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Expressivity and completeness for public update logics via reduction axioms

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ABSTRACT. *In this paper, we present several extensions of epistemic logic with update operators modelling public information change. Next to the well-known public announcement operators, we also study public substitution operators. We prove many of the results regarding expressivity and completeness using so-called reduction axioms. We develop a general method for using reduction axioms and apply it to the logics at hand.*

KEYWORDS: *dynamic epistemic logic, reduction axioms, speech acts, updates*

1. Introduction

There are many scientific theories about information, for instance information theory, probability theory, statistics, computer science, philosophy of science, and logic. The branch of logic called epistemic logic deals with information explicitly. It was initially developed by Hintikka (1962), whose main goal was a conceptual analysis of knowledge and belief. In epistemic logic the focus is on statements such as 'I know that p', 'I know that you know that p' and 'I know that he knows that we know that p'. Epistemic logic is especially useful when applied to situations involving more than one agent. One can model the information an agent has about the bare facts of the world and the information an agent has about other agents' information, i.e., *higher-order information*. This ability to model higher-order information distinguishes epistemic logic from other scientific theories about information.

The focus on higher-order information has led to investigations into group notions of information of which common knowledge is a prime example. A proposition p is *common knowledge* among a group of agents iff everybody in the group knows that

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p, everybody knows that everybody knows that p, and so on *ad infinitum*. This notion is of crucial importance if one wants to understand communication, because common knowledge is often exactly what communication aims to achieve. Epistemic logic with temporal operators has been applied to the analysis of Internet communication protocols and it has been used in formal specifications of multi-agent systems (Fagin *et al.*, 1995; Meyer *et al.*, 1995). There are also dynamic epistemic logics, where change is not modelled by the passage of time, but with update operations. These logics were developed specifically to analyse *change* of higher-order information. It has been a very active research field in the past years (Plaza, 1989; Gerbrandy *et al.*, 1997; Gerbrandy, 1998; Baltag *et al.*, 1999; van Ditmarsch, 2000; Baltag, 2002; Kooi, 2003; van Ditmarsch *et al.*, 2003; Baltag *et al.*, 2004; Renardel de Lavalette, 2004; van Benthem *et al.*, 2006).

In epistemic logic, the information the agents have is modelled by Kripke models. In dynamic epistemic logic, information change is modelled by manipulating these Kripke models. The focus has mostly been on information change due to communication. One of the characteristics of communication is that it does not change the bare facts of the world, but only the information agents have about the world and each other. Hence, the issue of information change due to changes of facts has mostly been left out of consideration. Notable exceptions are papers by Renardel de Lavalette (2004) and van Ditmarsch *et al.* (2005b). In this paper, updates where the bare facts of the world can change are studied alongside updates that model communication.

The focus in this paper is not on full-fledged dynamic epistemic logics with operators for complex communicative updates. Instead the focus is on the simple case of *public updates*: events where all agents get the same information and where it is common knowledge (among all agents) that they get the same information. Such public updates can be of two forms: communicative or fact changing. The technical term for the former is *public announcement* and for the latter I use the term *public substitution*. Public announcements are public updates where all the agents commonly receive the information that a certain formula is true. In the semantics the effect of a public announcement is modelled by adapting the model such that all the worlds where that formula is false are no longer considered possible by the agents. This was first introduced by Plaza (1989) and independently by Gerbrandy and Groeneveld (1997). Public substitutions are public updates where all the agents commonly receive the information that the truth value of a certain propositional variable has changed to the truth value of a (possibly) complex formula. In the semantics the effect of a public substitution is modelled by adapting the model such that after the substitution the propositional variable is true in those worlds where the complex formula was true before the substitution.

A logic with both these kinds of operators was introduced by van Ditmarsch *et al.* (2005b), but the issues of axiomatisation and expressivity were not addressed in that paper. This led to the investigations reported in the present paper, concerning the axiomatisation and expressivity of a whole range of logics with these operators. As it turns out, the logic introduced by van Ditmarsch *et al.* (2005b) is more expressive than

the logic without public substitutions. Based on the observation that its expressivity is equal to the logic of *relativised common knowledge* in the present paper a sound and complete axiomatisation is obtained.

In Section 2, the languages and semantics of the logics that will be studied are introduced. In Section 3, I prove general theorems about expressivity and completeness via so-called reduction axioms. In Section 4, these results are applied to the logics introduced in Section 2. A case of special interest is studied separately in Section 5. In Section 6, conclusions are drawn and directions for further research are indicated.

2. Languages and semantics

We introduce a number of logical languages and their semantics that will be studied in this paper. Relativised common knowledge is also introduced, because it will turn out to be quite important when we look at the expressivity of epistemic logic with public announcements, substitutions, and common knowledge. I use the style of notation from propositional dynamic logic (PDL) for modal operators which was also used by van Benthem *et al.* (2006).

DEFINITION 1 (LANGUAGES). — *Let a finite set of agents $\mathcal{A}$ and a countably infinite set of propositional variables $\mathcal{P}$ be given. The language $\mathscr{L}_{APSCR}$ is given by the following Backus-Naur Form (where φ are formulas, α are modalities, and σ are public substitutions):*

$$\begin{array}{lcl} \varphi & ::= & p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid [\alpha]\varphi \\ \alpha & ::= & a \mid \varphi \mid \sigma \mid B^{+} \mid (B; ?\varphi)^{+} \\ \sigma & ::= & p := \varphi \mid p := \varphi, \sigma \end{array}$$

where $p \in \mathcal{P}$, $a \in \mathcal{A}$, and $B \subseteq \mathcal{A}$. Besides the usual abbreviations $[B]\varphi$ will be used as an abbreviation of $\bigwedge_{a \in B}[a]\varphi$. Only substitutions σ such that any propositional variable p occurs at most once on the left side of a '$:=$' are considered. In this way σ can be seen as a finite, and hence partial, function from propositional variables to formulas. By abuse of language, I use $\sigma(p)$ to refer to the formula assigned to p if $p \in dom(\sigma)$, and to refer to p otherwise. Various sublanguages will be considered, where α is restricted. The subscripts of $\mathscr{L}$ below indicate whether Agents, Public announcements, Substitutions, Common knowledge, *or* Relativised common knowledge *are included. For instance $\mathscr{L}_{ASR}$ is the language with agents, substitutions and relativised common knowledge.*

The non-standard expressions in the definition above are read as follows:

$[a]\varphi$	Agent a knows that φ.
$[B]\varphi$	Everybody in group B knows that φ.
$[\varphi]\psi$	ψ is the case after the announcement that φ.
$[\sigma]\varphi$	φ is the case after the substitution σ.
$p := \varphi, q := \psi$	p changes to φ and simultaneously q changes to ψ.
$[B^+]\varphi$	φ is common knowledge among the members of group B.
$[(B; ?\varphi)^+]\psi$	ψ is common knowledge among the members of group B relative to φ.

The most difficult of these is relativised common knowledge. One can understand it in the same way one can understand ordinary common knowledge. "φ is common knowledge if everyone knows that φ is common knowledge" is a way of explaining what it means that something is common knowledge. The circularity of this explanation can be understood as a fixed point construction. In the same way we can characterise relativised common knowledge: "φ is common knowledge relative to ψ if everyone knows that if ψ, then φ is common knowledge relative to ψ."

Logics with substitution operators have been studied before. One of the epistemic actions considered by Baltag (2002) is a 'flip' action, where the extension of a propositional variable (the set of worlds in which the variable is true) changes to its complement. Renardel de Lavalette (2004) considers more general changes of truth values where the extension of a propositional variable can change to the extension of an arbitrary formula, but this logic does not contain a common knowledge operator. Simultaneous substitutions were added to action models by van Eijck (2004), and actions models with substitutions were adopted in the logic of communication and change (LCC) by van Benthem et al. (2006). Here they are studied as modal operators in themselves. One might expect that simultaneity adds expressivity, yet it does not make a difference in terms of expressivity (see Section 4). However, simultaneity does allow more succinct formulas.

Although the terms 'knowledge' and 'common knowledge' are used, I also consider belief and common belief. In fact the semantics given below is more suited for the case of belief. The results below also apply to the general modal case, where these operators do not even have an epistemic or doxastic interpretation. In order to keep things simple I only use the terms 'knowledge' and 'common knowledge'. The language is interpreted in multi-agent Kripke models.

