youth and old age (just as is possible

in the two-period Diamond (1965) model).

In Figure 3 we visualize the age profiles for the different variables at the cohort level. Cohort-level variables are obtained by multiplying individual outcomes for members of a given cohort by the relative population size of that cohort, e.g. for human wealth we have:

$$\hat{H}(u) \equiv l(v,t)\,\hat{\bar{h}}(u) = [\hat{w} - \hat{z}]\,\Omega(u,r),\tag{3.6}$$

where $l(v,t)$ is defined in (2.16) and $\Omega(u,\lambda)$ is given by:

$$\begin{aligned}\Omega(u,\lambda) &= be^{-nu-M(u)}\Delta(u,\lambda) \\ &= be^{(\lambda-n)u}\int_u^\infty e^{-\lambda\tau-M(\tau)}d\tau.\end{aligned}\tag{3.7}$$

Like $\Delta(u,\lambda)$, the $\Omega(u,\lambda)$-term depends critically on the parameters of the mortality process. In addition, however, $\Omega(u,\lambda)$ also depends on the birth rate b and the rate of population growth n because these parameters affect the population proportions of the cohorts.

The cohort-level values for consumption and financial wealth are defined as follows:

$$\hat{C}(u) \equiv l(v,t)\,\hat{\bar{c}}(u) = \frac{\hat{H}(0)}{\Delta(0,\theta)}e^{(r-\theta-n)u-M(u)},\tag{3.8}$$

$$\hat{A}(u) \equiv l(v,t)\,\hat{\bar{a}}(u) = \Delta(u,\theta)\hat{C}(u) - \hat{H}(u).\tag{3.9}$$

In the top right-hand panel of Figure 3 cohort-level human wealth is visualized for the different mortality models. For all models, cohort-level human wealth falls with the age of the cohort. This is not surprising since individual human wealth either stays the same (model 1) or falls (models 2-4) with age, and the population proportion falls with age (see top left-hand panel). As was the case for individual human wealth, the results for models 3-4 are very similar. This similarity also holds for the cohort-level results for consumption (bottom left-hand panel) and financial assets (bottom right-hand panel). Note that even for models 1 and 2, $\hat{A}(u)$ ultimately goes to zero for very old household as the decline in the population share starts to dominate the increase in individual asset holdings.

4 Visualizing Shocks with Realistic Demography

In this section we compute and visualize the effects on the different variables of a number of prototypical shocks affecting a small open economy.[22] The analytical ex-

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[22]These shocks do not have to be infinitesimal as no linearization techniques have been used.

pressions for the general demographic model are reported in the Appendix to this paper. To cut down on the number of illustrations, however, we restrict attention in this section to the visualization of the main contrasts between the standard Blanchard case and the PWL model. As was demonstrated above, the latter model captures the actual (expected) demography for the United States rather well.

4.1 Shocks

4.1.1 Balanced-budget fiscal policy

The first shock consists of an unanticipated and (believed to be) permanent increase in government consumption which is financed by means of lump-sum taxes (i.e. $d\hat{g} = d\hat{z} > 0$). The effects of this shock on individual human wealth ($\bar{h}(v,t)$) and financial assets ($\bar{a}(v,t)$) are illustrated in Figure 4. In that figure, the left-hand panels depict the Blanchard case whilst the right-hand panels illustrate the results for the PWL model.

In the Blanchard case, the increase in the lump-sum tax causes a once-off decrease in human wealth which is the same for all existing and future generations. In stark contrast, in the PWL model the fall in human wealth depends both on time and on the generations index. The top right-hand panel of Figure 4 shows the effects for two existing households (aged, respectively, 40 and 20 at the time of the shock) and two future households (born respectively one second and 40 years after the shock). As a result of the shock there is a *once-off* change in the age profile of human wealth. This profile itself does not depend on time because there is no transitional dynamics in after-tax wages.

In the bottom two panels of Figure 4 the paths for financial assets are illustrated. In the Blanchard case these assets rise monotonically over time for each household. The shock induces a slight kink (at time $t = 0$) in the profile for each generation. For the PWL model in the right-hand panel, the crowding-out effect due to the tax increase is much more visible. The peak in financial asset holdings is higher, the older the existing household is (compare, for example, the 40 and 20 year old households). The profiles for the future households born, respectively, in 0 and 40 years time are identical in shape (Again, this is because of the lack of transitional dynamics in after-tax wages).

4.1.2 Temporary tax cut

The second shock consists of a typical Ricardian equivalence experiment. At impact the lump-sum tax is reduced and deficit financing is used to balance the budget. As a result, the stock of government debt gradually increases over time. In order to en-

sure that government solvency is maintained, the tax is gradually increased over time and ultimately rises to a level higher than in the initial situation. The shock that is administered thus takes the following form (for $t \geq 0$):

$$dz(t) = -dz_0 e^{-\chi t} + d\hat{z}[1 - e^{-\chi t}], \tag{4.1}$$

where $0 < \chi \ll \infty$, $dz_0 > 0$, and $d\hat{z} = [(r-n)/\chi] dz_0 > 0$. At impact, the lump-sum tax *falls* by dz_0 but in the long run it *rises* by $d\hat{z}$. (The long-run effect on public debt equals $d\hat{d} = dz_0/\chi > 0$.) In the simulations, the persistence parameter is set at $\chi = 0.1$ implying that the tax reaches its pre-shock level only after about 13 to 14 years.[23]

The effects on human and financial wealth are illustrated for the two cases in Figure 5. In the Blanchard case, human wealth is age-independent. It nevertheless features transitional dynamics because the path of lump-sum taxes is time dependent. Human wealth increases at impact (because of the tax cut), but during transition it gradually falls again (because of the gradual tax increase). In the long run, the permanently higher taxes (needed to finance interest payments on accumulated debt) ensure that human wealth is less than before the shock.

In the PWL model, the effect on human wealth is both time- and age-dependent. At impact, all existing households experience an increase in their human wealth because of the tax cut. For each household, human wealth declines during transition both because of ageing (gradual increase in the annuity rate of interest) and because the tax rises over time. For the future household born 40 years after the shock, the human wealth profile is virtually in the steady state again as most of the shock has worn out by then.

In the bottom panels of Figure 5 the profiles for financial assets are illustrated. In the Blanchard case the tax cut causes a slight acceleration in asset accumulation at impact. This kink also occurs for the PWL model in the bottom right-hand right panel. The PWL case illustrates quite clearly that the Ricardian equivalence experiment re-distributes resources from distant future generations toward near future and existing generations. Especially members of the generation born at the time of the shock react strongly to the tax cut as far as their savings behaviour is concerned. Indeed, their maximum asset holding peaks at a much higher level than that of 40 year old existing

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[23]We compute time period t_0 such that $dz(t_0) = 0$. Using (4.1) we find:

$$t_0 = -\frac{1}{\chi} \ln \left(\frac{r-n}{r-n+\chi} \right).$$

For the piece-wise linear case $t_0 = 13.2$ years whilst for the Blanchard case we find $t_0 = 14.2$ years.

generations and generations born 40 years after the shock.[24]

4.1.3 Interest rate shock

The final shock analyzed in this paper consists of an unanticipated and permanent increase in the world interest rate (i.e. $dr > 0$ for $t \geq 0$). The effects of this shock on human and financial wealth are illustrated in Figure 6. In the Blanchard case the shock causes a once-off decrease in age-independent human wealth. The higher annuity rate of interest leads to stronger discounting of future after-tax wages. For the PWL model there is a once-off downward shift in the age profile of human wealth. Like the shock itself, this age profile displays no further transitional dynamics over time.

The bottom panels of Figure 6 illustrate the effects on financial assets. Whilst the effects for the Blanchard case speak for themselves, those for the PWL model warrant some further comment. For *future* generations, the age profile of financial assets features a once-off upward shift at impact and displays no further transitional dynamics thereafter. In contrast, for *existing* generations the time path of assets depends both on their age and on time. This transitional dynamics is caused by the fact that the consumption path for such generations depends on both t and v separately (see Appendix). Existing generations are affected by the interest rate hike both via their human wealth and via their accumulated financial assets which attract a higher rate of return after the shock.[25]

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[24]The following temporary productivity shock features results that are very simular to those of the Ricardian tax cut:

$$dw(t) = dw_0 e^{-\xi t}, \quad (\text{for } t \geq 0),$$

where $0 < \xi \ll \infty$ and $dw_0 > 0$. In the simulations (not shown), the persistence parameter is set at $\xi = 0.1$, implying a half-life of the adjustment of about $(1/\xi)\ln 2 = 6.93$ years. The equivalency between the two shocks is not surprising, of course, because the temporary wage increases boosts human wealth just as a temporary tax cut does.

[25]The bottom right-hand panel of Figure 6 also shows a slightly unattractive feature of the piece-wise linear model, namely that *individual* assets start to rise again after about age 100. This is due to the fact that the mortality rate does not rise sufficiently quickly after about age 85 for that model–see Figure 1. As a result, human wealth does not fall quickly enough (see Figure 2) and assets start to rise again at high ages. Figure 3 confims, however, that assets of the old *cohorts* approach zero for the piece-wise linear model. There are very few centenarians in the piece-wise linear model.

