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Practical encoders for controlling nonlinear systems under communication constraints

Claudio De Persis and Dragan Nešić

Abstract— **We introduce a new class of dynamic encoders for continuous-time nonlinear control systems which update their parameters only at discrete times. We prove that the information reconstructed from the encoded feedback can be used to deliver a piece-wise constant control law which yields semi-global practical stability. The result is achieved by assuming a property weaker than asymptotic stabilizability.**

I. Introduction

Controlling (nonlinear) systems via encoded feedback is of paramount importance in distributed control systems. For systems of the form

$$\dot{x} = f(x, u), \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m \qquad (1)$$

it has been very recently illustrated in the literature ([30], [18], [5]) how to design dynamic encoders which incorporate the model of the system,

$$\dot{\bar{x}}(t) = f(\bar{x}(t), u(t)) \qquad (2)$$

for $t \in [kT, (k+1)T)$ and $k \in \mathbb{Z}_+$, with a discrete reset of the state, that is

$$\bar{x}(kT) = \hat{x}(kT), \qquad (3)$$

and a discrete update of another fundamental parameter, namely the range of the quantization region (see below)

$$\ell((k+1)T) = \Lambda \ell(kT). \qquad (4)$$

These encoders are able to guarantee (semi-)global asymptotic stabilization via encoded feedback by assuming standard stabilizability. The result relies on – among other things - showing that the encoded feedback $\bar{x}(\cdot)$ is actually an asymptotic estimate of the state $x(\cdot)$. We will not explain in detail the functioning of (2)-(4), for which the reader is referred to [30], [18], [5] (for other contributions on control via encoded or quantized feedback, the reader is referred to the important papers [20], [29], [2], [6], [25], [11], [7], [17], [21], [16], [19], [15], [14], [8], [26], [10], to name a few). Nevertheless, a brief description of the quantities in (2)-(4) will be useful later on.

The two equations (2), (3) define a nonlinear jump system used to compute the evolution of the center of the quantization region (which here, for the sake of simplicity, is chosen

to coincide with a cube). At time kT, $k \in \mathbb{Z}_+$, the center is taken equal to ([30], [18]) $\bar{x}(kT^-) := \lim_{t \to kT^-} \bar{x}(t)$, with $\bar{x}(0^-) := 0$. Each edge of the quantization region at time kT has length $2\ell(kT)$. Under the assumption that $|x(0)|_\infty \le X$, for some $X > 0$, setting $\ell(0) = X$, the initial state is guaranteed to belong to the quantization region. Suppose this is true for all the times, that is

$$|x(kT) - \bar{x}(kT^-)|_\infty \le \ell(kT) \qquad (5)$$

for each $k \in \mathbb{Z}_+$. Then, denoted by B the integer representing the number of quantization levels used per each state component, and by $\hat{x}(kT)$ the quantized version of $x(kT)$ (see [30], [18], [5]), we have

$$|x(kT) - \hat{x}(kT)|_\infty \le \ell(kT)/B \qquad (6)$$

for each $k \in \mathbb{Z}_+$.

The actual adoption of devices such as those described above in distributed control systems very much depends on the possibility of easing the computational burden involved in the solution of (2). In this note, we address such issues by proposing encoders which do not require a continuous update of their state and which are able to reconstruct an asymptotically practically correct estimate of the state starting from encoded information. We also illustrate the possibility of using this estimate to the purpose of stabilizing the system. In particular, we discuss here an approach to achieve practical stabilizability under an assumption weaker than asymptotic stabilizability. The approach discussed in this paper can also be viewed as a general framework in which many of the results available for quantized discrete-time systems can be interpreted or even translated for continuous-time systems, although this is not discussed here in detail. Other approaches are possible [22]. We mention dwell-time switching control laws to cope with the stabilization problem under limited data rate constraints [11], [10]. They represent a durable solution to the problem, due to the simplicity of its implementation. In the next section, we consider an *approximate* discrete-time version of the system (1) and design a simplified version of the encoder (2)-(4). Two solutions are proposed, each one with its distinguishing features. In both cases, we prove that the estimate of the state $x(\cdot)$ generated by the encoder is asymptotically practically correct at the sampling times. In Section III, we extend these results to the case in which only partial-state measurements are available. In Section IV, basically under the asymptotic controllability assumption, we study the evolution of system (1) in closed-loop with a piece-wise constant control law designed on the basis of the feedback generated by the encoders examined in Sections

II and III.
Most of the proofs are omitted for lack of space.

