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# Heinz Langer and his Work

Aad Dijksma and Israel Gohberg

## 1. Introduction

This volume is dedicated to Professor Heinz Langer to honor him for his outstanding contributions to mathematics. His results in spectral analysis and its applications, in particular in spaces with an indefinite metric, are fundamental. Five main themes emerge in Heinz Langer's work, some of them are closely connected or have much in common:

- (1) Spectral theory of operators in spaces with indefinite inner product.
- (2) Pencils of linear operators (nonlinear eigenvalue problems).
- (3) Extension theory of operators in spaces with indefinite inner product.
- (4) Block operator matrices.
- (5) One-dimensional Markov processes.

Heinz has written more than 130 research papers with 45 coauthors from 11 countries. He advised about 25 Ph.D. students and always enjoyed cooperation with colleagues, students and friends. As a teacher, Heinz has the ability to clarify connections and to point out the important. His work has numerous followers and great influence in the world centers of operator theory.

The occasion marking the origin of this book is Heinz Langer's sixtieth birthday. Two of his collaborators Martin Blümlinger and Fritz Vogl, with the help of Gabi Schuster, organized a two day Colloquium on Thursday and Friday, 12 and 13 October 1995, at the Technical University of Vienna. Friends and colleagues from all over the world attended. At the end of the conference it was decided to prepare this anniversary volume.

## 2. Biography of Heinz Langer

Heinz Langer was born on August 8, 1935 in Dresden. He went to school and Gymnasium there and attended the Technical University of Dresden from 1953 to 1958. Originally he wanted to study physics, but the selection principles in the then eastern part of divided Germany were against him. Thanks to some personal connections of a friend to a professor in mathematics, he was enrolled in

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mathematics. In 1958, after the diploma (with a thesis on perturbation theory of linear operators), staying at the TU Dresden as an assistant, he began studying linear operators in indefinite inner product spaces.

It was his teacher, Professor P. H. Müller, who recommended this topic: At a conference in Hungary he had listened to a lecture of János Bognár, and had encountered such questions in his investigations of operator polynomials. Heinz studied the fundamental papers by I. S. Iohvidov and M. G. Kreĭn, and, in the fall of 1959, he showed his results to Professor Szőkefalvi-Nagy from Szeged, who visited Dresden on his first trip abroad after the 1956 revolution in Hungary. Sz.-Nagy reacted positively but, since he did not consider himself a specialist in this field, recommended to send it to Professor M. G. Kreĭn. Heinz sent a handwritten (Kreĭn mentioned this later, and not only once!) manuscript to him. Its main result was a generalization of the Theorem of L. S. Pontrjagin about the existence of a maximal nonnegative invariant subspace of a self-adjoint operator in a Pontrjagin space to a Kreĭn space. This result attracted Kreĭn's attention, and so he invited Heinz to stay for one year in Odessa.

There existed exchange programs for graduate students between the German Democratic Republic (GDR) and the Soviet Union (mainly used by students from the GDR), and Heinz applied. Before being admitted one had to undergo a preparatory course at a special faculty of the University of Halle, which lasted one month in 1960. Heinz attended this course in the summer of 1960, but after two weeks he was sent home: He was found ideologically unsuitable for a longer stay in the Soviet Union.

In the fall of 1960 Heinz received his Ph.D. at the TU Dresden. In September 1961, after someone from the ministry had encouraged him to apply again for a stay in Odessa and the preparatory course had been shortened from one month to two weeks, students and graduate students separated, he was finally admitted to go to the University of Odessa on a post doc fellowship for one year. However, until he arrived in Odessa, Heinz did not know that M. G. Kreĭn did not work at the University of Odessa, but held the chair of Theoretical Mechanics at the Odessa Civil Engineering Institute.

At that time in Odessa each week there was a special lecture of M. G. Kreĭn and about three seminars at different institutions (regularly at the Civil Engineering Institute, the Pedagogical Institute and monthly at the House of Scientists). In the second part of his Ph.D. Thesis, Heinz had applied the Livšic-Brodskiĭ model for operators with finite-dimensional imaginary part in order to obtain a model for self-adjoint operators in Pontryagin spaces. So he was quite well acquainted not only with indefinite inner product spaces, but also with some of the other main topics of interest of the Odessa school of functional analysis when he arrived, and he could actively take part in it.

The intense mathematical life in the circle around M. G. Kreĭn to which his former students belonged, among them M. S. Brodskiĭ, M. L. Brodskiĭ, I. S. Iohvidov,

I. S. Kac, Ju. P. Ginzburg, Ju. L. Smul'jan, but also V. P. Potapov and L. A. Sakhnovic, deeply impressed Heinz and had a great influence on his entire carrier and interests. In the Introduction to his Habilitationsschrift Heinz describes M. G. Kreĭn's influence as follows: 'Wer das Glück hatte, eine längere Zeit in der Umgebung von Professor M. G. Krein in Odessa arbeiten zu können, weiß, welche Fülle von Gedanken und Anregungen er ständig ausstrahlt. Diese habe ich in wesentlich größerem Maße ausgenutzt, als in der Einleitung zum Ausdruck gebracht werden konnte.' Heinz's high regard for M. G. Kreĭn was reciprocated: M. G. Kreĭn considered Heinz one of his most brilliant students and collaborators.