4.2 Welfare effects

The Blanchard model is often used to investigate the intergenerational welfare effects of various policy measures.[26] In this section we visualize the intergenerational welfare effects associated with the three shocks studied above. For existing households, the change in welfare from the perspective of the shock period $t = 0$ is evaluated ($d\Lambda(v, 0)$ for $v \leq 0$) whereas for future agents the welfare change from the perspective of their birth date is computed ($d\Lambda(v, v)$ for $v > 0$). As is shown in the Appendix, the welfare effect for existing agents ($v \leq 0$) can be written as:

$$d\Lambda(v, 0) = dr \int_0^\infty \tau e^{-\theta\tau - M(\tau-v) + M(-v)} d\tau$$
$$+ \Delta(-v, \theta) \ln \Gamma_E(v), \quad \text{(for } v \leq 0\text{)}, \quad (4.2)$$

where $\Delta(-v, \theta)$ is defined in equation (2.12) above and where $\Gamma_E(v)$ is defined as:

$$\Gamma_E(v) \equiv \frac{\hat{\bar{a}}(-v) + \bar{h}(v, 0)}{\hat{\bar{a}}(-v) + \hat{\bar{h}}(-v)}, \quad \text{(for } v \leq 0\text{)}. \quad (4.3)$$

Intuitively, $\Gamma_E(v)$ captures the effect of the impact change in human wealth for existing generations. The welfare effect consists of two separate components. The first term on the right-hand side of (4.2) represents the *consumption growth effect* and is only relevant for the world interest rate shock (i.e., if $dr > 0$). Individual consumption growth is equal to $r - \theta$ and an increase in r leads to a steeper consumption time profile. The mortality process exerts a non-trivial influence on the consumption growth effect via the utility function. The second term on the right-hand side of (4.2) summarizes the welfare effect of the change in the *level* of consumption caused by the impact change in human wealth. This *human wealth effect* is relevant for all shocks and is equal to the product of $\ln \Gamma_E(v)$ (defined in (4.3)) and the inverse propensity to consume $\Delta(-v, \theta)$.

The welfare effect for future generations can be written as:

$$d\Lambda(v, v) = dr \int_0^\infty s e^{-[\theta s + M(s)]} ds + \Delta(0, \theta) \ln \Gamma_F(v), \quad \text{(for } v > 0\text{)}, \quad (4.4)$$

where $\Delta(0, \theta)$ is the inverse propensity to consume of a newborn and $\Gamma_F(v)$ is defined as:

$$\Gamma_F(v) \equiv \frac{\bar{h}(v, v)}{\hat{\bar{h}}(0)}, \quad \text{(for } v > 0\text{)}. \quad (4.5)$$

---

[26]See, for example, Bovenberg (1993, 1994) on capital taxation and investment subsidies, Bettendorf and Heijdra (2001a, 2001b) on product subsidies and tariffs under monopolistic competition, and Heijdra and Meijdam (2002) on government infrastructure. All these studies are set in the context of a small open economy.

Here, $\Gamma_F(v)$ represents the effect on the human wealth of a future newborn. Just as for existing generations, the welfare effect for future generations consists of a consumption growth effect (first term on the right-hand side of (4.5)) and a human wealth effect (second term).

The welfare effects of the different shocks are illustrated in Figure 7. The left-hand panels present the results for the Blanchard case whilst the right-hand panels visualize those for the PWL model. The welfare effects of balanced-budget fiscal policy are illustrated in the top panels. All present and future generations experience a reduction in human wealth and as a result the welfare effect is negative for all generations. The effect is the same for all future generations because there is no transitional dynamics in human wealth (see above). For existing generations the welfare loss declines with the age of the generation. The human wealth effect decreases with age because both the inverse propensity to consume ($\Delta(-v, \theta)$) and the relative importance of human wealth ($\ln \Gamma_E(v)$ in (4.2) above) decline with age. The Blanchard and PWL models thus give *qualitatively* similar welfare results for the spending shock. A key difference between the two models concerns the slope of the welfare profile for existing generations. In the PWL model (right-hand panel) the welfare effect is practically zero for all generations older than 100 years. In contrast, for the Blanchard case (left-hand panel) there is still a noticeable welfare effect for 200 year old generations. This low "generational adjustment speed" of the Blanchard model is also observed for the other shocks. Intuitively, in the Blanchard case, old generations are not killed off rapidly enough (see also the top panel of Figure 1).

The middle two panels of Figure 7 illustrate the welfare effects for the Ricardian tax cut experiment. All existing generations as well as future generations born close to the time of shock benefit at the expense of more distant future generations. For future generations the welfare loss is larger the later they are born. For existing generations the welfare profile is monotonically decreasing in age for the Blanchard case but non-monotonic for the PWL model. In the Blanchard case, $\Delta(-v, \theta) = \Delta(0, \theta) = 1/(\theta + \mu_0)$ is constant and $\ln \Gamma_E(v)$ declines monotonically with age. In contrast, for the PWL model, $\Delta(-v, \theta)$ decreases with age but $\ln \Gamma_E(v)$ is non-monotonic. Indeed, $\ln \Gamma_E(v)$ is increasing in age for all generations up to about 120 years and only decreases in age thereafter.[27] As a result, the welfare profile for existing generations

---

[27]Of course, there are virtually no centenarians predicted by the PWL model so the downward sloping part of the $\ln \Gamma_E(v)$ function is practically irrelevant. In contrast, the estimated Blanchard demography predicts that about 50 percent of newborns will still be alive at age 100. See the bottom panel of Figure 1.

displays a bump around the age of 60 in the middle right-hand panel of Figure 7. At that point, the drop in $\Delta\left(-v, \theta\right)$ just matches the increase in $\ln\Gamma_E\left(v\right)$.

In the bottom two panels of Figure 7 the welfare effects for the interest rate shock are illustrated. Since the shock induces no transitional dynamics in the age profile of human wealth for future generations, the welfare effect is the same for all future generations in both models. For existing generations the welfare effect increases with age in the Blanchard model, but is non-monotonic for the PWL model. For an interest shock both the consumption growth effect and the human wealth effect are relevant. The shock induces a decrease in $\ln\Gamma_E\left(v\right)$ which falls with age in both models. In the Blanchard case, the consumption growth effect is constant (and positive) for all generations. In contrast, for the PWL model, the consumption growth effect is positive and constant for future generations, but falling in age for existing generations. As a result, the total effect on welfare displays a bump around the age of 25 for the PWL model (see the bottom right-hand panel of Figure 7).

4.3 Aggregate effects

As was pointed out above, Blanchard (1985) assumes a constant mortality rate in order to allow for exact aggregation of the consumption function. With the more general mortality processes considered in this paper, only numerical aggregation is possible. This subsection visualizes the *aggregate* effects on the key variables of the three shocks considered above. To what extent do the aggregate results predicted by the Blanchard and PWL models differ?

In Figure 8 we illustrate the effects on human wealth (first row), consumption (second row), and financial assets (third row) for the spending shock (first column), the Ricardian tax cut (second column), and the interest rate shock (third column). To facilitate the comparisons between the two models, we report the percentage deviations from the steady state for all variables, i.e. $(h\left(t\right) - \hat{h})/\hat{h}$, $(c\left(t\right) - \hat{c})/\hat{c}$, are plotted $(a\left(t\right) - \hat{a})/\hat{a}$ in Figure 8.

For the spending shock, the results for human wealth are identical and those for consumption and financial assets are qualitatively very similar but differ in terms of the speed of adjustment towards the new steady state. The slow speed of convergence is also a feature of the Blanchard results for the other two shocks.

For the Ricardian tax cut, the effects on human wealth are again similar but those on consumption and financial wealth are not. For the PWL model, the impact effect on consumption is much larger, and the slope of the aggregate Euler equation is much steeper during transition, than for the Blanchard model. Similarly, the savings response

is much more pronounced for the PWL model.

Finally, for the interest rate shock the effect on human wealth is qualitatively the same for the two models, though the Blanchard model overestimates the fall in human wealth. The impact reduction in consumption is virtually the same for the two models but transition is much faster for the PWL model. Again, the savings response at impact is stronger for the PWL model.

4.4 Discussion

The key findings of this section are as follows. Incorporating a realistic demographic structure is quite feasible in the context of a small open economy facing a constant world interest rate. At the level of individual households, a realistic description of the mortality process reinstates the classic life-cycle consumption-saving insights of Modigliani and co-workers.

The welfare effects associated with the different shocks are also potentially affected in a non-trivial manner by the incorporation of a more realistic demography. Two key difference stand out between the Blanchard and PWL models. First, the PWL model predicts a much faster (and in our view more realistic) "generational convergence speed" of the welfare effects than the Blanchard model. Second, the PWL model incorporates more extensive age-dependency and as a result may give rise to non-monotonic welfare effect on existing generations—something which is impossible in the Blanchard case (for the shocks studied).

Finally, we have demonstrated that the demographic details do not "wash out" at the aggregate level. The impulse-response functions for the different shocks are quite different for the Blanchard and PWL models, especially the ones for per capita consumption and financial assets.