II. A CONSISTENT DISCRETIZED ENCODER

Under piece-wise constant control laws, let us introduce as in [1] the exact discrete-time model of (1), that is

$$x((k+1)T) = f_T^e(x(kT), u(kT)) \qquad (7)$$

with

$$f_T^e(x(kT), u(kT)) = x(kT) + \int_{kT}^{(k+1)T} f(x(s), u(kT))ds . \qquad (8)$$

This model is in general not available, and an approximate discrete-time model [1]

$$x^a((k+1)T) = f_T^a(x^a(kT), u(kT)) , \qquad (9)$$

must instead be taken into account. Following [23], [1], we consider approximate models (9) which are consistent with the exact model (7):

Assumption 1: The model $f_T^a(x, u)$ is consistent with $f_T^e(x, u)$, that is for each compact set $\Omega \subset \mathbb{R}^n \times \mathbb{R}^m$, there exists a class-$\mathcal{K}$ function $\varrho(\cdot)$ and a constant $T_0 > 0$ such that for all $(x, u) \in \Omega$ and all $T \in (0, T_0]$,

$$|f_T^e(x, u) - f_T^a(x, u)|_\infty \le T\varrho(T) .$$

Remark. Conditions under which model consistency holds are thoroughly discussed in [22], [1].

Inspired by [30], [18] and [3], we propose the following discrete-time implementation of the encoder (2)-(4):

$$\begin{array}{rcl} \bar{x}((k+1)T) & = & f_T^a(\hat{x}(kT), u(kT)) \\ \ell((k+1)T) & = & \Lambda\ell(kT) + T\varrho(T) , \end{array} \qquad (10)$$

with $\Lambda > 0$ a constant to design. As recalled in the previous section, vector $\hat{x}(kT)$ is such that $|\hat{x}(kT) - x(kT)|_\infty \le \ell(kT)/B$, whenever $|\bar{x}(kT) - x(kT)|_\infty \le \ell(kT)$.
Assume the following ([3]):

Assumption 2: For any pair of constants $X > 0$ and $U > 0$, there exists a number $Y > 0$ such that, if $|x(0)|_\infty \le X$ and $|u(kT)|_\infty \le U$ for each $k \in \mathbb{Z}_+$, then $|x(kT)|_\infty \le Y$ for each $k \in \mathbb{Z}_+$, where $x(kT)$ is the solution of (1) at time kT.

The result below shows a clear relation between the degree of accuracy achievable by the "asymptotic estimate" $\hat{x}(\cdot)$ of $x(\cdot)$, the parameter B and the sampling period T. It relies on the concept of consistency ([23], [1]) recalled above and employs arguments inspired by those in [3], proof of Proposition 1. In the statement, the symbol $\mathcal{C}_r(s)$, with $r > 0$ an integer and $s > 0$ a real number, denotes the cube in $\mathbb{R}^r$ centered around the origin and with edges of length $2s$.

Proposition 1: Let Assumptions 1 and 2 hold. Then for any $X > 0$ and for any $U > 0$ we can find $T^* > 0$ with the property that, for all $T \in (0, T^*]$, $|x(0)|_\infty \le X$ implies

$$|x(kT) - \hat{x}(kT)|_\infty \le \Lambda^k \frac{X}{B} + \frac{1}{B - F^a} T\varrho(T) ,$$

provided that $\bar{x}(0) = 0$, $\ell(0) = X$, $\Lambda := F^a/B$, and $B > F^a + 1$, with $F^a > 0$ the Lipschitz constant for which

$$|f_T^a(x, u) - f_T^a(\hat{x}, u)|_\infty \le F^a|x - \hat{x}|_\infty$$

for all $(x, u), (\hat{x}, u) \in \mathcal{C}_n(X + Y + T\varrho(T)) \times \mathcal{C}_m(U)$.

Proof: In Assumption 1, let Ω be $\mathcal{C}_n(Y) \times \mathcal{C}_m(U)$ and fix $\varrho(\cdot)$ and T_0 accordingly. Set $T^* = T_0$ and fix $T \in (0, T^*]$. Note that $|x(0)|_\infty \le X$, $\bar{x}(0) = 0$ and $\ell(0) = X$ imply $|\hat{x}(0)|_\infty \le X$ and $|x(0) - \hat{x}(0)|_\infty \le X/B$. Assume that, for some $k \in \mathbb{Z}_+$, $|x(jT) - \bar{x}(jT)|_\infty \le \ell(jT)$ and $|x(jT) - \hat{x}(jT)|_\infty \le \ell(jT)/B$ for each $j = 0, 1, \ldots, k$. In particular, (5) (with $\bar{x}(kT^-)$ replaced by $\bar{x}(kT)$) and (6) hold. The evolution of $\ell(\cdot)$ as given by the second equation in (10) is described by the relation

$$\begin{aligned} \ell(kT) & = & \Lambda^k\ell(0) + \sum_{j=0}^{k-1} \Lambda^{k-1-j}T\varrho(T) \\ & = & \Lambda^k X + \frac{1 - \Lambda^k}{1 - \Lambda}T\varrho(T) . \end{aligned}$$