This fruitful collaboration lasted for almost twenty five years and ended with the death of M. G. Kreĭn. M. G. Kreĭn has worked with many mathematicians. Of his joint publications, most are written with I. Gohberg, next in number come those with Heinz. In Odessa Heinz also met I. Gohberg for the first time. The friendship with him and with the doctoral students of M. G. Kreĭn of that time (V. M. Adamjan, D. S. Arov, V. A. Javrjan and S. N. Saakjan) lasts until today!

During this year in Odessa Heinz completed the main result of his thesis by adding a statement about the location of the spectrum of the restriction of the selfadjoint operator in this invariant subspace which was now the full generalization of Pontryagin's theorem. Jointly with M. G. Kreĭn he proved the existence of a spectral function (with certain critical points) for a selfadjoint operator in a Pontryagin space. In the following years (until 1965) Heinz showed the existence of a spectral function for the more general situation of a definitizable operator in a Kreĭn space. Thus, besides the existence of a maximal nonnegative invariant subspace, a second cornerstone of the spectral theory of selfadjoint and other classes of operators in Kreĭn spaces was laid.

In 1955 R. J. Duffin proved a result in connection with network theory about strongly damped selfadjoint second order matrix pencils  $\lambda^2 I + \lambda B + C$ , which M. G. Kreĭn ingeniously interpreted as the existence of a solution  $Z$  of the quadratic matrix equation  $Z^2 + BZ + C = 0$ . Heinz realized that the main result of his Ph.D. thesis, which was proved just as an abstract generalization without any application in mind, could easily be applied in order to get a corresponding result for the infinite dimensional case. This was worked out by M. G. Kreĭn and Heinz in the summer of 1962, and thus finally the spectral theory of selfadjoint operators in Kreĭn spaces found an important application. It should be added that at about the same time M. G. Kreĭn was working with I. Gohberg on the two books on nonselfadjoint operators and quite a few results of this theory also turned out to be useful for second order pencils.

The main results of the years 1961–1965 were summed up in Heinz's Habilitationsschrift. This Habilitationsschrift became well-known among people interested in spaces with an indefinite inner product and had a big impact on the development of the spectral theory of operators in such spaces. The results about the spectral function for definitizable operators were published without proof only in

1971 in [22], and the original proofs were published only seventeen years later in the Lecture Notes [63].

At this time, Heinz received an offer to work at the Mathematical Institute of the Academy of Sciences of the GDR, but he preferred to remain at the Technical University of Dresden as this also involved teaching and working with Ph.D. and post doctoral students, which he always liked and still likes to do.

The next important period in Heinz's development was his one year stay in Canada during the academic year 1966/67. After having met him at an operator theory conference in Balatonföldvár in Hungary in 1964, Professor I. Halperin invited Heinz to spend the academic year 1965/66 with a fellowship of the National Research Council of Canada at the University of Toronto. As was to be expected in these years of the Berlin wall, the authorities of the GDR did not allow Heinz to accept this invitation. However, Halperin insistently renewed it for the following year, and so Heinz could spend the academic year 1966/67 at the University of Toronto. Shortly before, Heinz had married, and in May 1967 his only daughter Henriette was born. In Canada he also met Peter Lancaster, with whom he shared interests not only in operator pencils, but also personal ones like skiing and hiking in the mountains.

After returning in 1967, Heinz was appointed Professor at the Technical University of Dresden. Shortly afterwards in 1968, with the '3rd Hochschulreform' in the GDR, research at the universities was reorganized. It turned out that officially there should be no research group in analysis at the Technical University of Dresden, but only groups in 'Numerik', 'Mathematische Kybernetik und Rechen-technik' and 'Stochastik'. Heinz joined the last group and became interested in semigroup theory and Markov processes, in particular one-dimensional Markov processes. A fairly wide class of such processes, which contains diffusion processes and birth and death processes, can be described by a second order generalized or Kreĭn-Feller derivative, which Heinz had come across in Odessa. Together with his students he considered in particular processes with nonlocal boundary conditions and the time reversal of such processes. Nevertheless, he continued to be interested in operators in indefinite inner product spaces. The disadvantage of the situation was that he could not lecture about the topics he liked best. Instead, he lectured on 'Semigroups', 'Spectral theory of Kreĭn-Feller differential operators', 'Markov processes' etc and the topics for graduate students also had to have a probabilistic touch. At this point, it turned out to be useful that Heinz was known abroad: From 1970 on some mathematicians from other countries came to Dresden to do their Ph.D. work with him (Pekka Sorjonen, Björn Textorius, Karim Daho, and later Branko Ćurgus, Muhamed Borogovac), which was, of course, in operator theory, the topic which always was his favourite.

