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Extension of Kalman-Yakubovich-Popov Lemma to Descriptor Systems

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Abstract— This paper studies concepts of passivity and positive realness for regular descriptor systems. A complete analogue of the well-known Kalman-Yakubovich-Popov lemma is presented. Some of the earlier related results are recovered from the provided results.

I. INTRODUCTION

The notion of passivity has always been of interest in various problems of systems and control theory. It is intimately related to the notion of positive realness. The relation between this two properties has been under investigation ever since Kalman's introduction of state space approach. The very well-known Kalman-Yakubovich-Popov (KYP) lemma is among the classical results of systems theory. For more than four decades, many researchers have investigated passivity/positive realness and their various extensions within the framework of state space systems. As an encyclopedic account of this vast literature, we refer to [1].

One line of research consists of efforts to extend the available literature for state space systems to descriptor systems. Despite the considerable contributions of numerous papers, a full analogue of KYP lemma for descriptor systems has not appeared yet to the best of authors' knowledge. A most majority of the related studies (see e.g. [2], [3] and [4]) is concentrated on strict versions of positive realness and/or works under extra assumptions (e.g. impulse-freeness, sign conditions on the feed-through term).

This paper aims at providing the extension of KYP lemma to descriptor systems. To do so, we first formulate passivity in terms of the so-called dissipation inequality by following Jan Willems' conceptual framework that is introduced in the seminal paper [5]. This will be followed by necessary and sufficient LMI conditions for passivity. Our treatment is highly inspired by the approach of [3]. In [3], the authors present some sufficient conditions under an additional assumption on the feed-through term. Another interesting contribution of [3] is a refinement of Weierstrass form for realizations of positive real transfer matrices. This refinement serves as one of the key tool in our development. Another key tool that we borrow from the literature (see e.g. [6]) is the characterization of "smooth" solutions of descriptor systems.

II. NOTATIONS AND CONVENTIONS

The following notations and conventions will be in force. The symbols $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{C}$ and $\mathbb{C}_+$ denote the sets of real numbers, nonnegative real numbers, complex number and complex number with positive real part, respectively. The notation $\mathbb{R}^{n \times m}$ denotes the set of $n \times m$ matrices with real elements and $\mathbb{R}^{n \times m}(s)$ the set of $n \times m$ matrices of rational functions. For any set $\mathscr{S}$, the n tuples of elements of $\mathscr{S}$ will be denoted by $\mathscr{S}^n$. For a complex number s, $\mathrm{Re}(s)$ stands for the real part. For complex vector v, the conjugate, the transpose and the conjugate transpose are denoted, respectively, $\bar{v}$, v^T and v^H. These conventions are used for matrices in the obvious manner. Let M be a matrix. The image of M is denoted by $\mathrm{im}\, M$ and kernel of M by $\ker M$. Let P be a square matrix. The matrix P is said to be *symmetric* if $P = P^T$. We say that P (not necessarily symmetric) is *positive semi-definite* if $v^T P v \geqslant 0$ for all vectors v. It is said to be *positive definite* if it is positive semi-definite and $v^T P v = 0$ implies $v = 0$. We write $P \geqslant 0$ and $P > 0$ by meaning that P is positive semi-definite and positive definite, respectively. Negative (semi-)definiteness is defined in a similar fashion. Given two vectors u and v, the notation $\mathrm{col}(u, v)$ denotes the vector obtained by stacking u over v. The identity matrix will be denoted by I, while the zero matrix by 0. A rational matrix $G(s)$ is said to be *proper* if $\lim_{s \to \infty} G(s)$ is finite.

III. PRELIMINARIES

In what follows we introduce/review some of the concepts that will be used later.

A. Descriptor Systems

Consider the descriptor system

$$E\dot{x}(t) = Ax(t) + Bu(t) \tag{1a}$$

$$y(t) = Cx(t) + Du(t) \tag{1b}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the input, $y \in \mathbb{R}^p$ is the output, and the matrices (A, B, C, D, E) are of appropriate sizes with E and A being square. We denote (1) by $\Sigma(E, A, B, C, D)$. Throughout the paper, we assume that (1) is regular, i.e. $(sE - A)$ is invertible as a polynomial matrix. We say that a descriptor system is *minimal* if there is no other descriptor system with less number of states yielding the same transfer matrix. A full characterization of minimality can be found in the Appendix.