Hence,

$$\frac{\ell(kT)}{B} \le \Lambda^k \frac{X}{B} + \frac{1}{B - F^a}T\varrho(T) .$$

As $|x(kT)|_\infty \le Y$ for all $k \in \mathbb{Z}_+$, relation (6) guarantees that $|\hat{x}(kT)|_\infty \le \ell(kT)/B + Y \le X + Y + T\varrho(T)$. Consider now the following chain of relations:

$$|x((k+1)T) - \bar{x}((k+1)T)|_\infty =$$
$$|f_T^e(x(kT), u(kT)) - f_T^a(\hat{x}(kT), u(kT))|_\infty =$$
$$|f_T^e(x(kT), u(kT)) - f_T^a(x(kT), u(kT)) +$$
$$f_T^a(x(kT), u(kT)) - f_T^a(\hat{x}(kT), u(kT))|_\infty \le$$
$$T\varrho(T) + F^a|x(kT) - \hat{x}(kT)|_\infty \le T\varrho(T) + F^a\ell(kT)/B =$$
$$\Lambda\ell(kT) + T\varrho(T) = \ell((k+1)T) .$$

This implies that $x((k+1)T)$ belongs to the quantization region at time $(k+1)T$, and hence $|x((k+1)T) - \hat{x}((k+1)T)|_\infty \le \ell((k+1)T)/B$. By induction we conclude that both (5) and (6) hold for each $k \in \mathbb{Z}_+$. Bearing in mind that $\Lambda < 1$ and (6), we obtain

$$|x(kT) - \hat{x}(kT)|_\infty \le \Lambda^k \frac{X}{B} + \frac{1}{B - F^a}T\varrho(T) ,$$

that is the thesis. ∎

Allowing a more complex dynamics for the encoder, it is possible to remove the presence of the term $T\varrho(T)$. This can be done under the following assumption [23]:

Assumption 3: For each compact set $\Omega \subset \mathbb{R}^n \times \mathbb{R}^m$, there exist $K > 0$ and $T^* > 0$ such that, for all (x, u) and (z, u) in Ω, and all $T \in (0, T^*]$,

$$|F_T(x, u) - F_T(z, u)| \le (1 + KT)|x - z| ,$$

where F_T is either f_T^a or f_T^e.

We recall the following result (cf. Lemma 2, Lemma 3 and Remark 2 in [23]), which states that model consistency propagates through arbitrarily large intervals of time:

Lemma 1: Let Assumptions 1, 2, and 3 hold. Then, for any $X > 0$, $U > 0$, $L > 0$ and $\eta > 0$, there exists $\hat{T} > 0$ such that $T \in (0, \hat{T}]$ and $x(0) = x^a(0)$ imply

$$|x(kT) - x^a(kT)| \leq \eta , \quad \forall k : kT \in [0, L] .$$

The proposed modified encoder is as follows:

$$\begin{aligned}
x^a((k+1)T) &= f_T^a(x^a(kT), u(kT)) \\
\bar{x}((k+1)T) &= f_T^a(\hat{x}(kT), u(kT)) \\
\ell((k+1)T) &= \Lambda \ell(kT) ,
\end{aligned} \quad (11)$$

Remark. The decoder will implement only the last two equations of (11).

The main difference lies in the fact that it also implements the equations describing the evolution of the approximate model. Hence, in the present case, $\hat{x}(kT)$ is a vector for which (6), with $x(kT)$ replaced by $x^a(kT)$, holds, provided that (5), again with $x(kT)$ replaced by $x^a(kT)$, holds as well. Then the following result is true:

Proposition 2: Let Assumptions 1, 2, and 3 hold. Then for any $X > 0$, $U > 0$, $L > 0$ and $\eta > 0$ we can find a $T^* > 0$ with the property that, for all $T \in (0, T^*]$, $|x(0)|_\infty \leq X$ implies

$$|x(kT) - \hat{x}(kT)|_\infty \leq \Lambda^k \frac{X}{B} + \eta \quad \forall k : kT \in [0, L] ,$$

provided that $x^a(0) = x(0)$, $\bar{x}(0) = 0$, $\ell(0) = X$ and $\Lambda := F^a/B < 1$, with $F^a > 0$ the Lipschitz constant for which

$$|f_T^a(x, u) - f_T^a(\hat{x}, u)|_\infty \leq F^a |x - \hat{x}|_\infty$$

for all (x, u), $(\hat{x}, u) \in \mathcal{C}_n(X + Y + \eta) \times \mathcal{C}_n(U)$.