DEFINITION 2 (MULTI-AGENT KRIPKE MODELS). — *Let a finite set of agents $\mathcal{A}$ and a countably infinite set of propositional variables $\mathcal{P}$ be given. A multi-agent Kripke model M is a triple (W, R, V) such that*

- *W is a non-empty set of worlds,*
- *$R : \mathcal{A} \to \wp(W \times W)$ assigns an accessibility relation to each agent a,*
- *$V : \mathcal{P} \to \wp(W)$ assigns a set of worlds to each propositional variable.*

A multi-agent Kripke model M with a distinguished world $w \in W$ is called a pointed model (M, w). Below we will also refer to pointed models as models.

The accessibility relation assigned to an agent in these models is interpreted epistemically: $(w, v) \in R(a)$ indicates that if w is the actual world, then agent a cannot rule out that world v is the actual world on the basis of its information.

Since the results below do not depend on whether the accessibility relations be reflexive, transitive, or euclidean, these extra requirements are not imposed. The language is interpreted in pointed models, where the distinguished world is taken to be the actual world.

DEFINITION 3 (SEMANTICS). — *Let a multi-agent Kripke model (M, w) with $M = (W, R, V)$ be given. Let $a \in \mathcal{A}$, $B \subseteq \mathcal{A}$, and $\varphi, \psi \in \mathscr{L}_{APSCR}$.*

$$
\begin{array}{lll}
(M,w) \models p & \textit{iff} & w \in V(p) \\
(M,w) \models \neg\varphi & \textit{iff} & (M,w) \not\models \varphi \\
(M,w) \models \varphi \wedge \psi & \textit{iff} & (M,w) \models \varphi \textit{ and } (M,w) \models \psi \\
(M,w) \models [a]\varphi & \textit{iff} & (M,v) \models \varphi \textit{ for all } v \textit{ such that } (w,v) \in R(a) \\
(M,w) \models [\varphi]\psi & \textit{iff} & (M^{\varphi},w) \models \psi \\
(M,w) \models [\sigma]\varphi & \textit{iff} & (M^{\sigma},w) \models \varphi \\
(M,w) \models [B^{+}]\varphi & \textit{iff} & (M,v) \models \varphi \textit{ for all } v \textit{ such that } (w,v) \in R(B)^{+} \\
(M,w) \models [(B;?\varphi)^{+}]\psi & \textit{iff} & (M,v) \models \psi \textit{ for all } v \textit{ such that} \\
 & & \quad (w,v) \in (R(B) \cap (W \times [\![\varphi]\!]_M))^{+}
\end{array}
$$

The updated model $M^{\varphi} = (W, R^{\varphi}, V)$ is defined by restricting the accessibility relations to those worlds where φ holds. $[\![\varphi]\!]_M$ denotes the set $\{v \in W | M, v \models \varphi\}$. Now

$$R^{\varphi}(a) =_{def} R(a) \cap (W \times [\![\varphi]\!]_M) (= \{(w,v) \in R(a) \mid (M,v) \models \varphi\}).$$

The updated model $M^{\sigma} = (W, R, V^{\sigma})$ is defined by changing the valuation accordingly.

$$V^{\sigma}(p) =_{def} [\![\sigma(p)]\!]_M$$

In the clauses for $[B^{+}]\varphi$ and $[(B;?\varphi)^{+}]\psi$ we use $R(B)$ to denote $\bigcup_{a \in B} R(a)$ and the superscript $+$ denotes the transitive closure. (The transitive closure of a binary relation R is the smallest transitive relation that contains R.)

A formula φ is a tautology iff φ is true in all models: $(M, w) \models \varphi$ for all (M, w). This is denoted as $\models \varphi$.

The semantics differs a little from the semantics given by van Ditmarsch *et al.* (2005b), where only the S5 case was considered. In order to preserve S5 under public announcements it was required that the announced formula is true, otherwise the announcement cannot be executed, and $R^{\varphi}(a) = R(a) \cap [\![\varphi]\!]^2_M$. Definition 3 provides the semantics for the general modal case where the public update merely restricts access to the worlds where φ is true, but φ may be false in the actual world. In a belief

setting, a public announcement represents the event where the agents simply *take* the information to be true, even though they may be wrong.

Many performative speech acts classified by Austin (1962) as *exercitives* are examples of public substitutions. For example:

1) You're disqualified.
2) I choose George.
3) You're fired.
4) I sentence you to death.
5) I pronounce you husband and wife.

When the sentences above are uttered in the right circumstances, their utterance makes them true. So, all these examples could be expressed in our logical language as '$p := \top$' (or as '$p := \bot$'). Such performative speech acts cannot be modelled as public announcements. Public announcements, considered as speech acts, could be classified as *expositives*, where the utterance of a sentence merely informs the listeners that the sentence is true.

The following is another simple example of a public substitution. Suppose there are two agents a and b in a room. Agent a is blind, and can therefore not see whether the light in the room is on. Agent b is not visually impaired, and can therefore see whether the light is on. All this is common knowledge among the agents. Let p be the proposition 'the light is on'. Suppose that now the light switch is flicked. Neither agent is deaf and this is also common knowledge among both agents. So, it is common knowledge among the agents that the substitution '$p := \neg p$' has occurred. Agent a still does not know whether the light is in fact on or not, but does know that the truth value of p has changed. Agent b does know whether p. This public substitution is illustrated by Figure 1. This example shows that one might want to substitute using complex formulas rather than just $\top$ or $\bot$. It is also clear that if more than one fact changes at once, then one wants to model this using simultaneous substitutions.

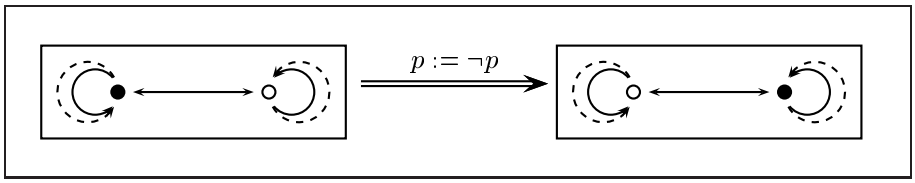

Figure 1. *Two Kripke models: the left one represents the situation before the public substitution $p := \neg p$; the one on the right represents the situation after the public substitution $p := \neg p$. A world where p is true is represented by a solid bullet. A world where p is false is represented by an open bullet*

As a final example of how public substitutions can be used, consider the Sum and Product puzzle. Mr. Sum and Mr. Product do not know the length or width of a room. They do know that these are natural numbers between 2 and 99 and that the length is at least as large as the width ($2 \leq w \leq l \leq 99$.) The sum of these numbers is given

to Mr. Sum, and their product is given to Mr. Product. All this is common knowledge among Mr. Sum and Mr. Product. Now the following conversation takes place:

Mr. Product: I don't know the numbers.
Mr. Sum: I knew you didn't know. I don't know either.
Mr. Product: Now I know the numbers.
Mr. Sum: Now I know them too.

The length and width of the room can be deduced from the dialogue by an outsider.[1] The original formulation and solution of the problem was given by Freudenthal (1969; 1970) in Dutch. The formulation above is by McCarthy (1990). This problem has been analysed using $\mathscr{L}_{APC}$ by van Ditmarsch *et al.* (2005a). The utterance 'I knew you didn't know' poses a problem for this approach. The past tense cannot be represented in $\mathscr{L}_{APC}$. Van Ditmarsch *et al.* (2005a) solve this by noting that the first announcement is superfluous given the second: the dialogue might just as well start with Mr. Sum saying 'I know that you don't know what the number are.' However, such solutions are not generally available in all scenarios where a past tense occurs.

In $\mathscr{L}_{APSC}$ there is a more natural way to represent past tenses (although it would be quite unsatisfactory to a linguist). Suppose that after the announcement that φ, one learns that ψ was the case before the update. The formula

$$[p := \psi][\varphi][p]\chi$$

where p does not occur in φ, ψ or χ, expresses this. It is as if the truth value of ψ has been put into an envelope before the update, and the envelope is opened publicly afterwards, thereby making it common knowledge what the old truth value of ψ is. Using this general approach one could show with the semantics of $\mathscr{L}_{APSC}$ that the adaptation of the scenario proposed by van Ditmarsch *et al.* (2005a) is indeed correct. Another approach to announcements involving the past tense is to extend the language with temporal operators. This is investigated by Yap (2006).