In some applications of our model, it may the case that individual behaviour depends in part on aggregate variables so that knowledge of the latter is crucial. For example, if the revenue of a consumption tax (t_C) is recycled in a lump-sum fashion to households (i.e. $\bar{z}(t) = z(t) = -t_C c(t)$) then *individual* consumption, human wealth, and financial assets will all depend on the *aggregate* tax revenue. This complication can be easily dealt with by using an iterative procedure in the simulations. In the first step the initial tax revenue and implied lump-sum transfer are guessed and individual and aggregate consumption levels are computed. In subsequent steps, the aggregate information is used to update the guess for transfers until convergence is achieved.

5 Extensions and Conclusion

The framework developed in this paper can be extended in a number of directions, all of which we plan to pursue in the near future. First, in order to investigate the effects of demographic change, it is necessary to generalize the stochastic distribution for expected remaining lifetimes. Two possibilities can be distinguished. *Embodied* demographic change can be studied by writing the density function as $\phi(v, s)$, so that both the cumulative distribution, $\Phi(v, s)$, and the instantaneous mortality rate, $m(v, s)$, are generation specific. In contrast, *disembodied* demographic change can be modelled by writing the functions as $\phi(t, s)$, $\Phi(t, s)$, and $m(t, s)$, i.e. by postulating a time-dependent mortality process.

Second, the age profile for individual consumption could be generalized by introducing shift factors in the utility function. In the current model (with $r > \theta$) consumption is increasing in the age of the household. There are reasons to believe that in reality consumption is hump-shaped, i.e. $\bar{c}(v, t)$ features a rising time profile early on in life followed by a falling profile later on. A simple way to capture this effect is to assume that a household's "needs" get smaller the older they get. In the diminishing-needs model, lifetime utility is given by:

$$\Lambda(v, t) \equiv e^{M(t-v)} \int_t^{\infty} \left[\frac{\bar{e}(v, \tau)^{1-1/\sigma} - 1}{1 - 1/\sigma} \right] e^{-[\theta(\tau - t) + M(\tau - v)]} d\tau, \qquad (5.1)$$

where $\sigma > 0$ is the intertemporal substitution elasticity and $\bar{e}(v, \tau)$ is *effective* consumption:

$$\bar{e}(v, \tau) \equiv \bar{c}(v, \tau) \exp\left\{ \frac{\zeta_0 (\tau - v)^{1+\zeta_1}}{1 + \zeta_1} \right\}, \qquad (5.2)$$

with $\zeta_0 > 0$ and $\zeta_1 > 0$. According to (5.2), a given amount of *actual* consumption, $\bar{c}(v, \tau)$, yields more effective consumption (featuring in the felicity function), the older the household is. Using this specification of preferences, it is straightforward to show that the individual consumption Euler equation (2.9) is generalized to:

$$\frac{\dot{\bar{c}}(v, \tau)}{\bar{c}(v, \tau)} = \sigma(r - \theta) - (1 - \sigma)\zeta_0 (\tau - v)^{\zeta_1}. \qquad (5.3)$$

For the empirically relevant case (with $0 < \sigma < 1$), consumption rises during the early phase of life ($\tau - v$ low) and falls during the later stages of life ($\tau - v$ high).

A third extension endogenizes the household's labour supply and retirement decisions. The introduction of a leisure choice decision is straightforward. Focusing on a

unitary intertemporal substitution elasticity, lifetime utility is written as:

$$\Lambda(v,t) \equiv e^{M(t-v)} \int_t^\infty \ln U\left[\bar{c}(v,\tau), 1 - \bar{n}(v,\tau)\right] e^{-[\theta(\tau-t)+M(\tau-v)]} d\tau, \quad (5.4)$$

where $U[\cdot]$ is subfelicity depending on consumption, $\bar{c}(v,\tau)$, and labour supply, $\bar{n}(v,\tau)$. The time endowment equals 1. Of course, labour supply features the restriction $0 \leq \bar{n}(v,\tau) \leq 1$, with the lower bound reflecting the retirement decision.[28] With endogenous labour supply, the household budget identity (2.7) is modified to:

$$\dot{\bar{a}}(v,\tau) = [r + m(\tau - v)]\bar{a}(v,\tau) + \bar{w}(\tau)\bar{n}(v,\tau) - \bar{z}(v,\tau) - \bar{c}(v,\tau). \quad (5.5)$$

where $\bar{w}(\tau)\bar{n}(v,\tau)$ is wage income and $\bar{z}(v,\tau)$ represents an age-dependent lump-sum tax (e.g. a pay-as-you-go pension system). For a small open economy facing a constant world interest rate it is straightforward to compute the optimal *retirement age* implied by the model and to study how it is affected by various shocks.[29] The most interesting shocks that can be studied with this extended model are ageing shocks and pension reform.

Whereas the first three extensions are relatively straightforward, the fourth and final one is not. The introduction of a realistic mortality process in a closed economy is complicated by the fact that exact aggregation of the consumption function is impossible (see above). Of course, the steady state can still be characterized analytically quite easily (see Subsection 2.3 above). The transitional and long-run effects of various shocks are, however, much more difficult to compute due to the fact that equilibrium factor prices will generally change. In the near future we wish to investigate whether *approximate* aggregation of the key behavioral relationships is feasible for particular shock parameterizations. If that fails, numerical methods will be employed to characterize transitional dynamics.

In conclusion, we express the sincere hope that the Blanchard-Yaari-Modigliani model constructed in this paper will prove to be a useful addition to the toolbox of both theoretical economists and policy practitioners alike. At least in the context of a small open economy, there is no justification whatsoever to use models based on a blatantly unrealistic description of demography. Had mortality not caught up with him, Benjamin Gompertz would probably support that conclusion!

---

[28]Under the twin assumptions that (i) consumption and leisure are both normal goods and (ii) that $\bar{n}(v,v) < 1$ (newborns consume some leisure), the upper bound can be ignored as it is always satisfied.

[29]The retirement *date* is that time period, t^R, for which $\bar{n}(v,t^R)$ just becomes equal to zero. The retirement *age* is then defined as $t^R - v$. Provided $\bar{n}(v,t)$ is decreasing in $t - v$, all agents older that $t^R - v$ are retired also.

Appendix

In this brief appendix we derive some key results used in the paper. More detailed derivations are presented in Heijdra and Romp (2005).

Proof of Lemma 1

By definition, $M(u) \equiv \int_0^u m(s)\, ds$ so that $M(0) = 0$, $M'(u) = m(u) \geq 0$, and $M''(u) = m'(u) \geq 0$. Since $M(s)$ is a convex function of s we have $M(s) \geq M(u) + m(u)[s-u]$ and thus:

$$\Delta(u, \lambda) \leq \tilde{\Delta}(u, \lambda) \equiv e^{\lambda u + M(u)} \int_u^\infty e^{-[\lambda s + m(u)(s-u) + M(u)]}\, ds$$
$$= \frac{1}{\lambda + m(u)}. \tag{A.6}$$

This establishes part (iii). Part (i) follows by straightforward differentiation:

$$\frac{\partial \Delta(u, \lambda)}{\partial \lambda} = -e^{\lambda u + M(u)} \int_u^\infty [s-u] e^{-[\lambda s + M(s)]}\, ds < 0. \tag{A.7}$$

Similarly, part (ii) is obtained by differentiating $\Delta(u, \lambda)$ with respect to u:

$$\frac{\partial \Delta(u, \lambda)}{\partial u} = [\lambda + m(u)] \Delta(u, \lambda) - 1 < 0, \tag{A.8}$$

where the sign follows from (A.6). Parts (iv)-(v) are obvious. Q.E.D.

Macroeconomic shocks

All the shocks studied (or mentioned) in Section 4 of the paper can be expressed in terms of the following functions:

$$w(t) = \begin{cases} \hat{w} & \text{for } t < 0 \\ \hat{w} + dw_0 e^{-\xi t} & \text{for } t \geq 0 \end{cases}, \tag{A.9}$$

$$r(t) = \begin{cases} r & \text{for } t < 0 \\ r_N \equiv r + dr & \text{for } t \geq 0 \end{cases}, \tag{A.10}$$

$$g(t) = \begin{cases} 0 & \text{for } t < 0 \\ d\hat{g} & \text{for } t \geq 0 \end{cases}, \tag{A.11}$$

$$z(t) = \begin{cases} 0 & \text{for } t < 0 \\ -dz_0 e^{-\chi t} + d\hat{z}[1 - e^{-\chi t}] & \text{for } t \geq 0 \end{cases}. \tag{A.12}$$

Government consumption $d\hat{g}$ and the path of government debt are related to the other parameters according to:

$$d\hat{g} = \frac{\chi d\hat{z}}{r-n+\chi} - \frac{(r-n)\,dz_0}{r-n+\chi}, \tag{A.13}$$

$$d(t) = \frac{d\hat{g}+dz_0}{\chi}[1-e^{-\chi t}]. \tag{A.14}$$

The time at which the shock occurs is normalized to zero.

The three shocks explicitly studied in the text are:

- Unanticipated and permanent balanced-budget increase in government consumption: $g(t)$ set as in (A.11), $z(t)$ set according to (A.12) and (A.13) with $\chi \to \infty$, i.e. $d\hat{g} = d\hat{z}$. No debt financing occurs, i.e. $d(t) = 0$ for all $t \geq 0$.