Proof: The proof is omitted for lack of space. ∎

Remark. A modification of the structure of the practical encoders described in this section may lead to encoders which employ lower data rates. See [30], [19] and [4] for details.

III. OBSERVER-BASED PRACTICAL ENCODERS

The previous section has focused on the case in which full state was available for measurements. Here we consider the case in which the system (1) is endowed with a readout map which is different from the identity, namely

$$y = h(x) \in \mathbb{R}^p . \quad (12)$$

In this scenario, the design of the encoders is based on observers ([1]). A common approach to the design of sampled-data observer lies on a suitable discretization of a continuous-time observer. This is examined in the next subsection. Another approach, which typically exhibits a better performance in simulations, consists of designing the discrete-time observer directly. This approach is studied in Subsection III.B.

A. Encoder design by emulation

In this subsection, we assume that a continuous-time observer

$$\dot{\sigma}(t) = g(\sigma(t), y(t), u(t)) \quad (13)$$

is actually available, and consider its zero order hold equivalent ([13]):

$$\sigma((k+1)T) = g_T^a(\sigma(kT), y(kT), u(kT)) . \quad (14)$$

Namely, we assume the following ([1]):

Assumption 4: System (14) is a semi-global practical observer, i.e. there exists a class-$\mathcal{KL}$ function $\omega(\cdot, \cdot)$ such that, for any $X > \chi > 0$ and any $Y > 0$ and $U > 0$, we can find a $T^* > 0$ such that, for all $T \in (0, T^*]$,

$$|x(0)|_\infty \leq X , \quad |\sigma(0) - x(0)|_\infty \leq 2X ,$$

and

$$|x(k)|_\infty \leq Y , \; |u(k)|_\infty \leq U ,$$

for each $k \in \mathbb{Z}_+$, imply

$$|\sigma(kT) - x(kT)| \leq \omega(|\sigma(0) - x(0)|, kT) + \chi , \quad (15)$$

for each $k \in \mathbb{Z}_+$.

Remark. There are precise conditions under which the assumption above is fulfilled, and these are investigated in [1]. Suppose the following hypotheses hold true:
(i) The model $g_T^a(\cdot, \cdot, \cdot)$ is consistent with the model $g(\cdot, \cdot, \cdot)$.
(ii) There exist a continuously differentiable function $V(x, \sigma)$ and class-$\mathcal{K}_\infty$ functions $\alpha_1(\cdot)$, $\alpha_2(\cdot)$, $\alpha_3(\cdot)$ such that

$$\alpha_1(|x - \sigma|) \leq V(x, \sigma) \leq \alpha_2(|x - \sigma|)$$

$$\frac{\partial V}{\partial x} f(x, u) + \frac{\partial V}{\partial \sigma} g(\sigma, y, u) \leq -\alpha_3(|x - \sigma|) .$$

Then, by [1], Theorem 3, Assumption 2 implies Assumption 4.

In the sequel, it will be useful to single out a part of the observer (14) not affected by the output:

Assumption 5: Map $g_T^a(\sigma(kT), y(kT), u(kT))$ can be decomposed as

$$g_T^a(\sigma(kT), y(kT), u(kT)) = g_{T1}^a(\sigma(kT), u(kT)) +$$
$$g_{T2}^a(\sigma(kT), y(kT), u(kT)) .$$

Furthermore, there exist a class-$\mathcal{KL}$ function $\tilde{\omega}(\cdot, \cdot)$ and a constant $\tilde{\chi}$ such that, $|\sigma(kT) - x(kT)| \leq \tilde{\omega}(|\sigma(0) - x(0)|, kT) + \tilde{\chi}$ implies

$$|g_{T2}^a(\sigma(kT), y(kT), u(kT))| \leq \tilde{\omega}(|\sigma(0) - x(0)|, kT) + \tilde{\chi} .$$