In 1969 Heinz again visited M.G. Kreĭn for one month. By then the famous papers of Adamjan, Arov and Kreĭn were finished. Because of these results, it seemed to be necessary and promising to generalize the extension theory for

symmetric operators in Hilbert space to Pontryagin spaces. In fact, it was clear that this would give an operator theoretic approach to the Adamjan, Arov, Kreĭn results. This turned out to be an interesting and fruitful program: Already the generalization of the classical von Neumann-Kreĭn-Naimark extension theory showed new and interesting features in the indefinite case. In the following years also M. G. Kreĭn's theory of generalized resolvents, resolvent matrices and entire operators was extended to the indefinite situation.

In the seventies and early eighties Heinz visited Odessa quite regularly, sometimes officially, sometimes not quite officially, and then it was difficult to get the permission for the stay in Odessa. Sometimes the Rector of the Odessa Civil Engineering Institute could help. Another time, Heinz, without permission, just stayed at the apartment of Ju. L. Smul'jan, which was of course completely illegal and certainly a risk for the Smul'jan family.

During these years, besides the abstract lines of extension theory of symmetric operators, applications to indefinite moment problems, interpolation problems, to the continuation problem for hermitian functions with a finite number of negative squares, and to boundary eigenvalue problems were also studied. In the abstract results as well as in the applications the classical Hilbert space results were sometimes also completed, that is, the role of the  $Q$ -function in extension theory was worked out. These were the main topics of Heinz's work in these years, often done jointly with students and friends, mostly from outside Dresden or even outside the GDR. In addition, he also returned to work on operator pencils in the seventies. One of the main results he proved was the equivalence of a factorization of an  $n$ -th order pencil with the existence of a properly supported invariant subspace of the companion operator. At that time operator pencils were also studied by A. S. Markus and V. M. Matsaev. While Heinz used results from operator theory in indefinite inner product spaces as a tool, Markus and Matsaev applied results from factorization theory of analytic operator functions by I. Gohberg and J. Leiterer. It happened that on the same day in Kreĭn's seminar both methods were presented in a kind of friendly competition. However, only after Markus and Matsaev had moved to Israel and Heinz to Vienna, did the three of them start working together.

In the seventies Heinz could travel to the West almost every year, keeping the number of trips in balance with trips to the East. He lectured at the Universities of Jyväskylä, Uppsala, Linköping, Antwerp, and the KTH Stockholm during stays of a few weeks. Nevertheless, his applications for journeys to the West were not always successful, and it was impossible until the middle of the eighties to go to conferences there. In one case the permission to attend was granted only after the conference was over. While abroad he usually contacted Israel Gohberg and other friends (which was forbidden by the GDR authorities). Once Israel sent a letter for Heinz to Sweden, which arrived only after he had left. It was forwarded by the secretary to Dresden. Luckily someone gave him the open letter before it was read

by some department officials (which was the rule for mail from abroad), and so it did not cause Heinz any problems.

In the eighties an intense collaboration developed with Aad Dijksma and Henk de Snoo from Groningen. They studied thoroughly the classes of analytic functions which arose in the extension theory of symmetric operators in spaces with indefinite inner product and applied this theory in order to get a unified treatment for selfadjoint boundary eigenvalue problems of ordinary differential operators containing the eigenvalue parameter in the boundary condition. On the invitation of Rien Kaashoek, he also regularly visited the Vrije Universiteit Amsterdam. All these contacts with colleagues from operator theory, which usually grew into friendship, were very important for Heinz's work. Also at that time, a result of R. Beals appeared about the half range completeness of Sturm-Liouville operators with an indefinite weight. Under the influence of Åke Pleijel, Heinz had considered such problems with Karim Daho already in the seventies in the context of the Kreĭn space generated by the weight function. So he understood that Beals' result could be interpreted as the regularity of the critical point infinity of the spectral function of the selfadjoint operator which can be associated with the problem in this Kreĭn space. This was further elaborated in cooperation with Branko Ćurgus.

In January 1988 Heinz was allowed for the first time to accept an invitation to West Germany: Reinhard Mennicken had invited him to spend one month at the University of Regensburg. They started joint work on the connections of operator pencils and special functions. Following an idea of F. W. Schäfke, certain systems of special functions were interpreted as eigenfunction systems of pencils of differential operators. They also began studying block operator matrices, which has been another main topic of Heinz's interests since then. The problem is to express the spectral properties of an operator, acting in the product of two spaces and given as a block operator matrix, by the properties of the entries of this matrix.

At the beginning of October 1989, shortly before the fall of the Berlin wall, Heinz fled from the GDR and went to West Germany. His first contact point was Regensburg. The decision to leave had ripened for a longer time and was certainly hard: a secure position at the university, pupils, friends and a part of his life had to be left behind. However, the pressure was stronger. Thanks to the assistance and intercession of Albert Schneider, Heinz obtained first a one year position as a professor at the University of Dortmund, and then, with the support of Reinhard Mennicken, a professorship at the University of Regensburg. Since August 1991 Heinz has held a chair in 'Anwendungsorientierte Analysis' at the Technical University of Vienna. Released from the psychological tension of life in the GDR, Heinz's life and work has come to a new blossoming. Within the last seven years he has organized three workshops on operator theory and its applications in Vienna, one of them in cooperation with the Schrödinger Institute, and he enjoys attending conferences all over the world. He has created a center

of active research in operator theory in Vienna, attracting visitors from many countries, and still keeping a nice balance between those coming from the West and those from the East.