- Ricardian equivalence experiment, temporary tax cut: $g(t) = 0$, $z(t)$ set according to (A.12) and (A.13) with $0 < \chi \ll \infty$, and the (stable) path of debt is set according to (A.14).

- Unanticipated and permanent increase in the world interest rate: $dr > 0$ for $t \geq 0$.

A fourth shock is only mentioned because its effects are very similar to those of the temporary tax cut:

- Temporary productivity shock: $g(t) = z(t) = d(t) = 0$, $w(t)$ set according to (A.9) with $0 < \xi \ll \infty$.

Post-Shock Profiles

The steady-state age profiles for the different variables before the shock occurs ($t < 0$) are defined for individual households in (2.24)-(2.26) and for cohort-level variables in (3.6) and (3.8)-(3.9). After the shock occurs ($t \geq 0$), the paths for individual and cohort-level human wealth are, respectively,

$$\bar{h}(v,t) = \hat{w}\Delta(t-v,r_N) + dw_0 e^{-\xi t}\Delta(t-v,r_N+\xi)$$
$$- d\hat{z}\Delta(t-v,r_N) + [dz_0 + d\hat{z}]e^{-\chi t}\Delta(t-v,r_N+\chi), \tag{A.15}$$

and:

$$H(v,t) = \hat{w}\Omega(t-v,r_N) + dw_0 e^{-\xi t}\Omega(t-v,r_N+\xi)$$
$$- d\hat{z}\Omega(t-v,r_N) + [dz_0 + d\hat{z}]e^{-\chi t}\Omega(t-v,r_N+\chi). \tag{A.16}$$

For households who were born before the shock ($v < 0$), the age index at the time of the shock is $-v > 0$. For such households, the paths for consumption and asset holdings (at individual and cohort level) after the shock ($t \geq 0$) are given by:

$$\bar{c}_E(v,t) = \frac{\hat{\bar{a}}(-v) + \bar{h}(v,0)}{\Delta(-v,\theta)} e^{(r_N - \theta)t}, \tag{A.17}$$

$$\bar{a}_E(v,t) = \Delta(t-v,\theta)\bar{c}_E(v,t) - \bar{h}(v,t), \tag{A.18}$$

$$C_E(v,t) = e^{M(-v)} \frac{\hat{A}(-v) + H(v,0)}{\Delta(-v,\theta)} e^{(r_N - \theta - n)t - M(t-v)}, \tag{A.19}$$

$$A_E(v,t) = \Delta(t-v)C_E(v,t) - H(v,t), \tag{A.20}$$

where the subscript "E" denotes *existing* households (at the time of the shock).

For households that are born after the shock ($v \geq 0$), the relevant age index at time t ($\geq v$) is defined as $t - v$. For such households the paths for consumption and asset holdings (at individual and cohort level) at time t (≥ 0) are given by:

$$\bar{c}_F(v,t) = \frac{\bar{h}(v,v)}{\Delta(0,\theta)} e^{(r_N - \theta)(t-v)}, \tag{A.21}$$

$$\bar{a}_F(v,t) = \Delta(t-v,\theta)\bar{c}_F(v,t) - \bar{h}(v,t), \tag{A.22}$$

$$C_F(v,t) = \frac{H(v,v)}{\Delta(0,\theta)} e^{(r_N - \theta - n)(t-v) - M(t-v)}, \tag{A.23}$$

$$A_F(v,t) = \Delta(t-v,\theta)C_F(v,t) - H(v,t), \tag{A.24}$$

where the subscript "F" denotes *future* households.

Welfare Effects

The welfare effects of the different shocks are illustrated in Figure 7 in the text. For existing agents the change in welfare from the perspective of the shock period $t = 0$ is evaluated ($d\Lambda(v,0)$ for $v \leq 0$) whereas for future agents the welfare change from the perspective of their birth date is computed ($d\Lambda(v,v)$ for $v > 0$).

Existing generations

Equation (4.2) is derived as follows. The effect on welfare of existing agents at $t = 0$ can be written as a function of their age at that moment ($-v$):

$$d\Lambda(v,0) = \int_0^\infty [\ln \bar{c}_E(v,\tau) - \ln \hat{\bar{c}}(v,\tau)] e^{-\theta\tau - M(\tau-v) + M(-v)} d\tau. \tag{A.25}$$

Consumption after the shock can be written in terms of pre-shock consumption:

$$
\begin{aligned}
\bar{c}_E(v,\tau) &= \frac{\hat{\bar{a}}(-v) + \bar{h}(v,0)}{\Delta(-v,\theta)} e^{(r_N - \theta)\tau} \\
&= e^{(r_N - r)\tau} \left[\frac{\hat{\bar{a}}(-v) + \hat{\bar{h}}(-v)}{\Delta(-v,\theta)} e^{(r-\theta)\tau} + \frac{\bar{h}(v,0) - \hat{\bar{h}}(-v)}{\Delta(-v,\theta)} e^{(r-\theta)\tau} \right] \\
&= e^{(r_N - r)\tau} \left[\frac{\hat{\bar{a}}(-v) + \hat{\bar{h}}(-v)}{\Delta(-v,\theta)} + \frac{\bar{h}(v,0) - \hat{\bar{h}}(-v)}{\Delta(-v,\theta)} \right] e^{(r-\theta)\tau} \\
&= e^{(r_N - r)\tau} \left[1 + \frac{\bar{h}(v,0) - \hat{\bar{h}}(-v)}{\hat{\bar{a}}(-v) + \hat{\bar{h}}(-v)} \right] \hat{\bar{c}}(-v) e^{(r-\theta)\tau} \\
&= e^{(r_N - r)\tau} \left[\frac{\hat{\bar{a}}(-v) + \bar{h}(v,0)}{\hat{\bar{a}}(-v) + \hat{\bar{h}}(-v)} \right] \hat{\bar{c}}(v,\tau). \tag{A.26}
\end{aligned}
$$

By taking logarithms of (A.26) and rewriting we obtain:

$$
\ln \bar{c}_E(v,\tau) - \ln \hat{\bar{c}}(v,\tau) = (r_N - r)\,\tau + \ln \Gamma_E(v), \tag{A.27}
$$

where $\Gamma_E(v)$ is defined in (4.3). By substituting (A.27) into (A.25) and splitting the integral we get:

$$
\begin{aligned}
d\Lambda(v,0) &= dr \int_0^\infty \tau e^{-\theta\tau - M(\tau - v) + M(-v)} d\tau \\
&\quad + \left[e^{M(-v)} \int_0^\infty e^{-\theta\tau - M(\tau - v)} d\tau \right] \ln \Gamma_E(v) \\
&= dr \int_0^\infty \tau e^{-\theta\tau - M(\tau - v) + M(-v)} d\tau \\
&\quad + \left[e^{-\theta v + M(-v)} \int_{-v}^\infty e^{-[\theta s + M(s)]} ds \right] \ln \Gamma_E(v) \\
&= dr \int_0^\infty \tau e^{-\theta\tau - M(\tau - v) + M(-v)} d\tau + \Delta(-v,\theta) \ln \Gamma_E(v). \tag{A.28}
\end{aligned}
$$

Equation (A.28) coincides with (4.2) in the text.

Future generations

Equation (4.4) is derived as follows. For future households the welfare effect at birth is defined as:

$$
d\Lambda(v,v) = \int_v^\infty [\ln \bar{c}_F(v,\tau) - \ln \hat{\bar{c}}(v,\tau)] e^{-\theta[\tau - v] - M(\tau - v)} d\tau. \tag{A.29}
$$

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Next we express the post-shock consumption path in terms of the pre-shock path as:

$$
\begin{aligned}
\bar{c}_F(v, \tau) &= \frac{\bar{h}(v, v)}{\Delta(0, \theta)} e^{(r_N - \theta)(\tau - v)} \\
&= e^{(r_N - r)(\tau - v)} \frac{\hat{h}(v, v)}{\hat{\bar{h}}(0)} \frac{\hat{\bar{h}}(0)}{\Delta(0, \theta)} e^{(r - \theta)(\tau - v)} \\
&= e^{(r_N - r)(\tau - v)} \frac{\bar{h}(v, v)}{\hat{\bar{h}}(0)} \hat{\bar{c}}(v, \tau).
\end{aligned}
\tag{A.30}
$$

By taking logarithms of (A.30) and rewriting we obtain:

$$
\ln \bar{c}_F(v, \tau) - \ln \hat{\bar{c}}(v, \tau) = (r_N - r)(\tau - v) + \ln \Gamma_F(v),
\tag{A.31}
$$

where $\Gamma_N(v)$ is defined in (4.5). By substituting (A.31) into (A.29) and splitting the integral we get:

$$
\begin{aligned}
d\Lambda(v, v) &= dr \int_v^\infty (\tau - v) e^{-\theta(\tau - v) - M(\tau - v)} d\tau \\
&\quad + \left[\int_v^\infty e^{-\theta(\tau - v) - M(\tau - v)} d\tau \right] \ln \Gamma_F(v) \\
&= dr \int_0^\infty s e^{-[\theta s + M(s)]} ds + \left[\int_0^\infty e^{-[\theta s + M(s)]} ds \right] \ln \Gamma_F(v) \\
&= dr \int_0^\infty s e^{-[\theta s + M(s)]} ds + \Delta(0, \theta) \ln \Gamma_F(v).
\end{aligned}
\tag{A.32}
$$

Equation (A.32) coincides with (4.4) in the text.