Throughout the paper, we are interested in a particular type of solutions for (1). Let $L_{2,\mathrm{loc}}$ denote the locally square integrable functions. Let AC denote the absolutely continuous functions. We say that a triple $(x, u, y) \in AC^n \times L_{2,\mathrm{loc}}^{m+p}$ is a *solution (or trajectory)* if it satisfies (1) for almost all $t \in \mathbb{R}$.

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B. Weierstrass form

A useful tool in the analysis of descriptor systems is Weierstrass form. If (1) is regular, there exist two square invertible matrices S and T such that the system (1) is transformed to the Weierstrass canonical form

$$\tilde{E}\dot{x}(t) = \tilde{A}x(t) + \tilde{B}u(t) \tag{2a}$$

$$y(t) = \tilde{C}x(t) + Du(t) \tag{2b}$$

with:

$$\tilde{E} = SET = \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \tilde{A} = SAT = \begin{bmatrix} A_1 & 0 \\ 0 & I \end{bmatrix},$$

$$\tilde{B} = SB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \tilde{C} = CT = \begin{bmatrix} C_1 & C_2 \end{bmatrix},$$

where $A_1 \in \mathbb{R}^{n_1 \times n_1}, B_i \in \mathbb{R}^{n_i \times m}, C_i \in \mathbb{R}^{n_i \times m}$ and $N \in \mathbb{R}^{n_2 \times n_2}$ is nilpotent, i.e. $N^q = 0$ for some integer $q \geqslant 0$. We denote the smallest of such integers by k.

C. Properties of solutions

Consider the set

$$\mathcal{W} = \{\xi \mid \text{there exists a trajectory of (1) and}$$
$$t \in \mathbb{R} \text{ such that } \xi = \text{col}(\dot{x}(t), x(t), u(t))\}.$$

The following lemma gives a characterization of the set $\mathcal{W}$.

Lemma III.1 *For the descriptor system* (1)*, there exist matrices* $V \in \mathbb{R}^{n \times \ell}$, $F \in \mathbb{R}^{\ell \times \ell}$, *and* $U \in \mathbb{R}^{m \times \ell}$ *with* $\ell = n + (k+1)m$ *such that*

$$\mathcal{W} = \text{im} \begin{bmatrix} VF \\ V \\ U \end{bmatrix}.$$

Moreover, $EVF = AV + BU$.

Proof. Without loss of generality, we can assume that (E, A, B, C) is in the Weierstrass form. Then, any trajectory of (1) is given by (see for instance [6])

$$\dot{x}_1(t) = A_1 x_1(t) + B_1 u(t) \tag{3}$$

$$x_2(t) = -\sum_{i=0}^{k-1} \frac{d^i}{dt^i}(N^i B_2 u(t)). \tag{4}$$

By differentiating the second equation, we get

$$\dot{x}_2(t) = -\sum_{i=0}^{k-1} \frac{d^{(i+1)}}{dt^{(i+1)}}(N^i B_2 u(t)). \tag{5}$$

By putting (3) and (5) together, we get equation (6) where $v_i(t)$ are functions satisfying $N^i B_2 v_i(t) = \frac{d^i}{dt^i}(N^i B_2 u(t))$ for $i = 0, 1, \ldots, k$. Then, the choices

$$V = \begin{bmatrix} I & 0 & 0 & \cdots & 0 & 0 \\ 0 & -B_2 & -NB_2 & \cdots & -N^{k-1}B_2 & 0 \end{bmatrix} \tag{7}$$

$$F = \begin{bmatrix} A_1 & B_1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & I \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \tag{8}$$

$$U = \begin{bmatrix} 0 & I & 0 & \cdots & 0 & 0 \end{bmatrix} \tag{9}$$

prove the first part. The rest readily follows from the definition of $\mathcal{W}$ as $\mathcal{W} \subseteq \ker \begin{bmatrix} E & -A & -B \end{bmatrix}$. ∎

D. Passivity and Positive Realness

Following Willems [5], [7], we formulate the notion of passivity via the so-called dissipation inequality.