### 3. Some main results

In this section we explain some of Heinz's main results in detail. We focus on some theorems from the first three themes mentioned in the Introduction and relate them to the work of others, but first we recall some definitions.

An inner product space  $(\mathcal{K}, [\cdot, \cdot])$  is called a Kreĭn space if  $\mathcal{K}$  is a complex linear space which has a fundamental decomposition with respect to the inner product  $[\cdot, \cdot]$ , that is, a decomposition of the form

$$\mathcal{K} = \mathcal{K}_+ [\dot{+}] \mathcal{K}_-,$$

where  $[\dot{+}]$  denotes the direct  $[\cdot, \cdot]$ -orthogonal sum and  $(\mathcal{K}_\pm, \pm[\cdot, \cdot])$  are Hilbert spaces. The fundamental decomposition induces a Hilbert space inner product on  $\mathcal{K}$ , given by

$$(x, y) = [x_+, y_+] - [x_-, y_-], \quad x = x_+ + x_-, \quad y = y_+ + y_-, \quad x_\pm, y_\pm \in \mathcal{K}_\pm.$$

The operator  $J = P_+ - P_-$ , where  $P_\pm$  is the  $(\cdot, \cdot)$ -orthogonal projection onto  $\mathcal{K}_\pm$ , is called the fundamental symmetry corresponding to the fundamental decomposition. Note  $(x, y) = [Jx, y]$ ,  $x, y \in \mathcal{K}$ . Although the fundamental decomposition is not unique, different ones generate equivalent Hilbert space norms. Topological notions refer to this Hilbert space topology. For example, a subspace of  $\mathcal{K}$  is a linear manifold in  $\mathcal{K}$  which is closed and continuity of an operator means continuity with respect to this norm topology, etc. We denote by  $\mathbf{L}(\mathcal{K})$  the set of bounded linear operators on  $\mathcal{K}$ .

The numbers  $\dim \mathcal{K}_\pm$ , each either a nonnegative integer or infinity, do not depend on the fundamental decomposition  $\mathcal{K} = \mathcal{K}_+ [\dot{+}] \mathcal{K}_-$  of  $\mathcal{K}$ . If  $\dim \mathcal{K}_+ = 0$   $\mathcal{K}$  is sometimes called an anti-Hilbert space. The Kreĭn space  $(\mathcal{K}, [\cdot, \cdot])$  is called a  $\pi_\kappa$ -space or a Pontryagin space of index  $\kappa$  if  $\kappa := \min(\dim \mathcal{K}_+, \dim \mathcal{K}_-) < \infty$ . In the sequel we consider only Pontryagin spaces for which  $\kappa = \dim \mathcal{K}_-$ .

A linear subset is called nonnegative if its elements  $x$  have a nonnegative self inner product:  $[x, x] \geq 0$ ; a nonpositive subset is defined in a similar way.

The linear operators, which we consider in the Kreĭn space  $\mathcal{K}$ , will in general be densely defined and closed or closable. If  $A$  is a densely defined operator then its adjoint  $A^+$  is defined as follows:  $\text{dom } A^+$  is the set of all  $u \in \mathcal{K}$  for which there exists a  $v \in \mathcal{K}$  with

$$[Ax, u] = [x, v] \text{ for all } x \in \text{dom } A,$$

and in this case  $A^+u = v$ . We have  $A^+ = JA^*J$ , where  $A^*$  is the adjoint of  $A$  with respect to the Hilbert space inner product  $[J\cdot, \cdot]$ . The operator  $A$  in the Kreĭn space  $\mathcal{K}$  is called selfadjoint if  $A = A^+$ , symmetric if  $A \subseteq A^+$ , unitary if



$A^+A = AA^+ = I$ , isometric if  $A^+A = I$ , contractive if  $[Ax, Ax] \leq [x, x]$  and expansive if  $[Ax, Ax] \geq [x, x]$ , for all  $x \in \mathcal{K}$ .

As mentioned before Heinz started his work in indefinite inner product spaces by studying the two articles by I. S. Iokhvidov and M. G. Kreĭn that had appeared in 1956 and 1959. Later these papers were incorporated in the joint work [57] with Heinz. It is a clear and comprehensive introduction to spectral theory in Pontryagin spaces and contains the basic results, not only on hermitian and isometric operators, but also on contractive and expansive operators in Pontryagin spaces. The material about the latter classes of operators has recently been reconsidered and completed in the joint publication [105] with Tomas Azizov.

From this early period also dates the Hilbert space result called ‘‘Langer’s Lemma’’ (terminology from N.K. Nikol’skiĭ, *Treatise on the shift operator*, *Grundlehren der mathematischen Wissenschaften* 273, Springer-Verlag, Berlin, 1985) about the orthogonal decomposition of a Hilbert space contraction into a unitary part and a completely nonunitary part; see [2]. The same result was obtained independently by B. Sz.-Nagy and C. Foiaş.