3. Reduction

In the completeness proofs of many of the logics introduced in Section 2 *reduction axioms* play an important role. A typical example of a reduction axiom is

$$[\varphi][a]\psi \leftrightarrow [a](\varphi \rightarrow [\varphi]\psi)$$

This is called a reduction axiom because going from the left of the equivalence to the right the complexity of the formula to which the announcement operator is applied *reduces*. These reduction axioms also play an important role in results about the expressivity of the logics under consideration. If the reduction can be continued

1. The numbers are four and thirteen.

depending on the logical form of ψ until no announcement operators remain, one can show that the language with announcement operators is just as expressive as the language without them. The method of proving completeness and equal expressivity for dynamic epistemic logic using reduction axioms has been used many times in the literature (Plaza, 1989; Gerbrandy, 1998; Baltag *et al.*, 1999; van Benthem *et al.*, 2006). Here we provide a uniform setup, that provides such a general perspective on reduction axioms that it can be applied to many logics. In this section I provide this general method, which is applied to the logics under consideration in Section 4.

The general setup is given by two logical languages $\mathscr{L}_1$ and $\mathscr{L}_2$ such that $\mathscr{L}_1$ is a sublanguage of $\mathscr{L}_2$. The only difference is that $\mathscr{L}_2$ contains additional operators. In order to show that the languages are equally expressive one needs to be able to translate each formula φ from $\mathscr{L}_2$ to an equivalent formula ψ in $\mathscr{L}_1$. This translation procedure is captured by the reduction axioms. These axioms make φ and ψ *provably* equivalent. In this way one can obtain completeness for $\mathscr{L}_2$ via completeness for $\mathscr{L}_1$. After giving a general definition of reduction axioms in Section 3.1, I prove a general theorem about expressivity and reduction axioms in Section 3.2, and prove a general theorem about completeness and reduction axioms in Section 3.3.

3.1. *Depth and reduction axioms*

Reduction axioms allow one to reduce the depth of the formulas to which the additional operators apply. In the proof of Theorem 10 (which states sufficient conditions for two languages to be equally expressive) three notions of depth are needed, namely: (ordinary) depth, O depth, and O reduction depth. The main induction is on the O depth, and in the induction step of this proof another induction on the O reduction depth is embedded. The definition of a reduction axiom is given in terms of the O reduction depth. Let us first define the notion of ordinary depth precisely.

DEFINITION 4 (DEPTH). — *Let a logical language $\mathscr{L}$ be given. The depth $d : \mathscr{L} \to \mathbb{N}$ is given inductively as follows:*

$$\begin{array}{lll} d(\varphi) & =_{def} & 0 \textit{ if no logical operators occur in } \varphi \\ d(\Box(\varphi_1, \ldots, \varphi_n)) & =_{def} & 1 + \max(\{d(\varphi_i) \mid 1 \leq i \leq n\}) \end{array}$$

where $\Box$ is some n-ary operator.

This is a very abstract way of looking at logical language. For a concrete language one has to specify what the logical operators are and what their arity is. The language $\mathscr{L}_{APSCR}$ contains formulas and other expressions. It is clear that, for instance, conjunction is a binary operator. We take $[a]$ to be unary operator. An announcement operator is a binary operator. For instance in the formula $[\varphi]\psi$, the two arguments are φ and ψ. A substitution operator $[\sigma]$ is an $(n+1)$-ary operator, where n is the cardinality of dom(σ): for instance a formula of the form $[p := \varphi, r := \psi]\chi$ takes φ, ψ, and χ as arguments. Therefore $d([p := \varphi, r := \psi]\chi) = 1 + \max(d(\varphi), d(\psi), d(\chi))$.

A limit case would be a nullary operator. Since a nullary operator has no arguments its depth is 1 (the maximum depth of formulas in the empty set is 0).

For the operators one wants to eliminate from the language, a special notion of depth is needed, which indicates to what extent the extra operators are nested.

DEFINITION 5 (O DEPTH). — *Let O be a set of operators in $\mathscr{L}$. The O depth $Od : \mathscr{L} \to \mathbb{N}$ is given inductively as follows:*

$$\begin{array}{lll} Od(\varphi) & =_{def} & 0 \text{ if no logical operators occur in } \varphi \\ Od(\Box(\varphi_1, \ldots, \varphi_n)) & =_{def} & \left\{ \begin{array}{ll} \max(\{Od(\varphi_i) \mid 1 \leq i \leq n\}) & \text{if } \Box \notin O \\ 1 + \max(\{Od(\varphi_i) \mid 1 \leq i \leq n\}) & \text{if } \Box \in O. \end{array} \right. \end{array}$$

Below we will take O to be the set of logical operators that occur only in $\mathscr{L}_2$, i.e. the language to be reduced. The third notion of depth is called the O-reduction depth, which indicates how complex the formulas are to which an outermost O operator applies.

DEFINITION 6 (O REDUCTION DEPTH). — *Let O be a set of operators in $\mathscr{L}$. The O reduction depth $Ord : \mathscr{L} \to \mathbb{N}$ is defined inductively as follows.*

$$\begin{array}{lll} Ord(\varphi) & =_{def} & 0 \text{ if no logical operators occur in } \varphi \\ Ord(\Box(\varphi_1, \ldots, \varphi_n)) & =_{def} & \left\{ \begin{array}{ll} \max(\{Ord(\varphi_i) \mid 1 \leq i \leq n\}) & \text{if } \Box \notin O \\ 1 + \sum_{i=1}^{n} d(\varphi_i) & \text{if } \Box \in O. \end{array} \right. \end{array}$$

Note that in the second case of the second clause of this definition the ordinary notion of depth is used. A general definition of reduction axioms can be given in terms of O reduction depth.

DEFINITION 7 (REDUCTION AXIOMS). — *Given are two languages $\mathscr{L}_1$ and $\mathscr{L}_2$ such that $\mathscr{L}_1$ is a sublanguage of $\mathscr{L}_2$, because $\mathscr{L}_2$ contains more logical operators, assembled in a set of operators O. A* reduction axiom *is a formula of the form $\varphi \leftrightarrow \psi$ such that $Ord(\varphi) > Ord(\psi)$.*

Of course, such axioms are only useful if they are sound and the proof system actually allows one to perform substitutions. The rule one wants to use in this case is the rule of *substitution of equivalents*. In a proof system this rule allows one to infer from $\varphi \leftrightarrow \psi$, that $\chi \leftrightarrow \chi'$, where χ' can be obtained from χ by substituting an occurrence of φ by ψ.

3.2. *Equal expressivity via reduction*

Let us clarify what it means for one logical language to be more expressive than another. Let us first distinguish the *richness* of a language from its *expressivity*. When one language contains more logical operators than another, the one language is richer.

In many cases a new operator is added to enrich a language because there is an important concept that is not yet captured in the language. This does not imply that the expressivity is actually extended. When one language can make more distinctions in the class of models in which it is interpreted than another, then the one language is more expressive than the other. In propositional logic, disjunction is an important concept. However, when one adds it to the language that already contains conjunction and negation it does not add any expressivity. Let us define expressivity formally.

DEFINITION 8 (EXPRESSIVITY). — *Let two logical languages $\mathscr{L}_1$ and $\mathscr{L}_2$ that are interpreted in the same class of models be given.*

– *$\mathscr{L}_1$ is* at least as expressive as *$\mathscr{L}_2$ iff for every formula $\varphi_2 \in \mathscr{L}_2$ there is a formula $\varphi_1 \in \mathscr{L}_1$ such that φ_1 and φ_2 are true in the same models. This is denoted as $\mathscr{L}_1 \succeq \mathscr{L}_2$.*

– *$\mathscr{L}_1$ and $\mathscr{L}_2$ are equally expressive iff $\mathscr{L}_1 \succeq \mathscr{L}_2$ and $\mathscr{L}_2 \succeq \mathscr{L}_1$. This is denoted as $\mathscr{L}_1 \equiv \mathscr{L}_2$.*

– *$\mathscr{L}_1$ is more expressive than $\mathscr{L}_2$ iff $\mathscr{L}_1 \succeq \mathscr{L}_2$ and $\mathscr{L}_2 \not\equiv \mathscr{L}_1$. This is denoted as $\mathscr{L}_1 \succ \mathscr{L}_2$.*

Note that this definition focuses on the expressivity of formulas. One could just as well focus on the expressivity of modalities and see which relations on the set of worlds and on the class of models can be expressed. Here we focus on the expressivity of formulas.

The presence of reduction axioms for a set of operators suggests that the language with the additional operators is just as expressive as the language without them. In this section and the next we will give very general conditions under which the presence of reduction axioms yields two equally expressive languages and general conditions under which these axioms can provide a complete proof system for the richer language. One of the conditions is that $\leftrightarrow$ gets its usual interpretation, and that hence the rule of substitution of equivalents is valid. The following lemma is used in the induction step of the main theorem regarding expressivity (Theorem 10).