Definition III.2 The system (1) is *passive* if there exists a nonnegative-valued function $V : \mathbb{R}^n \to \mathbb{R}_+$ such that

$$V(x(t_0)) + \int_{t_0}^{t_1} u^T(t)y(t) \, dt \geqslant V(x(t_1)).$$

for all t_0, t_1 with $t_1 \geqslant t_0$ and (u, x, y) satisfying (1). If exists, V is called a *storage function*.

An intimately related concept is positive realness.

Definition III.3 A rational matrix $G(s) \in \mathbb{R}^{m \times m}(s)$ is positive real if the following conditions are satisfied:

- G is analytic in $\mathbb{C}_+$;
- $G(\bar{s}) = \overline{G(s)}$ for all $s \in \mathbb{C}$;
- $G(s) + G^H(s) \geqslant 0$ for all $s \in \mathbb{C}_+$.

E. Kalman-Yakubovich-Popov Lemma

When $E = I$, the following theorem summarizes well-known relationship between passivity of a system and positive realness of its transfer matrix.

Theorem III.4 *Consider the system* (1) *with* $E = I$ *and* $m = p$. *Among the statements*

1) *The system* $\Sigma(I, A, B, C, D)$ *is passive with a quadratic storage function.*
2) *The linear matrix inequalities*

$$K = K^T \geqslant 0 \tag{10a}$$

$$\begin{bmatrix} A^T K + KA & KB - C^T \\ B^T K - C & -(D + D^T) \end{bmatrix} \leqslant 0 \tag{10b}$$

have a solution K.
3) *The transfer matrix* $D + C(sI - A)^{-1}B$ *is positive real.*
4) *The quadruple* (I, A, B, C) *is minimal.*
5) *The pair* (C, A) *is observable.*
6) *The matrix* K *is positive definite.*

the following implications hold:

A) *1 ⇔ 2.*
B) *2 ⇒ 3.*
C) *3 and 4 ⇒ 2.*
D) *2 and 5 ⇒ 6.*

$$
\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \hdashline x_1(t) \\ x_2(t) \\ \hdashline u(t) \end{bmatrix} = \begin{bmatrix} A_1 & B_1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -B_2 & \cdots & -N^{k-2}B_2 & -N^{k-1}B_2 \\ \hdashline I & 0 & 0 & \cdots & 0 & 0 \\ 0 & -B_2 & -NB_2 & \cdots & -N^{k-1}B_2 & 0 \\ \hdashline 0 & I & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ u(t) \\ v_1(t) \\ \vdots \\ v_{k-1}(t) \\ v_k(t) \end{bmatrix}.
\tag{6}
$$

IV. MAIN RESULTS

The main contribution of the paper is the following complete analogue of KYP lemma for descriptor systems.

Theorem IV.1 *Consider the system* (1) *with* $m = p$. *Let* V, F, U *be as in Lemma III.1. Among the statements*

1) *The system* $\Sigma(E, A, B, C, D)$ *is passive with a quadratic storage function.*
2) *The linear matrix inequalities*

$$K = K^T \geqslant 0 \tag{11a}$$

$$\begin{bmatrix} VF \\ V \\ U \end{bmatrix}^T \begin{bmatrix} 0 & K & 0 \\ K & 0 & -C^T \\ 0 & -C & -(D+D^T) \end{bmatrix} \begin{bmatrix} VF \\ V \\ U \end{bmatrix} \leqslant 0 \tag{11b}$$

   have a solution K.
3) *The transfer matrix* $D + C(sE - A)^{-1}B$ *is positive real.*
4) *The quadruple* (E, A, B, C) *is minimal.*

the following implications hold:

A) *1 ⇔ 2.*
B) *2 ⇒ 3.*
C) *3 and 4 ⇒ 2.*

Proof. *A :* This follows from Lemma (III.1) by employing the arguments of the standard case.