(1) The maximal nonnegative invariant subspace theorem in Kreĭn spaces proved by Heinz reads as follows: If  $A$  is a selfadjoint operator in the Kreĭn space  $\mathcal{K}$  and for some fundamental symmetry  $J = P_+ - P_-$ ,  $\text{ran } P_- \subset \text{dom } A$  and  $P_+AP_-$  is compact, then  $A$  has a maximal nonnegative invariant subspace. In particular, a (bounded or unbounded) selfadjoint operator  $A$  in a Pontryagin space has a maximal nonnegative and a maximal nonpositive invariant subspace. Since the latter is finite dimensional,  $A$  has eigenvalues with corresponding nonpositive eigenvectors. This statement can be made more precise by considering multiplicities and the location of the eigenvalues with respect to the real axis, see [57].

The spectral theory of definitizable operators in Kreĭn spaces developed in the Habilitationsschrift ‘‘Spektraltheorie linearer Operatoren in  $J$ -Räumen und einige Anwendungen auf die Schar  $L(\lambda) = \lambda^2 + \lambda B + C$ ’’, TU-Dresden, 1965, is a cornerstone in the operator theory in spaces with an indefinite metric. The spectral function of a definitizable operator and the description of its behavior in the critical points are powerful tools in the abstract theory as well as for the investigation of operators in function spaces, such as differential operators. We review the main definitions and results:

The selfadjoint operator  $A$  in the Kreĭn space  $\mathcal{K}$  is called definitizable if the resolvent set  $\rho(A)$  of  $A$  is nonempty and there exists a polynomial  $p$  such that

$$[p(A)x, x] \geq 0, \quad x \in \text{dom } A^k,$$

where  $k$  is the degree of  $p$ ;  $p$  is called a definitizing polynomial of  $A$ . The set  $c(A)$  of critical points of  $A$  is the set of all  $\lambda \in \mathbf{R}$  such that  $p(\lambda) = 0$  for each definitizing polynomial  $p$  of  $A$ , and  $\infty \in c(A)$  if one (and hence each) definitizing polynomial is of odd degree and  $\sigma(A)$  contains arbitrarily large positive and negative numbers. It can be shown that  $c(A) \subseteq \sigma(A)$ , the spectrum of  $A$ . We denote by  $\mathcal{R}_A$  the

Boolean algebra generated by all intervals of  $\mathbf{R} \cup \{\infty\}$  whose endpoints are not in  $c(A)$ . Heinz proved that there exists a unique mapping  $E : \mathcal{R}_A \rightarrow \mathbf{L}(\mathcal{K})$  with the following properties ( $\Delta$  and  $\Delta'$  are arbitrary elements of  $\mathcal{R}_A$ ):

- (a)  $E(\Delta) = E(\Delta)^+$ ,  $E(\emptyset) = 0$ .
- (b)  $E(\Delta \cap \Delta') = E(\Delta)E(\Delta')$  (in particular,  $E(\Delta)$  is a projection).
- (c)  $E(\Delta \cup \Delta') = E(\Delta) + E(\Delta') - E(\Delta)E(\Delta')$ .
- (d)  $E(\mathbf{R}) = 1_{\mathcal{K}} - E_0$ , where  $E_0$  is the Riesz-Dunford projection associated with the nonreal spectrum  $\sigma(A) \setminus \mathbf{R}$  of  $A$ .
- (e) If  $p|_{\Delta} > 0$  (or  $p|_{\Delta} < 0$ ) for some definitizing polynomial  $p$  of  $A$  then  $E(\Delta)\mathcal{K}$  is a positive (negative, respectively) subspace.
- (f)  $E(\Delta) \in \{(A - z)^{-1}\}''$ , the double commutant of the resolvent  $(A - z)^{-1}$  of  $A$ ,  $z \in \rho(A)$ .
- (g) If  $\Delta$  is bounded then  $E(\Delta)\mathcal{K} \subseteq \text{dom } A$ ; if  $\Delta$  is unbounded then  $E(\Delta)\mathcal{K} \cap \text{dom } A$  is dense in  $E(\Delta)\mathcal{K}$ . In both cases  $\sigma(A|_{E(\Delta)\mathcal{K}}) \subset \Delta^c$ .
- (h) If  $A$  is bounded then

$$p(A) = \int_{\mathbf{R}} p(\lambda)E(d\lambda) + N,$$

where  $N$  is a bounded nonnegative operator in  $\mathcal{K}$  with  $N^2 = 0$ . (The integral here is improper with respect to the points  $c(A)$  as at these points  $E$  is not defined.)