Table 1: Estimated Survival Functions

	$\hat{\mu}_0$	$\hat{\mu}_1$	$\hat{\mu}_2$	$\hat{\bar{u}}$	$\hat{\sigma}$	$\hat{n}(b)$	$1 - \widehat{\Phi}(100)$
1. *Constant* $M(u) = \mu_0 u$	0.7026×10^{-2} (4.92)	–	–	–	0.2277	0.80	49.53
2. *Linear* $M(u) = \mu_0 u + \mu_1^2 u^2$	-0.8970×10^{-2} (−3.83)	0.0152 (12.29)	–	–	0.1199	–	–
	–	0.0104 (13.66)	–	–	0.1595	0.49	34.05
3. *Piece-wise linear (PWL)* $M(u) = \mu_0 u + \delta(u)\mu_1^2(u - \bar{u})^2$ $\delta(u) = \begin{cases} 0 & \text{for } 0 < u < \bar{u} \\ 1 & \text{for } u \geq \bar{u} \end{cases}$	0.1544×10^{-2} (6.41)	0.0410 (16.12)	–	60.85 (43.08)	0.0294	0.37	6.57
4. *Gompertz-Makeham (GM)* $M(u) = \mu_0 u + (\mu_1/\mu_2)[e^{\mu_2 u} - 1]$	0.5834×10^{-3} (24.76)	0.3419×10^{-4} (27.01)	0.0928 (193.71)	–	0.0018	0.37	1.69
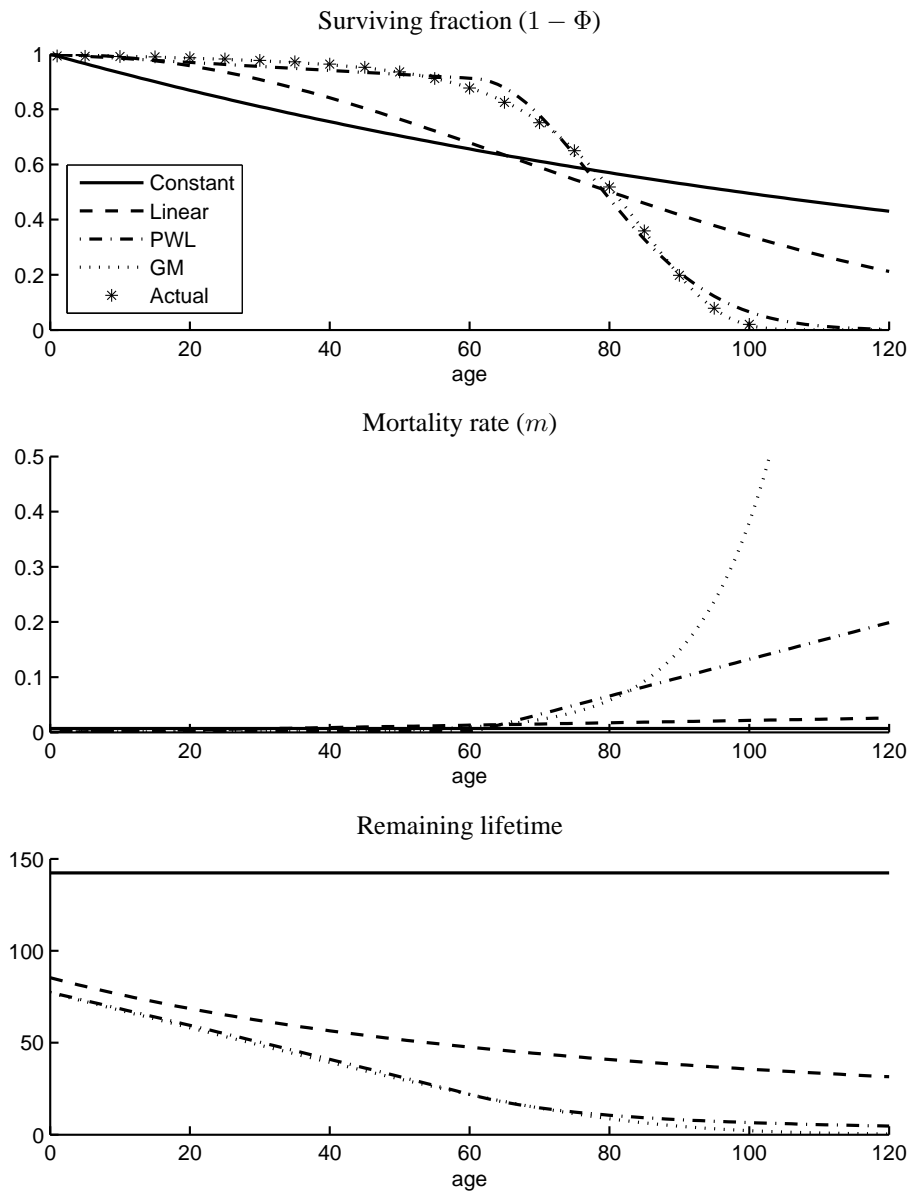
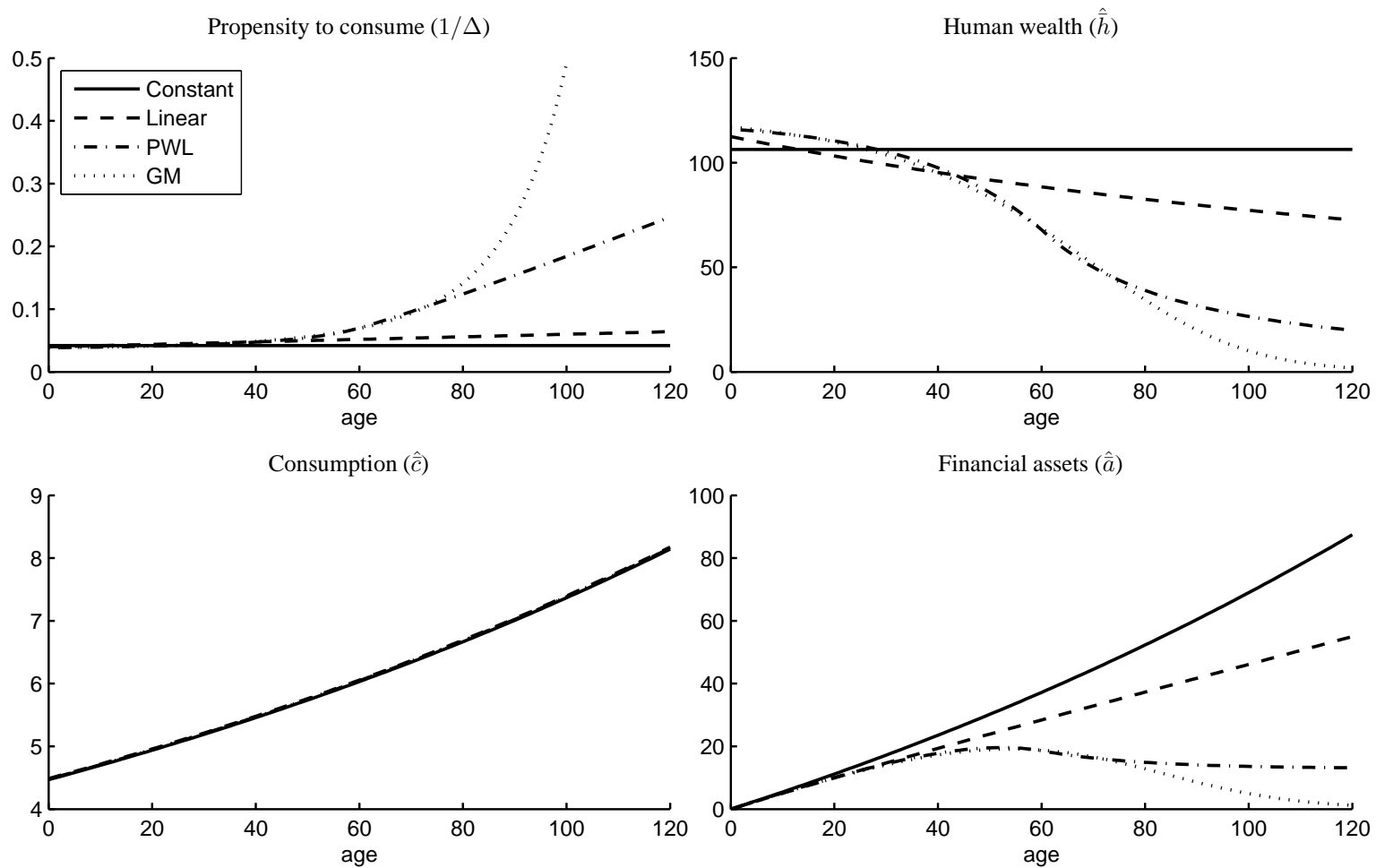Figure 1: Actual and Estimated Survival Rates