B: Let $\xi \in \ker V \cap \mathbb{C}^\ell$. Define $X(s) = V(sI - F)^{-1}\xi$, $U(s) = U(sI - F)^{-1}\xi$ and $w = \text{col}(sX(s), X(s), U(s))$. Since $\xi \in \ker V$, we get

$$
\begin{aligned}
sX(s) &= sV(sI - F)^{-1}\xi - V\xi = V[s(sI - F)^{-1} - I]\xi \\
&= VF(sI - F)^{-1}\xi.
\end{aligned}
$$

Thus $w \in \mathcal{W}$. Then

$$sEX(s) = AX(s) + BU(s). \tag{12}$$

Note that

$$
\begin{aligned}
0 \geqslant w^H &\begin{bmatrix} 0 & K & 0 \\ K & 0 & -C^T \\ 0 & -C & -(D+D^T) \end{bmatrix} w \\
&= \text{Re}(s)X^H(s)KX(s) - X^H(s)C^TU(s)+ \\
&\quad - U^H(s)CX(s) - U^H(s)(D+D^T)U(s).
\end{aligned} \tag{13}
$$

Since $s \in \mathbb{C}_+$, and K is positive semi-definite, (13) results in

$$U^H(s)[CX(s) + DU(s)] + [CX(s) + DU(s)]^HU(s) \geqslant 0$$

By solving $X(s)$ from (12), we get

$$U^H(s)[G(s) + G^H(s)]U(s) \geqslant 0 \tag{14}$$

To conclude the proof, we need to show that $U(sI - F)^{-1}(\ker V \cap \mathbb{C}^\ell) = \mathbb{C}^m$. This can be achieved by assuming that (1) is given in Weierstrass form and using (8) and (9). Now, suppose that $G(s)$ has a pole $s_0 \in \mathbb{C}_+$. This means that the condition (14) holds in a pointed neighborhood of s_0 which is free of any pole. This, however, would contradict to the fact that s_0 is a pole. Thus, $G(s)$ does not have any pole in $\mathbb{C}_+$ and (14) holds for all $s \in \mathbb{C}_+$. Note that $G(\bar{s}) = \overline{G(s)}$ for all $s \in \mathbb{C}$ as matrices in equation (E, A, B, C, D) are real. So, $G(s)$ is positive real.

C: In view of Proposition A.4 statement 2, we can assume without loss of generality

$$
(E, A) = \left(\begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_1 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \right)
$$

$$
(B, C) = \left(\begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix}, \begin{bmatrix} C_1 & C_2 & C_3 \end{bmatrix} \right).
$$

Since $G(s)$ is positive real and (E, A, B, C) is minimal, we can assume without loss of generality that (E, A, B, C) is in Weierstrass form of (26). Then, straightforward calculations yield that

$$
\begin{bmatrix} VF \\ V \\ U \end{bmatrix} = \begin{bmatrix} A_1 & B_1 & 0 & 0 \\ 0 & 0 & -B_2 & -B_3 \\ 0 & 0 & -B_3 & 0 \\ I & 0 & 0 & 0 \\ 0 & -B_2 & -B_3 & 0 \\ 0 & -B_3 & 0 & 0 \\ 0 & I & 0 & 0 \end{bmatrix}.
$$

Let K be a symmetric positive semi-definite matrix and be partitioned as

$$
K = \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{12}^T & K_{22} & K_{23} \\ K_{13}^T & K_{23}^T & K_{33} \end{bmatrix} = K^T \geqslant 0 \tag{15}
$$

where $K_{ij} \in \mathbb{R}^{n_i \times n_i}$. Then, it can be verified that equation (16) holds. Take K_{12}, K_{13}, K_{22} and K_{23} as zero matrices with the corresponding sizes. Since C_2B_3 is negative semi-definite and B_3 is of full row rank due to Proposition A.4, there exists a symmetric positive semi-definite matrix K_{33} such that $B_3^T K_{33} + C_2 = 0$. With these choices, (16) becomes (18). Since $G(s)$ is positive real and