The mapping  $E$  is called the spectral function (with critical points) of the definitizable operator  $A$ . The statement (e) implies that  $\mathcal{K}_{\Delta} = E(\Delta)\mathcal{K}$  is a Hilbert space or anti-Hilbert space if  $p$  is positive or negative on  $\Delta \cap \sigma(A)$  for some definitizing polynomial  $p$ . This space  $\mathcal{K}_{\Delta}$  reduces  $A$  and  $A|_{\mathcal{K}_{\Delta}}$  is a bounded or unbounded and densely defined Hilbert space selfadjoint operator in  $\mathcal{K}_{\Delta}$ . Thus, with the exception of the (finitely many) points in  $c(A)$ , the definitizable operator  $A$  has locally the same spectral properties as a selfadjoint operator in a Hilbert space. The critical points  $\lambda \in c(A)$  can be characterized as follows:

$$\Delta \in \mathcal{R}_A, \lambda \in \Delta \Rightarrow \text{the inner product } [\cdot, \cdot] \text{ is indefinite on } \mathcal{K}_{\Delta}.$$

In [22] and [12] the results from the Habilitationsschrift in a completed form were published. It was proved in [22] among other more general results that a bounded definitizable selfadjoint operator in a Kreĭn space has a maximal dual pair of invariant subspaces. The question of uniqueness of such pairs was considered in the note [18].

The paper [12] concerns selfadjoint operators in Kreĭn spaces that arise from a fundamentally reducible selfadjoint operator by perturbations of Matsaev class. Under some additional assumptions it was proved that these operators possess a local spectral function. This paper is the starting point for the study of locally definitizable operators in papers by Peter Jonas and in the joint work [125] with Alexander Markus and Vladimir Matsaev. In the latter paper the sign classification of the spectrum of a selfadjoint operator in a Kreĭn space is obtained with the help of approximating eigensequences. This new approach is applied to the study

of bounded and compact perturbations of selfadjoint operators in Kreĭn spaces. The main result in [125] on compact perturbations contains the result from [12] and has applications to the block operator matrices.

The papers [49] with Peter Jonas and [65] with Branko Najman deal with the perturbation theory of definitizable operators. In the first paper, for example, it is shown that within the class of selfadjoint operators, finite-rank perturbations of the resolvent preserve definitizability and that the emerging new critical points are of a special type. In the second paper, stability properties of the spectral function and its critical points are studied. In the paper [103] with Peter Jonas and Björn Textorius a model for an arbitrary selfadjoint operator in a Pontrjagin space is established. The model is closely related to selfadjoint differential operators with inner singularities arising in mathematical physics, presently under investigation with Aad Dijksma and Yuri Shondin.

(2) The spectral theory of selfadjoint operator pencils, of which Heinz is co-founder, is closely connected with indefinite metrics. The maximal nonnegative invariant subspace theorem has a large number of applications, for example, to the existence of an operator root and hence to the factorization of operator pencils. The joint papers [8] and [7] contain new ideas and methods which determined the development of this area for decades, and lead to new publications in spectral theory and in applications to mechanics and physics. To be more specific, with the pencil  $L(\lambda) = \lambda^2 + \lambda B + C$  there is associated the quadratic operator equation

$$Z^2 + BZ + C = 0$$

and M.G. Kreĭn and Heinz looked for a root  $Z$  whose spectrum coincides with a specified part of the spectrum of the pencil. This problem is closely connected with the problem of factorizing the pencil, that is, the problem of representing it in the form

$$L(\lambda) = (\lambda I - Y)(\lambda I - Z).$$

This approach can be used even when the pencil is not selfadjoint. But in the selfadjoint case  $B = B^*$ ,  $C = C^*$ , they proved that the quadratic equation has a root with the help of the invariant subspace theorem mentioned above applied to a certain companion matrix for the pencil. In the seventies Heinz proved general yet strong results on the factorization of operator polynomials of arbitrary degree; see [27], [30], [33], and [38]. For example, the operator polynomial

$$L(\lambda) = \lambda^n I + \lambda^{n-1} A_{n-1} + \cdots + \lambda A_1 + A_0$$

with operators  $A_j$  in a Hilbert space  $(\mathcal{H}, (\cdot, \cdot))$  admits a factorization

$$L(\lambda) = N(\lambda)M(\lambda), \quad M(\lambda) = \lambda^k I + \sum_{j=0}^{k-1} \lambda^j B_j, \quad N(\lambda) = \lambda^{n-k} I + \sum_{j=0}^{n-k-1} \lambda^j C_j,$$

if and only if the companion operator

$$\tilde{A} = \begin{pmatrix} -A_{n-1} & \dots & -A_1 & -A_0 \\ 1 & & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & & 1 & 0 \end{pmatrix}$$

acting in  $\tilde{\mathcal{K}} = \mathcal{H}^n$  has a specific invariant subspace. This result has been applied by many authors both for the selfadjoint and for the nonselfadjoint case. When the operators  $A_j$  are selfadjoint, the companion operator is selfadjoint with respect to the  $\tilde{G}$ -inner product  $(\tilde{G}\cdot, \cdot)$  on  $\tilde{\mathcal{K}}$ , where

$$\tilde{G} = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & & 1 & A_{n-1} \\ \vdots & \vdots & & & \vdots \\ 0 & 1 & & & A_2 \\ 1 & A_{n-1} & \dots & A_2 & A_1 \end{pmatrix}.$$

In examples the operators  $A_0, \dots, A_{n-1}$  are often unbounded. Sometimes  $\tilde{A}$  can also be considered in this situation, sometimes by a simple transformation the given pencil can be transformed into one with bounded operators.