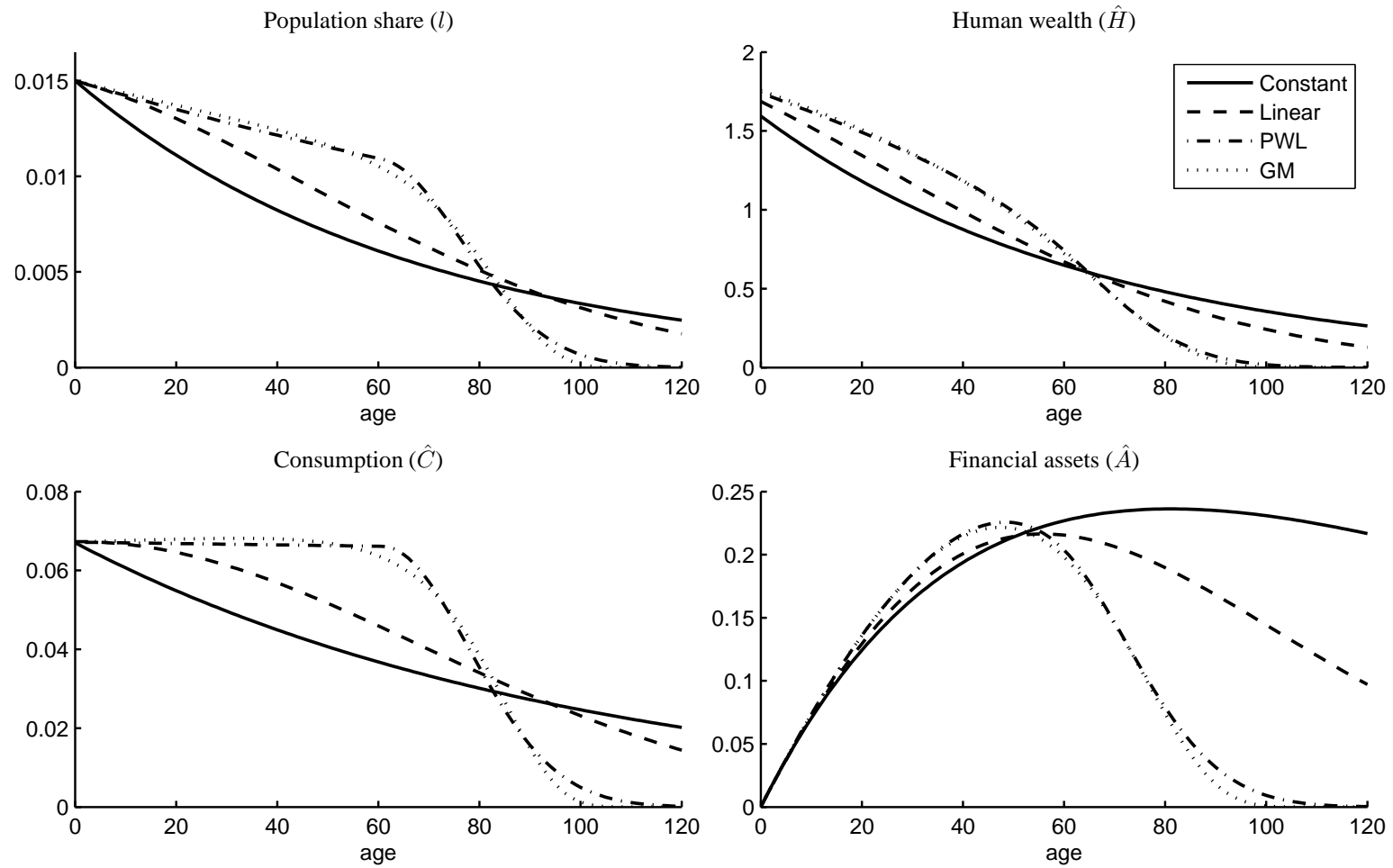Figure 2: Steady-State Profiles for Individuals
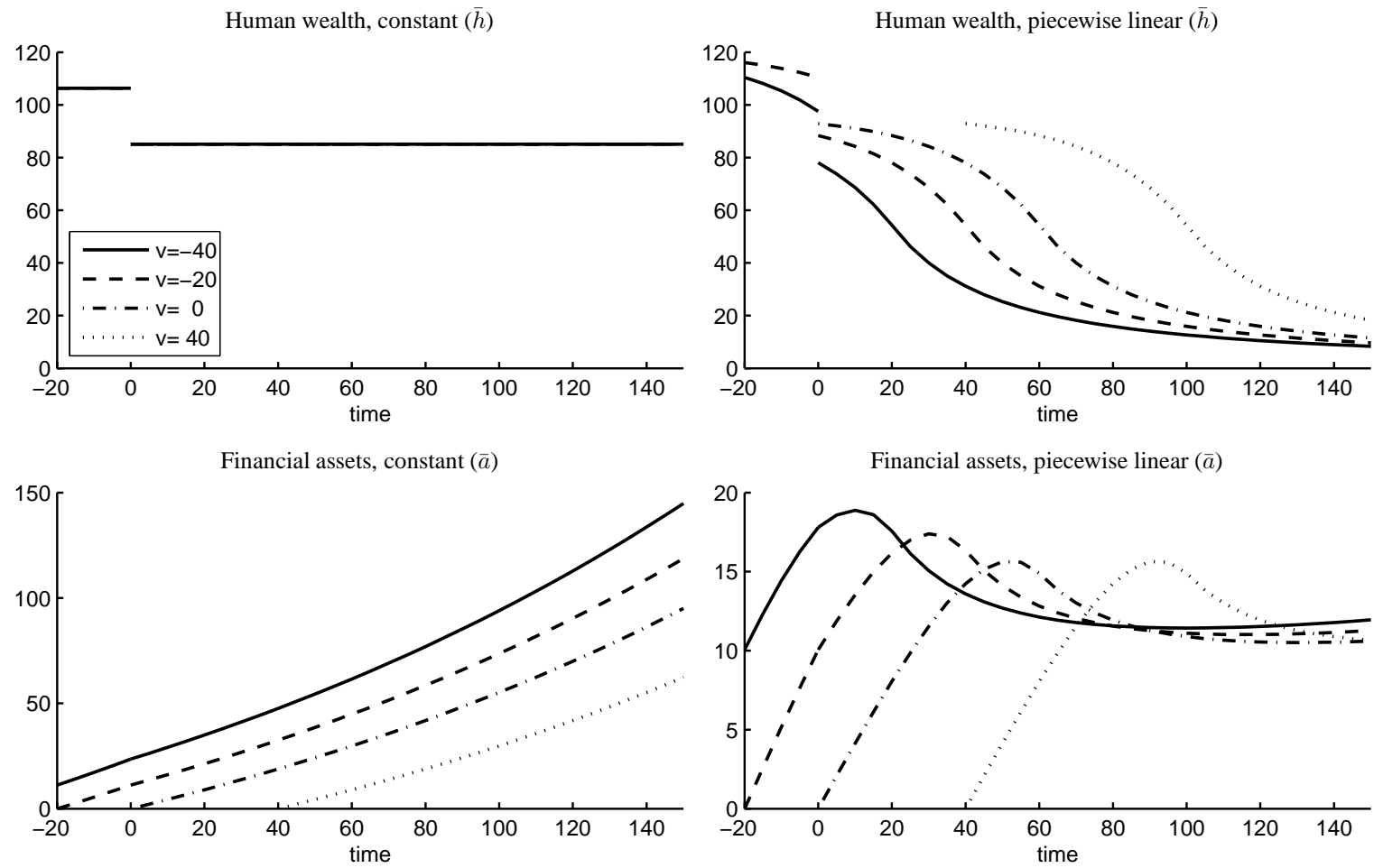
Figure 3: Steady-State Profiles for Cohorts