LEMMA 9. — *Given are two languages $\mathscr{L}_1$ and $\mathscr{L}_2$ such that $\mathscr{L}_2$ is an extension of $\mathscr{L}_1$ with a set of logical operators O. Moreover, $\mathscr{L}_2$ contains $\leftrightarrow$. Given is also a semantics for $\mathscr{L}_2$ (and hence a semantics for $\mathscr{L}_1$) in some class of models. Finally a set* A *of reduction axioms for O is given such that every formula which is not in $\mathscr{L}_1$ has at least one subformula φ such that there is a formula ψ and $\varphi \leftrightarrow \psi$ is in* A*. If the reduction axioms* A *and the rule of substitution of equivalents are sound for $\mathscr{L}_2$, then for all $\varphi \in \mathscr{L}_2$ with $Od(\varphi) = 1$, there is a formula $\psi \in \mathscr{L}_1$ such that $\models \varphi \leftrightarrow \psi$.*

PROOF. — Suppose that $Od(\varphi) = 1$. The remainder of the proof is by induction on $Ord(\varphi)$. Suppose $Ord(\varphi) = 0$. Therefore φ contains no operators in O, and so $\varphi \in \mathscr{L}_1$. Since $\models \varphi \leftrightarrow \varphi$, we are done.

Suppose as induction hypothesis that for every φ such that $Ord(\varphi) \leq n$, there is a formula $\psi \in \mathscr{L}_1$ such that $\models \varphi \leftrightarrow \psi$.

Suppose that $Ord(\varphi) = n + 1$. Therefore φ contains at least one formula of the form $\Box(\chi_1, \ldots, \chi_k)$ where $\Box \in O$ and $Ord(\Box(\chi_1, \ldots, \chi_k) = n+1$. According to our assumption $\Box(\chi_1, \ldots, \chi_k)$ has at least one subformula such that there is a reduction axiom for it. But, since the O depth of $\Box(\chi_1, \ldots, \chi_k)$ equals 1 by assumption, the only formula for which that can be true is $\Box(\chi_1, \ldots, \chi_k)$ itself. So there must be a formula ξ such that $\Box(\chi_1, \ldots, \chi_k) \leftrightarrow \xi \in \mathsf{A}$ and $Ord(\Box(\chi_1, \ldots, \chi_k)) > Ord(\xi)$. Now, the induction hypothesis applies to ξ and therefore there is a formula $\xi' \in \mathscr{L}_1$ that is equivalent to $\Box(\chi_1, \ldots, \chi_k)$. There is such a formula for each subformula of φ which has the form $\Box(\chi_1, \ldots, \chi_k)$ where $Ord(\Box(\chi_1, \ldots, \chi_k) \leq n+1$. By repeatedly applying the rule of substitution of equivalents one can obtain a formula $\psi \in \mathscr{L}_1$. Since the reduction axioms are sound and the rule of substitution of equivalents is sound it follows that $\models \varphi \leftrightarrow \psi$. ■

This lemma will be used in the induction step of the following theorem.

THEOREM 10. — *Given are two languages $\mathscr{L}_1$ and $\mathscr{L}_2$ such that $\mathscr{L}_2$ is an extension of $\mathscr{L}_2$ with a set of logical operators O. Moreover, $\mathscr{L}_2$ contains $\leftrightarrow$. Given is one semantics for $\mathscr{L}_2$ in some class of models. Given is a set* A *of reduction axioms for O such that every formula which is not in $\mathscr{L}_1$ has at least one subformula φ such that there is a formula ψ and $\varphi \leftrightarrow \psi$ is in* A. *If $\varphi \leftrightarrow \varphi$, the reduction axioms* A *and the rule of substitution of equivalents are sound for $\mathscr{L}_2$, then $\mathscr{L}_1$ and $\mathscr{L}_2$ have equal expressivity.*

PROOF. — It is given that $\mathscr{L}_1$ is a sublanguage of $\mathscr{L}_2$. So it is clear that $\mathscr{L}_2 \succeq \mathscr{L}_1$. In order to show that $\mathscr{L}_1 \succeq \mathscr{L}_2$ we have to prove that for every formula $\varphi \in \mathscr{L}_2$, there is a formula $\psi \in \mathscr{L}_1$ such that $\models \varphi \leftrightarrow \psi$. We show this by induction on the O depth. If the O depth is 0, then $\varphi \in \mathscr{L}_1$. It is clear that $\models \varphi \leftrightarrow \varphi$.

Suppose as induction hypothesis that for every $\varphi \in \mathscr{L}_2$ with $Od(\varphi) \leq n$, then there is a $\psi \in \mathscr{L}_2$ such that $\models \varphi \leftrightarrow \psi$.

Suppose that $Od(\varphi) = n + 1$. Therefore φ contains at least one subformula of the form $\Box(\chi_1, \ldots, \chi_k)$ where $\Box \in O$. For all χ_i it holds that $Od(\chi_i) \leq n$. Therefore, by the induction hypothesis for each χ_i there is a $\xi_i \in \mathscr{L}_1$ such that $\models \chi_i \leftrightarrow \xi_i$. By repeatedly applying the rule of substitution of equivalents one can show that $\models \Box(\chi_1, \ldots, \chi_k) \leftrightarrow \Box(\xi_1, \ldots, \xi_k)$. The O depth of $\Box(\xi_1, \ldots, \xi_k)$ is 1. Now by Lemma 9 there is a formula $\xi \in \mathscr{L}_1$ such that $\models \Box(\xi_1, \ldots, \xi_k) \leftrightarrow \xi$. Since an arbitrary subformula of φ was taken, one can repeatedly apply the rule of substitution of equivalents and find a formula in $\psi \in \mathscr{L}_1$ such that $\models \varphi \leftrightarrow \psi$. ■

3.3. *Completeness via reduction*

In the previous section it was shown how reduction axioms can be used to show that two languages are equally expressive: for every formula in the one language there exists an equivalent formula in the other language. The proof via reduction axioms was quite constructive. Given a set of reduction axioms one can find an equivalent

formula in the poorer language in a systematic way by repeatedly substituting subformulas according to reduction axioms, all the time decreasing the O reduction depth. Therefore if the reduction axioms and the rule of substitution of equivalents are added to a complete proof system for the poorer language, one obtains a complete proof system for the richer language, because the reduction can now take place within the proof system. In this way a *provably* equivalent formula is found. The proof of completeness is quite similar to the case of expressivity.

THEOREM 11. — *Given are two languages $\mathscr{L}_1$ and $\mathscr{L}_2$ such that $\mathscr{L}_2$ is an extension of $\mathscr{L}_1$ with a set of logical operators O. Moreover, $\mathscr{L}_2$ contains $\leftrightarrow$. Given is one semantics for $\mathscr{L}_2$ in some class of models. Given is a Hilbert style proof system PS which is sound and complete for $\mathscr{L}_1$ with respect to the given semantics and class of models. Given is a set A of reduction axioms for O such that every formula which is not in $\mathscr{L}_1$ has at least one subformula φ such that there is a formula ψ and $\varphi \leftrightarrow \psi$ is in A. If the proof system $\mathsf{PS} + \mathsf{A}$ together with $\varphi \leftrightarrow \varphi$ and the rule of substitution of equivalents (which we also refer to as $\mathsf{PS} + \mathsf{A}$) is sound for $\mathscr{L}_2$, then it is also complete for $\mathscr{L}_2$.*

PROOF. — Analogous to the proof of Theorem 10, we can show that by for every formula $\varphi \in \mathscr{L}_2$, there is a formula $\psi \in \mathscr{L}_1$ such that $\vdash_{\mathsf{PS+A}} \varphi \leftrightarrow \psi$. The proof is by induction on $Od(\varphi)$, where the induction step is an induction on $Ord(\varphi)$. We do not provide details.