Remark. If $g(\cdot, \cdot, \cdot)$ is a continuously differentiable function, and the Lyapunov function in the Remark following Assumption 4 holds with $V(x, \sigma) = V(x - \sigma) =: V(e)$, then it implies that $x = \sigma$ must be an invariant manifold for system (1), (12), (13) and therefore $g(x, h(x), u) = f(x, u)$ which in turn implies [28]

$$\dot{\sigma}(t) = f(\sigma(t), u(t)) + \tilde{g}(\sigma(t), y(t), u(t))(y(t) - h(\sigma(t))) ,$$

with $\tilde{g}(\cdot, \cdot, \cdot)$ a suitable continuous function. Using e.g. Euler discretization for the latter system and letting $h(\cdot)$ be Lipschitz continuous, then Assumption 4 implies:

$$|g_{T2}^a(\sigma(kT), y(kT), u(kT))| = |\tilde{g}(\sigma(kT), y(kT), u(kT)) \cdot$$

$$\cdot (y(kT) - h(\sigma(kT)))| \leq \tilde{G}(\omega(|\sigma(0) - x(0)|, kT) + \chi) ,$$

for some constant $\tilde{G} > 0$.

Following [30] (see also [24]), we propose the following observer-based encoder:

$$
\begin{aligned}
\sigma((k+1)T) &= g_T^a(\sigma(kT), y(kT), u(kT)) \\
\varsigma((k+1)T) &= g_{T1}^a(\hat{\varsigma}(kT), u(kT)) \\
\ell((k+1)T) &= \Lambda \ell(kT) + \tilde{\omega}(2\sqrt{n}X, kT) + \tilde{\chi} .
\end{aligned}
\tag{16}
$$

Remark. If the function $\tilde{\omega}(\cdot, \cdot)$ is not available, then it can be replaced by a suitable constant. This replacement will affect the accuracy of the encoding procedure carried out by the device (16).

Notice that, differently from the state feedback case, here the signal $\hat{\varsigma}(\cdot)$ represents not the encoded state of the process $x(\cdot)$ but the encoded state of the observer $\sigma(\cdot)$. Furthermore, $\varsigma(\cdot)$ represents the center of the quantization region. Hence, analogously to the state feedback case, $|\sigma(kT) - \varsigma(kT)|_\infty \leq \ell(kT)$ implies $|\sigma(kT) - \hat{\varsigma}(kT)|_\infty \leq \ell(kT)/B$. It is worth stressing that the decoder at the other end of the channel will implement only the last two equations in (16) ($y(\cdot)$ is not available to the decoder). Now, we introduce the constant:

$$Z := \omega(2\sqrt{n}X, 0) + \tilde{\omega}(2\sqrt{n}X, 0) + \chi + \tilde{\chi} + X + Y .$$

The main result of this subsection is as follows:

Proposition 3: Let Assumptions 2 and 4-5 hold. Then for any $X > 0$ and for any $U > 0$ we can find a $T^* > 0$ with the property that, for all $T \in (0, T^*]$, $|x(0)|_\infty \leq X$ implies

$$|x(kT) - \hat{\varsigma}(kT)|_\infty \leq \omega(|\sigma(0) - x(0)|, kT) + \chi + \ell(kT)$$

provided that $\varsigma(0) = 0$, $|\sigma(0)|_\infty \leq X$, $\ell(0) = X$, $\Lambda := G^a/B$, and $B > G^a + 1$, with $G^a > 0$ the Lipschitz constant for which

$$|g_{T1}^a(\sigma, u) - g_{T1}^a(\hat{\sigma}, u)|_\infty \leq G^a|\sigma - \hat{\sigma}|_\infty$$

for all $(\sigma, u), (\hat{\sigma}, u) \in \mathcal{C}_n(Z) \times \mathcal{C}_m(U)$.

Remark. It is easily seen that the estimation error $x(kT) - \hat{\varsigma}(kT)$ asymptotically converges to a square with edges of length $\chi + \tilde{\chi}/(B - G^a)$.