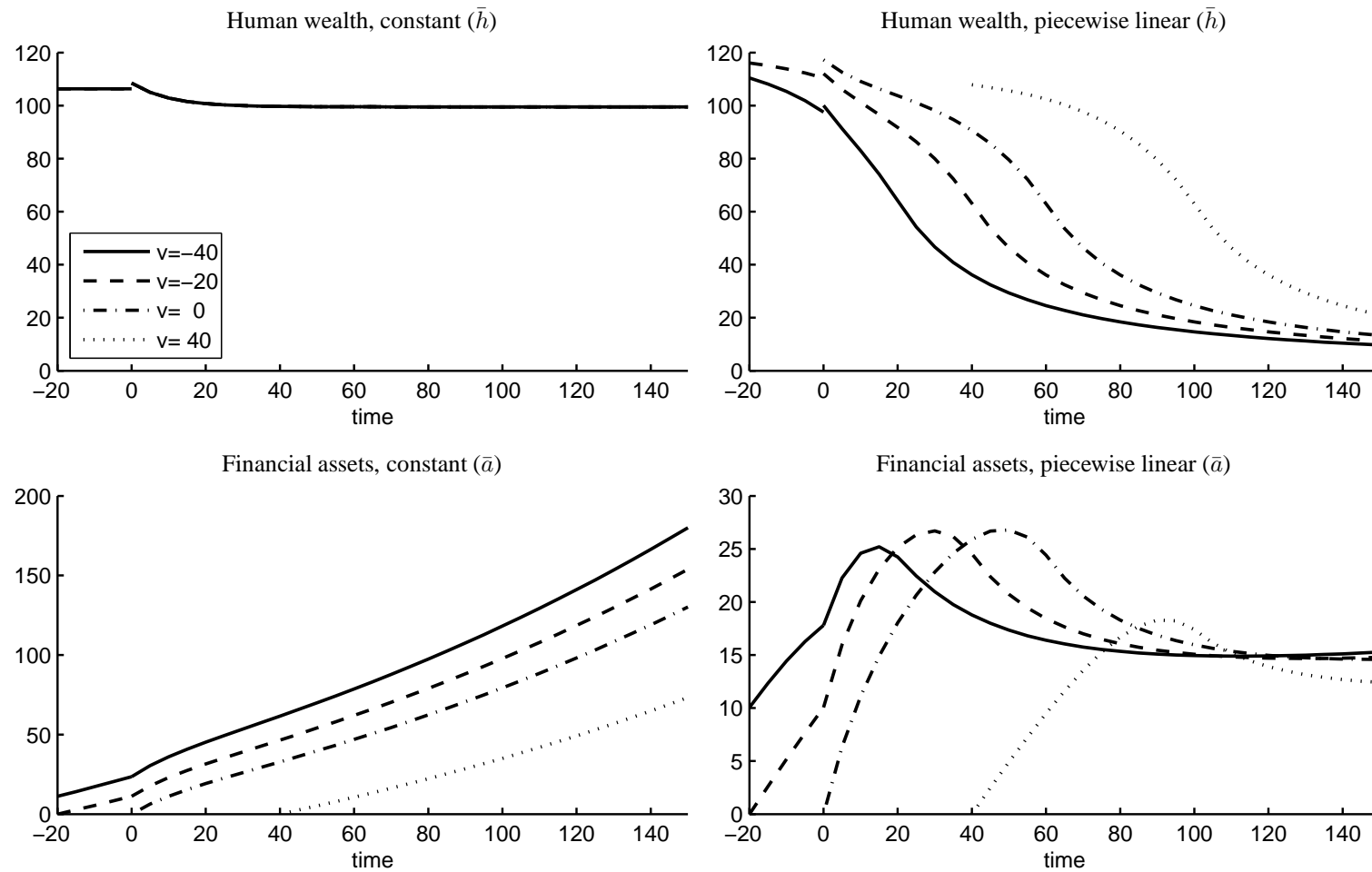Figure 4: Balanced-Budget Fiscal Policy

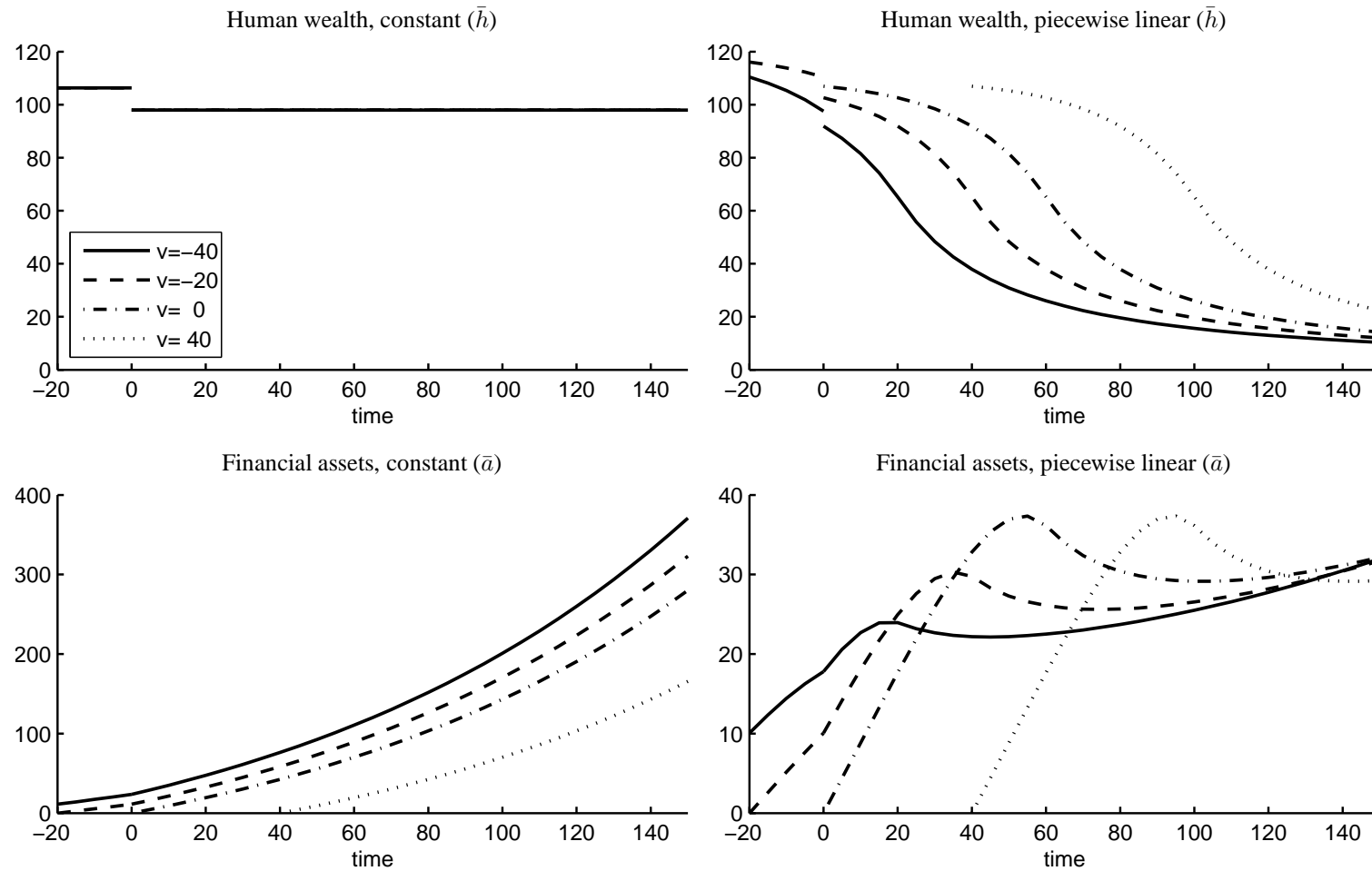Figure 5: Ricardian Equivalence Experiment: Temporary Tax Cut
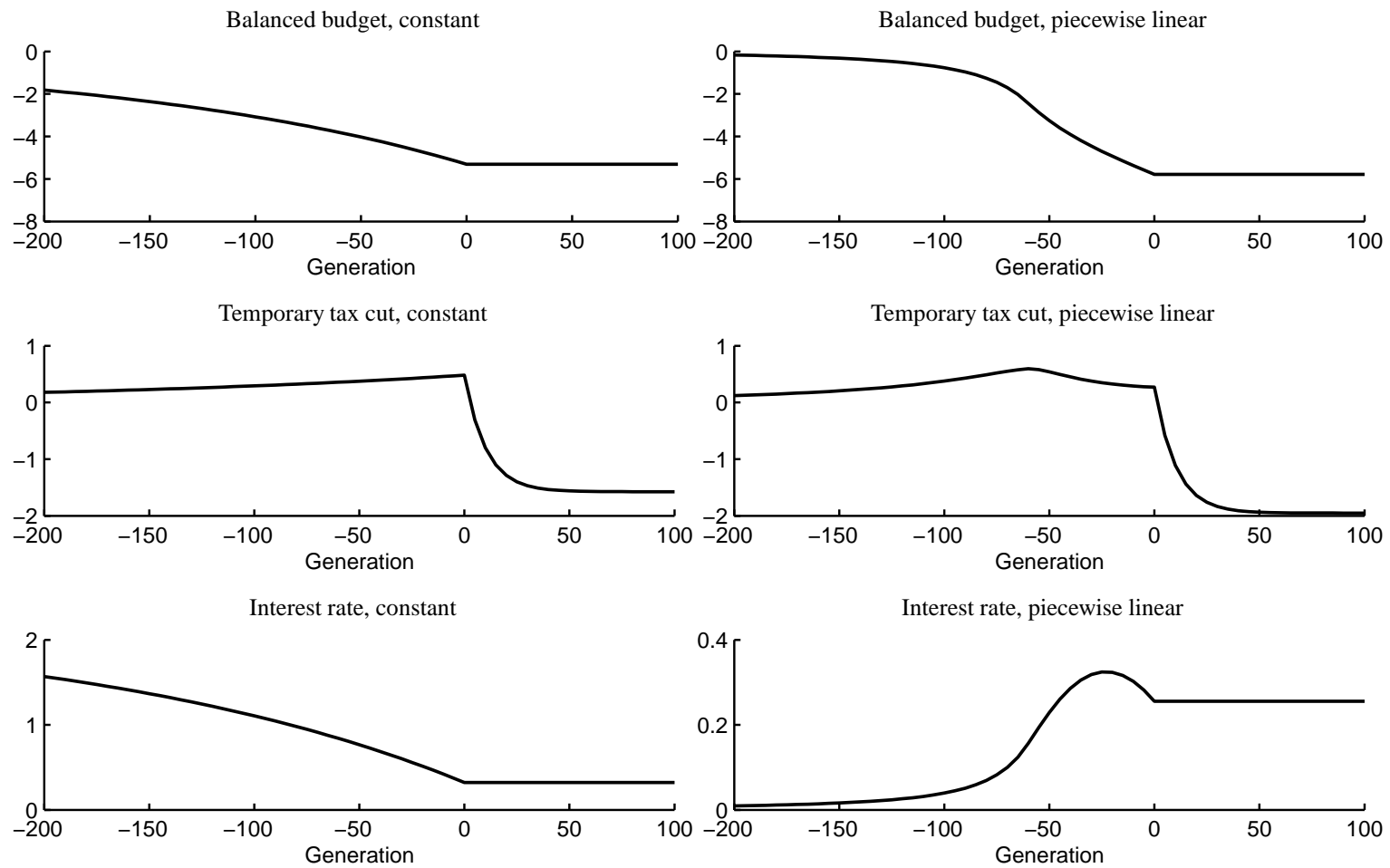
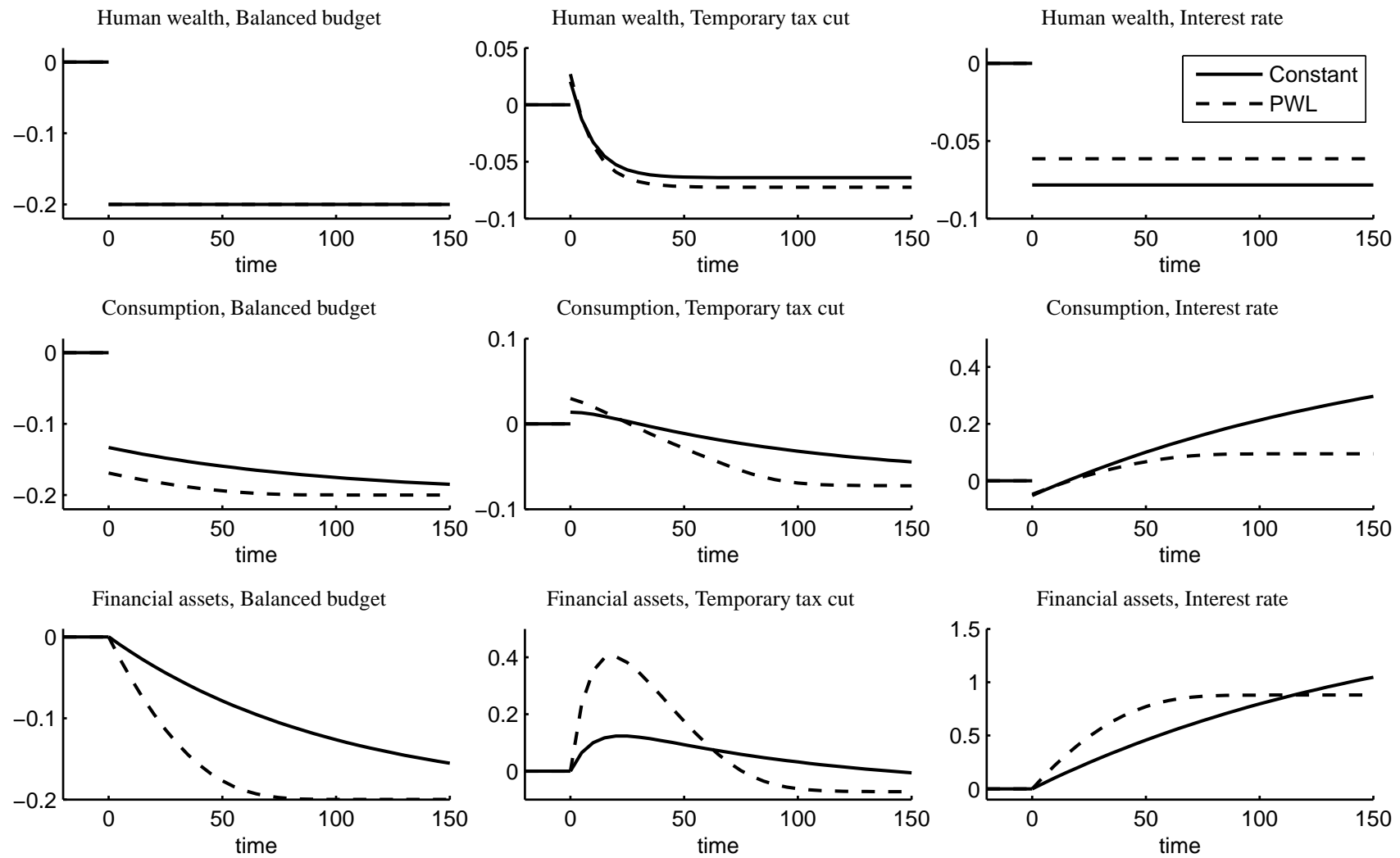Figure 6: Increase in the World Interest Rate