$$\begin{bmatrix} VF \\ V \\ U \end{bmatrix}^T \begin{bmatrix} 0 & K & 0 \\ K & 0 & -C^T \\ 0 & -C & -(D+D^T) \end{bmatrix} \begin{bmatrix} VF \\ V \\ U \end{bmatrix} = \left[\begin{array}{cc:cc} A_1^T K_{11} + K_{11} A_1 & \mathcal{M}_{12} & \mathcal{M}_{13} & -K_{12}B_3 \\ \mathcal{M}_{12}^T & \mathcal{M}_{22} & \mathcal{M}_{23} & \mathcal{M}_{24} \\ \hdashline \mathcal{M}_{13}^T & \mathcal{M}_{23}^T & \mathcal{M}_{33} & B_3^T K_{22} B_3 \\ -B_3^T K_{12}^T & \mathcal{M}_{24}^T & B_3^T K_{22} B_3 & 0 \end{array}\right] \tag{16}$$

where

$$\begin{aligned}
\mathcal{M}_{12} &= K_{11} - A_1^T K_{12} B_2 - A_1^T K_{13} B_3 - C_1^T \\
\mathcal{M}_{13} &= -K_{12}B_2 - K_{13}B_3 - A_1^T K_{12} B_3 \\
\mathcal{M}_{22} &= -B_2^T K_{12}^T B_1 - B_3^T K_{13}^T B_1 - B_1^T K_{12} B_2 - B_1^T K_{13} B_3 + C_2 B_2 + C_3 B_3 + B_2^T C_2^T + \\
&\quad + B_3^T C_3^T - (D + D^T) \\
\mathcal{M}_{23} &= B_2^T K_{22} B_2 + B_3^T K_{23}^T B_2 + B_2^T K_{23} B_3 + B_3^T K_{33} B_3 - B_1^T K_{12} B_3 + C_2 B_3 \\
\mathcal{M}_{33} &= B_3^T K_{22} B_2 + B_3^T K_{23} B_3 + B_2^T K_{22} B_3 + B_3^T K_{23}^T B_3 \\
\mathcal{M}_{24} &= B_2^T K_{22} B_3 + B_3^T K_{23} B_3.
\end{aligned}$$

(A_1, B_1, C_1) is minimal due to Proposition A.4, we can use Kalman-Yakubovich-Popov lemma to find a symmetric positive semi-definite K_{11} such that LMIs (19) holds. This means that

$$K = \begin{bmatrix} K_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & K_{33} \end{bmatrix}$$

is a solution of the LMIs (11). ∎

The arguments of the proof of the last statement can be employed to parameterize all solution of the LMIs (11).

Lemma IV.2 *Consider descriptor system* (1) *with* $m = p$. *Suppose that*

1) $C(sE - A)^{-1}B + D$ *is positive real.*
2) (E, A, B, C) *is minimal.*
3) (E, A, B, C) *is given in Weierstrass form of* (26).

Then, all solutions of the LMIs (11) *are given by*

$$K = \begin{bmatrix} K_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & K_{33} \end{bmatrix}$$

where $K_{ii} \in \mathbb{R}^{n_i \times n_i}$ *such that*

A) K_{11} *is a solution of the LMIs* (10) *for*

$$(A, B, C, D) = (A_1, B_1, C_1, D - C_2 B_2 - C_3 B_3).$$

B) K_{33} *is the unique solution of* $B_3^T K_{33} = C_2$.