Because of the formula

$$L(\lambda)^{-1} = Q(\tilde{A} - \lambda)^{-1}P, \quad P = \begin{pmatrix} I \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad Q = ( 0 \quad \dots \quad 0 \quad I ),$$

where  $P$  is a mapping from  $\mathcal{H}$  into  $\tilde{\mathcal{K}}$  and  $Q$  maps  $\tilde{\mathcal{K}}$  into  $\mathcal{H}$ , the companion matrix  $\tilde{A}$  is sometimes called the linearization of the pencil  $L(\lambda)$ . In the lecture series [120] other eigenvalue problems whose linearization lead to selfadjoint operators in Kreĭn spaces are discussed.

We also mention the following natural and beautiful result, which has a simple formulation but a complicated proof. We use the same notation as above. If  $L(\lambda)$  is a selfadjoint polynomial and for some segment  $[a, b]$  on the real axis,

$$L(a) \ll 0, \quad L(b) \gg 0, \quad L'(\lambda) \gg 0 \quad (a < \lambda < b),$$

then  $L(\lambda)$  admits the above factorization with  $k = 1$ ,  $M(\lambda) = \lambda I - Z$  and the spectrum of the operator  $Z$  lies in  $(a, b)$ . The operator  $Z$  not only has a real spectrum, but it is also similar to a selfadjoint operator.

Finally, in 1971–1973 Heinz studied the important class of weakly hyperbolic selfadjoint operator polynomials of arbitrary degree (or polynomials with real zeros) and proved theorems about their so-called spectral zones and factorizations. For the quadratic case this class is the class of “strongly damped pencils” and was considered earlier jointly with M.G. Kreĭn in [7] and [8].

(3) In the four “Fortsetzungsprobleme” papers [35], [40], [54], and [75] M.G. Kreĭn and Heinz formulate and study indefinite analogues of interpolation, moment and continuation problems. These papers contain a wealth of interesting results, which have subsequently been generalized by many authors. The indefiniteness comes in by requiring that certain kernels have  $\kappa$  negative squares,  $\kappa \in \{0, 1, \dots\}$ . A kernel  $\overline{K}$  on a nonempty set  $\Omega$  is a function  $K : \Omega \times \Omega \rightarrow \mathbf{C}$  which is hermitian:  $K(z, w) = \overline{K(w, z)}$ . It has  $\kappa$  negative squares on  $\Omega$  if for every natural number  $n$  and arbitrary points  $z_1, z_2, \dots, z_n \in \Omega$ , the hermitian matrix  $(K(z_i, z_j))_{i,j=1}^n$  has at most and for at least one choice of  $n, z_1, \dots, z_n$  exactly  $\kappa$  negative eigenvalues counting multiplicities. Special kernels yield special classes of functions; we mention two examples from [35] and [75]:

(a) A function  $Q$  belongs to the class  $N_\kappa$  of generalized Nevanlinna functions if it is meromorphic on  $\mathbf{C}^+$  and the kernel

$$N_Q(z, w) = \frac{Q(z) - \overline{Q(w)}}{z - \bar{w}}$$

has  $\kappa$  negative squares. For  $\kappa = 0$ , the class  $N_0$  coincides with the class of Nevanlinna functions; by definition these functions are holomorphic on  $\mathbf{C}^+$  and have a nonnegative imaginary part there. By  $N_\kappa^+$  we denote the set of  $Q \in N_\kappa$  for which  $zQ(z) \in N_0$ .

Like Nevanlinna functions, the functions in class  $N_\kappa$  have an operator and an integral representation, they are given in [35]. The latter is rather complicated because  $N_\kappa$ -functions have singularities which account for the negative squares; those at a nonreal point are just poles, but the ones on the real axis may be embedded.

(b) A function  $f$  belongs to the class  $\mathcal{P}_\kappa$  if it is defined and continuous on  $\mathbf{R}$ ,  $f(t) = \overline{f(-t)}$ , and the kernel  $H_f(s, t) = f(s - t)$  has  $\kappa$  negative squares.

From the many interpolation, moment and continuation problems studied by M.G. Kreĭn and Heinz, we single out the following two. The Stieltjes moment problem:

Given a sequence  $(s_j)_{j=0}^\infty$  of complex numbers such that of the Hankel forms

$$\sum_{j,k} s_{j+k} x_j \bar{x}_k, \quad \sum_{j,k} s_{j+k+1} x_j \bar{x}_k$$

the first has  $\kappa$  negative squares and the second is nonnegative, find all  $Q \in N_\kappa^+$  such that

$$Q(z) \sim -\frac{s_0}{z} - \frac{s_1}{z^2} - \dots, \quad z = iy, \quad y \rightarrow \infty,$$

and the continuation problem:

Given the continuous function  $f : [-2a, 2a] \rightarrow \mathbf{C}$  such that  $f(t) = \overline{f(-t)}$ ,  $t \in [-2a, 2a]$  and  $H_f(s, t)$  has  $\kappa$  negative squares on  $[-a, a]$ , find all  $\tilde{f} \in \mathcal{P}_\kappa$  such that

$$\tilde{f}(t) = f(t), \quad t \in [-2a, 2a].$$

For  $\kappa = 0$ , these problems were studied before by A.I. Akhiezer and M.G. Kreĭn, but even when restricted to this case some of the results in the Fortsetzungsprobleme were new.