Proof: The proof is omitted for lack of space. ∎

B. Encoder design by approximate discrete-time models

In this subsection, we pursue another approach to the design of practical encoders for system (1) with output map (12), namely we assume the existence of a discrete-time observer for the approximate model (9). We have [1]:

Assumption 6: System

$$\xi((k+1)T) = f_T^a(\xi(kT), u(kT)) + g_T(\xi(kT), y(kT), u(kT))$$

is a semi-global practical observer for system (1), (12), i.e. there exists a class-$\mathcal{KL}$ function β such that, for any $0 <$

$\chi < X$ and any $Y > 0$, $U > 0$, it is possible to find a $T^* > 0$ such that, for all $T \in (0, T^*]$,

$$|x(0)|_\infty \leq X , \quad |x(0) - \xi(0)|_\infty \leq 2X ,$$

and

$$|x(kT)|_\infty \leq Y , \quad |u(kT)|_\infty \leq U$$

for each $k \in \mathbb{Z}_+$, imply

$$|x(kT) - \xi(kT)| \leq \beta(|x(0) - \xi(0)|, kT) + \chi ,$$

for each $k \in \mathbb{Z}_+$.
Having set

$$Z = Y + \beta(2\sqrt{n}X, 0) + \chi ,$$

we pick the constant $\tilde{F}^a$ for which

$$|f_T^a(x, u) - f_T^a(\hat{x}, u)|_\infty \leq \tilde{F}^a|x - \hat{x}|_\infty$$

for all $(x, u), (\hat{x}, u) \in \mathcal{C}_n(Z) \times \mathcal{C}_m(U)$, and set

$$\mu(r, k, T) = \beta(r, (k+1)T) + \chi + \tilde{F}^a(\beta(r, kT) + \chi) + T\varrho(T) .$$

We also set $\Lambda = F^a/B$, with F^a the Lipschitz constant of $f_T^a(x, u)$ over the set $\mathcal{C}_n(\mu(2\sqrt{n}X, 0, T) + X + Z) \times \mathcal{C}_m(U)$. The encoder is as follows:

$$
\begin{aligned}
\xi((k+1)T) &= f_T^a(\xi(kT), u(kT)) + \\
&\qquad g_T(\xi(kT), y(kT), u(kT)) \\
\bar{\xi}((k+1)T) &= f_T^a(\hat{\xi}(kT), u(kT)) \\
\ell((k+1)T) &= \Lambda \ell(kT) + \mu(2\sqrt{n}X, k, T) ,
\end{aligned}
$$

where $|\hat{\xi}(kT) - \xi(kT)|_\infty \leq \ell(kT)/B$ provided that $|\bar{\xi}(kT) - \xi(kT)|_\infty \leq \ell(kT)$.

Remark. As in the previous subsection, the decoder will implement only the last two equations above. To correctly encode the observer state ξ, the decoder needs an estimate of the term $g_T(\xi(kT), y(kT), u(kT))$. Part of the proof of the proposition below deals with deriving this estimate.

We are now ready to state the main result of this subsection:

Proposition 4: Let Assumptions 1, 2 and 6 hold. Then for any $X > 0$ and for any $U > 0$ we can find a $T^* > 0$ with the property that, for all $T \in (0, T^*]$, $|x(0)|_\infty \leq X$ implies

$$|x(kT) - \hat{\xi}(kT)|_\infty \leq \beta(|\xi(0) - x(0)|, kT) + \chi + \frac{\ell(kT)}{B}$$

provided that $\bar{\xi}(0) = 0$, $|\xi(0)|_\infty \leq X$, $\ell(0) = X$, and $\Lambda := F^a/B$, with $B > F^a + 1$.

Proof: The proof is omitted for lack of space. ∎

IV. PRACTICAL STABILIZATION

The attention is now turned to the design of the controller, for which we follow very closely [12]. We shall refer to Proposition 1 for the state feedback case, and to Proposition 3, for the output feedback case. Analogous results can be given using Proposition 2 and, respectively, Proposition 4. A number of notions from [12] are now introduced. The positive numbers $r < R$ and $r_m < R_m$ are given and the symbol $\mathcal{V}(S)$ denotes the level set $\{x : V(x) \leq S\}$.

Assumption 7: There exists a continuous Lyapunov function $V(\cdot) : \mathbb{R}^n \to \mathbb{R}_+$ for which:

- There exist class-$\mathcal{K}_\infty$ functions $\nu(\cdot), \rho(\cdot)$ such that $|V(x_1) - V(x_2)| \leq \rho(|x_1 - x_2|)$. Moreover, $V(x) \geq \nu(|x|)$.
- There exist a feedback function $\kappa(\cdot) : \mathbb{R}^n \to \mathbb{R}^m$ and constants $T > 0$, $c > 0$ such that the solution of $\dot{x}(t) = f(x(t), \kappa(x(0)))$ with $x(0) \in \mathcal{V}(R)$ satisfies:
  (S1) $V(x(T)) \leq \max\{V(x(0)) - c, r\}$
  (S2) $V(x(t)) \leq \max\{V(x(0)), r\} + r_m$, $\forall t \in [0, T]$.