Proof. The hypotheses guarantee the solvability of LMIs (11). Let K be given as in (15). Note that the right hand side of (16) is a negative semi-definite matrix. Since $\mathcal{M}_{44} = 0$, it follows from elementary linear algebra that all the elements on the corresponding rows and columns must be zero. In other words,

$$B_3^T K_{22} B_3 = 0 \tag{17a}$$
$$\mathcal{M}_{24} = 0 \tag{17b}$$
$$-K_{12}B_3 = 0. \tag{17c}$$

Due to minimality, B_3 is of full row rank. Then, (17a)-(17c) yield that $K_{12} = 0$, $K_{22} = 0$ and $K_{23} = 0$. As such, we get $\mathcal{M}_{33} = 0$. This yields $K_{13} = 0$ and $K_{33} B_3 + C_2 = 0$. Since B_3 is of full row rank and $C_2 B_3$ is symmetric and negative semi-definite due to hypotheses, there always exists a unique symmetric positive semi-definite K_{33} satisfying $K_{33} B_3 + C_2 = 0$. Therefore, the right hand side of (16) boils down to the right hand side of (18). Since $G(s)$ is positive real, so is its proper part

$$G_1(s) = C_1(sI - A_1)^{-1}B_1 + D - C_2 B_2 - C_3 B_3.$$

Moreover, (I, A_1, B_1, C_1) is minimal due to minimality of (E, A, B, C). Hence, one can find K_{11} with the desired properties. ∎

A. Comparison with previous results

In what follows, we will compare our results with the available results in the literature.

Remark IV.3 (KYP lemma) When $E = I$, it can be verified that

$$\mathcal{W} = \text{im} \begin{bmatrix} A & B \\ I & 0 \\ 0 & I \end{bmatrix}.$$

Since

$$\begin{bmatrix} A & B \\ I & 0 \\ 0 & I \end{bmatrix}^T \begin{bmatrix} 0 & K & 0 \\ K & 0 & -C^T \\ 0 & -C & -(D+D^T) \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \\ 0 & I \end{bmatrix}$$
$$= \begin{bmatrix} A^T K + KA & KB - C^T \\ B^T K - C^T & -(D + D^T) \end{bmatrix},$$

Kalman-Yakubovich-Popov lemma is recovered as a special case from Theorem IV.1.

Remark IV.4 (Kablar [8]) In [8, Corollary 4.1], it is claimed that a minimal descriptor system (1) with $m = p$

$$\begin{bmatrix} VF \\ V \\ U \end{bmatrix}^T \begin{bmatrix} 0 & K & 0 \\ K & 0 & -C^T \\ 0 & -C & -(D+D^T) \end{bmatrix} \begin{bmatrix} VF \\ V \\ U \end{bmatrix}$$

$$= \left[\begin{array}{cc:cc} A_1^T K_{11} + K_{11} A_1 & K_{11} B_1 - C_1^T & 0 & 0 \\ B_1^T K_{11} - C_1 & C_2 B_2 + C_3 B_3 + B_2^T C_2^T + B_3^T C_3^T - (D+D^T) & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]. \quad (18)$$

$$\begin{bmatrix} A_1^T K_{11} + K_{11} A_1 & K_{11} B_1 - C_1^T \\ B_1^T K_{11} - C_1 & B_2^T C_2^T + C_2 B_2 + B_3 C_3^T + B_3 C_3 - (D+D^T) \end{bmatrix} \leqslant 0. \quad (19)$$

---

is passive if, and only if, the LMIs

$$P = P^T > 0 \quad (20a)$$

$$\begin{bmatrix} A^T P E + E^T P A & P B - C^T \\ B^T P - C & -(D+D^T) \end{bmatrix} \leqslant 0 \quad (20b)$$

admit a solution. However, this result cannot hold in this generality as $D+D^T$ is not necessarily positive semi-definite for passive descriptor system. As an example, consider the system

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - u.$$

Since $D = -1$, the LMIs (20) do not admit a solution. However, the dissipation inequality holds for the storage function $V(x) = \frac{1}{2} x_2^2$. To see this, note that $x_2 + u = 0 \Rightarrow u = -x_2$ and $y = x_1 - x_2 - u \Rightarrow y = x_1$. Hence, one gets $\frac{d}{dt} V = \dot{x}_2 x_2 = -x_1 x_2 = uy$. Therefore, the dissipation inequality holds as an inequality.

Remark IV.5 The following theorem summarizes the extension of Kalman-Yakubovich-Popov lemma to descriptor system that is proposed in [3].