To prove completeness, suppose that $\models \varphi$ for a formula in $\mathscr{L}_2$. There is a $\psi \in \mathscr{L}_1$ such that $\vdash_{\mathsf{PS+A}} \varphi \leftrightarrow \psi$. By the soundness of $\mathsf{PS} + \mathsf{A}$ it follows that $\models \psi$. By completeness for $\mathscr{L}_1$ of PS it follows that $\vdash_{\mathsf{PS}} \psi$. Since a proof in PS is also a proof in $\mathsf{PS} + \mathsf{A}$, it follows that $\vdash_{\mathsf{PS+A}} \psi$ as well. By the rule of substitution of equivalents it follows that $\vdash_{\mathsf{PS+A}} \varphi$. ■

4. Reducing public updates

In this section I will apply the results obtained in the previous section to some of the logics that were defined in Section 2. In order to apply the results we need:

1) semantics for the relevant sublanguages of $\mathscr{L}_{APSCR}$,
2) sound and complete Hilbert style proof systems for the relevant sublanguages of $\mathscr{L}_{APSCR}$,
3) soundness of the rule of substitution of equivalents, and
4) a set of reduction axioms.

The semantics for the entire language $\mathscr{L}_{APSCR}$ has been provided in Section 2, and thereby also for all its sublanguages. Fortunately, the literature provides Hilbert style proof systems for the logics without public updates. See (Fagin *et al.*, 1995; Meyer *et al.*, 1995) for systems for $\mathscr{L}_A$ and $\mathscr{L}_{AC}$, and see (Kooi *et al.*, 2004) for a proof system for $\mathscr{L}_{AR}$. So all that remains to be shown is that the rule of substitution of equivalents is sound. Moreover, we need to provide a set of reduction axioms, especially for the public announcement operator $[\varphi]$ and the substitution operator $[\sigma]$.

Let us briefly discuss the earlier completeness and expressivity results regarding these logics. Plaza (1989) introduced $\mathscr{L}_{AP}$ and provided a sound and complete proof system for it. Indeed Plaza used reduction axioms and showed that $\mathscr{L}_{A}$ and $\mathscr{L}_{AP}$ are equally expressive, thus obtaining an easy completeness proof via completeness for $\mathscr{L}_{A}$. The fact that $\mathscr{L}_{AC}$ is more expressive than $\mathscr{L}_{A}$ is folklore. A complete proof system for $\mathscr{L}_{AC}$ was obtained by adapting the results on propositional dynamic logic, of which the most readable completeness proof is considered to be Kozen and Parikh (1981). Baltag *et al.* (1999) showed, contrary to what was expected given Plaza's result, that $\mathscr{L}_{APC}$ is more expressive than $\mathscr{L}_{AC}$. This makes a completeness proof for $\mathscr{L}_{APC}$ much harder, and one cannot make do with just reduction axioms. Yet a proof system for $\mathscr{L}_{APC}$ is provided by Baltag *et al.* (1999). Kooi and van Benthem (2004) provided a complete proof system for $\mathscr{L}_{AR}$, also based on the paper by Kozen and Parikh (1981), and it was shown that $\mathscr{L}_{AR}$ and $\mathscr{L}_{APR}$ are equally expressive by reduction axioms. It was established that $\mathscr{L}_{AR}$ is more expressive than $\mathscr{L}_{APC}$ by van Benthem *et al.* (2005). These results are shown in Figure 2 together with the new results obtained in this section.

All the new results regarding expressivity and completeness of these logics except completeness for $\mathscr{L}_{APSC}$ (see Section 5) will be dealt with using the following reduction axioms.

DEFINITION 12 (REDUCTION AXIOMS). —

1) $[\varphi]p \leftrightarrow p$
2) $[\varphi]\neg\psi \leftrightarrow \neg[\varphi]\psi$
3) $[\varphi](\psi \wedge \chi) \leftrightarrow ([\varphi]\psi \wedge [\varphi]\chi)$
4) $[\varphi][a]\psi \leftrightarrow [a](\varphi \rightarrow [\varphi]\psi)$
5) $[\varphi][(B;?\psi)^{+}]\chi \leftrightarrow [(B;?(\varphi \wedge [\varphi]\psi))^{+}][\varphi]\chi$
6) $[\sigma]p \leftrightarrow \sigma(p)$
7) $[\sigma]\neg\varphi \leftrightarrow \neg[\sigma]\varphi$
8) $[\sigma](\varphi \wedge \psi) \leftrightarrow ([\sigma]\varphi \wedge [\sigma]\psi)$
9) $[\sigma][a]\varphi \leftrightarrow [a][\sigma]\varphi$
10) $[\sigma][B^{+}]\varphi \leftrightarrow [B^{+}][\sigma]\varphi$
11) $[\sigma][(B;?\varphi)^{+}]\psi \leftrightarrow [(B;?[\sigma]\varphi)^{+}][\sigma]\psi$
12) $[B^{+}]\varphi \leftrightarrow [(B;?\top)^{+}]\varphi$
13) $[(B;?\varphi)^{+}]\psi \leftrightarrow [p := \psi][\varphi][B^{+}]p$ *where p does not occur in φ.*

Although these axioms are called reduction axioms, they are not reduction axioms in themselves, but, following Definition 7, only relative to some set of logical operators. Indeed, in some cases (such as in the proof system for $\mathscr{L}_{APC}$) they cannot be construed as reduction axioms. Below it will be clear that in the proper context they are reduction axioms for their leftmost logical operator. One can immediately see that, in that case, the O reduction depth is strictly less on the right hand side of the equivalence. Axioms 1, 6, 12 and 13 are unlike the other reduction axiom in that they directly reduce the O depth (thereby reducing the O reduction depth). Axioms 1 and 6 might well be dubbed elimination axioms, since there is one less modal operator on the right hand side. Remember that in axiom 6 we abuse the language such that $\sigma(p)$

refers to the formula assigned to p if $p \in \mathrm{dom}(\sigma)$, and to refer to p otherwise. One might say axioms 12 and 13 are translation axioms, because operators are replaced. That p does not occur in φ is called a *freshness condition*. This kind of condition also occurs in the axioms for quantifiers in first order logic. In order to apply the theorems of the previous section, it needs to be established that these axioms are sound.

LEMMA 13. — *All reduction axioms are sound.*

PROOF. — For the soundness of reduction axioms 1–4 I refer to Plaza (1989). For the soundness of reduction axiom 5 I refer to Kooi and van Benthem (2004). In all the proofs below we use the semantics provided in Definition 3.

6) $(M, w) \models [\sigma]p$ iff $(M^\sigma, w) \models p$. The latter is the case iff $w \in V^\sigma(p)$. This is the case iff $(M, w) \models \sigma(p)$.

7) $(M, w) \models [\sigma]\neg\varphi$ iff $(M^\sigma, w) \models \neg\varphi$. The latter is the case iff $(M^\sigma, w) \not\models \varphi$. This is the case iff $(M, w) \not\models [\sigma]\varphi$, which is equivalent to $(M, w) \models \neg[\sigma]\varphi$.

8) $(M, w) \models [\sigma](\varphi \wedge \psi)$ iff $(M^\sigma, w) \models (\varphi \wedge \psi)$. The latter is the case iff $(M^\sigma, w) \models \varphi$ and $(M^\sigma, w) \models \psi$, which is equivalent to $(M, w) \models [\sigma]\varphi$ and $(M, w) \models [\sigma]\psi$. This is equivalent to $(M, w) \models [\sigma]\varphi \wedge [\sigma]\psi$.

9) $(M, w) \models [\sigma][a]\varphi$ iff $(M^\sigma, w) \models [a]\varphi$. The latter is the case iff $(M^\sigma, v) \models \varphi$ for all v such that $(w, v) \in R(a)$, which is equivalent to $(M, v) \models [\sigma]\varphi$ for all v such that $(w, v) \in R(a)$. This is equivalent to $(M, w) \models [a][\sigma]\varphi$.

10) $(M, w) \models [\sigma][B^+]\varphi$ iff $(M^\sigma, w) \models [B^+]\varphi$. The latter is the case iff $(M^\sigma, v) \models \varphi$ for all v such that $(w, v) \in R(B)^+$, which is equivalent to $(M, v) \models [\sigma]\varphi$ for all v such that $(w, v) \in R(B)^+$. This is equivalent to $(M, w) \models [B^+][\sigma]\varphi$.

11) $(M, w) \models [\sigma][(B; ?\varphi)^+]\psi$ iff $(M^\sigma, w) \models [(B; ?\varphi)^+]\psi$. The latter is the case iff $(M^\sigma, v) \models \psi$ for all v such that $(w, v) \in (R(B) \cap (W \times [\![\varphi]\!]_{M^\sigma})^+$, which is equivalent to $(M, v) \models [\sigma]\psi$ for all v such that $(w, v) \in (R(B) \cap (W \times [\![[\sigma]\varphi]\!]_M)^+$. This is equivalent to $(M, w) \models [(B; ?[\sigma]\varphi)^+][\sigma]\psi$.