The conditions on the data are necessary and sufficient for the existence of a solution, and there is either one solution or there are infinitely many solutions.

If the moment problem has infinitely many solutions, a  $2 \times 2$  matrix function  $W(z) = (w_{ij}(z))_{i,j=1}^2$  exists such that the formula

$$Q(z) = \frac{w_{11}(z)N(z) + w_{12}(z)}{w_{21}(z)N(z) + w_{22}(z)}$$

gives a one-to-one correspondence between all solutions  $Q(z)$  and all functions  $N(z) \in N_0^+ \cup \{\infty\}$ .

A similar result holds for the continuation problem, but extra assumptions on the function  $f$  are needed: Assume that (i)  $f$  has an accelerant, that is, there is a hermitian function  $H \in L^2(-2a, 2a)$  such that

$$f(t) = f(0) - \frac{1}{2}|t| - \int_0^t (t-s)H(s)ds, \quad t \in [-2a, 2a],$$

and that (ii)  $-1$  does not belong to the spectrum of the integral operator  $\mathbf{H}$  on  $L^2(0, 2a)$  defined by

$$\mathbf{H}\varphi(t) = \int_0^{2a} H(t-s)\varphi(s)ds, \quad t \in [0, 2a].$$

Then if the continuation problem has infinitely many solutions, a  $2 \times 2$  matrix function  $\tilde{W}(z) = (\tilde{w}_{ij}(z))_{i,j=1}^2$  exists such that the formula

$$-i \int_0^\infty e^{-izt} \tilde{f}(t)dt = \frac{w_{11}(z)N(z) + w_{12}(z)}{w_{21}(z)N(z) + w_{22}(z)}, \quad \text{Im } z \leq -\gamma,$$

for some  $\gamma \geq 0$  gives a one-to-one correspondence between all solutions  $\tilde{f}$  and all functions  $N(z) \in N_0 \cup \{\infty\}$ .

Suitably normalized, the resolvent matrices  $W(z)$  and  $\tilde{W}(z)$  are unique; their entries are entire and have finite order. The matrix  $W(z)$  coincides essentially with the transmission matrix of a string that can be associated with the moment problem. The string has a special structure: besides positive masses, it also has a finite number of negative masses and certain new elements called dipoles; see [54], part II. Under certain conditions on the accelerant, the matrix  $\tilde{W}(z)$  is a solution of a Hamiltonian system of differential equations; see [75]. These results are closely related to a theorem of Louis de Branges in his theory of Hilbert spaces of entire functions.

The method M.G. Kreĭn and Heinz used to obtain the above fractional linear transformation representation of the solutions is based on the extension theory of a symmetric operator or isometric operator in a Pontryagin space, developed in for example [20],[21], [28], and [40]: The data of the problem at hand give rise to a symmetric operator  $S$  in a Pontryagin space  $\mathcal{P}$  of index  $\kappa$  and an element  $u$  from

$\mathcal{P}$ . The solutions correspond 1–1 to the  $u$ -resolvents  $[(\tilde{A} - z)^{-1}u, u]$  of  $S$ , where  $\tilde{A}$  runs through the class of selfadjoint extensions of  $S$  with nonempty resolvent set acting in spaces of the form  $\tilde{\mathcal{P}} = \mathcal{P} \oplus \mathcal{H}$ ,  $\mathcal{H}$  a Hilbert space. These  $u$ -resolvents can be written as a fractional linear transformation over the functions from the class  $N_0^+ \cup \{\infty\}$  or  $N_0 \cup \{\infty\}$ , depending on the problem.

Extension theory also entails the study of Straus extensions of a symmetric operator and the description of these involves unitary colligations and characteristic functions. For the indefinite case this has been worked out in, for example, [79], [80] and [82] with Aad Dijksma and Henk de Snoo and [91] also with Branko Ćurgus, and applied to the study of nonstandard boundary eigenvalue problems associated with Sturm-Liouville and Hamiltonian systems of differential operators; see [68], [85], [87], and [104]. Here nonstandard means that the boundary conditions contain the eigenvalue parameter. Earlier on eigenfunction expansions were obtained for the Hilbert space case in [62], [69], [74] with Björn Textorius using Kreĭn's method of directing functionals. Basis properties of the eigenfunctions for certain classes of boundary eigenvalue problems have been obtained recently with Reinhard Mennicken and Christiane Tretter in [121], [128] and [129].

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