Figure 7: Welfare Effects

Figure 8: Aggregate Effect of the Shocks

References

Arias, E., Anderson, R., Hsiang-Ching, K., Murphy, S., & Kochanek, K. (2003). *Deaths: Final data for 2001*. Hyattsville, MD: National Center for Health Statistics. ((National Vital Statistics Report, Vol. 52, No. 3))

Bettendorf, L. J. H., & Heijdra, B. J. (2001a). Intergenerational and international welfare leakages of a product subsidy in a small open economy. *International Tax and Public Finance*, *8*, 705-729.

Bettendorf, L. J. H., & Heijdra, B. J. (2001b). Intergenerational welfare effects of a tariff under monopolistic competition. *Journal of Economics*, *73*, 313-346.

Blanchard, O.-J. (1985, April). Debts, deficits, and finite horizons. *Journal of Political Economy*, *93*, 223-247.

Blanchard, O. J., & Fischer, S. (1989). *Lectures on macroeconomics*. Cambridge, MA: MIT Press.

Bovenberg, A. L. (1993, May). Investment promoting policies in open economies: The importance of intergenerational and international distributional effects. *Journal of Public Economics*, *51*, 3-54.

Bovenberg, A. L. (1994). Capital taxation in the world economy. In F. van der Ploeg (Ed.), *Handbook of international macroeconomics*. Oxford: Basil Blackwell.

Buiter, W. H. (1988, June). Death, birth, productivity growth and debt neutrality. *Economic Journal*, *98*, 279-293.

Calvo, G. A., & Obstfeld, M. (1988, March). Optimal time-consistent fiscal policy with finite lifetimes. *Econometrica*, *56*, 411-432.

Couperus, L. (1918). *Old people and the things that pass*. New York: Dodd, Mead, and Co. (Translated from the 1906 Dutch original by Alexander Teixeira de Mattos. Reissued in 1963 by Sijthoff, Leiden and Heinemann, London)

Diamond, P. A. (1965). National debt in a neoclassical growth model. *American Economic Review*, *55*, 1126-1150.

Faruqee, H. (2003). Debt, deficits, and age-specific mortality. *Review of Economic Dynamics*, *6*, 300-312.

Faruqee, H., & Laxton, D. (2000). *Life cycles, dynasties, saving: Implications for closed and small, open economies* (IMF Working Paper No. WP/00/126). International Monetary Fund, Washington, D.C.

Fisher, I. (1930). *The theory of interest*. New York: Macmillan. (Reprinted by Augustus M. Kelley, Fairfield, NJ, 1986)

Gompertz, B. (1825). On the nature of the function expressive of the law of human

mortality. *Philosophical Transactions of the Royal Society of London, Series A*, *115*, 513-580.

Heijdra, B. J., & Meijdam, A. C. (2002, May). Public investment and intergenerational distribution. *Journal of Economic Dynamics and Control*, *26*, 707-735.

Heijdra, B. J., & Romp, W. E. (2005). *Old people and the things that pass: Mathematical appendix.* (Mimeo, University of Groningen)

Heijdra, B. J., & van der Ploeg, F. (2002). *Foundations of modern macroeconomics*. Oxford: Oxford University Press.

Hooker, P. F. (1965). Benjamin Gompertz: 5 March 1779-14 July 1865. *Journal of the Institute of Actuaries*, *91*, 203-212.

Kreyszig, E. (1988). *Advanced engineering mathematics* (Sixth ed.). New York: John Wiley.

Makeham, W. M. (1860). On the law of mortality and the construction of annuity tables. *Assurance Magazine*, *8*, 301-310.

Matsuyama, K. (1987, November). Current account dynamics in a finite horizon model. *Journal of International Economics*, *23*, 299-313.

Preston, S. H., Heuveline, P., & Guillot, M. (2001). *Demography: Measuring and modelling population processes*. Oxford: Blackwell.

Weil, P. (1989). Overlapping families of infinite-lived agents. *Journal of Public Economics*, *38*, 183-198.

Yaari, M. E. (1965). Uncertain lifetime, life insurance, and the theory of the consumer. *Review of Economic Studies*, *32*, 137-150.