12) Note that $R(B) \subseteq (W \times W)$ and that $[\![\top]\!] = W$. Therefore $R(B)^+ = (R(B) \cap (W \times [\![\top]\!]))^+$. $(M, w) \models [B^+]\varphi$ iff $(M, v) \models \varphi$ for all v such that $(w, v) \in R(B)^+$. Given the observation above, the latter is equivalent to $(M, v) \models v$ for all v such that $(w, v) \in (R(B) \cap (W \times [\![\top]\!]))^+$. This is equivalent to $(M, w) \models [(B; ?\top)^+]\varphi$.

13) Since p does not occur in φ, the substitution $p := \psi$ does not affect the extension of φ. Therefore $[\![\varphi]\!]_M = [\![\varphi]\!]_{M^{p:=\psi}}$. So $(M, w) \models [(B; ?\varphi)^+]\psi$ iff $(M, v) \models \psi$ for all v such that $(w, v) \in (R(B) \cap (W \times [\![\varphi]\!]_{M^{p:=\psi}}))^+$. Note that the relation $(R(B) \cap (W \times [\![\varphi]\!]_{M^{p:=\psi}}))^+$ is identical to $R^\varphi(B)^+$. Note also that $\models \psi \leftrightarrow [p := \psi]p$. Therefore $(M, w) \models [(B; ?\varphi)^+]\psi$ iff $(M, v) \models [p := \psi]p$ for all v such that $(w, v) \in R^\varphi(B)^+$, which is equivalent to $(M^{p:=\psi}, v) \models p$ for all v such that $(w, v) \in R^\varphi(B)^+$. This is equivalent to $(M^{p:=\psi}, w) \models [\varphi][B^+]p$, which is equivalent to $(M, w) \models [p := \psi][\varphi][B^+]p$. ■

Note that the rule of substitution of equivalents is sound for all the logics under consideration.

LEMMA 14. — *The rule of substitution of equivalents is sound.*

The proof of this lemma is left to the reader. It is not that difficult to show that this rule is derivable in K (see Hughes and Cresswell (1996, p.32)). The lemma follows by the soundness of the proof systems. It is also possible to show that this rule is derivable in all the systems we are going to consider (if we have necessitation and distribution for α), but since this would distract from the main line of the paper, we just add it to the proof systems.

4.1. *Expressivity of public updates*

Now that the soundness of the reduction axioms and the rule of substitution of equivalents is established, it is easy to obtain expressivity results for a great number of logics using the reduction axioms. See Figure 2 for a graphic representation of these results together with previously established results.

In this paper only the equal expressivity of languages is directly shown. The fact that some languages are more expressive than others follows from these new results combined with previously obtained results.

THEOREM 15. —

1) $\mathscr{L}_A \equiv \mathscr{L}_{AP} \equiv \mathscr{L}_{AS} \equiv \mathscr{L}_{APS}$
2) $\mathscr{L}_{AC} \equiv \mathscr{L}_{ASC}$
3) $\mathscr{L}_{AR} \equiv \mathscr{L}_{APR} \equiv \mathscr{L}_{ASR} \equiv \mathscr{L}_{APSR} \equiv \mathscr{L}_{ACR} \equiv \mathscr{L}_{ASCR} \equiv \mathscr{L}_{APCR} \equiv \mathscr{L}_{APSCR} \equiv \mathscr{L}_{APSC}$

PROOF. — In all three cases above Theorem 10 applies. We have one semantics for $\mathscr{L}_{APSCR}$, and all languages under consideration are sublanguages of it. We already showed that all reduction axioms are sound as well as the rule for substitutions of equivalents (Lemma 13 and 14). All that remains to be shown is that for each formula in the richer language which is not in the poorer language there is a subformula for which there is a reduction axiom.

1) To see that $\mathscr{L}_A \equiv \mathscr{L}_{AP}$, let the set of reduction axioms A be reduction axioms 1–4 of Definition 12. It is easy to see that each formula in $\mathscr{L}_{AP}$ that is not in $\mathscr{L}_A$ contains a subformula for which there is a reduction axiom. An innermost nested occurrence of an announcement operator precedes a formula which is either a propositional variable, a negation, a conjunction, or a knowledge formula. For each of these cases there is a reduction axiom. Therefore, by Theorem 10, $\mathscr{L}_A \equiv \mathscr{L}_{AP}$.
To see that $\mathscr{L}_A \equiv \mathscr{L}_{AS}$, let the set of reduction axioms A be reduction axioms 6–9 of Definition 12. Again, it is easy to see that each formula in $\mathscr{L}_{AS}$ that is not in $\mathscr{L}_A$ contains a subformula for which there is a reduction axiom. Therefore, by Theorem 10, $\mathscr{L}_A \equiv \mathscr{L}_{AS}$.
To see that $\mathscr{L}_A \equiv \mathscr{L}_{APS}$, we simply take the union of the sets of reduction axioms above. Now one simply takes one of the innermost nested occurrences of a substitution *or* a public announcement operator to see that every formula in $\mathscr{L}_{APS}$ which is not in $\mathscr{L}_A$ contains a subformula for which there is a reduction axiom. Therefore, by Theorem 10, $\mathscr{L}_A \equiv \mathscr{L}_{APS}$.

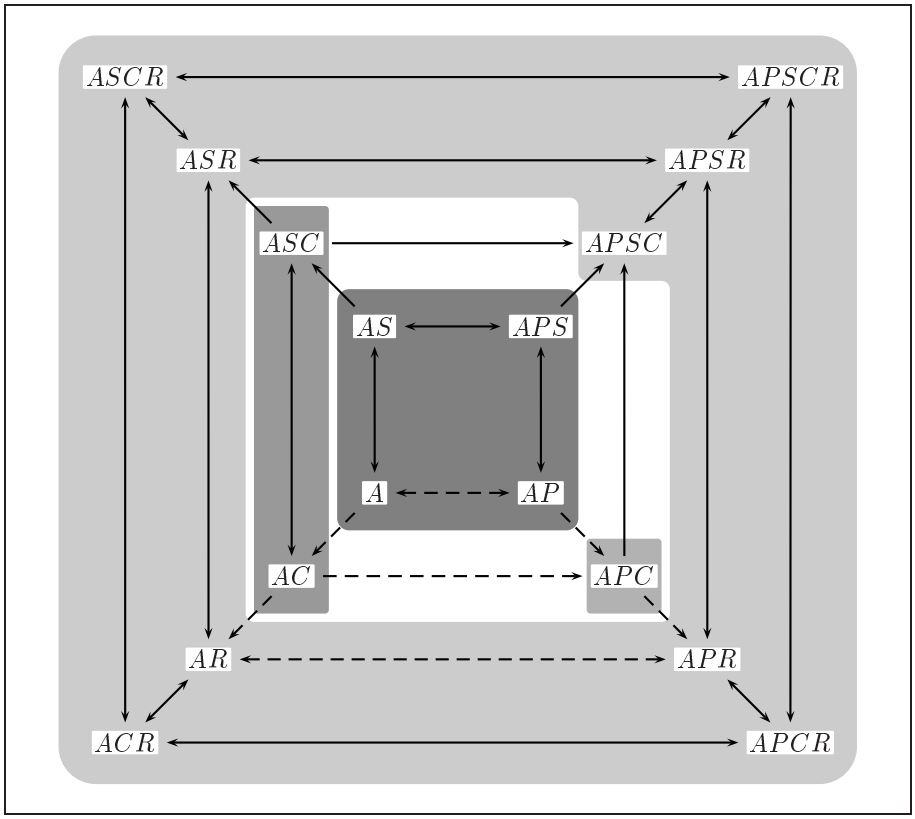

Figure 2. *An arrow $S \to T$ indicates that $\mathscr{L}_T$ is more expressive than $\mathscr{L}_S$. A double arrow $S \leftrightarrow T$ indicates that $\mathscr{L}_S$ is equally expressive as $\mathscr{L}_T$. The dashed arrows indicate previously established results. The black arrows indicate new results. For the sake of readability all reflexive arrows are omitted and not all transitive arrows are shown. The differently shaded areas indicate the equivalence classes. The lighter gray the area is, the more expressive the languages in it are*

2) Here we take reduction axioms 6–10 of Definition 12. From Theorem 10 it follows that $\mathscr{L}_{AC} \equiv \mathscr{L}_{ASC}$ by similar reasoning as above.