We also introduce the following:

Assumption 8: Set $\mathcal{U} := \{u \in \mathbb{R}^m : u = \kappa(x) , x \in \mathcal{V}(R + R_m)\}$.

1) There exists $M \geq 0$ such that $|f(x, u)| \leq M$ for all $x \in \mathcal{V}(R + R_m)$, for all $u \in \mathcal{U}$.

2) There exists $L_{fx}, L_{fu} > 0$ such that $|f(x_1, u_1) - f(x_2, u_2)| \leq L_{fx}|x_1 - x_2| + L_{fu}|u_1 - u_2|$, for all $x_1, x_2 \in \mathcal{V}(R + R_m)$, for all $u_1, u_2 \in \mathcal{U}$.

Remark. Assumptions 7 and 8 are discussed in [12].

We recall the following statement from [12]:

Theorem 1: Under Assumptions 7 and 8, consider

$$\dot{x}(t) = f(x(t), \kappa(x(0))) + d(t)$$

where $x(0) \in \mathcal{V}(R)$. Let $\sigma \in [0, R_m - r_m)$. If the disturbance $d(\cdot)$ satisfies

$$\max_{t \in [0,T]} \left| \int_0^t d(s)ds \right| \leq \rho^{-1}(\sigma)\mathrm{e}^{-L_{fx}T}$$

then for all $t \in [0, T]$, the solution $x(\cdot)$ exists and satisfies

$$V(x(T)) \leq \max\{V(x(0)) - c, r\} + \sigma$$
$$V(x(t)) \leq \max\{V(x(0)), r\} + (r_m + \sigma) .$$

We now apply this theorem and the results in the previous section to show practical stabilization when using the control law

$$u(t) = \kappa(\hat{x}(kT)) , \quad t \in [kT, (k+1)T) ,$$

where the samples $\hat{x}(\cdot)$ are generated by the decoder (10). To proceed, fix $r = r_m < R, R_m$. Set

$$\nu(X) = R , \quad \nu(Y) = R + R_m , \quad U = \max_{|x|_\infty \leq Y} |\kappa(x)|_\infty ,$$

so that $x \in \mathcal{V}(R)$ implies $|x| \leq X$ and $x \in \mathcal{V}(R + R_m)$ implies $|x| \leq Y$. Choose the constant F^a in Proposition 1 accordingly. Finally, in (10) set $\Lambda = F^a/B$. The result below proves that under the assumptions just stated, and despite of the quantization error, a control law exists which keeps the state confined in $\mathcal{V}(R + R_m)$, where, applying the results established in the previous section, an increasingly accurate estimation of the state is possible. Practical stability of the resulting closed-loop system is then concluded.

Proposition 5: Let Assumptions 1, 7 and 8 hold. Let $T \in (0, T_0]$ with T and T_0 as in Assumption 7 and, respectively, Assumption 1. Let $\sigma \in [0, \min\{R_m - r_m, R - r, c/4\})$. Set

$$\frac{1}{\sqrt{n}} E_k := \Lambda^k \frac{X}{B} + \frac{1}{B - F^a} T\varrho(T) , \quad k \in \mathbb{Z}_+ ,$$

and choose $B > F^a + 1$ so that:

$$E_0 \in \left[0, \min \left\{ \frac{\rho^{-1}(\sigma)}{2 + L_{fx}T} \mathrm{e}^{-L_{fx}T}, \rho^{-1}(R - r - \sigma) \right\} \right) . \tag{17}$$

Then, the solution of the closed-loop system

$$\dot{x}(t) = f(x(t), \kappa(\hat{x}(kT))) , \quad t \in [kT, (k+1)T) , \quad k \in \mathbb{Z}_+$$

from the initial condition $x(0) \in \mathcal{V}(R - 2\rho(E_0))$ exists for all $t \geq 0$ and satisfies

$$V(x(kT)) \leq \max\{V(x(0)) - \frac{(3k-1)c}{4}, r\} + \rho(E_k) + \sigma , \quad \forall k \in \mathbb{Z}_+$$

$$V(x(t)) \leq \max\{V(x(kT)) + \rho(E_k), r\} + r + \sigma + \rho(E_k) , \quad \forall t \in [kT, (k+1)T) , \forall k \in \mathbb{Z}_+ .$$

Remark. From the first inequality above, we see that the state of the closed-loop system at the sampling times asymptotically converges to the level set

$$\left\{ x \in \mathbb{R}^n : V(x) \leq r + \sigma + \rho\left(\frac{1}{B - F^a} T\varrho(T) \right) \right\} .$$