Theorem IV.6 *[3, Theorem 1, 2 and 3] Consider the descriptor system* (1) *with* $m = p$. *Let* $G(s) = C(sE - A)^{-1} B + D$. *The following statement hold.*

1) *If the LMIs*

$$E^T X = X^T E \geqslant 0 \quad (21a)$$

$$\begin{bmatrix} A^T X + X^T A & X^T B - C^T \\ B^T X - C & -(D+D^T) \end{bmatrix} \leqslant 0 \quad (21b)$$

admit a solution, then $G(s)$ *is positive real.*

2) *If the LMIs*

$$E^T X E = E^T X^T E \geqslant 0 \quad (22a)$$

$$\begin{bmatrix} A^T X E + E^T X^T A & E^T X^T B - C^T \\ B^T X E - C & -(D+D^T) \end{bmatrix} \leqslant 0 \quad (22b)$$

admit a solution, then $G(s)$ *is positive real.*

3) *Suppose that* $G(s)$ *is positive real and* $G(s) = G_1(s) + sG_0$ *where* $G_1(s)$ *is proper. If* (E, A, B, C) *is minimal*

and $D + D^T \geqslant G_1(\infty) + G_1^T(\infty)$ *then the LMIs* (21) *admit a solution.*

By taking $K = E^T X$ and using $EVF = AV + BU$, we get

$$\begin{bmatrix} VF \\ V \\ U \end{bmatrix}^T \begin{bmatrix} 0 & K & 0 \\ K & 0 & -C^T \\ 0 & -C & -(D+D^T) \end{bmatrix} \begin{bmatrix} VF \\ V \\ U \end{bmatrix}$$

$$= \begin{bmatrix} V \\ U \end{bmatrix}^T \begin{bmatrix} A^T X + X^T A & X^T B - C^T \\ B^T X - C & -(D+D^T) \end{bmatrix} \begin{bmatrix} V \\ U \end{bmatrix}. \quad (23)$$

Similarly, by taking $K = E^T X E$ and using $EVF = AV + BU$, we get

$$\begin{bmatrix} VF \\ V \\ U \end{bmatrix}^T \begin{bmatrix} 0 & K & 0 \\ K & 0 & -C^T \\ 0 & -C & -(D+D^T) \end{bmatrix} \begin{bmatrix} VF \\ V \\ U \end{bmatrix}$$

$$= \begin{bmatrix} V \\ U \end{bmatrix}^T \begin{bmatrix} A^T X E + E^T X^T A & E^T X^T B - C^T \\ B^T X E - C & -(D+D^T) \end{bmatrix} \begin{bmatrix} V \\ U \end{bmatrix}. \quad (24)$$

It follows from (23) and (24) that the LMIs (11) admit a solution whenever one of the LMIs (21) and (22) admits a solution.

To see that the last statement follows from our main result, assume that (E, A, B, C) is given in Weierstrass form of (26). Note that $G_1(\infty) = D - C_2 B_2 - C_3 B_3$ and also that

$$\begin{bmatrix} A_1^T K_{11} + K_{11} A_1 & K_{11} B_1 - C_1^T \\ B_1^T K_{11} - C_1 & -(D+D^T) \end{bmatrix}$$

$$\leqslant \begin{bmatrix} A_1^T K_{11} + K_{11} A_1 & K_{11} B_1 - C_1^T \\ B_1^T K_{11} - C_1 & -(G_1(\infty) + G_1^T(\infty)) \end{bmatrix}$$

as $D + D^T \geqslant G_1(\infty) + G_1^T(\infty)$. Then, it follows from Lemma IV.2 that the LMIs (21) admits a solution with

$$X = \begin{bmatrix} K_{11} & 0 & 0 \\ 0 & 0 & K_{33} \\ 0 & 0 & 0 \end{bmatrix}.$$

V. CONCLUSIONS

This paper presents a complete analogue of Kalman-Yakubovich-Popov lemma for regular descriptor systems. Unlike previous work, we do not make any additional assumptions such as impulse-freeness or any condition on the feed-through term. After establishing our main result,

some of the earlier results are covered as special cases. Our future research topic is to characterize strict versions of passivity/positive realness in the spirit of the current paper.