3) To see that $\mathscr{L}_{AR} \equiv \mathscr{L}_{APR} \equiv \mathscr{L}_{ASR} \equiv \mathscr{L}_{APSR}$ is completely analogous to the case $\mathscr{L}_A \equiv \mathscr{L}_{AP} \equiv \mathscr{L}_{AS} \equiv \mathscr{L}_{APS}$, except now axioms 5 and 11 of Definition 12 are used as well.
Using axiom 12 of Definition 12 it can be shown that $\mathscr{L}_{AR} \equiv \mathscr{L}_{ACR}$, that $\mathscr{L}_{APR} \equiv \mathscr{L}_{APCR}$, that $\mathscr{L}_{ASR} \equiv \mathscr{L}_{ASCR}$, and that $\mathscr{L}_{APSR} \equiv \mathscr{L}_{APSCR}$.
To see that $\mathscr{L}_{APSC}$ also belongs to this set of languages, observe that it can be shown that $\mathscr{L}_{APSC} \equiv \mathscr{L}_{APSCR}$ with axiom 13 of Definition 12. ■

The most surprising of these results is that $\mathscr{L}_{AR} \equiv \mathscr{L}_{APSC}$. The logic of relativised common knowledge was introduced in (Kooi *et al.*, 2004) for a rather technical

reason. The aim was to have a reduction axiom for public announcements and common knowledge. Now it turns out to correspond to a quite natural logic.

From Theorem 15 together with earlier results, it follows that $\mathscr{L}_{APSC} \prec \mathscr{L}_{APC}$, since $\mathscr{L}_{AR} \prec \mathscr{L}_{APC}$ and $\mathscr{L}_{AR} \equiv \mathscr{L}_{APSC}$. As can be seen in Figure 2 this is the only case where adding public substitutions to a language extends its expressive power. This is also quite surprising.

As an aside, observe that the substitution in translation axiom 13 is just one substitution, i.e. we do not need to change more propositional variables simultaneously. This raises the question whether one can just make do with single substitutions. This is indeed the case. Consider the scheme $[p := \varphi, \sigma]\psi \leftrightarrow [q := \varphi][\sigma][p := q]\psi$ where q does not occur in $[\sigma]\psi$. This formula is a tautology, and allows one to show that simple substitutions are equally expressive as simultaneous substitutions.

4.2. *Completeness for public updates*

There are two problems for a direct approach to proving completeness for update logics: modal logics with update operators are not *normal* modal logics[2] and modal logics with a transitive closure operator (such as (relativised) common knowledge) are not *compact*, i.e. it is not the case that an infinite set of formulas is satisfiable, if every finite subset of that infinite set.

Modal logics with update operators are not normal because the rule of uniform substitution is no longer sound. This rule allows one to substitute a propositional variable for an arbitrary formula uniformly. The idea behind uniform substitution is that if a formula is a tautology, then it is true in every model no matter what the extension of the propositional variables in the formula is. Therefore one can uniformly substitute a propositional variable for a complex formula, which also has a certain extension. In public update logics propositional variables play a special role. Their truth value is not effected by public announcements, although the truth value of complex formulas can be effected by them. Examples of such formulas are so-called *unsuccessful updates*: formulas that become false by their announcement (Gerbrandy, 1998; van Ditmarsch *et al.*, 2006), a concept closely related to Moore's paradox. Consider the tautology $[p][a]p$. If we replace p with $(p \wedge \neg[a]p)$ the result is the formula $[(p \wedge \neg[a]p)][a](p \wedge \neg[a]p)$. This is not a tautology. Hence the uniform substitution is unsound in this case. In the case of public substitution propositional variables also play a special role. Only the extension of propositional variables can be changed directly, not of complex formulas. Moreover, given that the extension of a propositional variable can be set to the extension of a complex formula by a public substitution, the extension of propositional variables cannot be seen as being arbitrary within the scope of a public assignment. Consider the tautology $[p := \top]q \leftrightarrow q$, where p and q are

2. See of Hughes and Cresswell (1996, p.25) or Blackburn *et al.* (2001, p.33) for the definition of normal modal logics.

different propositional variables. If we replace q with p, we get $[p := \top]p \leftrightarrow p$, which is not a tautology. So, the rule of uniform substitution is also unsound in this case. General methods for proving completeness for a modal logic are geared towards normal modal logics (for example Blackburn *et al.* (2001)). Therefore one cannot apply these methods directly to dynamic epistemic logics.

The other difficulty in providing completeness results for (dynamic) epistemic logics, is that when (relativised) common knowledge is in the language, the logic is no longer compact. Therefore one cannot easily construct a canonical model where the worlds are maximal consistent sets of formulas, because it can occur that an infinite set of formulas is consistent, but not satisfiable. This problem also occurs in propositional dynamic logic, where it is solved by making a finite canonical model, depending on the particular formula one is interested in (Kozen *et al.*, 1981). In this way only weak completeness is attained[3]. One can adopt a similar method for dynamic epistemic logics with common knowledge, as was done by Baltag *et al.* (1999).

Compared to a direct approach to completeness for dynamic epistemic logics, an approach with reduction axioms is much more straightforward. And given the generality of the approach we can easily deal with many logics simultaneously. We will reduce the logics under consideration to three base languages: $\mathscr{L}_A$, $\mathscr{L}_{AC}$ and $\mathscr{L}_{AR}$. As we remarked earlier, for these there are known complete Hilbert-style proof systems. Table 1 shows which reduction axioms for the additional operators should be added to which base system. The numbers refer to the reduction axioms in Definition 12. The extensions that are not considered are left blank[4].

Table 1. *The table indicates which reduction axioms are to be added to the base proof systems*

	P	S	C
$\mathscr{L}_A$	1–4	6–9	
$\mathscr{L}_{AC}$		6–10	
$\mathscr{L}_{AR}$	1–5	6–9,11	12

THEOREM 16. —

3. Strong completeness of a proof system PS with respect to a class of frames F is the property that $\Gamma \models_F \varphi$ implies that $\Gamma \vdash_{\mathsf{PS}} \varphi$ for every set of formulas Γ and every formula φ. This generalises weak completeness, where Γ is empty.

4. The cell in the upper right of the table is left blank, because adding common knowledge to $\mathscr{L}_A$ yields $\mathscr{L}_{AC}$, which is dealt with in the second row. The cell below is left blank because adding common knowledge to a language that already contains common knowledge does not make a difference. The cell in the middle left column of the table is left blank because adding public announcements to the language with common knowledge, yields a more expressive language, which can therefore not be dealt with using reduction axioms.

1) The proof system for $\mathscr{L}_A$ together with the appropriate reduction axioms from Table 1 and the rule of substitution of equivalents is complete for $\mathscr{L}_{AP}$, for $\mathscr{L}_{AS}$ and for $\mathscr{L}_{APS}$.

2) The proof system for $\mathscr{L}_{AC}$ together with reduction axioms 6–10 and the rule of substitution of equivalents is complete for $\mathscr{L}_{ASC}$.

3) The proof system for $\mathscr{L}_{AR}$ together with the appropriate reduction axioms from Table 1 and the rule of substitution of equivalents is complete for $\mathscr{L}_{APR}$, $\mathscr{L}_{ASR}$, $\mathscr{L}_{APSR}$, $\mathscr{L}_{ACR}$, $\mathscr{L}_{APCR}$, $\mathscr{L}_{ASCR}$, and $\mathscr{L}_{APSCR}$.

PROOF. — In order to prove all these results Theorem 11 is applied. We already showed that all the reduction axioms and the rule of substitution of equivalents are sound. From the literature, complete proof systems for $\mathscr{L}_A$, $\mathscr{L}_{AC}$ and $\mathscr{L}_{AR}$ were obtained. In the same way as was shown in the proof of Theorem 15, we can show that in each case a formula in the richer language contains a subformula to which a reduction axioms applies. Therefore by Theorem 11 all the proof systems are complete. ■

5. A complete proof system for $\mathscr{L}_{APSC}$

The only new result that cannot be obtained using the reduction axioms given in the previous section is a complete proof system for $\mathscr{L}_{APSC}$. In the proof that $\mathscr{L}_{AR} \equiv \mathscr{L}_{APSC}$ I showed that $\mathscr{L}_{APSC} \equiv \mathscr{L}_{APSCR}$ where $\mathscr{L}_{APSCR}$ was reduced to $\mathscr{L}_{APSC}$. Since $\mathscr{L}_{APSCR}$ also reduces to $\mathscr{L}_{AR}$ it followed that $\mathscr{L}_{AR} \equiv \mathscr{L}_{APSC}$. So $\mathscr{L}_{APSC}$ was *not reduced* to $\mathscr{L}_{AR}$. Such a reduction is in fact impossible, since neither language is a sublanguage of the other. This example leads to a more general question how one might obtain a complete proof system for one language by using a known proof system for an equally expressive logic, but neither is a sublanguage of the other. In Section 6 we return to this question. In this section we solve a particular problem of this kind.