Proof: The result is an application of [12], Proposition 1, and its proof is basically the same. The differences are as follows: We explicitly take into account the fact that the "measurement noise" (i.e. the quantization error) is vanishing and we take care of the fact that the measurement noise itself is not known *a priori* unless we guarantee that the state is confined within the quantization region. For lack of space, details are omitted. ∎

Mutatis mutandis, an output-feedback version of the previous result can also be stated. In the following proposition, the solution $x(\cdot)$ we refer to is the solution of the process (1) in closed loop with the "output feedback" control law:

$$u(t) = \kappa(\hat{\varsigma}(kT)) , \quad t \in [kT, (k+1)T) ,$$

where the samples $\hat{\varsigma}(\cdot)$ are generated by the encoder (16). Fix the constants r, R, r_m, R_m and X as before, set $\nu(Y) = R + R_m$,

$$Z = \omega(2\sqrt{n}X, 0) + \tilde{\omega}(2\sqrt{n}X, 0) + \chi + \tilde{\chi} + X + Y ,$$

$$U = \max_{|x|_\infty \leq Z} |\kappa(x)|_\infty ,$$

and let $G^a > 0$ be such that

$$|g_{T1}^a(\sigma, u) - g_{T1}^a(\hat{\sigma}, u)|_\infty \leq G^a |\sigma - \hat{\sigma}|_\infty$$

for all $(\sigma, u), (\hat{\sigma}, u) \in \mathcal{C}_n(Z) \times \mathcal{C}_m(U)$. Also let $\Lambda = G^a/B$ in the encoder (16). Then we can state:

Proposition 6: Let Assumptions 1, and 4-8 hold. Let $T \in (0, T_0]$ with T and T_0 as in Assumption 7 and, respectively, Assumption 1.

Set

$$\frac{1}{\sqrt{n}} E_k := \Lambda^k \frac{X}{B} + \frac{1}{B - F^a} \tilde{\chi} + \sum_{j=0}^{k-1} \Lambda^{k-1-j} \frac{\tilde{w}(2\sqrt{n}X, jT)}{B} ,$$

$k \in \mathbb{Z}_+$, and choose $B > G^a + 1$ so that:

$$E_0 \in \left[0, \min\left\{ \frac{\rho^{-1}(\sigma)}{2 + L_{fx}T} \mathrm{e}^{-L_{fx}T}, \rho^{-1}(R - r - \sigma) \right\}\right) . \tag{18}$$

Then, the solution of the closed-loop system

$$\dot{x}(t) = f(x(t), \kappa(\hat\varsigma(kT))) \ , \ t \in [kT, (k+1)T) \ , \ k \in \mathbb{Z}_+$$

from the initial condition $x(0) \in \mathcal{V}(R - 2\rho(E_0))$ exists for all $t \geq 0$ and satisfies

$$
\begin{aligned}
V(x(kT)) &\leq \max\{V(x(0)) - \frac{(3k-1)c}{4}, r\} + \\
&\quad + \rho(E_k) + \sigma \ , \ \forall k \in \mathbb{Z}_+ \\
V(x(t)) &\leq \max\{V(x(kT)) + \rho(E_k), r\} + r + \sigma + \\
&\quad + \rho(E_k) \ , \ \forall t \in [kT, (k+1)T) \ , \forall k \in \mathbb{Z}_+ \ .
\end{aligned}
$$

Proof: The proof is omitted. ∎

V. Conclusion

The paper deals with the design of encoders for continuous-time nonlinear systems via their approximate discrete-time models. This approach has several advantages. With respect to previous dynamic encoding schemes presented in the literature, the encoder designed in this way allows to achieve (semi-global practical) stability with less computational effort. Moreover, the methods presented in the paper allow to extend the results available for the quantized control of discrete-time systems to continuous-time systems, and to overcome the drawbacks in connections with some existing methods. In the results established in the paper, decrease in sampling time improves the performance of the system. However, this may not be possible due to communication constraints. The introduction of another parameter in addition to the sampling period T allows to refine the model independently of T, and thus to achieve the same results while fulfilling the communication constraints. Although the analysis of the role of such additional parameter has not been included in the paper for lack of space, it is very important for further developments. The paper has also shown how to apply the results of [12] to the study of the robustness of nonlinear systems in the presence of quantization errors. Applications of the techniques presented in the paper to more general problems of control under communication constraints [9], [27] may represent another interesting line of research to pursue.

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