<center>APPENDIX</center>
<center>KNOWN RESULTS</center>

We quote the following well-known theorem that states necessary and sufficient conditions for minimality.

Theorem A.1 *[9], [10] Let*

$$G(s) = C(sE - A)^{-1}B + D \qquad (25)$$

be a rational function where E and A are square matrix with dimension n. Then, (25) is a minimal realization of $G(s)$ if, and only if, the following conditions are satisfied:

- rank $\begin{bmatrix} A - sE & B \end{bmatrix} = n$ *for all $s \in \mathbb{C}$ (Finite controllability).*
- rank $\begin{bmatrix} E & B \end{bmatrix} = n$ *(Infinite controllability).*
- rank $\begin{bmatrix} A^T - sE^T & C^T \end{bmatrix} = n$ *for all $s \in \mathbb{C}$ (Finite observability).*
- rank $\begin{bmatrix} E^T & C^T \end{bmatrix} = n$ *for all $s \in \mathbb{C}$ (Infinite observability).*
- $A \ker E \subseteq \operatorname{im} E$ *(Absence of non dynamics modes).*

A characterization of positive realness can be found in the following theorem:

Theorem A.2 *[11, Theorem 2.7.2] A real rational function $G : \mathbb{C} \mapsto (\mathbb{C} \cup \infty)^{m \times m}$ is positive real if, and only if, the following conditions are satisfied:*

- *G has no poles in $\mathbb{C}_+$;*
- *$G(i\omega) + G^H(i\omega) \geqslant 0$ for all $\omega \in \mathbb{R}$ with $i\omega$ not a pole of G;*
- *If $i\omega$ or ∞ is a pole of G, then it is a simple pole and the associated residue matrix is positive semi-definite.*

Weierstrass form plays a key role in the analysis of descriptor systems. The following proposition imposes a particular structure on Weierstrass form of the systems of interests in this paper.

Proposition A.3 *Let $(E, A, B, C) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m} \times \mathbb{R}^{m \times p}$ be given such that*

1) *(E, A, B, C) is minimal.*
2) *If $s = \infty$ is a pole of $C(sE - A)^{-1}B$ then it is a simple pole*

Then, there exist matrices $(S, T) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n}$ such that

$$SET = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{bmatrix} \quad SAT = \begin{bmatrix} A_1 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \qquad (26a)$$

$$SB = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} \quad CT = \begin{bmatrix} C_1 & C_2 & C_3 \end{bmatrix} \qquad (26b)$$

where $A_1 \in \mathbb{R}^{n_1 \times n_1}, B_i \in \mathbb{R}^{n_i \times m}, C_i \in \mathbb{R}^{n_i \times m}$ and all other matrices involved are of appropriate sizes.

Proof. The same result is obtained by [3] where the second condition is replaced by positive realness. Its proof is based on Proposition 2 of [3]. However, this proposition still holds if one replaces positive realness by the condition 2. This observation concludes our proof. ∎

Proposition A.4 *Let (E, A, B, C) given such that $G(s) = D + C(sE - A)^{-1}B$ is positive real. The following statements hold.*

1) *$G(s) = G_1(s) + G_0 s$ where G_1 is proper and positive real and $G_0 = G_0^T \geqslant 0$.*
2) *(E, A, B, C) admits the Weierstrass form (26) and $G_0 = -C_2 B_3$.*
3) *If (E, A, B, C) is minimal, then (A_1, B_1, C_1) is minimal and B_3 is of full row rank.*

Proof. *1:* Follows from Theorem A.2.
2: Follows from Theorem A.2 and Proposition A.3.
3: Note that $G(s) = C_1(sI - A_1)^{-1}B_1 + D - C_2 B_2 - C_3 B_3 - sC_2 B_3$. Hence, $G_1(s) = C_1(sI - A_1)^{-1}B_1 + D - C_2 B_2 - C_3 B_3$. Minimality of (I, A_1, B_1, C_1) follows from Theorem A.1. ∎

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