A complete proof system for $\mathscr{L}_{APSC}$ can also be constructed based on the observation that $\mathscr{L}_{APSC}$ is equally expressive as $\mathscr{L}_{AR}$. The way to do it is as follows. There is a complete proof system for $\mathscr{L}_{AR}$ that is also complete for $\mathscr{L}_{APSCR}$ if it is extended with the appropriate reduction axioms. The difference between the language $\mathscr{L}_{APSCR}$ and $\mathscr{L}_{APSC}$ is that the latter does not contain relativised common knowledge, but there is a reduction axiom for it (reduction axiom 13). The idea is that if we apply this reduction axiom to the proof system for $\mathscr{L}_{APSCR}$ we obtain a complete proof system for $\mathscr{L}_{APSC}$. In other words, we let $[p := \psi][\varphi][B^+]p$ play the role of $[(B; ?\varphi)^+]\psi$ and thus adapt the proof system for $\mathscr{L}_{APSCR}$. Every occurrence of $[(B; ?\varphi)^+]\psi$ is replaced by $[p := \psi][\varphi][B^+]p$ and the freshness of p is set as a side condition. In this way the following proof system presents itself.

DEFINITION 17. — *The proof system* APSC *consists of reduction axioms 1–4 and 6–9 from Definition 12 together with the rule of substitution of equivalents and the following axioms and rules.*

1) all instantiations of propositional tautologies

2) $[\alpha](\varphi \to \psi) \to ([\alpha]\varphi \to [\alpha]\psi)$
3) $[p := \psi][\varphi][B^+]p \leftrightarrow [B](\varphi \to (\psi \wedge [p := \psi][\varphi][B^+]p))$
where p does not occur in φ.
4) $[p := (\psi \to [B](\varphi \to \psi))][\varphi][B^+]p \to ([B](\varphi \to \psi) \to [p := \psi][\varphi][B^+]p)$
where p does not occur in φ.
5) $[\varphi][p := \psi][\chi][B^+]p \leftrightarrow [p := [\varphi]\psi][\varphi \wedge [\varphi]\chi][B^+]p))$
where p does not occur in $[\varphi]\psi$.
6) $[\sigma][p := \varphi][\psi][B^+]p \leftrightarrow [p := [\sigma]\varphi][[\sigma]\psi][B^+]p))$
where p does not occur in $[\sigma]\psi$.
7) $[B^+]\varphi \leftrightarrow [p := \varphi][\top][B^+]p$
8) From φ and $\varphi \to \psi$, infer ψ
9) From φ, infer $[\alpha]\varphi$

Axioms 3 and 4 look really difficult, but close examination reveals that they are direct translations of the mix axiom and the induction axiom for relativised common knowledge[5] respectively. Axioms 5, 6 and 7 are direct translations of the reduction axioms 5, 11 and 12 from Definition 12 respectively.

THEOREM 18 (COMPLETENESS). — *For every $\varphi \in \mathscr{L}_{APSC}$ if $\models \varphi$, then $\vdash_{\mathsf{APSC}} \varphi$.*

PROOF. — Suppose $\models \varphi$, where $\varphi \in \mathscr{L}_{APSC}$. This formula is also in $\mathscr{L}_{APSCR}$. Therefore, by Theorem 16, there is a proof of this formula in the proof system for $\mathscr{L}_{APSCR}$ using the proof system for $\mathscr{L}_{AR}$ with the appropriate reduction axioms. With the proof system for $\mathscr{L}_{APSC}$ one can simulate this proof by replacing every expression of the form $[(B; ?\varphi)^+]\psi$ with $[p := \psi][\varphi][B^+]p$. So, indeed $\vdash_{\mathsf{APSC}} \varphi$. ■

6. Conclusion and further questions

In this paper dynamic epistemic logics with public announcements and public substitutions were studied. With these logics one can study speech acts and model other kinds of public information change, including learning information about the past. The focus of this paper is mainly on completeness and expressivity via reduction axioms. The general method given in Section 3 can actually be applied to other logics outside the field of dynamic epistemic logic as well. The results in Section 5 suggest that the method could also be extended to cases where one is presented with three languages $\mathscr{L}_1$, $\mathscr{L}_2$ and $\mathscr{L}_3$, where $\mathscr{L}_1 \subseteq \mathscr{L}_3$ and $\mathscr{L}_2 \subseteq \mathscr{L}_3$, and there are reduction axioms to reduce $\mathscr{L}_3$ both to $\mathscr{L}_1$ and $\mathscr{L}_2$. If a complete proof system is available for only $\mathscr{L}_1$, a complete proof system for $\mathscr{L}_2$ can be obtained by applying the reduction axioms for

5. The mix axiom and induction axiom are the following:

$$[(B; ?\varphi)^+]\psi \leftrightarrow [B](\varphi \to (\psi \wedge [(B; ?\varphi)^+]\psi))$$
$$[(B; ?\varphi)^+](\psi \to [B](\varphi \to \psi)) \to ([B](\varphi \to \psi) \to [(B; ?\varphi)^+]\psi)$$

See also (Kooi *et al.*, 2004).

$\mathscr{L}_2$ to the proof system for $\mathscr{L}_1$ extended with the reduction axioms that allowed the reduction of $\mathscr{L}_3$ to $\mathscr{L}_1$.

The method of using reduction axioms seems related to work on term rewriting systems, as is also indicated by Baltag *et al.* (1999). Reduction axioms can be seen as rewrite rules, and, interpreted in these terms Lemma 10 states that the term rewriting system terminates. In fact this follows from a general theorem from term rewriting that states that a term rewriting system terminates iff there exists a so-called reduction order. The order induced by the O reduction depth is such a reduction order. See Baader and Nipkow (1998, p.102–103) for a definition of reduction orders and the theorem. The connection between reduction axioms and term rewriting should be further explored.

As the results show, the logic $\mathscr{L}_{APSC}$ is really more expressive than $\mathscr{L}_{APC}$. Remarkably, this is the only example where the language with public substitutions is more expressive than the language without public substitutions. In all other cases the expressivity remained the same. It is still the case however, as the examples in Section 2 show, that it is very convenient to have these operators in the language.

It would be interesting to study the relation between the logics presented in this paper and the notion of update as it is studied in the field of belief revision (Katsuno *et al.*, 1992; Herzig *et al.*, 1999), where the term 'update' is given quite a different meaning than in dynamic epistemic logic. One receives the information that a formula φ has become true, and one has to adapt one's information state to accommodate this information. In terms of the logics presented in this paper such an update can best be conceived of as an announcement that some private substitution has occurred of which the *postcondition* is φ. In dynamic epistemic logic, announced formulas are taken as *preconditions* of the announcements.

If one were to generalise the notion of substitution to include private substitution and further enrich the language, it seems that the statement $\psi \in K \star \varphi$ regarding updates[6] in the belief revision literature would correspond to $[\bigcup\{\sigma \mid [\sigma]\varphi\}][a]\psi)$, i.e. after you learn that the world has somehow changed such that φ is now true, you know that ψ. When it is assumed that this change is minimal, the corresponding formulation would be $[\mu\{\sigma \mid [\sigma]\varphi\}][a]\psi)$, i.e. after you learn that the smallest change has occurred such that φ has become true, you know that ψ.

This perspective shows that there are different questions one may want to answer when the world changes.

- Given some preconditions and an action, what are the postconditions?
- Given an action and some postconditions, what are the preconditions?
- Given some preconditions and some postconditions, what actions enable this?

Dynamic epistemic logic tries to answer the first question. It seems that the approaches to update in the belief revision literature try to answer the last question: a question

6. The expression $\psi \in K \star \varphi$ indicates that ψ is in the knowledge base K after it has been updated with φ.

at the centre of computer science. Given an algorithmic problem, one knows what desired output is given the input, but not which algorithm implements the transition. The second question seems interesting from the point of diagnostics. It is known which program is running and what the results are, and one has to figure out what the initial conditions were. A systematic integrated account of all three questions certainly seems worthwhile.

7. References

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